Regularization meshless method for solving multiple scattering problems

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Abstract

In this paper, we employ the regularized meshless method (RMM) to search for multiple scattering problems. The solution is represented by using the double layer potentials. The source points can be located on the physical boundary not alike MFS after using the proposed technique to regularize the singularity and hypersingularity of the kernel functions. The troublesome singularity in the MFS methods is desingularized and the diagonal terms of influence matrices are determined by employing the subtracting and adding-back technique. Fictitious frequencies are filtered out by using singular value decomposition (SVD) updating term technique. The accuracy and stability of the RMM are verified through the numerical experiments of the Dirichlet and Neumann problems. The method is found to perform pretty well in comparison with analytical solutions and numerical results of boundary element method and the finite element method.

Keywords: Regularized meshless method, Helmholtz, subtracting and adding-back technique, fictitious frequencies, SVD updating term, Method of fundamental solution.
一、Introduction

In the scattering problem of infinite domain, the boundary element method (BEM) [24] had been widely used. Chen et al. [1] presents the occurring mechanism why irregular (fictitious) frequencies are imbedded in the exterior acoustics using the dual boundary element method. It is found that the irregular values depend on the kernels in the integral representation solution. Grote et al. [5] to solve multiple scattering problems employed FDM combined single DtN and FEM combined multiple DtN, respectively. In 2005, Chen et al. [2] employed boundary integer equations (BIE) and BEM to solve exterior Helmholtz equation subject to the mixed-type boundary condition. The CHIEF technique and Burton and Miller method were adopted to suppress the occurrence of the fictitious Frequency. Chen et al. [3] study radiation and scattering problems by using the null-field integral equations in conjunction with degenerate kernel and Fourier series. The semi-analytical solutions can be obtained in 2007. Also, fictitious frequencies were examined in the MFS for scattering problems.

In the MFS [7], the solution is approximated by a set of fundamental solutions of the governing equations which are expressed in terms of sources located outside the physical domain. However, the MFS is still not a popular method because of the debatable artificial boundary (fictitious boundary) distance of source location in numerical implementation especially for a complicated geometry. Young et al. [8, 9] proposed the novel meshless method, namely regularized meshless method (RMM), to deal with 2-D problems including the Laplace problem and Helmholtz problem of exterior acoustics. The subtracting and adding-back technique [4, 8, 9] can regularize the singularity and hypersingularity of the kernel functions. The diagonal terms of the influence matrices can be extracted out by using the proposed technique.

Following the success of previous applications [9], we investigate the scattering problem by using the RMM in this chapter. The rationale for choosing double-layer potential instead of the single-layer potential as used in the RMM for the form of RBFs is to take the advantage of the regularization of the subtracting and adding-back technique. A general-purpose program was developed to solve the multiple scattering problems of Laplace operator. Fictitious frequencies are filtered out by using SVD updating term technique [6]. Furthermore, the results will be compared with analytical solutions and those of BEM and FEM to show the validity of our method.

二、Governing equation and boundary conditions
Consider a boundary value problem with an acoustic pressure field \( u(x) \), which satisfies the Helmholtz equation as follows:

\[
(\nabla^2 + k^2)u(x) = 0, \quad x \in D,
\]

subject to boundary conditions,

\[
u(x) = \bar{u}, \quad x \in \bar{B}_p^0, \quad p = 1, 2, 3, \ldots, m
\]

\[
t(x) = \bar{t}, \quad x \in \bar{B}_q^i, \quad q = 1, 2, 3, \ldots, m
\]

where \( \nabla^2 \) is Laplacian operator, \( k \) is wave number, \( D \) is the domain of the problem, \( t(x) = \partial u(x)/\partial n_x \), \( m \) is the total number of boundaries including \( m-1 \) numbers of inner boundaries and one outer boundary (the \( m \)th boundary), \( \bar{B}_p^0 \) is the essential boundary (Dirichlet boundary) of the \( p \)th boundary in which the potential is prescribed by \( \bar{u} \) and \( \bar{B}_q^i \) is the natural boundary (Neumann boundary) of the \( q \)th boundary in which the flux is prescribed by \( \bar{t} \). Both \( \bar{B}_p^0 \) and \( \bar{B}_q^i \) construct the whole boundary of the domain \( D \).

### 三、Regularized meshless method

By employing the RBF technique \([6, 14]\), the representation of the solution for multiply-connected problem as shown in Fig. 1 (a) can be approximated in terms of the \( \alpha_j \) strengths of the singularities at \( s_j \) as

\[
u(x_i) = \sum_{j=1}^{N} \bar{T}(s_j, x_i) \alpha_j
\]

\[
= \sum_{j=1}^{N_1} \bar{T}(s_j, x_i) \alpha_j + \sum_{j=N_1+1}^{N_1+N_2} \bar{T}(s_j, x_i) \alpha_j + \cdots + \sum_{j=N_1+N_2+\cdots+N_{m-1}+1}^{N} \bar{T}(s_j, x_i) \alpha_j,
\]

\[
t(x_i) = \sum_{j=1}^{N} \bar{M}(s_j, x_i) \alpha_j
\]

\[
= \sum_{j=1}^{N_1} \bar{M}(s_j, x_i) \alpha_j + \sum_{j=N_1+1}^{N_1+N_2} \bar{M}(s_j, x_i) \alpha_j + \cdots + \sum_{j=N_1+N_2+\cdots+N_{m-1}+1}^{N} \bar{M}(s_j, x_i) \alpha_j,
\]

where \( x_i \) and \( s_j \) represent the \( i \)th observation point and the \( j \)th source point, respectively, \( \alpha_j \) are the \( j \)th unknown coefficients (strength of the singularity), \( N_1, N_2, \ldots, N_{m-1} \) are the numbers of source points on \( m-1 \) numbers of inner boundaries, respectively, \( N_m \) is the number of source points on the outer boundary, while \( N \) is the total numbers of source points.
\( N = N_1 + N_2 + \cdots + N_m \) and \( \overline{M}(s_j, x_j) = \partial \overline{T}(s_j, x_j) / \partial n_{x_j} \). The coefficients \( \{\alpha_j\}_{j=1}^N \) are determined so that BCs are satisfied at the boundary points. The distributions of source points and observation points are shown in Fig. 1 (a) for the MFS. The chosen bases are the double layer potentials \([28]\) as

\[
\overline{T}(s_j, x_j) = -\frac{i\pi k}{2} H_1^{(1)}(kr_j) \frac{n_k y_k}{r_{ij}},
\]

\[
\overline{M}(s_j, x_j) = \frac{i\pi k}{2} \{kH_2^{(1)}(kr_j) \frac{y_k y_j n_k}{r_{ij}^2} - H_1^{(1)}(kr_j) \frac{n_k n_j}{r_{ij}}\},
\]

where \( H_2^{(1)}(kr_j) \) is the second-order Hankel function of the first kind, \( r_{ij} = |s_j - x_j| \), \( n_j \) is the normal vector at \( s_j \), and \( \overline{n}_j \) is the normal vector at \( x_j \).

The fictitious distance between the fictitious (auxiliary) boundary (\( B' \)) and the physical boundary (\( B \)), defined by \( d \), shown in Fig. 1 (a) needs to be chosen deliberately. To overcome the above mentioned shortcoming, \( s_j \) is distributed on the real boundary, as shown in Fig. 1 (b). When the collocation point \( x_i \) approaches the source point \( s_j \), the potentials in Eqs. (6) and (7) are approximated by:

\[
\lim_{x_i \to x_j} \overline{T}(s_j, x_j) = T(s_j, x_j) = \frac{n_k y_k}{r_{ij}^2},
\]

\[
\lim_{x_i \to x_j} \overline{M}(s_j, x_j) = M(s_j, x_j) + \frac{k^2}{4} i = \left(2 \frac{y_k y_j n_k n_j}{r_{ij}^4} - \frac{n_k n_j}{r_{ij}^2}\right) \frac{k^2}{4} i,
\]

by using the limiting form for small arguments and the identities form the generalized function as shown below \([1]\)

\[
\lim_{\epsilon \to 0} H_1^{(1)}(kr_{ij}) = \frac{1}{\pi k r_{ij}},
\]

\[
\lim_{\epsilon \to 0} H_2^{(1)}(kr_{ij}) = \frac{1}{8} \left(\frac{k r_{ij}}{\pi}\right)^2 i.
\]

The kernels in Eqs. (8) and (9) have the same singularity order as the Laplace equation. Therefore, Eqs. (4) and (5) for the multiply-connected domain problems can be regularized by using the regularization of subtracting and adding-back technique \([12, 27, 28]\) as follows:

\[
u(x_i^j) = \sum_{j=1}^{N_i} \overline{T}(s_j^i, x_i^j) \alpha_j + \cdots + \sum_{j=N_i+1}^{N+1} \overline{T}(s_j^i, x_i^j) \alpha_j
\]
Similarly, the boundary flux is obtained as

\[
\begin{align*}
\sum_{j=1}^{N_r} \overline{\mathbf{T}}(s_j', x_j') \alpha_j + \cdots + \sum_{j=N_r+1}^{N} \overline{\mathbf{T}}(s_j', x_j') \alpha_j \\
+ \sum_{j=1}^{N_r} \overline{\mathbf{M}}(s_j', x_j') \alpha_j - \sum_{j=N_r+1}^{N} \overline{\mathbf{M}}(s_j', x_j') \alpha_j \\
+ \sum_{j=N_r+1}^{N} \overline{\mathbf{M}}(s_j', x_j') \alpha_j - \sum_{j=N_r+1}^{N} \overline{\mathbf{M}}(s_j', x_j') \alpha_j \\
\end{align*}
\]

\[x_j' \in B_p, \ p = 1, 2, 3, \ldots, m - 1.\]

By collocated $N$ observation points to match with the BCs from Eq. (12) for the Dirichlet problem, the linear algebraic equation is obtained

\[
\{ \tilde{\mathbf{T}} \} = [\mathbf{M}] \{ \alpha \}. \tag{14}
\]

For the Neumann problem, Eq. (13) yield

\[
\{ \tilde{\mathbf{F}} \} = [\mathbf{M}] \{ \alpha \}. \tag{15}
\]

### 四、Treatments of fictitious frequency

Fictitious frequencies are well as true one will be examined. In order to sort out the fictitious frequencies, the SVD updating term is utilized [6, 9]. We can combine Eqs. (14) and (15) by using the SVD updating term as follows:

\[
[\mathbf{P}] \{ \tilde{\mathbf{F}} \} = [\mathbf{M}] \{ \alpha \} = 0.
\tag{16}
\]

The rank of the matrix $[\mathbf{P}]$ must be smaller than $2N$ to have a fictitious mode [6]. By using the SVD technique, the matrix in Eq. (16) can be decomposed into

\[
[\mathbf{P}] = \begin{bmatrix}
\Phi_{\tilde{\mathbf{F}}} & 0 \\
0 & \Phi_{\mathbf{M}}
\end{bmatrix}
\begin{bmatrix}
\Sigma_{\tilde{\mathbf{F}}} & 0 \\
0 & \Sigma_{\mathbf{M}}
\end{bmatrix}
\begin{bmatrix}
\Psi_{\tilde{\mathbf{F}}} & 0 \\
0 & \Psi_{\mathbf{M}}
\end{bmatrix}^H,
\tag{17}
\]

Based on the equivalence between the SVD technique and the Least-squares method, we
extract out the fictitious frequencies by detecting zero singular value for $[P]$ matrix.

五、Numerical examples

For the scattering problem subject to the incident wave, this problem can be decomposed into two parts, (a) incident wave field and (b) radiation field. By matching the boundary condition, the radiation boundary condition in part (b) is obtained as the minus quantity of incident wave function, e.g. $t^n = -t^l$ for hard scatter or $u^n = -u^l$ for soft scatter, respectively. In order to show the accuracy and validity of the proposed method, the multiple scattering problems subjected to the Dirichlet and Neumann BCs are considered.

5.1 Single scatter

Plane wave scattering for a circular cylinder (Dirichlet or Neumann boundary condition) is considered in Fig. 2. The analytical solution of single scattering for Dirichlet and Neumann types, respectively, is shown below

$$u(r, \theta) = -\frac{J_0(ka)}{H_0^{(1)}(ka)} H_0^{(1)}(kr) - 2 \sum_{n=1}^{\infty} i^n \frac{J_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(kr) \cos(n\theta)$$  \hspace{1cm} (18)

$$u(r, \theta) = -\frac{J'_0(ka)}{H_0^{(1)}(ka)} H_0^{(1)}(kr) - 2 \sum_{n=1}^{\infty} i^n \frac{J'_n(ka)}{H_n^{(1)}(ka)} H_n^{(1)}(kr) \cos(n\theta)$$  \hspace{1cm} (19)

The results of Dirichlet boundary condition are shown in Fig. 3, when $ka = \pi$. Compared with Fig. 3 (a)–(c), the better results can be obtained. For the Neumann boundary condition, the numerical results of $ka = 4\pi$ are shown in Fig. 4 by using analytical solution, RMM and BEM, respectively. The fictitious frequency can be extraction employing SVD updating term technique and plotted in Fig. 5.

5.2 Double elliptic scatters

In this case, the two elliptic scatters under plane wave with incidence angle $\gamma = 3\pi/8$ is consider as shown in Fig. 6. The semi-major and semi-minor axes of the ellipses were chosen 0.75, 0.5 and 0.375, 0.25, respectively. In the Fig. 7, the numerical solution obtained by using our RMM. Good agreement is obtained after comparing with FEM combined with single-DtN condition and BEM.

六、Conclusions
In this study, we used the RMM to solve the multiple scattering problems subject to the Dirichlet and Neumann BCs, respectively. Only the boundary nodes on the physical boundary are required. The perplexing fictitious boundary in the MFS is then circumvented. Despite the presence of singularity and hypersingularity of double layer potentials, the finite values of the diagonal terms of the influence matrix can be extracted out by employing subtracting and add-back techniques. The fictitious frequencies have been extracted out by using SVD techniques of updating term. The numerical results were obtained by applying the developed program to three examples with different BCs and shapes of domain. Numerical results agreed very well with analytical solutions and those of BEM and FEM.

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References

Fig. 1 The distribution of the source points and observation points and definitions of $r, \theta, \rho, \phi$ by using the conventional MFS and the RMM for the multiply-connected problems: (a) Conventional MFS, (b) RMM.
Fig. 2 Problem sketch for case 1.

Fig. 3 (a) Fig. 3 (b) Fig. 3 (c)
Fig. 3 Single scatter with Dirichlet B.C. by using (a) Exact solution, (b) RMM, and (c) BEM, respectively.

Fig. 4 (a) Fig. 4 (b) Fig. 4 (c)
Fig. 4 Single scatter with Neumann B.C. by using (a) Exact solution, (b) RMM, and (c) BEM, respectively.

Fig. 5 (a) Fig. 5 (b)
Fig. 5 Fictitious frequency analysis for the case 1. (a) Dirichlet BC, (b) Nuemann BC, (c) SVD Updating term.

Fig. 6 Problem sketch for case 2.

Fig. 7 Results of double elliptic scatters by using (a) FEM combined with single-DtN condition, (b) RMM, (c) BEM.