The Method of Fundamental Solutions
– A Meshless Method

Edited by

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Dynamic Publishers
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Published by Dynamic Publishers
Preface

The first international workshop devoted entirely to the Method of Fundamental Solutions (MFS) was held in the resort town of Ayia Napa, Cyprus from 11-13 June 2007. The ideas behind the MFS have been around for many years and were developed primarily by V. D. Kupradze and M. A. Alexidze in the late 1950s and early 1960s. However, the method was proposed as a computational technique much later by R. Mathon and R. L. Johnston in the late 1970s. A number of papers by Mathon, Johnston and Graeme Fairweather with applications followed, and slowly but surely the MFS became a useful tool for the solution of a large variety of physical and engineering problems. A major obstacle was overcome when, in the 1990s, M. A. Golberg and C. S. Chen extended the method to deal with inhomogeneous equations and time-dependent problems. In addition, recent developments indicate that the MFS can also be used to solve partial differential equations with variable coefficients. The MFS has proved particularly effective for certain classes of problems such as inverse problems and free-boundary problems. The method today is applied to disciplines as varied as computer-aided design and biomedical engineering. The MFS is known by many different names in the literature. Among these are the charge simulation method, the superposition method, the desingularized method, the indirect boundary element method and the virtual boundary element method, to name a few.

The great advantage of the MFS over other methods is the ease with which it can be implemented for problems in three-dimensional and irregular domains. This factor is very important to people who actually compute, but unfortunately it is often overlooked. The MFS has become more widely used in the last decade because of the popularity gained by meshless methods in general. Clearly, there are still many limitations in the range of applications of the method as well as in its theoretical foundations and development. We felt, however, that the MFS had sufficiently matured as a numerical technique to deserve a meeting devoted entirely to
it. The purpose of this book is to present some of the recent significant developments in the MFS which were presented at the Ayia Napa meeting.

The international workshop, entitled MFS 2007, was attended by thirty-two participants from thirteen countries from as far away as the USA and Taiwan. Among all participants, there was an even mix of mathematicians and engineers. Thus, the MFS 2007 provided an excellent platform to bridge the gap between theory and practice.

The programme included three plenary talks which were delivered by Graeme Fairweather (Colorado School of Mines, USA), Alex H.-D. Cheng (University of Mississippi, USA), and R. Schaback (Universität Göttingen, Germany). We would like to take this opportunity to thank all participants for their contributions. We would also like to thank our sponsors, namely the University of Cyprus, the University of Southern Mississippi, the Electricity Authority of Cyprus, and the Cyprus Tourism Organization for their generous support.

After the end of the workshop, we invited all participants to contribute papers to be considered for publication as chapters in this book. After a careful refereeing process, fourteen papers were selected. In this selection, we tried to maintain a balance among theoretical analysis, numerical algorithms, and engineering applications of the MFS.

The first part of this book is focused on the theoretical analysis and the development of new numerical algorithms. In the MFS, one of the outstanding problems is how to choose the locations of the source points. In Chapter 1 Schaback contributes to this question by developing a greedy adaptive technique for the automatic choice of suitable source points. In Chapter 2 Li provides an improved error bound for solving the Laplace equation in a bounded simply connected domain. In Chapter 3 Alves and Martin discuss the application of the MFS to an inverse potential problem that consists of detecting inclusions or cavities using a single boundary measurement on an external boundary. They also develop an iterative MFS approach to inverse problems. In Chapter 4 Chen, Fan, and Monroe extend the MFS to solving general elliptic PDEs with variable coefficients. Despite its effectiveness, the MFS has so far been restricted to a small class of PDEs with constant coefficients. The new developments presented in this chapter are expected to generate more interest for solving a large class of science and engineering problems using the MFS. In Chapter 5 Kara-georghis, Mogilevskaya, and Stolarski develop an efficient complex MFS.
algorithm for the solution of 2D potential and linear elasticity problems. In Chapter 6 Karageorghis and Lesnic propose an MFS approach for the solution of steady-state nonlinear heat conduction problems. In Chapter 7 Gáspár presents a regularization technique for the MFS. The computational problems caused by the singularity of the fundamental solution can be avoided by introducing the fundamental solution of a higher order auxiliary problem. As a result, the use of a large, dense, and ill-conditioned matrix can be avoided.

The second part of this book provides a collection of papers on engineering applications which include fluid dynamics, solid mechanics, and wave propagation. In Chapter 8 Uscilowska uses the MFS for solving axisymmetric isothermal gas flow in a porous medium. Picard iteration is used to treat the nonlinearity. In Chapter 9 Young, Chen, and Fan propose to combine the MFS and an Eulerian-Lagrangian method to solve the unsteady nonlinear Navier-Stokes equations for low Reynold number flows with a moving rigid body. In Chapter 10 Kolodziej and Klekiel apply the MFS and radial basis functions for solving the fully developed laminar flow of non-Newtonian fluids inside ducts with constant but arbitrary cross-sections. Picard iteration is also used to tackle the non-linear governing equation. In Chapter 11 Wen, Chen, and Liu investigate the application of the MFS in the shear deformable plate problem under either static or dynamic loads. Durbin's inversion method for Laplace transform is employed for dynamic loads. In Chapter 12 Fujisaki focuses on the accuracy of the MFS by calculating the maximum stress of a circle and an elliptical hole under internal pressure. To verify the effectiveness of the MFS, the effect of a point load and its shape are investigated. In Chapter 13 Mohareb, Rashed, and Akl propose a new MFS approach using the concept of dipoles for Reissner’s plate theory. In Chapter 14 Gudinho, Tadeu, and Mendes study the applicability of the MFS for the computation of the rotational motion generated by a line load in an elastic medium. In Chapter 15 Särler extends the desingularised modified method of fundamental solutions to solution of potential flow problems. In Chapter 16 Rodriguez et al proposed to use the MFS to analyze micro strip patch antennas of arbitrary shape.

This first MFS workshop was dedicated to Professor Graeme Fairweather, in honor of his 65th birthday and in recognition of his pioneering work in the development of the method.
In 1965 Graeme Fairweather was appointed Lecturer of Applied Mathematics at the University of St Andrews where he remained until 1969. Then he joined the faculty of the Department of Mathematics at Rice University in Houston as an Assistant Professor. In 1971 he moved to the University of Kentucky where he remained until 1994. While at Kentucky he also served as Acting Director and Associate Director at the Center for Computational Sciences of the University. Since 1994 he has been the head of the Department of Mathematical and Computer Sciences of the Colorado School of Mines. Graeme has always been a keen traveler and has held visiting appointments at a number of places. He is the author of one book and over 100 journal articles on numerical analysis and scientific computing. He is also the recipient of numerous NSF grants.

Graeme’s earlier contributions were primarily in the numerical solution of partial differential equations by finite differences. Later he also worked on finite element Galerkin methods for the same class of problems. This interest resulted in the publication of his book Finite Element Galerkin Methods for Differential Equations published by Marcel Dekker in 1978. From his work on finite difference and finite element methods resulted interesting contributions in other areas of numerical analysis such as numerical integration and numerical linear algebra. While at Kentucky, Graeme also collaborated with Frank Rizzo and David Shippy on the boundary integral equation method. While on sabbatical in Toronto, Graeme collaborated with Laurie Johnston on a new idea which was to use approximations with linear combinations of fundamental solutions. Laurie had worked on it previously with R. Mathon and the idea appeared promising. Their collaboration led to Graeme’s first two papers on the MFS in the early 80’s. Graeme remained interested in the method and has since been instrumental in its development. In recent years, he has also worked on the numerical solution of partial differential equations with more conventional methods, notably on spline collocation methods with B. Bialecki and A. Karageorghis.

On behalf of the MFS community, we thank Graeme Fairweather for the inspiration he has offered to us all and are looking forward to more of his contributions in all areas of numerical mathematics.

The production of this book required the help of a number of individuals whose efforts we would like to acknowledge. We first thank all the authors whose work has made this volume possible. We would also like to
thank all the referees who contributed their time to review all these papers. Finally, we are grateful to Dr. John Dudley at the University of Southern Mississippi who provided services for proofreading some of the submitted manuscripts.

While this book was in the last stage of preparation, we received the sad news that Dr. Michael Golberg had passed away after a long illness. We deeply regret that we lost a colleague and friend whose work has inspired much of our research on the MFS.

C. S. Chen
A. Karageorghis
Y.S. Smyrlis
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CHAPTER 1

Adaptive Numerical Solution of MFS Systems

R. Schaback

Abstract. The linear systems arising from MFS calculations share certain numerical effects with other systems involving radial basis functions. These effects concern approximation error and stability, which are closely related, and they can already be studied for simple interpolation problems without PDEs. In MFS calculations, they crucially depend on the position and density of the source points and the collocation points. In turn, the choice of these points must depend on the smoothness and possible singularities of the solution. This contribution provides an adaptive method which chooses good source points automatically. A series of examples shows that the adaptive choice of source points follows the theoretical predictions quite well.

1.1 Introduction

The Method of Fundamental Solutions (MFS) solves a homogeneous boundary value problem via approximation of the boundary data by traces of fundamental solutions centered at source points outside the domain in question. The method has been used extensively in recent years, and there are excellent surveys [6, 8, 5]. However, this contribution focuses on the linear systems arising in MFS calculations and ignores applications in engineering and science. Since our observations will easily generalize to other

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cases, we keep the presentation and the examples simple by restricting ourselves to the homogeneous Poisson problem

$$\Delta u = 0 \quad \text{in } \Omega \subset \mathbb{R}^2$$
$$u = \varphi \quad \text{on } \Gamma := \partial \Omega$$

with the Laplace operator. In this case, the fundamental solution (up to a multiplicative constant) is the singular radial kernel function

$$\Phi(x, y) := \log \|x - y\|_2^2, \quad x, y \in \mathbb{R}^2.$$ 

The source points will be taken from a curve $\Sigma$ outside $\overline{\Omega}$ which is often called the “fictitious” boundary. In particular, users normally choose $N$ points $y_1, \ldots, y_N \in \Sigma$ and take linear combinations

$$s(x) := \sum_{j=1}^{N} \alpha_j \log \|x - y_j\|_2^2, \quad x \in \Omega$$

of fundamental solutions as trial functions being homogeneous solutions of the Laplacian, i.e. harmonic functions. Of course, other homogeneous solutions can also enrich the trial space, and there are plenty of such possibilities, including harmonic polynomials. Methods like this date back to Trefftz [16] in much more general form, and are currently revived under the name of boundary knot methods [4].

1.2 Error Bounds

Whatever homogeneous solutions the trial functions $s$ are composed of, the maximum principle will under mild assumptions on the regularity of the domain and the boundary data [11] imply that the true solution $u$ and the trial approximation $s$ satisfy the error bound

$$\|u - s\|_{\infty, \Omega} \leq \|u - s\|_{\infty, \partial \Omega}.$$ 

This means that users only have to worry about the $L_\infty$ approximation error on the boundary. If a fixed space of general linear combinations

$$s(x) := \sum_{j=1}^{N} \alpha_j s_j(x), \quad x \in \Omega$$
of smooth homogeneous solutions $s_j$ are admitted, the natural numerical approach induced by the Maximum Principle would be to minimize the $L_\infty$ norm of the error on the boundary. This is a semi–infinite linear optimization problem

\[
\text{Minimize } \eta \quad \text{subject to } -\eta \leq \varphi(x) - \sum_{j=1}^{N} \alpha_j s_j(x) \leq \eta, \quad x \in \Gamma
\]  

(1.3)

with $N + 1$ variables $\eta, \alpha_1, \ldots, \alpha_N$ and infinitely many affine–linear constraints. The literature on optimization deals with such problems [10, 9], but in many cases it suffices to come up with a cheap but suboptimal approximation. We shall focus on this situation and give examples later.

### 1.3 Linear Systems

In particular, users often try to get away with picking $N$ collocation points $x_1, \ldots, x_N$ on the boundary $\Gamma$ and setting up an $N \times N$ linear system

\[
\sum_{j=1}^{N} \alpha_j s_j(x_k) = \varphi(x_k), \quad 1 \leq k \leq N
\]  

(1.4)

for interpolation at these points. This works well in many cases, but the main theoretical problem with such systems is that the coefficient matrix with entries $s_j(x_k)$ may be singular. This clearly occurs for $N > 1$ and the MFS, because the determinant of the $N \times N$ system with matrix entries

\[
s_j(x_k) = \log \|y_j - x_k\|_2^2, \quad 1 \leq j, k \leq N
\]

will be a smooth function of the source points $y_j$, and swapping two source points will change the sign of the determinant. Thus there are plenty of configurations of source and test points where the system is necessarily singular. Confining source points to curves may help in 2D cases, but not in 3D if source points are restricted to surfaces.

Consequently, it does not make any sense to head for theorems proving nonsingularity of the above systems. The same holds for other unsymmetric collocation–type techniques like the one introduced by E. Kansa [12, 13] for general PDE problems in strong form, or the meshless local Petrov–Galerkin method of S.N. Atluri and collaborators [1, 2].
Instead, systems like (1.4) should not be expected to be solvable exactly. In view of the maximum principle and the semi–infinite optimization problem (1.3) one can take many more collocation points than source points and solve the overdetermined linear system

\[ \sum_{j=1}^{N} \alpha_j s_j(x_k) = \varphi(x_k), \quad 1 \leq k \leq M \geq N \]  

(approximately, e.g. by a standard least–squares solver. We shall focus on such systems from now on.

1.4 Choice of Test and Collocation Points

If a good linear combination \( s \) of the form (1.2) is found by any method whatsoever, users will check the maximum boundary error \( \| \varphi - s \|_\infty, \Gamma \) by evaluating the error in sufficiently many test points on the boundary. Though this test also needs a thorough mathematical analysis in order to be safe, we ignore it here. We just remark that users will need very many test points in case of steep gradients of the trial functions, and this inevitably occurs if the MFS is used with source points close to the boundary. Adding more test points still is computationally cheap if \( N \) is not too large, and most users will be satisfied with a simple plot of the boundary errors evaluated at test points guaranteeing graphic accuracy, i.e. at most 1000 points per plot. We shall use this rule–of–thumb in later examples.

Choosing \( M \) collocation points for setting up the system (1.4) is somewhat more difficult, but it will always stabilize the system if more points are taken. Independent of the choice of trial functions, users can repeat the calculation with more or other collocation points, if they are not satisfied with the first result. This is a simple way of introducing adaptivity into the numerical solution strategy:

Adaptivity of Testing:

If the evaluation of the boundary error on certain test points yields values that are intolerably large, take these test points as collocation points and repeat the calculation.

As long as the trial space \( S \) is not changed, this can improve the results, but if the trial space is poorly chosen, the final boundary error cannot be
less than
\[ \inf_{s \in S} \| \phi - s \|_{\infty, \Gamma} \] (1.6)
no matter how collocation and testing is done and how many points are used.

But there is another argument that needs consideration. If the linear optimization problem (1.3) is solved for a large but finite subset \( \Gamma_0 \) of the boundary instead of the full boundary, the Karush–Kuhn–Tucker conditions applied to the dual reformulation [3] of a linear minimax problem will imply that there is a subset \( \Gamma_1 \) of \( \Gamma_0 \) consisting of at most \( N + 1 \) points such that
\[ \inf_{s \in S} \| \phi - s \|_{\infty, \Gamma_0} = \inf_{s \in S} \| \phi - s \|_{\infty, \Gamma_1}. \]
This is related to the notion of support vectors in support vector machines, and it has the following implication:

Reducibility of collocation points:
If a system (1.4) with \( M \gg N \) has a good approximate solution, it even has a good approximate solution determined already by a subset of at most \( N + 1 \) collocation points.

Unfortunately, these collocation points are not known beforehand, but users should be aware of the fact that a large system with a good approximate solution will have a much smaller subsystem with an equally good solution. This fact will reappear later, and we shall provide examples.

1.5 Choice of Trial Space
The lower bound (1.6) for the achievable boundary error reveals that the main design problem consists in picking good trial functions, or, in case of the MFS, in picking good source points.

Let us postpone the MFS for a while. Users can take all homogeneous solutions as trial functions, and this will work well in certain examples we shall look at later. For the Laplace operator in 2D, the real part of any differentiable function of a complex variable will be harmonic and can serve as a possible trial function. The standard fundamental solution just is a special case of a real part of a complex function with a singularity, but there are many others without singularities, e.g. harmonic polynomials or entire functions like \( f(x,y) := \exp(y) \cos(x) \).
How to choose? We shall later let an algorithm decide adaptively, but there is a general though trivial rule:

Take trial functions with similar analytic properties as the expected solution. In particular, be careful when the solution or one of its derivatives will necessarily have singularities somewhere.

1.6 Harmonic Polynomials

We explain this first for the case of using harmonic polynomials. If the solution $u$ of the given Poisson problem is itself a real part of a function of a complex variable without singularities anywhere, it can be well approximated by harmonic polynomials on any curve, namely by the real part of its partial sums of its power series. The shape of the domain does not matter at all, and the background PDE problem is completely irrelevant because we only have to recover a partial power series. A full power series of an analytic function is determined by values on any countable set with an accumulation point, and thus recovery of globally harmonic functions from point evaluation data will work almost anywhere.

By analogy to certain theorems on polynomial approximations to analytic functions [7], the rate of approximation can be expected to be spectral, i.e. the error should behave like $C \lambda^n \to \infty$ as a function of the degree $n$ of the harmonic polynomials used, and $\lambda > 0$ can be arbitrarily small. Then the choice of harmonic polynomials should be superior to all choices of fundamental solutions. In many engineering applications where MFS users report that source points of the MFS taken far away from the domain work best, the special examples usually have solutions without singularities anywhere, but users tend to ignore that harmonic polynomials will do even better in such situations.

If the solution, when viewed as a global function, is still harmonic but has a singularity at a positive distance to the boundary $\Gamma$, the rate of approximation will again be like $C \lambda^n \to \infty$, but with $\lambda < 1$ now being bounded below, and related to the distance of the singularity to the boundary, with $\lambda \to 1$ if the singularity moves towards the boundary. Again, the shape and smoothness of the boundary is irrelevant. The crucial quantity is the distance of the closest singularity of the solution from the boundary, when the solution is extended harmonically as far as possible. Again, this
case is hard to beat by the MFS, if the singularity is sufficiently far away from the domain.

The situation gets serious if the solution or one of its derivatives has a singularity directly on the boundary $\Gamma$. Note that this case occurs whenever the boundary data, however smooth, are given by a function which is not itself harmonic. In such a case, the rate of approximation of boundary values by harmonic polynomials can be very poor, depending on the smoothness of the solution when restricted to the boundary. The upshot of this discussion of harmonic polynomials is that the MFS makes sense only if the boundary data come from a non–harmonic function or if there is no harmonic extension of the solution without singularities close to the boundary. Users working in application areas do not seem to be aware of this fact.

### 1.7 Rescaling Fundamental Solutions

Before we go over to the problem of choosing good source points for the MFS, let us consider the case of far–away source points $y \in \mathbb{R}^2$ while the evaluation of a fundamental solution $\log \|x - y\|_2^2$ is at $x \in \mathbb{R}^2$ with a relatively small value of $\|x\|_2$. In such cases, the functions $\log \|x - y\|_2^2$ will not differ much if $y$ varies, and consequently the resulting matrix gets a bad condition. But we can rewrite the function for large $\|y\|_2 \neq 0$ as

$$
\log \|x - y\|_2^2 = \log \left( \|x\|_2^2 - 2(x, y) + \|y\|_2^2 \right)
= \log \left( \|y\|_2^2 \left( \frac{\|x\|_2^2}{\|y\|_2^2} - 2 \left( \frac{x}{\|y\|_2^2} \right) \right) + 1 \right)
= \log \|y\|_2^2 + \log \left( 1 + \left( \frac{\|x\|_2^2}{\|y\|_2^2} - 2 \left( \frac{x}{\|y\|_2^2} \right) \right) \right)
.$$ 

If we use the expansion

$$
\log(1 + z) = \sum_{j=1}^{\infty} (-1)^{j-1} \frac{z^j}{j}
$$
Adaptive Numerical Solution of MFS Systems

for $|z| < 1$, we get for sufficiently large $\|y\|_2$ the expansion

$$
\log \|x - y\|_2^2 - \log \|y\|_2^2
= \log \left( 1 + \left( \frac{\|x\|_2^2}{\|y\|_2^2} - 2 \left( x, \frac{y}{\|y\|_2} \right)_2 \right) \right)
= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \left( \frac{\|x\|_2^2}{\|y\|_2^2} - 2 \left( x, \frac{y}{\|y\|_2} \right)_2 \right)^j
= \sum_{j=1}^{\infty} \frac{(-1)^{j-1}}{j} \sum_{m=0}^{j} \binom{j}{m} \frac{1}{\|y\|_2^{2j-2m}} \|x\|_2^{2j-2m} \left( -2 \left( x, \frac{y}{\|y\|_2} \right)_2 \right)^m
= \sum_{k=1}^{\|y\|_2^2} \sum_{1 \leq j \leq k} \frac{(-1)^{j-1}}{j} \left( \frac{\|x\|_2^{2j-2k}}{\|y\|_2^{2j-2k}} - 2 \left( x, \frac{y}{\|y\|_2} \right)_2 \right)^{2j-k}
= \sum_{k=1}^{\|y\|_2^2} \frac{1}{\|y\|_2^{2k}} p_k(x, y)
$$

of the fundamental solution at $y$ into harmonic polynomials

$$
p_k(x, y) := \sum_{k/2 \leq j \leq k} \frac{(-1)^{j-1}}{j} \sum_{m=0}^{j} \binom{j}{2j-k} \|x\|_2^{2k-2j} \left( -2 \left( x, \frac{y}{\|y\|_2} \right)_2 \right)^{2j-k}
$$

with respect to $x$ of degree $k$. If we push the source point $y$ to infinity by writing it as $y = rz$ for large $r > 0$ and fixed $z \in \mathbb{R}^2$ with $\|z\|_2 = 1$, we get

$$
\log \|x - rz\|_2^2 = 2 \log r + \sum_{k=1}^{m} \frac{1}{r^k} p_k(x, z)
$$

and this is something like a “far field expansion” of the fundamental solution. Note that $z$ and $r$ are considered to be fixed, and thus users are strongly advised to include constants into the space of trial functions in order to cope with the $2 \log r$ term.

Now let us look at the span of fundamental solutions based on points $y_j = rz_j$ on a circle of radius $r$ for large $r$. We want to find functions which are in the span when taking the limit $r \to \infty$, and we call this the
“asymptotic span”. The linear combinations are

\[
    s_r(x) = \sum_{j=1}^{N} \alpha_j(r) \left( 2 \log r + \sum_{k=1}^{\infty} \frac{1}{r^k} p_k(x, z_j) \right) 
\]

and thus have specific expansions in terms of harmonic polynomials. If constants are not added to the span, and if the MFS works at all for large \( r \) in a specific case, the sum of the coefficients \( \alpha_j(r) \) will tend to zero for \( r \to \infty \) while the coefficients themselves cannot stay all bounded. In all “pure MFS” examples with far-away source points, the sum of coefficients will always be close to zero while the sum of the absolute values will be extremely large.

To avoid computational crimes, we now add the constant 1 to the span of trial functions and use a coefficient \( \alpha_0 \) for it. Then we have a span of

\[
    s_r(x) = 1 \left( \alpha_0(r) + 2 \log r \sum_{j=1}^{N} \alpha_j(r) \right) + \sum_{k=1}^{\infty} \frac{1}{r^k} \sum_{j=1}^{N} \alpha_j(r) p_k(x, z_j) 
\]

which we can analyze somewhat easier. We have the constants in the span, of course, but for arbitrary \( \alpha_1(r), \ldots, \alpha_N(r) \) we can always set

\[
    \alpha_0(r) := -2 \log r \sum_{j=1}^{N} \alpha_j(r) 
\]

to cancel the first term. Now \( s_r(x) \) must be in the span, and this asymptotically is in the span of the \( p_1(x, z_j) \), \( 1 \leq j \leq N \), which necessarily is a subspace \( V_1 \) of the linear polynomials. To proceed inductively, we now look at the subspace \( A_1 \) of coefficient vectors \( \alpha \in \mathbb{R}^N \) with

\[
    \sum_{j=1}^{N} \alpha_j p_1(x, z_j) = 0. 
\]

If we take a vector \( \alpha \in A_1 \) and form the functions \( r^2 s_r(x) \), we find that the asymptotic span of the fundamental solutions contains the polynomial space

\[
    V_2 := \left\{ \sum_{j=1}^{N} \alpha_j p_2(x, z_j) : \alpha \in A_1 \right\} 
\]
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of maximally second–degree polynomials. Inductively we can define \( A_0 := \mathbb{R}^N \) and

\[
A_m := \left\{ \alpha \in \mathbb{R}^N : \sum_{j=1}^{N} \alpha_j p_i(x, z_j) = 0, \ 1 \leq i \leq m \right\}
\]

for all \( m \geq 1 \) and use it for defining a space

\[
V_m := \left\{ \sum_{j=1}^{N} \alpha_j p_m(x, z_j) : \alpha \in A_{m-1} \right\}
\]

of polynomials of degree at most \( m \). The spaces \( A_m \) form an inclusion chain

\[
\mathbb{R}^N = A_0 \supseteq A_1 \supseteq A_2 \supseteq \cdots
\]

and if we take an appropriate orthogonal basis for that chain, we get

**Theorem 1.** The asymptotic span for \( r \to \infty \) of fundamental solutions with source points of the form \( y_j = rz_j \) for fixed points \( z_j \) on the unit circle is a space of harmonic polynomials spanned by constants and the union of all \( V_m \). □

Unfortunately, it seems to be difficult to calculate the dimension of that space, because it will depend on the number and the geometry of the points \( z_j \).

The upshot of all of this is that the MFS for far–away source points, if it works at all, is asymptotically nothing else than a fit of the boundary data by specific harmonic polynomials. Thus the MFS should not be used at all for far–away source points, but rather be replaced by use of harmonic polynomials. For this reason, we do not elaborate the above argument any further, though it would result in a way of preconditioning MFS matrices for far–away points. It does not make sense to precondition a matrix one should not use.

However, a rather primitive but still somewhat useful change of basis induced by the above argument is to add constants to the MFS span and replace the fundamental solution at \( y \neq 0 \) by

\[
(\log ||x - y||_2^2 - \log ||y||_2^2) ||y||_2
\]

behaving like a linear polynomial in \( x \) when \( y \) is far away from \( x \). A full preconditioning will use such basis changes plus coefficient vectors from
an orthogonal basis of $\mathbb{R}^N$ which is compatible with the chain of the $A_m$ spaces. Details can be worked out similarly to [19]. As an aside, we remark that it is no problem to replace the standard fundamental solutions by rational trial functions arising when taking derivatives of $\log \|x - ry\|_2^2$ with respect to $r$.

Finally, we present an example supporting the results of this and the previous section. In Fig. 1.1 we show the $L_\infty$ error $\varepsilon_\infty(r)$ on the full circle when we recover the harmonic function $f(x,y) = e^x \cos(y)$ from boundary values only on a half circle. We collocate at 100 test points on the right half unit circle, using 20 source points on the right half circle of radius $r$. We stopped the calculation when the numerical rank of the $100 \times 20$ collocation matrix, as given by MATLAB® was less than 20, and this occurred for $r \approx 5.5$ already. The error decreases nicely with increasing $r$, because the setting converges towards harmonic polynomials for $r \to \infty$, as was shown in this section, and since the discussion in the previous section showed that recovery by harmonic polynomials should work on any arc.

![Figure 1.1: $L_\infty$ error as function of $r$](image-url)
1.8 Choice of Source Points

It should be clear by now that a good placement of source points will crucially depend on the distance of the closest singularity arising when extending the solution harmonically outside the domain. In many cases, the user normally does not have this information, but there are a few guidelines.

We start by an upside–down argument. If the MFS works for sufficiently many source points on a fixed curve $\Sigma$, and if the results are getting better when taking more source points, the solution will have a harmonic extension up to $\Sigma$, because the MFS constructs it. But if there necessarily is a singularity inside $\Sigma$ for some reason or other, the MFS cannot work satisfactorily on $\Sigma$.

We now have to find a–priori indicators for singularities close to the boundary. The first and simplest case arises when the known boundary data are such that there is no $C^\infty$ extension locally into $\mathbb{R}^2$. This always happens if the boundary data are not $C^\infty$ on smooth parts of the boundary. If users know where the “boundary points of data nonsmoothness” are, source points should be placed close to those. Unfortunately, there currently is no general way to guess the type of singularity beforehand, even if the position is known. Thus this case usually must be handled experimentally.

A second and partially independent case arises for incoming corners of the domain. Even if the boundary data have a $C^\infty$ extension to $\mathbb{R}^2$, e.g. if they are non–harmonic polynomials, users must expect a singularity at the boundary, but the type of singularity is known, depending on the boundary angle. Again, users should either add the correct type of singularity or place source points close to corners in such cases. But the situation is different if the data come from an extendable harmonic function, even if corners are present. Then the MFS can ignore the corners. Note that MFS examples on domains with corners are useless as long as they consider specific boundary data which are values of functions with a harmonic extension.

Finally, the convergence rate of the MFS when adding more and more source points will be strongly influenced by the smoothness of both the data function and the MFS trial functions on the boundary. Approximation theory proves in many situations that convergence rates are completely
controlled by the minimal smoothness of the data function and the trial functions. Thus smooth boundary data on smooth boundaries will lead to good convergence rates improving with the smoothness properties. If source points can be kept at a fixed positive minimal distance from the boundary (this requires the solution to have a harmonic extension), then the trial functions are analytic and the convergence rate will be completely determined by the smoothness of the data on the boundary. But then the approximation by harmonic polynomials on the boundary will also have a good convergence rate depending on the smoothness of the boundary data, and it is not easy to predict superiority of the MFS over approximation by harmonic polynomials.

If singularities of derivatives are on the boundary or if there are incoming corners, the convergence rate of approximation of the boundary data by harmonic polynomials will deteriorate seriously, and the MFS can be competitive by placing source points closer and closer to the singularities. A general rule is not known, but there are certain adaptive techniques [18, 14, 15, 6] to handle this case. We shall provide a simple adaptive method in the next section.

1.9 Greedy Adaptive Techniques

Overdetermined systems like (1.5) can be approximately solved by a stepwise adaptive techniques even if they are huge. We applied the method of [17] to MFS problems, but it turned out to be less stable than the algorithm we describe now, because the previous one did not keep all collocation points under control.

The basic idea can be formulated independent of the MFS in terms of solving a linear unsymmetric over- or underdetermined $m \times n$ system of the form $Ax = b$. The goal is to pick useful columns of the $m \times n$ matrix $A$ in a data-dependent way without cutting the number of rows down. This is also done by any reasonable solution algorithm, e.g. by the backslash operator in MATLAB©, and we shall present examples later. However, standard $QR$ routines do not account for the right-hand side $b$, and they do not stop early when only a few columns of the matrix suffice to reproduce the right-hand side with small error. To maintain stability, we use orthogonal transformations like in any $QR$ decomposition, but we make the choice of columns dependent on the right-hand side.
In short, our adaptive algorithm for selecting good columns works as follows:

1. Pick the column of $A$ whose multiples approximate $b$ best.

2. Then transform the problem to the space orthogonal to that column and repeat.

If the algorithm has selected a number of columns this way, take this column selection for a trial space and use your algorithm of choice for solving the given problem on that trial space. For instance, in MFS applications one can use $L_\infty$ minimization of boundary errors after the selection process has provided a small set of useful source points.

The actual implementation of the algorithm needs some further explanation. Approximation of $b$ by multiples of a single nonzero vector $a$ is optimal in $L_2$, if the error vector has the form $b - a \cdot \frac{b^T a}{\|a\|^2}$, and its squared norm then takes the minimal possible value

$$\|b\|^2 - \frac{(b^T a)^2}{\|a\|^2}.$$

If we denote the columns of $A$ by $a^1, \ldots, a^n$, we thus can implement Step 1 by taking the maximum of

$$\frac{(b^T a_j)^2}{\|a_j\|^2}, \quad 1 \leq j \leq n, \quad \|a_j\|_2 \neq 0$$

to pick the best column for approximation of $b$. If we denote this column by $u$, we form the normalized vector $v := u/\|u\|_2$ and transform both $A$ and $b$ into

$$A_1 := A - v(v^T A) \quad b_1 := b - v(v^T b)$$

to let both $b_1$ and the columns of $A_1$ be orthogonal to $u$ and $v$. The new matrix has a zero column where once was $u$. To avoid roundoff problems, we insert exact zeros there, but we do not delete the zero column in order to avoid unnecessary storage transformations. Instead, we store the column index of $u$ for later use and proceed. The following steps will always automatically ignore the columns we already picked. Note that we do not care for the approximate solution of the system and about accumulation of
roundoff during the transformations. The $L_2$ norms of the vectors $b, b_1, \ldots$ will necessarily decrease, and this can be used for stopping the algorithm. Going for a strong error reduction will finally use the full matrix, while users can get away with just a few columns and a very simple numerical solution if they admit larger errors. A MATLAB® package is available from the author.

Application of this technique to MFS calculations is particularly appealing in cases where the user lets the algorithm decide which trial function to choose. One can offer harmonic polynomials up to a fixed degree and plenty of fundamental solutions at different distances to the boundary, and the algorithm will pick suitable ones without knowing background mathematics like harmonic extendibility of the solution. Running the algorithm several times will provide the user with information about hazardous places at the boundary, and the user can offer refined choices of source points close to these when preparing the next run. The actual calculation of the solution is done after column selection in order to keep the accumulated errors small.

1.10 Examples

To illustrate the mathematical issues of the previous sections, we now provide a series of examples, but we have to explain the notation in the tables and figures first. Following (1.4), the number of collocation points will be $M$, and $N$ will denote the number of source points offered to the algorithm. If only a smaller subset of source points is actually used for the calculation, we use $n \leq N$. Similarly, $K$ denotes the number of harmonic polynomials included into the trial space, and $k$ stands for the number actually arising in the solution. For approximate evaluation of the $L_\infty$ norm $\varepsilon_\infty$ of the error on the boundary we use $\max(1000, 5 \times N)$ points for graphic accuracy. The number $M$ of collocation points is always defined as $1 + \max(200, 2N + 2K)$. The approximate solution of the system will be done by algorithms labeled as

- $L2$: the MATLAB® backslash operator, i.e. a standard least-squares solver with internal selection of columns,
- $A2$: the same as above, but applied to the reduced matrix after adaptive column selection along the lines of the previous section,
$A^\infty$: an $L^\infty$ linear optimizer applied to the same adaptively reduced matrix.

Note that the solution algorithm will have quite some influence on the final error and the number of nonzero solution coefficients. The $L^2$ solver always exploits maximal machine accuracy, while our adaptive solvers $A^2$ and $A^\infty$ are tuned for a compromise between error and complexity.

We denote the $L^\infty$ error on our test points as $\varepsilon^\infty$. It is debatable to use the RMSE error measure at all, because the Maximum Principle is guiding the error behavior, but we also include it as $\varepsilon^2$. The curves we use for boundaries or source points consist of circles $C_r$ with radius $r$ around zero, or they are obtained from a polar coordinate representation of the standard boundary $B_0$ by adding a distance $h$ to get a source boundary $B_h$ at polar distance $h$.

The first example in Table 1.1 concerns a more or less trivial case where the data come from a globally harmonic function $f(x, y) = \exp(x)\cos(y)$ which is the real part of the entire complex function $\exp(z) = \exp(x + iy)$. The real part of the power series of $\exp(z)$ yields a sequence of perfect approximations by harmonic polynomials on each domain whatsoever, and the recovery by collocation via harmonic polynomials even works on open arcs anywhere. As expected, the MFS cannot outperform harmonic polynomials, independent of where the sources are. The domain is a lemniscate with an incoming corner as in Fig. 1.2, while the source points are on the circle of radius 4 around the origin.

<table>
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<th>$K$</th>
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Table 1.1: Recovery of harmonic function
The table should be interpreted as follows. The first three lines used no source points at all ($N = 0$), but $M = 201$ collocation points and allowed $K = 25$ harmonic polynomials. Only up to $k = 13$ nonzero polynomial coefficients were calculated due to the symmetry of the data function, and the recovery quality via the first terms of the power series is around $1.0e-10$, independent of the algorithm used. This happened for all domains tested. But the adaptive algorithms in lines 2 and 3 solve only a subproblem with 12 degrees of freedom after picking the most useful harmonic polynomials.

The next three lines are a pure MFS offering 200 source points on the circle. With the $L_2$ solver, the results are even better than for harmonic polynomials, and surprisingly the MATLAB backslash solver yields only 36 nonzero coefficients, i.e. only 36 source points were necessary. The adaptive solvers are satisfied with less accuracy, but also use a simpler approximation by 25 or 28 source points.
The final three lines offered the same 200 source points, but allowed also 25 harmonic polynomials. All algorithms prefer harmonic polynomials over fundamental solutions. This is to be expected, because the solution has no finite singularities. The 25 marked source points in Fig. 1.2 belong to the situation of the fifth line of Table 1.1. The adaptive $L_2$ algorithm picks these 25 source points with no connection to the domain corner, as is to be expected. However, the error is worse than for harmonic polynomials in this case.

The $L_\infty$ norm of the error using the $A_\infty$ solver in line 3 is only slightly better than the one from the $A_2$ solver in line 2. It cannot be worse because they use the same trial space. However, in some of the later cases, the $A_\infty$ algorithm, after starting from the same trial space as the $A_2$ algorithm, is less stable and often ends prematurely with a larger $L_\infty$ error than the $A_2$ solver. Future work should add a more sophisticated $L_\infty$ solver.

In Table 1.2 we present the same situation, but with boundary data given by the function $x^2y^3$. This is still smooth, but the domain has an incoming corner causing problems.

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<td>$A_\infty$</td>
<td>0</td>
<td>26</td>
<td>12</td>
<td>9.15e-004</td>
<td>6.46e-004</td>
</tr>
</tbody>
</table>

Table 1.2: Smooth boundary data on lemniscate, source points on circle

With source points on the circle, none of the methods outperforms harmonic polynomials seriously.

To demonstrate that the incoming corner is the culprit, we replace the lemniscate now by the unit circle and get Table 1.3. Again, harmonic polynomials do best. We now go back to the lemniscate, but place the source points at a polar radial distance of 0.2 outside the boundary (i.e. we
Table 1.3: Smooth boundary data on circle

calculate them by adding 0.2 to the radius of boundary points in polar co-
ordinates). This gives Table 1.4 and should be compared to Table 1.2. The

Table 1.4: Smooth boundary data on lemniscate, source points at 0.2

results are not better than for sources on the circle, but maybe the sources
are not close enough. Thus we go for a distance of 0.02 in Table 1.5, and
Fig. 1.3 shows the source point distribution (circles, while the offered but
unused source points are small dots) for the $A_2$ technique in the eighth line
of Table 1.5. Note that the source points are automatically picked close to
the singularity.
Going even closer does not pay off, unless we enhance the collocation resolution.

Now we admit source points at different distances, starting at 0.02 and ending at 10.24 after repeated multiplications by 2. At each distance, we choose random source points such that their total is roughly 201 as in the previous cases. Now the algorithms can pick source points at very different distances, and they do.

Finally, we allowed roughly 700 source points and varying distances from 0.002 to 2.048 to get Table 1.7. The distances were prescribed by multiplying 0.002 by powers of 2 until 2.048 was reached, while the points for fixed distance were uniformly sampled with respect to the parametrization of the boundary. Zooming in on the adaptively selected source point placements gives Fig. 1.4 for the first line of Table 1.7, while Figs 1.5 and 1.6 are close-ups for line 2 with the A2 algorithm. Note that the small dots are the offered source points, while circles indicate the selected source points.
Users may suspect that things are better if there is no incoming corner of the domain. Thus let us take the unit circle and prescribe the continuous boundary values $\phi(x, y) := \max(0, y)$. This leads to two derivative singularities of the harmonic solution at $(1, 0)$ and $(-1, 0)$. Numerical results are quite similar to the case on the lemniscate, and thus we confine our-
Adaptive Numerical Solution of MFS Systems

Figure 1.4: Lemniscate with source points, \( L_2 \) algorithm

selves to offering about 700 source points on circles with distances 0.002 to 2.048 in Table 1.8. The \( L_2 \) solver does not care about the right-hand side of the system, and thus it does not realize the two singularities. This is shown in a close-up in Fig. 1.7, while the \( A_2 \) algorithm (see Fig. 1.8) selects source points close to the problematic boundary locations. Note that this case is offered 729 degrees of freedom and uses maximally 490 of these.

<table>
<thead>
<tr>
<th>( N )</th>
<th>( M )</th>
<th>( K )</th>
<th>Alg</th>
<th>( n )</th>
<th>( m )</th>
<th>( k )</th>
<th>( L_\infty )</th>
<th>( L_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>704</td>
<td>1451</td>
<td>25</td>
<td>( L_2 )</td>
<td>471</td>
<td>1451</td>
<td>19</td>
<td>7.69e-004</td>
<td>3.41e-005</td>
</tr>
<tr>
<td>704</td>
<td>1451</td>
<td>25</td>
<td>( A_2 )</td>
<td>350</td>
<td>1451</td>
<td>13</td>
<td>8.13e-004</td>
<td>3.03e-005</td>
</tr>
<tr>
<td>704</td>
<td>1451</td>
<td>25</td>
<td>( A_\infty )</td>
<td>356</td>
<td>1449</td>
<td>14</td>
<td>6.92e-003</td>
<td>2.01e-003</td>
</tr>
</tbody>
</table>

Table 1.8: MFS results for max(0,\( y \))
If we drop the MFS completely and offer 751 harmonic polynomials on the circle instead, we get Table 1.9. Note that this performs slightly better than the MFS and uses 377 of the possible degrees of freedom.

<table>
<thead>
<tr>
<th>N</th>
<th>M</th>
<th>K</th>
<th>Alg</th>
<th>n</th>
<th>m</th>
<th>k</th>
<th>$L_\infty$</th>
<th>$L_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1503</td>
<td>751</td>
<td>$L^2$</td>
<td>0</td>
<td>1503</td>
<td>377</td>
<td>6.67e-004</td>
<td>4.08e-005</td>
</tr>
<tr>
<td>0</td>
<td>1503</td>
<td>751</td>
<td>$A^2$</td>
<td>0</td>
<td>1503</td>
<td>377</td>
<td>6.67e-004</td>
<td>4.08e-005</td>
</tr>
<tr>
<td>0</td>
<td>1503</td>
<td>751</td>
<td>$A^\infty$</td>
<td>0</td>
<td>379</td>
<td>377</td>
<td>1.47e-003</td>
<td>1.90e-004</td>
</tr>
</tbody>
</table>

Table 1.9: Results for max$(0, y)$, harmonic polynomials only

Finally, Table 1.10 shows how the sum of coefficients and the $L_1$ norm of coefficients vary with the radius $r$ of the distance of the source points to the boundary. To avoid symmetries, we took the boundary data function $f(x, y) := \max(0, |y|)$ on the unit circle, offered 200 source points on a
circle of radius $r$ and used 401 collocation points. The solution method was the standard backslash $L_2$ solver from MATLAB®. Note that the increase of condition is counteracted by the solver in a very nice way, using fewer and fewer source points. This effect is even more significant when using the $A2$ or $A\infty$ methods.

<table>
<thead>
<tr>
<th>$n$</th>
<th>$r$</th>
<th>$\varepsilon_\infty$</th>
<th>sum</th>
<th>$L_1$ norm</th>
</tr>
</thead>
<tbody>
<tr>
<td>57</td>
<td>1</td>
<td>1.13e-002</td>
<td>2.30e-001</td>
<td>7.23e+009</td>
</tr>
<tr>
<td>53</td>
<td>2</td>
<td>1.15e-002</td>
<td>1.45e-001</td>
<td>2.09e+011</td>
</tr>
<tr>
<td>35</td>
<td>4</td>
<td>1.86e-002</td>
<td>9.89e-002</td>
<td>6.44e+009</td>
</tr>
<tr>
<td>22</td>
<td>8</td>
<td>2.87e-002</td>
<td>7.24e-002</td>
<td>1.56e+011</td>
</tr>
<tr>
<td>18</td>
<td>16</td>
<td>3.50e-002</td>
<td>5.62e-002</td>
<td>3.33e+009</td>
</tr>
<tr>
<td>18</td>
<td>32</td>
<td>3.89e-002</td>
<td>4.57e-002</td>
<td>1.62e+012</td>
</tr>
</tbody>
</table>

Table 1.10: Sums of coefficients as functions of $r$
Figure 1.7: Source points selected by $L^2$ algorithm

References


Figure 1.8: Source points selected by A2 algorithm


CHAPTER 2

The Method of Fundamental Solutions for the Laplace Equation with Mixed Boundary Problems

Zi-Cai Li

Abstract. The method of fundamental solutions (MFS) was first used in Kupradze [16] in 1963. Since then, there have appeared numerous reports of MFS for computation, but there exist only a few for analysis. For the Dirichlet problem of Laplace’s equation on the bounded simply-domain S, when the source points of fundamental solutions (FS) are located uniformly on an outside circle of S, the polynomial convergence rates were proved in Bogomolny [2], and the exponential convergence rates were provided in Katsurada and Okumoto [14]. This paper is devoted to Laplace’s equation with mixed boundary problems, and bounds of both errors and condition numbers are first provided for bounded simply-connected domains. Numerical experiments are carried out for the benchmark of singularity problems, Motz’s problem. Moreover, the analysis of MFS is briefly viewed, to display the nature of MFS.

2.1 Introduction

The method of fundamental solutions (MFS) was first used in Kupradze [16] in 1963. Since then, there have appeared numerous reports of MFS for computation, see the review of MFS in Fairweather and Karageorghis [9], and Golberg and Chen [10], and a systemic introduction on MFS is given in Chen et al. [3]. To celebrate the progress of MFS, there held the first
The MFS for Laplace’s equation with mixed boundary conditions, and bounds of both errors and condition numbers are provided for bounded simply-connected domains. Numerical experiments are carried out for Motz’s problem, by adding singular solutions, or local refinements of collocation nodes. The values of Cond are huge, and those of Cond$_{eff}$ are moderately large. Moreover, the expansion coefficients obtained by MFS are oscillatingly large, to cause another kind of instability: subtraction cancelation in the final harmonic solutions. Hence for practical applications, the errors and the ill-conditioning must be balanced to each other.

This paper is organized as follows. In the next section, the algorithms of MFS are described, and stability is measured by Cond and Cond$_{eff}$. In Section 2.3, error analysis is briefly proved. In Section 2.4, numerical experiments are carried out for Motz’s problem by MFS, and in the last section, a few remarks for the analysis on MFS are made.

2.2 Algorithms and Stability

2.2.1 Algorithms

Consider Laplace’s equation with the mixed boundary problems of the Dirichlet and the Robin boundary conditions,

\[
\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad S, \quad \text{(2.1)}
\]

\[
u \frac{\partial u}{\partial \nu} + \alpha u = f \quad \text{on} \quad \Gamma_D, \quad \text{(2.2)}
\]

\[
u \frac{\partial u}{\partial \nu} + \alpha u = g \quad \text{on} \quad \Gamma_R, \quad \text{(2.3)}
\]

where $\alpha$ is a non-negative constant, $S$ is a bounded and simply-connected domain with the boundary $\partial S = \Gamma_D \cup \Gamma_R$, and $\nu$ is the exterior normal of $\Gamma$. 

Denote in Figure 2.1,

\[ r_{\text{max}} = \max_S r, \quad r_{\text{min}} = \max_{S_{\text{in}}(S \subseteq S)} r, \]  

(2.4)

where \( S_{\text{in}} \) is the maximal disk inside of \( S \). Let the source (charge) points \( Q \)

be located outside of \( S \), the fundamental solutions

\[ \phi(r, \theta) = \ln |\overline{PQ}|, \quad P \in S \cup \partial S \]  

(2.5)

are harmonic, where

\[ P = \{(x, y) \mid x = r \cos \theta, \; y = r \sin \theta\}. \]  

(2.6)

A circle surrounding \( S \) is given by

\[ \ell_R = \{(r, \theta) \mid r = R, \; 0 \leq \theta \leq 2\pi, \; R > r_{\text{max}}\}. \]  

(2.7)

Based on Bogomolny [2], the source points \( Q_i \) may be simply located uniformly on \( \ell_R \)

\[ Q_i = \{(x, y) \mid x = R \cos ih, \; y = R \sin ih\}, \]  

(2.8)
where $R > r_{\text{max}}$ and $h = \frac{2\pi}{N}$. We obtain the fundamental solutions
\[ \phi_i(P) = \ln|PQ_i|, \quad i = 1, 2, \ldots, N, \]
and the numerical solution is given by a linear combination
\[ u_N = \sum_{i=1}^{N} c_i \phi_i(P), \]
where $c_i$ are the unknown coefficients to be sought. Since $u_N$ satisfies Laplace’s equation in $S$ already, the coefficients $c_i$ can be sought by enforcing the boundary conditions (2.2) and (2.3) only. We will follow the Trefftz method (TM) [34] in [22, 27, 29], to seek $u_N$ (i.e., $c_i$). Denote the energy
\[ I(u) = \int_{\Gamma_D} (u - f)^2 + w^2 \int_{\Gamma_R} \left( \frac{\partial u}{\partial \nu} + \alpha u - g \right)^2, \]
where $w$ is a positive weight. We choose $w = \frac{1}{N}$ in our computations (see [22]). Also denote by $V_N$ the space of (2.10). Then the numerical solution $u_N$ can be obtained by
\[ I(u_N) = \min_{v \in V_N} I(v). \]

When the integrals in (2.11) involve approximation, denote
\[ \hat{I}(v) = \int_{\Gamma_D} (v - f)^2 + w^2 \int_{\Gamma_R} \left( \frac{\partial v}{\partial \nu} + \alpha v - g \right)^2, \]
where $\hat{\int_{\Gamma_D}}$ and $\hat{\int_{\Gamma_R}}$ are the numerical approximations of $\int_{\Gamma_D}$ and $\int_{\Gamma_R}$ by some quadrature rules, such as the central or the Gaussian rule. Hence, the numerical solution $\tilde{u}_N \in V_N$ is obtained by
\[ \hat{I}(\tilde{u}_N) = \min_{v \in V_N} \hat{I}(v). \]

We may establish the collocation equations directly from (2.2) and (2.3), to yield
\[ \sum_{i=1}^{N} c_i \phi_i(P_j) = f(P_j), \quad P_j \in \Gamma_D, \]
\[ \sum_{i=1}^{N} c_i \left[ \frac{\partial}{\partial \nu} \phi_i(P_j) + \alpha \phi_i(P_j) \right] = g(P_j), \quad P_j \in \Gamma_R. \]
First, let $\Gamma_D$ and $\Gamma_R$ be divided into small $\Gamma_j^D$ and $\Gamma_j^R$ with the mesh spacings $\Delta h_j$, i.e.,

$$\Gamma_D = \bigcup_{j=1}^{M_1} \Gamma_j^D, \quad \Gamma_R = \bigcup_{j=1}^{M_2} \Gamma_j^R.$$  \hspace{1cm} (2.17)

We obtain from (2.15) and (2.16) by multiplying different weights,

$$\sqrt{\Delta h_j} \sum_{i=1}^{N} c_i \phi_i(P_j) = \sqrt{\Delta h_j} f(P_j), \quad P_j \in \Gamma_j^D, \quad j = 1, 2, \cdots, M_1, \quad (2.18)$$

$$w \sqrt{\Delta h_j} \sum_{i=1}^{N} c_i \left\{ \frac{\partial}{\partial \nu} \phi_i(P_j) + \alpha \phi_i(P_j) \right\} = w \sqrt{\Delta h_j} g(P_j), \quad P_j \in \Gamma_j^R, \quad j = M_1 + 1, \cdots, M_1 + M_2, \quad (2.19)$$

where for simplicity, $P_j$ are the midpoints of $\Gamma_j^D$ and $\Gamma_j^R$. Following Lu et al. [29], Eqs (2.18) and (2.19) are just equivalent to (2.14), where the central rule is chosen for $\hat{\int}_{\Gamma_D}$ and $\hat{\int}_{\Gamma_R}$. In computation, we may choose the number of collocation points to be equal or larger than that of source points, i.e.,

$$M = M_1 + M_2 \geq N. \quad (2.20)$$

When the Gaussian rule is chosen, the following collocation equations are obtained.

$$\beta_j \sum_{i=1}^{N} c_i \phi_i(P_j) = \beta_j f(P_j), \quad P_j \in \Gamma_D, \quad (2.21)$$

$$\beta_j w \sum_{i=1}^{N} c_i \left\{ \frac{\partial}{\partial \nu} \phi_i(P_j) + \alpha \phi_i(P_j) \right\} = \beta_j w g(P_j), \quad P_j \in \Gamma_R, \quad (2.22)$$

where $P_j$ are the Gaussian nodes, the weights $\beta_j = O(\sqrt{\Delta h})$, and $\Delta h = \max_j \Delta h_j$. Eqs. (2.21) and (2.22) (i.e., (2.14)) are called the collocation Trefftz method (CTM) in [27].
2.2.2 Stability

Consider

\[ Fx = b, \quad (2.23) \]

where \( F \in \mathbb{R}^{m \times n} \) (\( m \geq n \)), \( x \in \mathbb{R}^n \) and \( b \in \mathbb{R}^m \). Assume that \( \text{rank}(F) = n \).

The condition number for (2.23) is defined by

\[ \text{Cond} = \frac{\sigma_1}{\sigma_n}, \quad (2.24) \]

where \( \sigma_1 \) and \( \sigma_n \) are the maximal and the minimal singular values of matrix \( F \), respectively. From [24], we may define the effective condition number

\[ \text{Cond}_{\text{eff}} = \frac{\|b\|}{\sigma_n \|x\|}. \quad (2.25) \]

The bound of Cond of MFS for bounded simply-connected domains is given in Li, et al. [25] by

**Theorem 2.2.1.** Let \( \mu \in [1, 2] \) and choose \( w = O\left(\frac{1}{N^{1/2}}\right) \). For (2.1) – (2.3) by the MFS, there exists the bound

\[ \text{Cond}(F) = \frac{\sigma_{\text{max}}(F)}{\sigma_{\text{min}}(F)} \leq CN^{(2+\bar{\delta})} \left( \frac{R}{r_{\min}} \right)^{\frac{
u}{2}}, \quad (2.26) \]

where \( \bar{\delta} = \max\{0, \frac{\mu-1}{2}\} \).

2.3 Error Estimates

In this section, the error analysis is made for Laplace’s equation by the method of fundamental method (MFS), based on [22, 27, 2, 8]. The error bounds are derived for the mixed boundary problems in bounded simply-connected domains. Since the MFS can be classified into the Trefftz method (TM) using the FS, we may follow the analysis of TM in [22, 27], and pay an attention of the errors between harmonic polynomials and the solutions of MFS in [2]. By our analysis, when the Laplace’s solutions are infinitely smooth, the exponential convergence rates can also be achieved as those of the TM in [27] (see Remark 3.1). However, when \( u \in H^p(S)(p > \frac{3}{2}) \), only the polynomial convergence rates are obtained. In this section, we have extended our analysis of the TM to that of the MFS. Then more interesting
results of the analysis of MFS may follow [22, 27]. The MFS is exactly the TM using FS.

Choose

\[ u_N = \sum_{i=1}^{N} c_i \phi_i(r, \theta), \quad (r, \theta) \in S, \tag{2.27} \]

where

\[ \phi_i(r, \theta) = \ln \sqrt{R^2 + r^2 - 2Rr\cos(\theta - \xi_i)}, \tag{2.28} \]

\[ \psi_i(r, \theta) = \frac{\partial}{\partial \nu} \phi_i(r, \theta) = -\frac{a \cos(\theta - \xi_i) - 1}{\rho(a^2 + 1 - 2a\cos(\theta - \xi_i))}, \tag{2.29} \]

with \( \xi_i = ih, h = \frac{2\pi}{N} \) and \( a = \frac{\rho}{\rho} > 1 \). In this section, only the error bounds of \( u_N \) by the TM are derived, and those by the CTM may follow [27, 29].

Let \( V_N \) denote the set of the admissible functions in (2.27). Denote the boundary norm

\[ \|v\|_B = \{\|v\|_{0, \Gamma_D}^2 + w^2 \|\frac{\partial v}{\partial \nu}\|_{0, \Gamma_R}^2\}_\frac{1}{2}, \tag{2.30} \]

where \( \|v\|_{0, \Gamma_D} \) is the Sobolev norm. The solution by the TM, (2.12), also satisfies

\[ \|u - u_N\|_B = \min_{v \in V_N} \|u - v\|_B. \tag{2.31} \]

In Figure 2.1, denote

\[ S_{\text{min}} \subset S \subset S_{\text{max}}, \tag{2.32} \]

where

\[ S_{\text{min}} = \{(r, \theta) | r \leq r_{\text{min}}, 0 \leq \theta \leq 2\pi\}, \]

\[ S_{\text{max}} = \{(r, \theta) | r \leq r_{\text{max}}, 0 \leq \theta \leq 2\pi\}, \tag{2.33} \]

where \( r_{\text{max}} \) and \( r_{\text{min}} \) are given in (2.4). Denote the fundamental solutions

\[ v_N = \sum_{i=1}^{N} c_i \ln |PQ_i|, \tag{2.34} \]

where \( P \in (S \cup \partial S) \), and \( Q_i \) are given in (2.8)

\[ Q_i = (R \cos ih, R \sin ih), \tag{2.35} \]

We obtain the following lemma.
Lemma 2.3.1. Let (2.32) and \( u \in H^p(S)(p > \frac{3}{2}) \) hold. Choose \( w = \frac{1}{N} \). Suppose that \( N \) satisfies
\[
2^{2q+1} \left( \frac{R}{r_{\text{max}}} \right)^{-2N} < 1 \tag{2.36}
\]
with \( q = 1 \) and
\[
\left( \frac{R}{r_{\text{max}}} \right)^{2n-N} \left( \frac{r_{\text{max}}}{r_{\text{min}}} \right)^n \leq n^{-p}, \tag{2.37}
\]
where \( C \) is a constant independent of \( N \). Then the solution \( u_N \) by the MFS has the error bound,
\[
\|u - u_N\|_B = O \left( \frac{1}{N^{p-\frac{1}{2}}} \right). \tag{2.38}
\]

Proof: We have
\[
u(r, \theta) = P_n(r, \theta) + R_n(r, \theta), \tag{2.39}
\]
where \( P_n(r, \theta) \) is the harmonic polynomials of order \( n \),
\[
P_n(r, \theta) = \frac{\alpha_0}{2} + \sum_{i=1}^n r^i (\alpha_i \cos i \theta + \beta_i \sin i \theta), \tag{2.40}
\]
\( \alpha_i \) and \( \beta_i \) are the Fourier coefficients, and \( R_n = R_n(r, \theta) \) is the the residual. From \( u \in H^p(S) \), there exists the bound of \( R_n \),
\[
\|R_n(r, \theta)\|_{q,S} = \|u(r, \theta) - P_n(r, \theta)\|_{q,S} = O\left( \frac{1}{n^{p-q}} \|u\|_{p,S} \right), \tag{2.41}
\]
For the given \( P_n(r, \theta) \), there exists the special solution of FS denoted as \( u_N = \Sigma_N(P_n) \in V_N \) given in Bogomolny [2]. Then we have from (2.31)
\[
\|u - u_N\|_B \leq \|u - u^*_N\|_B. \tag{2.42}
\]
Also from the embedding theorem [7], there exist the bounds,
\[
\|v\|_{0,\Gamma} \leq C \|v\|_{\frac{1}{2},S} , \quad \|\frac{\partial v}{\partial \nu}\|_{0,\Gamma} \leq C \|v\|_{\frac{3}{2},S}, \tag{2.43}
\]
where $C$ is a constant independent\(^2\) of $N$. Then we have from (2.30), (2.43) and $w = \frac{1}{N}$
\[
\|v\|_B \leq \|v\|_{0, \Gamma_D} + w \|\frac{\partial v}{\partial \nu}\|_{0, \Gamma_R} + w \alpha \|v\|_{0, \Gamma_R} \tag{2.44}
\]
\[
\leq (1 + w \alpha) \|v\|_{\frac{1}{2}, S} + w \|v\|_{\frac{3}{2}, S}
\]
\[
\leq C \left\{ \|v\|_{\frac{1}{2}, S} + w \|v\|_{\frac{3}{2}, S} \right\}.
\]

Hence, we obtain from (2.42), (2.39) and the triangle inequality
\[
\|u - u_N\|_B \leq C \left\{ \|u - u_N^*\|_{\frac{1}{2}, S} + w \|u - u_N^*\|_{\frac{3}{2}, S} \right\} \tag{2.45}
\]
\[
\leq C \left\{ \|R_N(v, \theta)\|_{\frac{1}{2}, S} + w \|R_N(v, \theta)\|_{\frac{3}{2}, S} \right\}
\]
\[
+ (\|P_n(v, \theta) - u_N^*\|_{\frac{1}{2}, S} + w \|P_n(v, \theta) - u_N^*\|_{\frac{3}{2}, S}) \right\}
\]
\[
:= C(T_1 + T_2).
\]

Since there exist the bounds from (2.41)
\[
\|R_n\|_{k, S} = O \left( \frac{1}{n^{p-k}} \right) \|u\|_{p, S}, \quad k = \frac{1}{2}, \frac{3}{2}, \tag{2.46}
\]
we have from $w = \frac{1}{N}$
\[
T_1 = \|R_N\|_{\frac{1}{2}, S} + w \|R_N\|_{\frac{3}{2}, S} = O \left( \frac{1}{n^{p-\frac{1}{2}}} \right). \tag{2.47}
\]

Next, for $\Delta v = 0$, we have from Oden and Reddy [30], p. 195,
\[
\|v\|_{k, S} \leq C \|v\|_{k-\frac{1}{2}, \partial S}. \tag{2.48}
\]

Then we have from (2.32), (2.48) and (2.43)
\[
T_2 = \|P_n(r, \theta) - u_N^*\|_{\frac{1}{2}, S} + w \|P_n(r, \theta) - u_N^*\|_{\frac{3}{2}, S} \tag{2.49}
\]
\[
\leq \|P_n(r, \theta) - u_N^*\|_{\frac{1}{2}, S_{\text{max}}} + w \|P_n(r, \theta) - u_N^*\|_{\frac{3}{2}, S_{\text{max}}}
\]
\[
\leq C \left\{ \|P_n(r, \theta) - u_N^*\|_{0, \partial S_{\text{max}}} + w \|P_n(r, \theta) - u_N^*\|_{1, \partial S_{\text{max}}} \right\}.
\]

\(^2\)In the following, $C$ is always a constant independent of $N$, but the values of $C$ may be different in different texts.
Under (2.36), there exists the special solution $u^*_N \in V_N$ of FS such that (see Bogomolny [2])

$$
\|P_n(r, \theta) - u^*_N\|_{q, \partial S_{\text{max}}} \leq CN^q \left( \frac{R}{r_{\text{max}}} \right)^{2n-N} \left( \frac{r_{\text{max}}}{r_{\text{min}}} \right)^n \sqrt{n} \|P_n(r, \theta)\|_{0, \Gamma}.
$$

(2.50)

When (2.36) with $q = 1$ is satisfied, we obtain from (2.49) and (2.50),

$$
T_2 \leq C \left( \frac{R}{r_{\text{max}}} \right)^{2n-N} \left( \frac{r_{\text{max}}}{r_{\text{min}}} \right)^n \sqrt{n} \|P_n(r, \theta)\|_{0, \Gamma}.
$$

(2.51)

From $u \in H^p(S)(p > \frac{3}{2})$, we conclude that $u \in C(S)$ and $P_n(r, \theta) \approx u$ such that $\|P_n(r, \theta)\|_{0, \Gamma} = O(1)$. Eq. (2.51) leads to

$$
T_2 \leq C \left( \frac{R}{r_{\text{max}}} \right)^{2n-N} \left( \frac{r_{\text{max}}}{r_{\text{min}}} \right)^n \sqrt{n}.
$$

(2.52)

Moreover, when Eq. (2.37) holds, i.e.,

$$
N = 2n + \frac{n \ln (\frac{r_{\text{max}}}{r_{\text{min}}}) + p \ln n}{\ln (\frac{R}{r_{\text{max}}})} \leq Cn,
$$

(2.53)

We have

$$
T_2 \leq C \frac{1}{n^{p-\frac{3}{2}}},
$$

(2.54)

Combining (2.45), (2.47) and (2.54) gives the desired result (2.38) by noting $\frac{1}{n} \leq C\frac{1}{N}$. This completes the proof of Lemma 2.3.1.

**Lemma 2.3.2.** Suppose that there exists a positive constant $\mu$ independent of $N$ such that

$$
|v|_{1, \Gamma_D} \leq CN^\mu \|v\|_{0, \Gamma_D}, \ v \in V_N,
$$

(2.55)

where $V_N$ is the set of (2.10). For $\Delta v = 0$, there exists the bound,

$$
\|v\|_{1, S} \leq C \left( N^2 + \frac{1}{w} \right) \|v\|_B.
$$

(2.56)

---

3In the recent study in [20], the factor $\sqrt{n}$ in (2.50) can be removed.
Proof: For the harmonic functions \( v \in V_N \) in \( S \), we have from Oden and Reddy [30]

\[
\|v\|_{1,S} \leq C \left\{ \|v\|_{\frac{1}{2},\Gamma_D} + \left\| \frac{\partial v}{\partial \nu} + \alpha v \right\|_{-\frac{1}{2},\Gamma_R} \right\}. \tag{2.57}
\]

Since (2.55) we have

\[
\|v\|_{\frac{1}{2},\Gamma_D} \leq C \{ \|v\|_{1,\Gamma_D} \|v\|_{0,\Gamma_D} \}^{\frac{1}{2}} \leq CN^\mu \|v\|_{0,\Gamma_D} \tag{2.58}
\]

and

\[
\left\| \frac{\partial v}{\partial \nu} + \alpha v \right\|_{-\frac{1}{2},\Gamma_R} \leq C \left\| \frac{\partial v}{\partial \nu} + \alpha v \right\|_{0,\Gamma_R}. \tag{2.59}
\]

Then we have from (2.57) – (2.59) for \( \Delta v = 0\),

\[
\|v\|_{1,S} \leq C \left\{ \|v\|_{\frac{1}{2},\Gamma_D} + \left\| \frac{\partial v}{\partial \nu} + \alpha v \right\|_{-\frac{1}{2},\Gamma_R} \right\} \tag{2.60}
\]

\[
\leq C \left\{ N^\mu \|v\|_{0,\Gamma_D} + \left\| \frac{\partial v}{\partial \nu} + \alpha v \right\|_{0,\Gamma_R} \right\}
\]

\[
\leq C \left( N^\mu + \frac{1}{w} \right) \|v\|_B.
\]

This completes the proof of Lemma 2.3.2.

From Lemmas 2.3.1 and 2.3.2, we have the following theorem.

Theorem 2.3.1. Let the conditions in Lemmas 2.3.1 and 2.3.2 hold. Then the numerical solutions by the MFS (i.e., the TM using FS) have the error bound,

\[
\|u - u_N\|_{1,S} = O \left( \frac{1}{N^{p-t}} \right), \tag{2.61}
\]

where \( t = \frac{1}{2} + \max\{1, \frac{\mu}{2}\} \).

To obtain the inverse inequality (2.55), we may also use a similar bound in Katsurada and Okamoto [14], Theorem A.1. In fact, a detailed analysis of (2.55) is given in [25], to show \( \mu \in [1,2] \) for some simple cases.
Remark 3.1. For the infinite smooth function \( u \), we have 
\[
 u = P_n(r, \theta) + R_n(r, \theta),
\]
where the residual \( R_n \) is exponential convergence,
\[
 \|R_n(r, \theta)\|_{1,S} = O(\sigma_1^n), \quad 0 < \sigma_1 \leq \bar{\sigma} < 1. \quad (2.62)
\]
Hence we may choose the \( N \) such that
\[
 n^2 \left( \frac{R}{r_{\text{max}}} \right)^{2n-N} \left( \frac{r_{\text{max}}}{r_{\text{min}}} \right)^n \sqrt{n} \leq \sigma_2^n, \quad 0 < \sigma_2 < \sigma \leq 1, \quad (2.63)
\]
i.e.,
\[
 N = 2n + \frac{n \ln \left( \frac{r_{\text{max}}}{r_{\text{min}}} \right) + \ln \left( \frac{1}{\sigma_2} + (q + \frac{1}{2}) \ln n \right)}{\ln \left( \frac{R}{r_{\text{max}}} \right)} \leq Cn. \quad (2.64)
\]
Hence, by following the proof of Theorem 2.3.1, we have the exponential convergence,
\[
 \|u - u_N\|_{B} \leq \|u - u_N^*\|_{B} \leq \|u_N^* - P_n(r, \theta)\|_{1,S} + \|R_n(r, \theta)\|_{1,S} 
\leq C \left\{ \sigma_1^n + \sigma_2^n \left( \frac{N}{n} \right)^q \right\}
\leq O(\sigma_1^n + \sigma_2^n).
\]

2.4 Numerical Experiments
Consider Motz’s problem (see Figure 2.2)
\( \Delta u = 0, \quad (2.66) \)
\( u = 500 \quad \text{on} \ \overline{AB}, \quad \text{on} \ \overline{DO}, \quad (2.67) \)
\( \frac{\partial u}{\partial n} = 0 \quad \text{on} \ \overline{BC} \cup \overline{CD} \cup \overline{OA}. \quad (2.68) \)
There exists a singularity at \( O \) due to the intersection of the Dirichlet and the Neumann boundary conditions. Since the fundamental functions (2.9) are smooth on \( \partial S \), we first add the singular functions into (2.10), to obtain
\[
 u_N = \sum_{i=0}^{L} D_i r^{i+1} \cos \left( i + \frac{1}{2} \right) \theta + \sum_{i=1}^{N} c_i \phi_i(P), \quad (2.69)
\]
where the singular particular solutions

\[ \sum_{i=1}^{L} D_i r^{i+\frac{1}{2}} \cos \left( i + \frac{1}{2} \right) \theta \]  

(2.70)

are given in [22]. To deal with the singularity, the admissible functions as in (2.69) are also in [1, 3] for an enrich MFS domain decomposition technique. Let \( M \) denote the number of collocation points along \( AB \). Then the total number of collocation points on \( \partial S \) in Figure 2.2 is \( m = 6M \). The number of unknown coefficients in (2.69) is \( L + N + 1 \). In our computation we choose

\[ m = 6M \geq L + N + 1 = n. \]  

(2.71)

From [29], using the Gaussian rule is beneficial to the accuracy of the singular coefficient \( D_i \) in (2.70). Then we choose the Gaussian rules.
Define the boundary errors

$$\| \varepsilon \|_B = \left\{ \int_{AB \cup DO} \varepsilon^2 + \int_{BC \cup CD \cup OA} \left( \frac{\partial \varepsilon}{\partial \nu} \right)^2 \right\}^{\frac{1}{2}},$$

where \( \varepsilon = u - \bar{u}_N \). We use the SVD to solve (2.18) and (2.19). Choose \( R = \sqrt{3} \) and \( L = 3 \) in (2.69), the errors and condition numbers are listed in Table 2.1. From Table 2.1, we can see the empirical rates for \( N \leq 70 \)

$$\| \varepsilon \|_B = O(0.8N), \quad \left| \frac{\Delta D_0}{D_0} \right| = O(0.8N), \quad (2.72)$$

and

$$\text{Cond} = O(1.8N), \quad \text{Cond}_{\text{eff}} = O(1.2N). \quad (2.73)$$

From (2.72) and (2.73), both errors and condition numbers are exponential, with respect to \( N \). Since \( R = \sqrt{3} \) and \( r_{\text{min}} = 0.5 \), we have from Theorem 2.2.1,

$$\text{Cond}(F) \leq CN^{(2+\tilde{\delta})} \left( \frac{\sqrt{3}}{0.5} \right)^N = CN^{(2+\tilde{\delta})}(1.86)^N, \quad (2.74)$$

where \( \tilde{\delta} = \max \{ 0, \mu^{-\frac{1}{2}} \} \) with \( \mu \in [1, 2] \). Evidently, Eqs. (2.73) and (2.74) are consistent with each other. Note that the growth rates of \( \text{Cond}_{\text{eff}} \) are lower than those of \( \text{Cond} \), to indicate that \( \text{Cond}_{\text{eff}} \) is a better estimate on the true stability of rounding errors, also see [24].

From Table 2.1 we can see that the norm \( \| x \| \) is large, which is caused by small \( \sigma_{\text{min}} \). After \( D_i \) and \( c_i \) have been obtained, the final harmonic solutions are computed by (2.69). Note that since \( R - r_{\text{max}} \geq c_0 > 0 \), \( |\phi(P)| = \ln |\overline{PQ_i}| \) is bounded. Also the harmonic solutions of (2.66) – (2.68) satisfy that \( |\mu_N| \leq 1 \). Then there must occur the severe subtraction cancelation in (2.69), which is another kind of instability. On the other hand, the \( \text{Cond} \) and the \( \text{Cond}_{\text{eff}} \) are used for the stability analysis for \( D_i \) and \( c_i \) by MFS, to evaluate the ill-conditioning of their rounding errors. Hence the final solutions (2.69) suffer in severe instability from both rounding errors and subtraction cancelation.
The MFS for Laplace Equation

<table>
<thead>
<tr>
<th>N</th>
<th>28</th>
<th>56</th>
<th>70</th>
<th>84</th>
<th>112</th>
</tr>
</thead>
<tbody>
<tr>
<td>M</td>
<td>20</td>
<td>40</td>
<td>50</td>
<td>60</td>
<td>80</td>
</tr>
<tr>
<td>||ε|B |</td>
<td>5.364(-5)</td>
<td>5.811(-8)</td>
<td>1.982(-8)</td>
<td>6.371(-9)</td>
<td>1.319(-9)</td>
</tr>
<tr>
<td>D0</td>
<td>401.159855</td>
<td>401.162417</td>
<td>401.162434</td>
<td>401.162446</td>
<td>401.162451</td>
</tr>
<tr>
<td>D1</td>
<td>87.5677622</td>
<td>87.6556309</td>
<td>87.6554638</td>
<td>87.6558065</td>
<td>87.6558657</td>
</tr>
<tr>
<td>D2</td>
<td>16.9236032</td>
<td>17.1955434</td>
<td>17.2084924</td>
<td>17.2189297</td>
<td>17.2271946</td>
</tr>
<tr>
<td>D3</td>
<td>-10.319633</td>
<td>-8.0661360</td>
<td>-8.1168024</td>
<td>-8.0720120</td>
<td>-8.0736890</td>
</tr>
<tr>
<td>β</td>
<td>1.572(-5)</td>
<td>7.370(-8)</td>
<td>-9.758(-9)</td>
<td>-4.665(-9)</td>
<td>-4.640(-10)</td>
</tr>
<tr>
<td>∥x∥</td>
<td>4.346(2)</td>
<td>1.297(4)</td>
<td>7.901(5)</td>
<td>1.660(8)</td>
<td>4.445(12)</td>
</tr>
<tr>
<td>∥b∥</td>
<td>1.118(2)</td>
<td>7.906(1)</td>
<td>7.071(1)</td>
<td>6.455(1)</td>
<td>5.590(1)</td>
</tr>
<tr>
<td>σ_{max}</td>
<td>9.967(-1)</td>
<td>9.787(-1)</td>
<td>9.752(-1)</td>
<td>9.729(-1)</td>
<td>9.700(-1)</td>
</tr>
<tr>
<td>σ_{min}</td>
<td>6.106(-7)</td>
<td>5.851(-12)</td>
<td>1.308(-14)</td>
<td>3.727(-17)</td>
<td>2.381(-22)</td>
</tr>
<tr>
<td>Cond</td>
<td>1.632(6)</td>
<td>1.672(11)</td>
<td>7.458(13)</td>
<td>2.610(16)</td>
<td>4.074(21)</td>
</tr>
<tr>
<td>Cond_{eff}</td>
<td>4.213(5)</td>
<td>1.042(9)</td>
<td>6.844(9)</td>
<td>1.043(10)</td>
<td>5.282(10)</td>
</tr>
<tr>
<td>Cond_{EE}</td>
<td>7.112(6)</td>
<td>1.073(9)</td>
<td>7.269(9)</td>
<td>1.384(10)</td>
<td>1.205(11)</td>
</tr>
</tbody>
</table>

Table 2.1: The errors and condition numbers for Motz’s problem by the MFS with (2.69) for \( R = \sqrt{3} \) and \( L = 3 \), where \( M \) is the number of Gaussian nodes on \( \overline{AB} \).

For the singularity at \( O \) in Figure 2.2, the second technique is to adopt the local refinements of \( P_j \) near \( O \) (see Figure 2.2),

\[
|\overline{OP_j}| = (jh)^q, \quad j = 1, 2, \ldots (2.75)
\]

where \( q \geq 1 \). Choose (2.10) with the FS only, where the coefficients are obtained from MFS. Afterwards, the leading coefficients in (2.70) can be evaluated by

\[
D_\ell = \frac{2}{\pi r^{\ell+\frac{1}{2}}} \int_0^\pi u_N(r, \theta) \cos \left( \ell + \frac{1}{2} \right) \theta \, d\theta, \quad \ell = 0, 1, 2, 32.76)
\]

The leading coefficient \( D_0 \) indicates the intensity factor, which is important in both theory and application.

The uniform collocation points with \( h = \frac{1}{M} \) are used on \( \partial S \), except those on \( \overline{OE} \cup \overline{OF} \) in Figure 2.2, where the collocation points \( P_j \) are given
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by (2.75). In computation, we choose $|OF| = |OE| = \frac{1}{8}$ and retain the invariant $M$ of collocation points on $OA$ and $OD$. The radius $R = \sqrt{3}$ in (2.7) and $q = 4$ in (2.75) are chosen, based on our numerical experiments.

For Motz’s problem, the errors and condition numbers are listed in Table 2.2, and the coefficients $c_i$ and $D_i$ in Table 2.3.

<table>
<thead>
<tr>
<th>$N$</th>
<th>28</th>
<th>42</th>
<th>56</th>
<th>70</th>
<th>84</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>20</td>
<td>30</td>
<td>40</td>
<td>50</td>
<td>60</td>
</tr>
<tr>
<td>$|\varepsilon|_B$</td>
<td>0.877(-2)</td>
<td>0.214(-3)</td>
<td>0.864(-5)</td>
<td>0.274(-6)</td>
<td>0.130(-7)</td>
</tr>
<tr>
<td>$D_0$</td>
<td>402.091842</td>
<td>405.634884</td>
<td>401.645653</td>
<td>403.9010585</td>
<td>401.5045143</td>
</tr>
<tr>
<td>$|\Delta D_0|_2$</td>
<td>0.232(-2)</td>
<td>0.112(-1)</td>
<td>0.120(-2)</td>
<td>0.683(-2)</td>
<td>0.853(-3)</td>
</tr>
<tr>
<td>$|x|_{\infty}$</td>
<td>85.4862012</td>
<td>83.1566289</td>
<td>86.5255205</td>
<td>84.9189393</td>
<td>86.8840334</td>
</tr>
<tr>
<td>$|\Delta D_1|_2$</td>
<td>0.248(-1)</td>
<td>0.513(-1)</td>
<td>0.129(-1)</td>
<td>0.683(-1)</td>
<td>0.881(-2)</td>
</tr>
<tr>
<td>$|D_1|_4$</td>
<td>17.2046114</td>
<td>16.3452515</td>
<td>16.7878829</td>
<td>16.3189306</td>
<td>16.63959418</td>
</tr>
<tr>
<td>$|\Delta D_2|_2$</td>
<td>0.193(-2)</td>
<td>0.518(-1)</td>
<td>0.261(-1)</td>
<td>0.533(-1)</td>
<td>0.347(-1)</td>
</tr>
<tr>
<td>$|D_2|_4$</td>
<td>-7.7106008</td>
<td>-8.1404903</td>
<td>-7.8077693</td>
<td>-8.0527179</td>
<td>-7.8482114</td>
</tr>
<tr>
<td>$|\Delta D_3|_2$</td>
<td>0.502(5)</td>
<td>0.214(-7)</td>
<td>0.935(8)</td>
<td>0.471(10)</td>
<td>0.220(12)</td>
</tr>
<tr>
<td>$|D_3|_4$</td>
<td>0.903(-1)</td>
<td>0.533(-7)</td>
<td>0.326(-1)</td>
<td>0.229(-2)</td>
<td>0.698(-6)</td>
</tr>
<tr>
<td>$|x|_{\infty}$</td>
<td>112</td>
<td>91.3</td>
<td>79.1</td>
<td>70.7</td>
<td>64.5</td>
</tr>
<tr>
<td>$\sigma_{\text{max}}$</td>
<td>0.865</td>
<td>0.865</td>
<td>0.865</td>
<td>0.865</td>
<td>0.865</td>
</tr>
<tr>
<td>$\sigma_{\text{min}}$</td>
<td>0.293(-6)</td>
<td>0.257(-9)</td>
<td>0.153(-11)</td>
<td>0.150(-15)</td>
<td>0.983(-19)</td>
</tr>
<tr>
<td>Cond</td>
<td>0.295(7)</td>
<td>0.337(10)</td>
<td>0.566(13)</td>
<td>0.576(16)</td>
<td>0.880(19)</td>
</tr>
<tr>
<td>Cond$_{\text{eff}}$</td>
<td>0.758(4)</td>
<td>0.166(6)</td>
<td>0.553(7)</td>
<td>0.999(8)</td>
<td>0.298(10)</td>
</tr>
<tr>
<td>Cond$_{\text{EE}}$</td>
<td>0.794(4)</td>
<td>0.171(6)</td>
<td>0.566(7)</td>
<td>0.101(9)</td>
<td>0.303(10)</td>
</tr>
</tbody>
</table>

Table 2.2: The error norms and condition numbers for Motz’s problem by the MFS with local refinements for $R = \sqrt{3}$ and $q = 4$.

From Table 2.2, the following empirical rates can be observed:

$$
\|\varepsilon\|_B = O(0.74^N), \quad \frac{\Delta D_0}{D_0} = O(0.97^N), \quad (2.77)
$$

$$
\sigma_{\text{max}} = O(1), \quad \sigma_{\text{min}} = O(0.487^N), \quad (2.78)
$$

$$
\text{Cond} = O(2.05^N), \quad \text{Cond}_{\text{eff}} = O(1.37^N), \quad (2.79)
$$

$$
\|x\| = O(1.47^N).
$$

Evidently, the Cond$_{\text{eff}}$ is much smaller than Cond. The norm $\|x\|$ is large. Note that some coefficients $c_i$ in Table 2.3 are huge and highly oscillating. Hence, the subtraction cancelation occurs in the final harmonic solution.
The MFS for Laplace Equation

| D₀ | 401.504514344611 | D₁ | 86.8840334693767 |
| D₂ | 16.6395941822623 | D₃ | -7.8482111479329 |
| ... | ... | ... | ... |
| c₁₇ | 5.12088955005D+07 | c₅₉ | 1.65586759379D+11 |
| c₁₈ | -7.1686899498D+07 | c₆₀ | -5.5761144558D+11 |
| c₁₉ | 8.95398609608D+07 | c₆₁ | 6.34402739441D+11 |
| c₂₀ | -1.9648035454D+07 | c₆₂ | -9.2121325213D+11 |
| c₂₁ | 9.86061783032D+07 | c₆₃ | 1.09312719485D+12 |
| c₂₂ | -8.670430339D+07 | c₆₄ | -1.059956431D+12 |
| c₂₃ | 6.7722646301D+07 | c₆₅ | 8.41202271679D+11 |
| c₂₄ | -4.6990352789D+07 | c₆₆ | -5.419461868D+11 |
| c₂₅ | 2.89846804493D+07 | c₆₇ | 2.9489130335D+11 |
| c₂₆ | -1.591631056D+07 | c₆₈ | -1.3193813281D+11 |
| c₂₇ | 7.80158417112D+06 | c₆₉ | 4.96146614829D+10 |
| ... | ... | ... | ... |

Table 2.3: All coefficients for Motz’s problem by the MFS with local refinements with $R = \sqrt{3}$, $q = 4$, $M = 60$ and $N = 84$, where the order of $c_i$ is shown as that of $Q_i$ in Figure 2.2.

From (2.79) we can see

$$\text{Cond} \approx \text{Cond}_{\text{eff}} \|x\|.$$  \hfill (2.80)

### 2.5 Concluding Remarks

For numerical PDEs, once a method is designed by numerical algorithms with some preliminary computation, three important issues of analysis need to be studied: (1) Convergence with error bounds, (2) Stability with bounds of condition numbers, (3) Combined algorithms with other numerical methods, \textit{since no method is perfect}. A numerical method, such as the finite element method (FEM), can stand as a competitive and popular method if its efficiency has been proven numerically, and if the theoretical analysis of the above three issues has been established. The book [22] focuses the third issue: combinations of different methods, accompanied with error and stability analysis. One important method in [22] is called the boundary approximation method (BAM), which is the very Trefftz method (TM).

Below we discuss the MFS. Since there have appeared numerous (even uncountable) reports on computations by MFS, and the first Int. Work-
I. Error Analysis. Since MFS is a special one of TM using fundamental solutions (FS) as basis functions (or admissible functions), the analytical framework of TM in [28, 21, 22] and the recent book [27] is also valid for MFS. Take Laplace’s equation for example. When the harmonic polynomials are chosen, the error bounds of TM are studied in [26]. Since the extra errors between FS and harmonic polynomials can be found from (2.50) in Bogomolny [2], the error bounds of the solutions by MFS are obtained. By following [2], the analysis of MFS for annular domains is given in [20], to retain the same polynomial convergence rates as in [2]. For Laplace’s equation, the error bounds of MFS are derived for the Dirichlet, the Neumann [23] and the Robin problems, as well as their mixed boundary problems in this paper. The error bounds can also be derived for the problems with interior boundary conditions, so that different FS with different source circles can be applied simultaneously, for complicated geometrical domains.

II. Stability Analysis. For MFS, the instability is the most severe issue of both computation and analysis, since both Cond and Cond_eff are huge, and since they grow exponentially as the number of fundamental solutions increases. From (2.72) & (2.73) and (2.77) & (2.79), we can see

\[
\text{Cond} = O\left(\left(\frac{1}{\|\epsilon\|_B}\right)^\alpha\right), \quad \alpha > 1. \tag{2.81}
\]

Eq. (2.81) indicates that the rate of Cond grows faster than that of the error diminishing. Usually, the Cond and Cond_eff are so huge that the computation of MFS for problems of PDE is difficult under double precision. Of course, we may solicit Mathematics using unlimited working digits, to produce useful numerical solutions. However, the more the working digits are used, the more the CPU time and the computer storage are needed. From our experiments in Chen [4] to seek the extremely \(D_0\) in (2.69) with more than 10000 significant digits, there exists the empirical relation,

\[
\text{CPU Time} = O(p^{2.4}), \tag{2.82}
\]

where \(p\) is the number of working digits.
Moreover, the ill-conditioning of MFS results from the infinitesimal \( \sigma_{\min} \) of the discrete matrix, which also causes the huge, oscillating coefficients \( c_i \) in (2.69). Hence, for the final harmonic solution \( u_N \) from (2.69), a new instability of substraction cancelation occurs, to lose \( \ln_{10}(\|x\|) \) significant digits. Hence, the MFS is a monster in instability, who needs to be controlled. To reduce the ill-conditioning, the truncated singular value decomposition (TSVD) and the Tikhonov regularization should be employed, in order to control the ill-conditioning of MFS. For stability analysis of MFS, bounds of condition numbers are derived for Dirichlet, Neumann and Robin problems, as well as their mixed boundary problems not only for circular domains but also for bounded, simply-connected domains. Theorem 2.1 is for the mixed boundary problems in bounded simply-connected domains; details are reported in [25, 23]

III. Combined Methods of MFS. From Section 2.3, for Laplace’s equation, the errors of MFS will not be smaller than those of TM using harmonic polynomials. It is also proven in Schaback[31] that when \( R \to \infty \), the FS goes to the harmonic polynomials. However, the instability of MFS (even with small \( R \)) is worse than that of TM using harmonic polynomials [27]. Hence, using TM with harmonic polynomials is superior to the MFS. Moreover, the MFS are not efficient for the corner singularity, either, except by some techniques, such as adding singular functions or local refinements in Section 4. On the other hand, the MFS is simple and easy for some domains with curved boundary. A nature strategy is to use the singular solutions to deal the corner singularity, and the FS for the rest of \( S \). Hence, the TM using both FS and the singular solutions should first be adapted in the TM. By following [22], the algorithms and error analysis of combinations of MFS can be established. For the non-homogeneous PDE, or those with variable coefficients, the FEM is efficient. Hence for the complicated problems, we may combine FMS with FEM. Of course, the MFS can also be combined with other methods, such as the finite difference method (FDM), the finite volume method (FVM), etc. Details are reported in [19].

References

References


CHAPTER 3

Reconstruction of Inclusions or Cavities in Potential Problems Using the MFS

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Abstract. We discuss the application of the Method of Fundamental Solutions (MFS) to an inverse potential problem that consists of detecting inclusions or cavities using a single boundary measurement on an external boundary. The application of the method of fundamental solutions presented here is closely related to a method introduced by Kirsch and Kress [12], in another context – obstacle detection in an exterior acoustic inverse scattering problem. We briefly address the identifiability questions on the shape reconstruction, presenting a counterexample for the case of Robin boundary conditions. Using fundamental solutions on auxiliary boundary curves we prove density results for separated data on the whole of the boundary or for Cauchy data on the accessible part of the boundary. This justifies the reconstruction of the solution from the Cauchy data using the MFS. Moreover, this connection relates the linear part of the Kirsch-Kress method to the direct MFS, using boundary operator matrices. Numerical examples are presented, showing good reconstructions for the shape and location with this MFS version of the Kirsch-Kress method. Although the method proved to be robust for noisy data, results can be improved using an iterative Quasi-Newton scheme here presented.

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3.1 Introduction
The Method of Fundamental Solutions has been mostly considered as a meshfree method for direct problems, since the first papers by Kupradze and Alekside [13], Oliveira [8] or Mathon and Johnston [16]. An excellent account of the MFS as a numerical method for direct problems can be found in the review paper by Fairweather and Karageorghis [9].

On the other hand the MFS is also being considered as a tool for inverse problems, mainly for Cauchy data reconstruction. An example of this application can be found in the early work by Kirsch and Kress [12], twenty years ago, where it was used to fit the boundary data in an external problem, in the context of obstacle detection for exterior acoustic inverse scattering problems. Recently, it has gained some popularity as a method to solve linear inverse problems, that involve Cauchy reconstruction (e.g. [14]).

In this work, we aim to reconstruct the shape and location of the object using a single boundary Cauchy measurement on an accessible part of the external boundary. The buried object is either a rigid inclusion (defined by a Dirichlet boundary condition), a cavity (defined by a Neumann type boundary condition) or a more general inclusion (defined by a Robin type boundary condition).

The inverse problem that consists of the identification of inclusions or cavities in potential problems using single external boundary measurements, is a problem in nondestructive testing. It has been addressed in the literature, with applications to thermal imaging and corrosion detection (see for instance [4], [10], [11]) and more recently in [5] where unknown Robin boundary conditions were also considered. The identifiability questions are discussed in Section 2. In Section 3 we consider the numerical resolution of the inverse problem using the MFS. Following the original ideas of the Kirsch-Kress Method (KKM) as in [2], the numerical scheme can be divided in two parts:

(i) linear part - uses the MFS as a solver for the Cauchy problem,
(ii) nonlinear part - recover the level set.

This connection is made clear by analysing the operator matrices with boundary potentials for both the MFS and the KKM.

Density results play an important role for the justification of the MFS approximation (e.g. [3] for boundary approximations, [1] for domain ap-
proximations). Unlike the classical case, the density results presented here allow for the approximation of both function and normal derivative, therefore justifying the Cauchy reconstruction.

In Section 3 we also present a method based on a reciprocity gap functional, introduced by Caldéron [6], to approximate the barycenter of a rigid inclusion.

Finally, in Section 4, several numerical examples show the potential of the method. It proved to be a fast numerical scheme that enables a good approximation of the location and shape of the unknown inclusion even with noisy data. To improve the approximations we use a Levenberg–Marquardt method, already presented in [15] (in the context of a Stokes problem), which lead to even more accurate results.

3.2 Direct and Inverse Problem

Consider a medium \( \Omega \subset \mathbb{R}^d \) (\( d = 2,3 \)) with inclusions or cavities represented by \( \omega \). We assume that \( \Omega \) is an open bounded simply connected set and \( \omega \) is open, bounded and the disjoint union of simply connected sets, with regular \( C^1 \) boundaries \( \Gamma = \partial \Omega \) and \( \gamma = \partial \omega \) such that \( \omega \subset \Omega \). We define the domain of propagation by

\[
\Omega_c := \Omega \setminus \bar{\omega}.
\]

Note that \( \partial \Omega_c = \Gamma \cup \gamma \).

**Direct problem:**

Given \( g \in H^{1/2}(\Gamma) \), determine \( \partial_n u \) on \( \Gamma \), such that \( u \) that satisfies

\[
(P_{\Omega_c}) \quad \begin{cases} 
\Delta u = 0 & \text{in } \Omega_c \\
 u = g & \text{on } \partial \Omega = \Gamma \\
 B u = 0 & \text{on } \partial \omega = \gamma
\end{cases}
\]

\( B \) stands for one of the classic boundary trace operators: \( B u = \tau_\gamma u \) (Dirichlet, \( \omega \) is a rigid inclusion) or \( B u = \tau_\gamma^u u \) (Neumann, \( \omega \) is a cavity) or \( B u = \tau_\gamma^u u + Z \tau_\mu u \) (Robin, \( \omega \) is an inclusion). This direct problem is well–posed with solution \( u \in H^1(\Omega_c) \) (for the Robin condition, we must have \( Z \in L^\infty(\gamma) \) and \( Z \geq 0 \)).

**Inverse problem:**
Reconstruction of inclusions or cavities using the MFS

From the known (input function) \( g \) and the measured (output function) \( \partial_n u \) on a part \( \Sigma \) of the external boundary \( \Gamma = \partial \Omega \), we aim to identify the location and shape of the internal boundary \( \gamma = \partial \omega \) (and therefore the inclusion or the cavity \( \omega \)). It is well known that this is an ill posed and non linear inverse problem.

**Remark 1.** The direct problem will also be addressed in the more general formulation

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega_c \\
\tilde{B}u = \tilde{g} & \text{on } \partial \Omega_c
\end{cases}
\]  

(3.1)

where \( \tilde{B} \) is the boundary operator defined by

\[
\tilde{B} = a \tau_{\partial \Omega_c}^n + \tilde{Z}_a \tau_{\partial \Omega_c}
\]  

(3.2)

with \( a \in \{0,1\} \) continuous on \( \partial \Omega_c \) and on the set \( \sigma = a^{-1}(0) \) we have \( \tilde{Z}_a|\sigma = 1 \). This latter restriction on \( \tilde{Z}_a \) gives the Dirichlet boundary condition on \( \Gamma \) or \( \gamma \). For a pure Neumann problem, i.e. \( a = 1 \) and \( \tilde{Z}_a = 0 \) both on \( \partial \Omega_c \), it is well–known that, in order to obtain uniqueness, a compatibility condition is required.

### 3.2.1 Uniqueness

As shown in [5], a single boundary measurement may not suffice to identify a part of the boundary with a Robin condition. For the particular case of identification of inclusions \( \omega \), we present the following example:

Consider the harmonic function in \( \mathbb{R}^2 \setminus \{0\} \) defined by

\[
u(x, y) = x - \frac{x}{x^2 + y^2}
\]  

(3.3)

and the annular domain \( \Omega_c = \Omega \setminus \omega \) where \( \Omega = B(0,P) = \{ x \in \mathbb{R}^2 : |x| < P \} \), \( \omega = B(0,\rho) \) and \( 0 < \rho < P \). On \( \gamma = \partial \omega \), we have

\[
\partial_n u|_\gamma = \frac{1 + \rho^2}{\rho^2} n.e_1 \wedge u|_\gamma = \frac{\rho^2 - 1}{\rho} n.e_1.
\]

Hence, \( u \) solves the direct problem

\[
\begin{cases}
\Delta u = 0 & \text{in } \Omega_c \\
u = g & \text{on } \Gamma \\
\partial_n u + Z_\rho u = 0 & \text{on } \gamma
\end{cases}
\]
where $g$ is the restriction of $u$ to $\Gamma = \partial \Omega$ and

$$Z_{\rho} \equiv \frac{1 + \rho^2}{\rho(1 - \rho^2)}.$$ 

The function $\rho \to \frac{1 + \rho^2}{\rho(1 - \rho^2)}$ is not injective for $0 < \rho < 1$ and we can conclude that at least two circular inclusions generate the same Cauchy data on $\Gamma$.

The previous example also shows that the identification, from a single boundary measurement, is not possible for non homogeneous boundary conditions on the inclusion/cavity.

We now address the identification of inclusions or cavities defined by homogeneous Dirichlet or Neumann conditions.

**Theorem 2.** Assume that the boundary condition on the inclusion or cavity $\gamma$ is defined by a homogeneous Dirichlet or Neumann condition. If $\Sigma \subset \Gamma$ is an open set in the topology of $\Gamma$ and $g$ is not constant (null for the case of inclusions), the Cauchy data on $\Sigma$ determines $\gamma$ uniquely.

**Proof.** Suppose that $\Omega^1_\epsilon$ and $\Omega^2_\epsilon$ are different non-disjoint propagation domains with boundaries

$$\partial \Omega^1_\epsilon = \Gamma \cup \gamma_1, \quad \partial \Omega^2_\epsilon = \Gamma \cup \gamma_2,$$

where $\gamma_j$ refer to the boundary of the inclusion/cavity $\omega_j$. Each $\omega_j$ ($j = 1, 2$) can be the disjoint union of several simply connected components

$$\omega_j = \bigcup \omega_{j,k} \quad \text{and therefore} \quad \gamma_j = \bigcup \gamma_{j,k}.$$

Denote by $u_1$ and $u_2$ the solutions of problems $(P_{\Omega^1_\epsilon})$ and $(P_{\Omega^2_\epsilon})$ respectively. We show that, if

$${\tau}_\Sigma u_1 = {\tau}_\Sigma u_2, \quad {\tau}_\Sigma^g u_1 = {\tau}_\Sigma^g u_2,$$

then $u_1$ is constant (null, for the case of inclusions) in $\Omega^\sim_\epsilon$, where $\Omega^\sim_\epsilon$ denotes the connected component of $\Omega^1_\epsilon \cap \Omega^2_\epsilon$ that contains $\Gamma$. By Holmgren’s theorem, the same Cauchy data on $\Sigma$ implies

$$u_1 = u_2 \text{ in } \Omega^\sim_\epsilon.$$
Now, $\partial\Omega^\sim = \Gamma \cup \gamma^\sim \cup \gamma_2$ with $\gamma_j^\sim \subset \gamma_j$ and $\gamma_1^\sim \cap \gamma_2^\sim = \emptyset$. Without loss of generality assume that $\gamma_2^\sim$ is not empty and let $\omega_{2,k}$ be a simply connected component such that $\gamma_2^\sim \cap \omega_{2,k}$ is not empty. If $\Omega_1^\sim \cap \Omega_2^\sim$ is connected, i.e., $\Omega_1^\sim \cap \Omega_2^\sim = \Omega^\sim$, then take $\sigma = \omega_{2,k} \setminus \overline{\omega_1} \subset \Omega_1^\sim$ which is an open set with boundary $\partial\sigma \subset \gamma_2^\sim \cup \gamma_1$. It is clear that $\Delta u_1 = 0$ in $\sigma$ and on $\gamma_1$ we have null Dirichlet/Neumann data. By analytic continuation, $u_1$ has also null Dirichlet/Neumann data on $\gamma_2^\sim$.

If $\Omega_1^\sim \cap \Omega_2^\sim$ is not connected, take $\sigma$ as the connected component of $\Omega_1^\sim \setminus \overline{\Omega_2^\sim}$ that intersects $\omega_{2,k}$. Again, $\partial\sigma \subset \gamma_2^\sim \cup \gamma_1$ and in both cases we have
\[
\begin{align*}
\Delta u_1 &= 0 \quad \text{in } \sigma, \\
\mathcal{B}u_1 &= 0 \quad \text{on } \partial\sigma,
\end{align*}
\]
where $\mathcal{B}$ is the trace or the normal trace operator. Thus, $u_1$ is constant on $\sigma$ (null, for the Dirichlet boundary condition) and we conclude, by analytic continuation, that $u_1$ is constant/null in $\Omega^\sim$. This implies $g = u_1|_\Gamma$ is constant/null and the contradiction with the hypothesis follows.

### 3.3 The MFS and its Connection to Kirsch-Kress Method

#### 3.3.1 Multi-connected Domain - Approximation by the MFS

Suppose that $\Omega_c \subset \mathbb{R}^d$ ($d = 2, 3$) is bounded, open and multi-connected. The complementary set $\mathbb{R}^d \setminus \overline{\Omega_c}$ has several connected components, one exterior $\Omega^c = \mathbb{R}^d \setminus \overline{\Omega_c}$ and $N$ interior connected components (inclusions or cavities) $\omega_1, \ldots, \omega_N \subset \Omega$ with boundaries $\gamma_i = \partial \omega_i$. Again, we will denote by $\omega$ the union of the components $\omega_i$ and $\gamma$ is $\partial \omega$, the union of their boundaries.

To apply the Method of Fundamental Solutions, we will consider artificial sets that will define the location of the point-sources. These admissible sets are $\hat{\gamma}_i = \partial \hat{\omega}_i$ internal regular boundaries of $\hat{\omega}_i$ simply connected open sets such that $\overline{\hat{\omega}_i} \subset \omega_i$. Again, we will denote by $\hat{\omega}$ the union of the components $\hat{\omega}_i$ and $\hat{\gamma}$ is $\partial \hat{\omega}$, the union of the artificial inner boundaries. Finally, we define an external boundary $\Gamma = \partial \hat{\Omega}$ with $\hat{\Omega}$ an open unbounded set $\overline{\hat{\Omega}} \subset \Omega^c$ with a boundary that encloses the domain, $\Omega \subset \mathbb{R}^d \setminus \hat{\Omega}$. (The case of a bounded $\hat{\Omega}$ is also possible, and theoretically simpler, but it gives worst numerical results when $\Gamma$ does not enclose the domain $\Omega$.)
Recall that the fundamental solution for the Laplacian is given by

\[ \Phi(x) = \begin{cases} \frac{-1}{4\pi} \log |x|, & \text{2D case} \\ \frac{1}{4\pi} |x|, & \text{3D case} \end{cases} \]

and define the source function \( \Phi_y(x) := \Phi(x - y) \).

Consider the single and double layer potential given in the integral form on a boundary \( B \),

\[
L_B \phi(x) = \int_B \Phi(x - y) \phi(y) dS_y, \quad x \in \mathbb{R}^d \setminus B,
\]

\[
M_B \psi(x) = \int_B \partial_n \Phi(x - y) \psi(y) dS_y, \quad x \in \mathbb{R}^d \setminus B,
\]

with \( \phi \in H^{-1/2}(B), \psi \in H^{1/2}(B) \). The boundary operator \( \mathcal{B}_B = a \tau_B + Z_a \tau_B \) is defined by the trace \( \tau \) and normal trace \( \tau^n = \partial_n \) on \( B \) as in (3.2).

Consider the solution of the direct problem \((\tilde{P}_{\Omega_c})\) represented in terms of the single layer potential

\[
u = L_\Gamma \chi + \sum_{k=1}^N L_{\gamma_k} \varphi_k. \tag{3.5}\]

On the boundary \( \Gamma \cup \gamma \), we separate the integral equations

\[
\mathcal{B}_\Gamma u = g, \quad \mathcal{B}_\gamma u = g_i \quad \text{where} \quad g = \tilde{g}|\Gamma; g_i = \tilde{g}_i|\gamma, \tag{3.6}\]

with \( g \in H^{(-1)d\Gamma/2}(\Gamma) \) and \( g_i \in H^{(-1)d\gamma_i/2}(\gamma) \). This leads to the matrix operator system

\[
\begin{bmatrix}
\mathcal{B}_\Gamma L_{\Gamma} & \mathcal{B}_\Gamma L_{\gamma_1} & \cdots & \mathcal{B}_\Gamma L_{\gamma_N} \\
\mathcal{B}_\gamma L_{\Gamma} & \mathcal{B}_\gamma L_{\gamma_1} & \cdots & \mathcal{B}_\gamma L_{\gamma_N} \\
\vdots & \vdots & \ddots & \vdots \\
\mathcal{B}_\gamma L_{\Gamma} & \mathcal{B}_\gamma L_{\gamma_1} & \cdots & \mathcal{B}_\gamma L_{\gamma_N}
\end{bmatrix}_{M(\Gamma, \gamma)} \begin{bmatrix}
\chi \\
\varphi_1 \\
\vdots \\
\varphi_N
\end{bmatrix} = \begin{bmatrix}
g \\
g_1 \\
\vdots \\
g_N
\end{bmatrix}. \tag{3.7}\]

**Theorem 3.** The set \( \mathcal{S} = \text{span} \left\{ (\mathcal{B}_\Gamma \Phi_y, \ldots, \mathcal{B}_\gamma \Phi_y) : y \in \hat{\Gamma} \cup \hat{\gamma} \right\} \)

is dense in \( H^{(-1)d\Gamma/2}(\Gamma)/\mathbb{R}^{p_0} \times \cdots \times H^{(-1)d\gamma_N/2}(\gamma)/\mathbb{R}^{p_N} \) (or \( H^{(-1)d\Gamma/2}(\Gamma) \times H^{(-1)d\gamma_N/2}(\gamma) \times \cdots \times H^{(-1)d\gamma_N/2}(\gamma) \)).
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\[ \ldots \times H^{(-1)^n/2}(\Gamma) \text{ in 3D}, \]

where

\[ p_k = \begin{cases} 
0 & \text{if } \int_{\gamma_k} Z_u(x) dS_x = 0 \\
1 & \text{otherwise} 
\end{cases} \]

**Proof.** We prove that \( \ker(\mathcal{M}(\Gamma \cup \gamma)^*) \) is null, and this implies the density of the range of \( \mathcal{M}(\Gamma \cup \gamma) \). Discretizing the smooth single layer potentials, the density of the linear span \( \mathcal{S} \) follows.

Note that \( (\mathcal{B}_\gamma L_\gamma)^* = \tau_\gamma (aM_\gamma + L_\gamma Z_\gamma) \) because

\[
\langle \mathcal{B}_\gamma L_\gamma \phi, \psi \rangle_{\gamma} = \langle (aM_\gamma + L_\gamma Z_\gamma) \phi, \psi \rangle_{\gamma}
= \int_{\gamma} (a \partial_n + Z_\gamma(x)) \int_{\gamma} \Phi_\gamma(x) \phi(y) dS_y \psi(x) dS_x \\
= \int_{\gamma} \int_{\gamma} (a \partial_n + Z_\gamma(x)) \Phi_\gamma(x) \psi(x) dS_x \psi(y) dS_y \\
= \langle aM_\gamma \psi + L_\gamma Z_\gamma \psi, \phi \rangle_{\gamma}.
\]

Hence,

\[
\mathcal{M}(\Gamma, \gamma)^* = \begin{bmatrix}
\tau_\Gamma(aM_\Gamma + L_\Gamma Z_\Gamma) & \tau_\Gamma(aM_\gamma + L_\gamma Z_\gamma) & \ldots & \tau_\Gamma(aM_{\gamma_n} + L_{\gamma_n} Z_{\gamma_n}) \\
\tau_\gamma(aM_\Gamma + L_\Gamma Z_\Gamma) & \tau_\gamma(aM_\gamma + L_\gamma Z_\gamma) & \ldots & \tau_\gamma(aM_{\gamma_n} + L_{\gamma_n} Z_{\gamma_n}) \\
\vdots & \vdots & \ddots & \vdots \\
\tau_{\gamma_n}(aM_\Gamma + L_\Gamma Z_\Gamma) & \tau_{\gamma_n}(aM_\gamma + L_\gamma Z_\gamma) & \ldots & \tau_{\gamma_n}(aM_{\gamma_n} + L_{\gamma_n} Z_{\gamma_n})
\end{bmatrix}.
\]

In 2D we consider

\[
\mathcal{H}_Z_\gamma(\gamma) = \left\{ \phi \in H^{-1/2}(\gamma) : \int_{\gamma_k} Z_u(x) \phi(x) dS_x = 0 \right\},
\]

which can be identified to \( H^{-1/2}(\gamma_k) / \mathbb{R}^{p_k} \).

Now, let \( \psi \in \mathcal{H}_Z_\gamma(\Gamma) \) and \( \phi_k \in \mathcal{H}_Z_\gamma(\gamma) \) (2D case) or \( \psi \in H^{-1/2}(\Gamma) \) and \( \phi_k \in H^{-1/2}(\gamma) \) (3D case) and define the harmonic function

\[
u(y) = (aM_\Gamma \psi + L_\Gamma Z_\gamma \psi + \sum_{k=1}^{N} (aM_{\gamma_k} \phi_k + L_{\gamma_k} Z_\gamma \phi_k))(y), \quad (y \in \mathbb{R}^d \setminus (\Gamma \cup \gamma)).\]
Assume that \( \psi = (\psi, \phi_1, \ldots, \phi_k, \ldots, \phi_N) \in \text{Ker}(\mathcal{M}(\Gamma, \gamma)^*) \). Then, \( \tau_{\Gamma} \cdot u = \mathcal{M}(\Gamma, \gamma)^* \psi \cdot e_1 = 0 \) and we can conclude that \( u \) is the solution of the well-posed exterior problem (cf. [7])

\[
\begin{cases}
\Delta u = 0 & \text{in } \hat{\Omega} \\
u = 0 & \text{on } \hat{\Gamma} \\
u(y) = \begin{cases} 
A_\omega \log |y| + \mathcal{O}(1) & \text{if } d = 2 \\
\mathcal{O}(|y|^{-1}) & \text{if } d = 3 
\end{cases} & |y| \to \infty
\end{cases}
\]

and uniqueness implies \( u = 0 \) in \( \hat{\Omega} \). In the 2D case, uniqueness follows by taking the asymptotic behavior with constant

\[
A_\omega = -\frac{1}{2\pi} \left( \int_{\Gamma} Z_\omega(x) \psi(x) dS_x + \sum_{k=1}^{N} \int_{\gamma_k} Z_\omega(x) \phi_k(x) dS_x \right),
\]

hence \( A_\omega = 0 \) because \( \psi \in \mathcal{H}(Z_\omega(\Gamma)) \), \( \phi_k \in \mathcal{H}(Z_\omega(\gamma_k)) \). In 3D the asymptotic behavior is immediately verified by the fundamental solution.

By analytic continuation, \( u = 0 \) in \( \hat{\Omega} \) implies \( u \) is null in \( \Omega^c \) and has null exterior traces \( u|_{\hat{\Gamma}^+} = 0 \). On the other hand, since

\[
u(y) = \int_{\Gamma} a \partial_\omega \Phi_\omega(x) \psi(x) dS_x + \int_{\Gamma} Z_\omega(x) \Phi_\omega(x) \psi(x) dS_x + \sum_{k=1}^{N} \left( \int_{\gamma_k} (a \partial_\omega \Phi_\omega(x) + Z_\omega(x) \Phi_\omega(x)) \phi_k(x) dS_x \right)
\]

the boundary jumps across \( \Gamma \) are given by \( [u]|_{\Gamma} = a \psi \wedge [\partial_\omega u]|_{\Gamma} = -Z_\omega \psi \) and we conclude that

\[
u]|_{\Gamma} = -a \psi \wedge [\partial_\omega u]|_{\Gamma} = Z_\omega \psi.
\]

Also, \( u \) is the solution of the well-posed interior problems for each \( j = 1, \ldots, N \),

\[
\begin{cases}
\Delta u = 0 & \text{in } \hat{\omega}_j, \\
u = 0 & \text{on } \hat{\gamma}_j = \partial \hat{\omega}_j,
\end{cases}
\]

hence, \( u = 0 \) in \( \hat{\omega}_j \). Again, by analytic continuation this implies that \( u = 0 \) in each \( \hat{\omega}_j \) and that the exterior traces \( u|_{\hat{\gamma}_j}^+, \partial_\omega u|_{\hat{\gamma}_j}^+ \) are null, thus

\[
u]|_{\hat{\gamma}_j} = -a \phi_j \wedge [\partial_\omega u]|_{\hat{\gamma}_j} = Z_\omega \phi_j.
\]
Therefore, $u$ is also a solution of the interior problem
\[
\begin{aligned}
\Delta u^- &= 0 \quad \text{in } \Omega_c \\
\partial_n u^- + \tilde{Z}_a u^- &= 0 \quad \text{on } \partial \Omega_c = \Gamma \cup \gamma
\end{aligned}
\]
and by the well–posedness of this problem we have $u^- = 0$ in $\Omega_c$. This implies that all traces $u^-|_{\Gamma \cup \gamma}$ and $\partial_n u^-|_{\Gamma \cup \gamma}$ are null and
\[
a \psi = [u]|_{\Gamma} = 0, \quad a \phi_j = [u]|_{\gamma_j} = 0, \quad Z_a \psi = -[\partial_n u]|_{\Gamma} = 0 \land Z_a \phi_j = -[\partial_n u]|_{\gamma_j} = 0.
\]
From the restrictions on $a$ and $\tilde{Z}_a$, we can conclude that $\psi = \phi_j = 0$.

In the MFS, the single layer representation is discretized, ie.,
\[
u(x) \approx \sum_{i=1}^{n} \alpha_i \Phi_{y_i}(x)
\]
for some source points $y_1, \ldots, y_n \in \hat{\Gamma} \cup \hat{\gamma}$. The coefficients $\alpha_i \in \mathbb{R}$ are computed by solving the discretized version of the previous matrix operator system which leads to the linear system
\[
\begin{equation}
\begin{pmatrix}
B_{\Gamma} \Phi_{y_1}(x_1) & \cdots & B_{\Gamma} \Phi_{y_n}(x_1) \\
\vdots & \ddots & \vdots \\
B_{\Gamma} \Phi_{y_1}(x_{l_0}) & \cdots & B_{\Gamma} \Phi_{y_n}(x_{l_0}) \\
\vdots & \ddots & \vdots \\
B_{\gamma} \Phi_{y_1}(x_{l_{i-1}+1}) & \cdots & B_{\gamma} \Phi_{y_n}(x_{l_{i-1}+1}) \\
\vdots & \ddots & \vdots \\
\vdots & \ddots & \vdots \\
B_{\gamma} \Phi_{y_1}(x_{N_{-1}+1}) & \cdots & B_{\gamma} \Phi_{y_n}(x_{N_{-1}+1}) \\
\vdots & \ddots & \vdots \\
B_{\gamma} \Phi_{y_1}(x_N) & \cdots & B_{\gamma} \Phi_{y_n}(x_N)
\end{pmatrix}
\begin{pmatrix}
\alpha_1 \\
\vdots \\
\alpha_n
\end{pmatrix}
= 
\begin{pmatrix}
g(x_1) \\
\vdots \\
g(x_{l_0}) \\
\vdots \\
g(x_{l_{i-1}+1}) \\
\vdots \\
g(x_N)
\end{pmatrix}
\end{equation}
\]
at some collocation points $x_1, \ldots, x_{l_0} \in \Gamma$ and $x_{l_{i-1}+1}, \ldots, x_N \in \gamma_i$. The previous system can be solved in a least squares sense, for $n < m$, or by interpolation using Gaussian elimination, for $n = m$. 

at some collocation points $x_1, \ldots, x_{l_0} \in \Gamma$ and $x_{l_{i-1}+1}, \ldots, x_N \in \gamma_i$. The previous system can be solved in a least squares sense, for $n < m$, or by interpolation using Gaussian elimination, for $n = m$. 

\[ \text{(3.8)} \]
We now consider the original framework \( (\mathcal{P}_{\Omega_c}) \) (recall that on \( \Gamma \) we are considering a Dirichlet boundary condition) and due to the identifiability results, we assume that \( B_\gamma = \tau_\gamma \) or \( B_\gamma = \tau^p_\gamma \).

The inverse problem now consists in obtaining the shape of the boundary \( \gamma \) from the Cauchy data on \( \Gamma \).

In the Kirsch-Kress Method, initially presented for acoustic scattering, one assumes some knowledge on \( \gamma \), such that we can prescribe an artificial boundary \( \hat{\gamma} \) inside \( \gamma \) and write the solution in terms of the inner boundary layer representation. In the acoustic scattering problem the unknown density for the artificial inner boundary layer was recovered fitting its far field pattern. In the bounded domain we need to fit the Cauchy data and it is clear that the inner boundary will not be sufficient to adjust both Dirichlet and Neumann data. An extra external boundary layer must be considered.

The adaptation of the KKM for the inverse bounded problem presented here uses the MFS and involves two steps:

(a) **linear part**: solving the system of integral equations

\[
\begin{bmatrix}
\tau_\Gamma L_\Gamma & \tau_\gamma L_\gamma \\
\tau^p_\Gamma L_\Gamma & \tau^p_\gamma L_\gamma
\end{bmatrix}
\begin{bmatrix}
\chi \\
\varphi
\end{bmatrix}
= \begin{bmatrix}
g \\
\partial_n u
\end{bmatrix},
\]

\( (3.9) \)

where \( \tau_\Gamma, \tau^p_\Gamma \) are, respectively, the trace and the normal trace on \( \Gamma \).

Note that \( \mathcal{K}(\Gamma, \Gamma) \) is \( \mathcal{M}(\Gamma, \Gamma) \) with \( B_\Gamma = \tau_\Gamma \) and \( B_\gamma = \tau^p_\gamma \), when \( \gamma = \Gamma \) (see Fig. 3.1).

(b) **nonlinear part**: the boundary \( \gamma \) will be given by the level set \( u^{-1}(0) = \{ x \in \Omega : u(x) = 0 \} \), for the Dirichlet problem (for the Neumann problem, an optimization scheme similar to the one presented in section 4.1 can be used).

The linear part of the KKM to solve the Cauchy problem is therefore connected to the MFS. It may use the same boundary layers on \( \hat{\gamma} \) and on \( \hat{\Gamma} \) to give the solution of both direct problem from the boundary conditions on \( \gamma \) and \( \Gamma \), and the reconstruction of the solution from the Cauchy data on \( \Gamma \). In fact, the first line of the system (3.9) would be the same – known Dirichlet data. The same densities will verify both systems (3.7) and (3.9).
Reconstruction of inclusions or cavities using the MFS

Figure 3.1: Limit situation where the operator matrix arising from the direct MFS (on the left), $M(\Gamma, \gamma)$, formally tends to $K(\Gamma, \Gamma)$, the matrix arising from the KKM, used as a Cauchy solver in a different region of interest (on the right).

We now prove that a pair of Cauchy data can be retrieved using this MFS version of the KKM method.

**Theorem 4.** The set

$$S_n = \text{span}\{ (\tau_{\Gamma} \Phi_y, \tau_{\Gamma}^n \Phi_y) : y \in \hat{\Gamma} \cup \hat{\gamma} \}$$

is dense in $H^{1/2}(\Gamma)/\mathbb{R} \times H^{-1/2}(\Gamma)$ (or $H^{1/2}(\Gamma) \times H^{-1/2}(\Gamma)$ in 3D).

**Proof.** We follow the proof of Theorem 3. The adjoint of the matrix operator $K(\Gamma, \Gamma)$ defined in (3.9) is given by

$$K(\Gamma, \Gamma)^* = \begin{bmatrix} \tau_{\Gamma} L_{\Gamma} & \tau_{\Gamma} L_{\Gamma} \\ \tau_{\Gamma}^n L_{\Gamma} & \tau_{\Gamma}^n L_{\Gamma} \end{bmatrix} = \begin{bmatrix} \tau_{\Gamma} L_{\Gamma} & \tau_{\Gamma} M_{\Gamma} \\ \tau_{\Gamma}^n L_{\Gamma} & \tau_{\Gamma}^n M_{\Gamma} \end{bmatrix}.$$ 

In the 2D case, again the space $\mathcal{H}_1(\Gamma)$ is identified as $H^{-1/2}(\Gamma)/\mathbb{R}$.

Let $\psi \in \mathcal{H}_1(\Gamma), \phi \in H^{1/2}(\Gamma)$ (2D case) or $\psi \in H^{-1/2}(\Gamma), \phi \in H^{1/2}(\Gamma)$ (3D case) and define the function

$$u(y) = (L_{\Gamma} \psi + M_{\Gamma} \phi)(y),$$
a combination of single and double layer potentials defined on \( \Gamma \). Following the same arguments as in Theorem 3, we prove that if \( u = 0 \) on \( \hat{\Gamma} \cup \hat{\gamma} \) then by analytic continuation of the unique null solution of the interior and exterior problems (now \( A_1 = -\frac{1}{\pi} \int_{\Gamma} \psi(x) dS_x = 0 \), because \( \psi \in \mathcal{H}_1(\Gamma) \)), we obtain \( u = 0 \) in \( \mathbb{R}^d \backslash \Gamma \). Then \( \psi = -[\partial_n u]_{\Gamma} = 0 \), \( \phi = [u]_{\Gamma} = 0 \) and the result follows.

3.3.3 Retrieving the Barycenter

In this paragraph, we present a method based on a reciprocity gap functional to obtain an approximation for the barycenter, for \( \omega \) simply connected. Recall that, the reciprocity gap functional is defined by

\[
\mathcal{R}(v) = \int_{\Gamma} (v \partial_n u - u \partial_n v),
\]

and that, by Green’s formula

\[
\mathcal{R}(v) = \int_{\Omega_c} (v \Delta u - u \Delta v) - \int_{\gamma} (v \partial_n u - u \partial_n v).
\]

Suppose that \( u \) solves the direct problem \((\mathcal{P}_{\Omega_c})\) with null Dirichlet boundary condition on \( \gamma \) and that \( v \) is harmonic. Then,

\[
\mathcal{R}(v) = \int_{\gamma} v \partial_n u,
\]

assuming that the normal direction on \( \gamma = \partial \omega \) points outwards with respect to the inclusion \( \omega \).

Prescribing positive data on \( \Gamma \) we have

\[
W := \partial_n u > 0 \text{ on } \gamma
\]

by the maximum principle (still assuming that the normal on \( \gamma \) points outward with respect to \( \omega \)).

Taking \( v_i = x_i \) we define

\[
c_i^* = \frac{\mathcal{R}(x_i)}{\mathcal{R}(1)} = \frac{\int_{\gamma} x_i W}{\int_{\gamma} W}, \quad i = 1, 2.
\]
Since \( W > 0 \) then these values act as weights in the previous sums and \( c^* = (c_1^*, c_2^*) \) belongs to the convex hull of \( \overline{O} \). We use the point \( c^* \) as an approximation for the barycenter of the inclusion (Note that for \( W \) constant we obtain the barycenter).

### 3.4 Numerical Simulations

We present three numerical simulations of the MFS–KKM applied to the reconstruction of a single (rigid) inclusion in 2D. The accessible part of the boundary, \( \Gamma \), is \( \partial B(0, 3.5) \) and the boundary of the inclusion is given by the parametrization

\[
\gamma(t) = c_i + j_i(t)(\cos t, \sin t), \quad 0 \leq t \leq 2\pi
\]

with

\[
c_1 = (1, -1), \quad j_1(t) = 1.0 + 0.3 \sin (4t),
\]
\[
c_2 = (0, 0), \quad j_2(t) = 0.8 + 1.8(\sin t \cos t/2),
\]

and

\[
\gamma_3(t) = (-0.2 + 0.9 \cos t + 0.4 \cos (2t))e_1 + 1.1 \sin (t)e_2 - 1.2 e_1 + 0.5 e_2
\]

(see Fig. 3.2). As input function, we considered

\[
g \equiv 1.
\]

The Neumann data on \( \Gamma \) were generated synthetically by solving numerically the direct problem with the MFS. Here, we used the artificial boundaries \( \tilde{\Gamma} = \partial B(0, 4.0) \) and \( \tilde{\gamma}(t) = 0.9(\gamma(t) - c_i) + c_i, \quad 0 \leq t \leq 2\pi \) and considered 400 points on \( \Gamma \cup \tilde{\gamma} \) (source points) and the same amount on \( \Gamma \cup \gamma \) (collocation points). For the presented simulations, artificial random noise was added to the data (up to 5%) and measured it at 80 uniformly distributed observation points, for all cases.

For the linear part of the MFS–KKM we started with the computation of the barycenter: In the first simulation we obtained \( c_1^* = (1.15, -1.14) \) as an approximation of the barycenter \( c_1 = (1, -1) \). The results for the other simulations are presented in Fig. 3.2.

Using the computed approximation of the barycenter \( c_i^* \), we considered as an artificial inner curve \( \tilde{\gamma}_i = \partial B(c_i^*, 0.3) \). This choice of artificial
inner curve is based on the assumption that no \textit{a priori} information regarding the dimension or shape of the inclusion is known. The other artificial curve was $\hat{\Gamma} = \partial B(0, 5.5)$ and we used 160 source points and 80 collocation points. In order to avoid the so called inverse crimes the considered artificial boundaries for both direct MFS and KKM–MFS were different. Thus, no information regarding $M(\Gamma, \gamma)$ was used to construct the matrix for the KKM–MFS, $K(\Gamma, \Gamma)$. The resulting system of linear equations was solved using the Tikhonov regularization procedure (the choice of the parameter was based on the L–curve analysis). Fig. 3.3 shows a comparison, for the third simulation, between the initial (noisy) Cauchy data on $\Gamma$ and the corresponding ones obtained with the MFS–KKM.

Figure 3.2: Geometry of the domains. Left– first test, middle– second test and right– third test. Red dots– computed barycenter, dashed curves– artificial inner curves $\hat{\gamma}$. 
Reconstruction of inclusions or cavities using the MFS

Figure 3.3: On the left– absolute error of the inverse problem’s solution on $\Gamma$, on the right– comparison between the noisy Cauchy data on $\Gamma$ (red dots) and the computed data (black line) for the third simulation.

For the reconstruction of the inclusion (non linear part of the MFS–KKM) we computed the level curve $u^{-1}(0)$ based on the maximum principle and a radial search: The first ensures that a harmonic function in the domain $\Omega_c$ with the Dirichlet data under consideration (recall that on $\Gamma$, we have $g = 1$ and on $\gamma$ zero) cannot be zero in that domain. Based on this property, we performed a search along several radial segments, joining $\Gamma$ to $\tilde{\gamma}$, and choose, on each, a point with image near zero (see Fig. 3.4). The set of these points is the approximation of $\gamma$.

The numerical simulations obtained with this procedure showed that this fast method produces accurate reconstructions for both the barycenter (Fig. 3.2) and the shape (Fig. 3.5). Note that the previous procedures based on the maximum principle to solve the non linear part of the MFS–
Figure 3.4: Left– Segment defined by two points on $\Gamma$ and $\hat{\gamma}_1$ (black dots). Right– Solution of the inverse problem along the line of the left plot. The red dot is the computed approximation.

KKM and to compute an approximation of the barycenter do not have a trivial extension to problems where this principle does not apply. Still, we were able to retrieve some information about the inclusion’s geometry in a linear elastoestatic framework (cf. [2]).

3.4.1 Iterative MFS Approach to the Inverse Problem

Minimization based methods for the reconstruction of boundaries (or part of them) usually rely on computing the ”best approximation” of the boundary in some space. By best approximation, we mean the curve that produces the closest Cauchy data to the available ones. These type of iterative
methods requires, at each step, several solutions of direct problems. This makes them slower than the MFS–KKM. They are, however, more accurate. In this paragraph, we present the minimization procedure, already tested with success in [15].

Consider, as space of representation the space of star like curves

$$Y_c = \{ r(t)(\cos t, \sin t) + c : r(t) > 0, t \in [0, 2\pi] \}$$

where $r$ is the distance to the point $c$. The distance function $r$ is represented

Figure 3.5: Reconstruction using the MFS–KKM. Full line– curve to reconstruct, red dots– computed approximation.
Reconstruction of inclusions or cavities using the MFS

in the subspace spanned by the trigonometric polynomials

\[ A_p = \text{span} \{1, \cos t, \sin t, \ldots, \cos (pt), \sin (pt)\}. \]

The goal is to search, iteratively, for a positive function \( r \in A_p \) such that the Cauchy data generated by the corresponding curve in \( \Upsilon_c \) is the best fit for the given data. In the discrete version we aim to solve the nonlinear optimization problem

\[ \arg \min_{\alpha \in \mathbb{R}^{2p+1}} F(\alpha) \]

where

\[ F(\alpha) := \left< \partial_n u, \partial_n u^{(k)} \right>_{L^2(\Gamma)} = \sum_{i=1}^k \left( \partial_n u(x_i) - \partial_n u^{(k)}(x_i) \right)^2, \quad x_i \in \Gamma. \]

The vector \([\partial_n u(x_i)]\) represents the measured (eventually noisy) data, \(u^{(k)}\) solves the problem

\[
\begin{cases}
\Delta u^{(k)} = 0 & \text{in } \Omega_c^{(k)} \\
u^{(k)} = g & \text{on } \Gamma \\
B u^{(k)} = 0 & \text{on } \gamma^{(k)}
\end{cases}
\]

and \(\gamma^{(k)}(t) = \gamma^{(k-1)}(t) + \delta \gamma^{(k-1)}(t)\) where the correcting term \(\delta \gamma^{(k-1)}\) is defined by the radius function

\[ r^{(k-1)}(t) = \alpha \cdot (1, \cos t, \sin t, \ldots, \cos (pt), \sin (pt)), \]

given \(\gamma^{(0)} \in \Upsilon_c\).

We use the Levenberg–Marquardt method (eg. [17]) to solve the optimization problem (the gradient is approximated by finite differences) and the MFS for the numerical solution of the direct problems. We present the results for the first two simulations, with the same noisy data used in the KKM. As a starting curve, we took the approximation computed in the MFS–KKM simulation and introduced the barycenter as an optimization variable. The iterative procedure stops at the first \(k\) such that

\[ \frac{|F_k - F_{k-1}|}{F_{k-1}} \leq 0.01 \]
where $F_{k-1}$ and $F_k$ are the values of the objective function $F$ at $\gamma^{(k-1)}$ and $\gamma^{(k)}$, respectively.

Fig. 3.6 shows the approximations in $\mathcal{A}_4$. For both cases, few iterations were sufficient to obtain very accurate reconstructions. We tested the procedure in $\mathcal{A}_8$, starting from the last curve computed in $\mathcal{A}_4$ and no valid iteration was obtained (no descent direction was encountered). However, it is expected that numerical instabilities arise with the increase of the number of degrees of freedom. As in [15], this phased approximation over several $\mathcal{A}_p$, starting with a small $p$ and using the last curve as a starting curve for bigger $p$, leads to stable reconstructions and helps to avoid the convergence to some local extremum.

3.5 Conclusions

In this work we discussed the question of the identification of inclusions/cavities for the Laplace equation, using a single boundary measurement. We proved the adequacy of the MFS to solve not only the direct problem, but also the inverse (Cauchy) problem with a MFS version of the Kirsch-Kress Method. We proposed two reconstruction methods: A method based on the maximum principle for the MFS–KKM method and a Quasi Newton method. The first revealed to be a fast method and gave good results. The Quasi Newton based method, relying on the computation of several direct problems, produced better results but required bigger computational effort. This effort was substantially reduced by combining the MFS-KKM with the iterative method. The use of the MFS for the solution of the direct problems arising in the iterative method provided fast and accurate solutions, making the extension of this work to the 3D case computational affordable.

Acknowledgment

The authors acknowledge the financial support of Fundação para a Ciência e Tecnologia (FCT), through projects POCI MAT/61792/2004, MAT/60863/2004 and ECM/58940/2004. N. Martins is also partially supported by FCT through the scholarship SFRH/BD/27914/2006. The authors also thank the comments made by the referees.
Figure 3.6: Approximation of $\gamma_i$ in $A_4$. Full line starting curve, dashed lines intermediate curves, dotted line last curve, full bold line $\gamma_i$.

References

References


References


CHAPTER 4

The Method of Fundamental Solutions for Solving Elliptic Partial Differential Equations with Variable Coefficients

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Abstract. In this paper we extend the method of fundamental solutions to solve general elliptic partial differential equations with variable coefficients through a new formulation that resembles both the dual reciprocity method (DRM) and Kansa’s method. As in the case of the DRM, the current approach keeps the Laplacian as the major differential operator. Unlike Kansa’s method using MQ as the sole basis function, we employed two basis functions which contain a radial basis function and the fundamental solution of Laplacian. In contrast to the traditional formulation which uses a two-stage process to find the particular solution and the homogeneous solution separately, the current approach only requires one matrix system. The particular solution and homogeneous solution can thus be obtained simultaneously.

Numerical results show the current approach is more stable and accurate using truncated singular value decomposition than LU decomposition. In particular, the results of the approximate derivatives are also excellent.

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4.1 Introduction

During the past decade, various types of meshless methods have emerged as effective numerical methods for solving problems in science and engineering which involve partial differential equations. The major attraction of meshless methods is to alleviate the difficulty of meshing the domain and boundary, particularly in a 3D case. Among all meshless methods, the method of fundamental solutions (MFS) has been proven to be a highly effective boundary meshless method when the fundamental solutions of the governing equations are available. The MFS was first introduced by Kupradze and Aleksidze [18] in 1964 as the method of generalized Fourier series. In 1977, Mathon and Johnston [20] further developed the MFS as a numerical technique. In the 1980’s, Fairweather and Karageorghis [9] made extensive contributions of the MFS through solving a large class of science and engineering problems. However, during this period of time, the MFS was essentially restricted to solving homogeneous partial differential equations. It was not until 1996 that Golberg et. al. [12] first proposed to extend the MFS to nonhomogeneous partial differential equations through the use of radial basis functions (RBFs). Many RBFs, such as the higher order conical functions, spline-types, multiquadric, compactly supported RBFs, etc., were introduced. Subsequently, the method has further extended to time-dependent problems, nonlinear problems, etc. Detailed reviews of the MFS can be found in several survey papers [8, 9, 10, 11]. The success of extending the MFS to various types of nonhomogeneous problems is largely due to the efficient evaluation of the particular solutions. Since the particular solution is not unique, which allows a wide variety of choices for its computation, many numerical techniques based on polynomial, trigonometric, and radial basis functions have been developed and are available in the literature [8, 11]. In 2001, Balakrishnan and Ramachandran [3] proposed to combine the two parts of the solutions, the homogeneous and inhomogeneous solutions, for solving nonlinear Poisson problems of the form $\Delta u = f(x, u)$. Following a similar technique, Wang and Qin [25] extended the method to solving Poisson-type problems $\Delta u = f(x, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$. In the above mentioned approaches, one can only handle Poisson-type equations with constant coefficients. Recently, Alves and Valtchev [1] developed a Kansa-type scheme [13] using the MFS to solve a certain class of elliptic partial differential equations.
The derivation of the closed form particular solutions for a general differential operator is a non-trivial task. Before 1998, the derivation of a particular solution was restricted to the cases of Laplacian or biharmonic operators due to the difficulty of deriving the closed form particular solution using RBFs. It was not until the work of Chen and Rashed [5] that the construction of a particular solution of Helmholtz-type operators could be efficiently treated. The ability of deriving a particular solution for non-homogeneous Helmholtz-type equations has made significant progress for effectively solving time-dependent problems via integral transformations or time-differencing [8, 11].

Once the particular solution of the given linear partial differential equation is available, the nonhomogeneous equation can be reduced to a homogeneous equation through the use of the method of particular solution [11]. A boundary meshless method such as the MFS can then be used to obtain the homogeneous solution. The original solution can be recovered by adding the particular solution and the homogeneous solution. This is a two-stage procedure that is commonly called the method of particular solutions (MPS) or splitting method. In this approach, we try to keep the nonhomogeneous term as simple as possible so that it can be approximated easily and accurately. It is known that the accuracy of the particular solution is directly related to the accuracy of interpolating the nonhomogeneous term of a given partial differential equation. On the other hand, by keeping the nonhomogeneous term simple, the differential operator will be more involved and the closed form particular solution will be more difficult to obtain. There are limited closed form particular solutions available beyond Helmholtz-type operators.

It is the purpose of this paper to develop a one-stage method similar to Balakrishnan and Ramachandran [3] and Wang and Qin [25] by combining the particular solution and homogeneous solution together in the solution process for solving partial differential equations with variable coefficients. Similar to the DRBEM [22, 23], the Laplacian is kept as the differential operator and all the other terms of the original differential operator are moved to the right hand side to become a part of the forcing term. As we
The MFS for PDEs with variable coefficients

shall see later, such a formulation requires two basis functions. The major advantage of such a formulation is that the closed form particular solutions for Laplacian are available for various RBFs and the fundamental solution for the Laplacian is simple. Unlike the formulation of Helmholtz-type operators where the basis functions are restricted to polyharmonic functions, the current approach allows us to use a wide variety of basis functions, particularly multiquadric (MQ) or inverse MQ for superior convergence in the evaluation of the particular solution. Furthermore, our proposed method resembles Kansa’s method [13] which is very popular in the meshless community. In [19], the authors investigated the performance of Kansa’s method and the combined two-stage MFS-MPS. They concluded that there is no difference between these two methods in terms of numerical accuracy. As in the case of Kansa’s method, our new approach can solve practically all types of linear partial differential equations with variable coefficients. The major difference between the two methods is that Kansa’s method employs MQ as the basis function and we adopt the RBF and the fundamental solution as the basis functions. It is clear that our approach with a two basis system has a more sound mathematical foundation. In our numerical tests, we will demonstrate that the current approach outperforms Kansa’s method in both stability and accuracy, particularly in the evaluation of partial derivatives.

The structure of the paper is as follows. In Section 4.2, we briefly describe the MPS, the MFS, and Kansa’s method. In Section 4.3, we add the idea of the formulation of the DRM and combine the methods mentioned in Section 2 to establish an effective numerical scheme that will allow us to solve elliptical PDEs with variable coefficients. In Section 4.4, we test four examples to demonstrate the convergence, stability, and high accuracy of current method. In Section 4.5, we give a list of computational advantages of our numerical technique and discuss possible directions of future research.

4.2 Standard Formulation

In this section, we briefly review the methods that will be used later for the formulation of our new proposed method.
4.2.1 The Method of Particular Solutions

Let $\mathcal{L}$ be a linear second order elliptic partial differential operator with constant coefficients, $\mathcal{B}$ be a boundary operator, and let a fundamental solution of $\mathcal{L}$ be known. We consider the following model boundary value problem (BVP)

$$
\mathcal{L}u(x) = f(x), \quad x \in \Omega, \quad (4.1)
$$
$$
\mathcal{B}u(x) = g(x), \quad x \in \partial \Omega, \quad (4.2)
$$

where $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, is a bounded domain with a sufficiently regular boundary $\partial \Omega$. We assume that the boundary value problem (4.1)–(4.2) has a unique solution $u$ for any given continuous nonhomogeneous term $f$ and the boundary data $g$.

In the MPS we split the solution, $u$, of (4.1)–(4.2) into a particular solution and a homogeneous solution [2]. Let $u = u_p + u_h$ where $u_p$ is a particular solution satisfying the nonhomogeneous equation

$$
\mathcal{L}u_p(x) = f(x), \quad (4.3)
$$

but does not necessarily satisfy the boundary condition in (4.2). Then the homogeneous solution, $u_h$, satisfies

$$
\mathcal{L}u_h(x) = 0, \quad x \in \Omega, \quad (4.4)
$$
$$
\mathcal{B}u_h(x) = g(x) - \mathcal{B}u_p(x), \quad x \in \partial \Omega. \quad (4.5)
$$

Since a fundamental solution of $\mathcal{L}$ is known, various boundary methods can be easily applied to solve (4.4)–(4.5). In order to exploit all the advantages of a mesh free boundary method, we employ the MFS for the solution of (4.4)–(4.5). We will elaborate on the MFS later.

The key issue here is how to compute an approximation to $u_p$ for a general forcing term, $f$, in (4.3). The classical way to evaluate $u_p$ is based on the Newtonian potential

$$
u_p(x) = \int_{\Omega} G(x, y)f(y)dV(y), \quad (4.6)$$

where $G(x, y)$ is the fundamental solution of $\mathcal{L}$.

To overcome the difficulty of domain integration in evaluating $u_p$, Nardini and Brebbia [22] introduced the DRM. The success of the DRM depends on how the right-hand side is approximated. Typically, this is done
by approximating $f$ with a finite series of basis functions

$$ f(x) \approx \hat{f}(x) = \sum_{j=1}^{n} a_j \phi(r_j), \quad (4.7) $$

where $r_j = \|x - x_j\|$ and $\{x_j\}_{j=1}^{n}$ are called the centers or trial points and $\phi : \mathbb{R}_+ \rightarrow \mathbb{R}$ is a univariate function. The coefficients $\{a_j\}_{j=1}^{n}$ are usually obtained by a collocation method; i.e., by solving

$$ \sum_{j=1}^{n} a_j \phi(r_{kj}) = f(x_k), \quad 1 \leq k \leq n, \quad (4.8) $$

where $r_{kj} = \|x_k - x_j\|$, $x_k \in \mathbb{R}^d$, $d = 2, 3$. An approximate particular solution, $\hat{u}_p$, to (4.3) is given by

$$ \hat{u}_p(x) = \sum_{j=1}^{n} a_j \Phi(r_j), \quad (4.9) $$

where $\Phi$ is obtained by analytically solving

$$ \mathcal{L} \Phi = \phi. \quad (4.10) $$

An accurate approximation of $u_p$ depends on how well $f$ is approximated. Consequently, the appropriate choice of basis function, $\phi$, is of considerable importance. On the other hand, $\mathcal{L}$ also affects how $\phi$ should be chosen. For instance, when $\mathcal{L} = \Delta$ or $\Delta^2$, there are a wide range of $\phi$ that can be chosen so that the closed form $\Phi$ can be obtained. Among them, MQ, inverse MQ, and polyharmonic splines are the most popular choices. When $\mathcal{L} = \Delta \pm \lambda^2$, it is not a trivial task to obtain $\Phi$ in (4.10) [21]. Up to date, it is known that $\Phi$ is available only if we choose $\phi$ to be the polyharmonic splines ($r^{2k} \ln r$ in 2D or $r^{2k-1}$ in 3D, $k \geq 1$) or compactly supported radial basis functions in 2D [5, 21]. Little is known for differential operators other than Helmholtz-type operators. The mathematical analysis of deriving $\Phi$ is very demanding. This is the major difficulty in the application of the MPS.

### 4.2.2 The Method of Fundamental Solutions

After the particular solutions have been evaluated, we solve the homogeneous equation in (4.4) and (4.5) by boundary methods. The MFS is an
efficient and flexible boundary method. We will briefly describe the MFS
which will serve as a foundation for our proposed new unified method de-
scribed in the next section.

The basic idea of the MFS is to approximate the homogeneous solu-
tion, \( u_h \), by \( \hat{u}_h \), which can be expressed as a linear combination of funda-
mental solutions

\[ \hat{u}_h(x) = \sum_{j=1}^{m} b_j G(\rho_j), \quad x \in \Omega \cup \partial \Omega, \]  

(4.11)

where \( \rho_j = \|x - y_j\| \) is the distance between \( x \) and the source point, \( y_j \). Note that \( G(\rho) \) is the known fundamental solution of the elliptic linear
differential operator \( L \) in (4.4). The source points in the MFS may be
considered as lying on a fictitious boundary \( \Gamma = \partial \hat{\Omega} \) of a region \( \hat{\Omega} \)
containing \( \Omega \). It is an important issue to determine the optimal location of the
fictitious boundary.

In general, there are two approaches used in choosing \( \{y_j\}_{j=1}^{m} \), fixed
and adaptive [9]. In this paper, we focus on the fixed method where \( \{y_j\}_{j=1}^{m} \)
are chosen \textit{a priori}. Much of the work in this direction has relied on the
approximation results of Bogomolny [4] and Cheng’s convergence results
[6] for the Dirichlet problem for Laplace’s equation when \( \Omega \) and \( \hat{\Omega} \)
are concentric circles. In their work, it was shown that the accuracy of the ap-
proximation improves as \( \Gamma \) is moved farther away from \( \partial \Omega \). Cheng’s result
was generalized by Katsurada and Okamoto [16, 17, 15, 14], who showed
that if \( \partial \Omega \) is a closed Jordan curve in the plane and data are analytic, then

\[ \|u - u_m\|_{\infty} \leq c(r/R)^m \]

where \( r \) and \( R \) are the diameters of \( \Omega \) and \( \hat{\Omega} \) respectively.

Once the source points have been chosen, the \( \{b_j\}_{j=1}^{m} \) in (4.11) are
generally obtained by collocation. That is, \( m \) points, \( \{x_i\}_{i=1}^{m} \), are chosen
on \( \partial \Omega \) and then \( \{b_j\}_{j=1}^{m} \) satisfies (4.5); i.e.,

\[ \sum_{j=1}^{m} b_j \mathcal{B}G(\rho_j) = g(x_i) - \mathcal{B}u_p(x_i), \quad 1 \leq i \leq m, \]  

(4.12)

where \( \rho_j = \|x_i - y_j\| \).

Despite the ill-conditioning of the matrix \( \mathcal{B}[G(\rho_{ij})] \), the accuracy of
the numerical solution is largely unaffected if the boundary conditions are
smooth and exact for the homogeneous equation [7, 11]. For nonhomogeneous problems, the error produced by the particular solution will have an impact on the homogeneous solution. In (4.12), the error in the evaluation of \( \mathcal{B}u_p \) has the same effect as noise added to the boundary. As a result, the ill-conditioning of the MFS will magnify the error introduced in the boundary. As such, instead of using Gaussian elimination, the truncated singular value decomposition (TSVD) is recommended in the solution process to reduce the effect of ill-conditioning in the MFS. For more details, we refer readers to Reference [7]. We will illustrate this issue through examples in the later section of numerical results.

4.2.3 Kansa’s Method

We consider the following elliptic partial differential equation in 2D

\[
\Delta u + \alpha(x,y) \frac{\partial u}{\partial x} + \beta(x,y) \frac{\partial u}{\partial y} + \gamma(x,y)u = f(x,y), \quad (x,y) \in \Omega, \quad (4.13)
\]
\[
\mathcal{B}u = g(x,y), \quad (x,y) \in \partial\Omega, \quad (4.14)
\]

where \( \alpha(x,y), \beta(x,y), \gamma(x,y), f(x,y), \) and \( g(x,y) \) are given functions. \( \mathcal{B} \) is a boundary differential operator.

Let \( \{(x_j, y_j)\}_{j=1}^n \) be \( n \) distinct collocation points in \( \Omega \) of which \( \{(x_j, y_j)\}_{i=1}^n \) are interior points and \( \{(x_j, y_j)\}_{i=n+1}^n \) are boundary points. MQ is one of the most widely adopted RBFs in Kansa’s method. Though other RBFs can be used, we consider only the MQ basis function:

\[
\varphi_j(x,y) = \sqrt{r_j^2 + c^2}, \quad r_j = \sqrt{(x-x_j)^2 + (y-y_j)^2},
\]

where \( c \) is a free parameter and often called the shape parameter of MQ. By direct differentiation, we have

\[
\frac{\partial \varphi_j}{\partial x} = \frac{x-x_j}{\sqrt{r_j^2 + c^2}}, \quad \frac{\partial \varphi_j}{\partial y} = \frac{y-y_j}{\sqrt{r_j^2 + c^2}},
\]
\[
\frac{\partial^2 \varphi_j}{\partial x^2} = \frac{(y-y_j)^2 + c^2}{(r_j^2 + c^2)^{3/2}}, \quad \frac{\partial^2 \varphi_j}{\partial y^2} = \frac{(x-x_j)^2 + c^2}{(r_j^2 + c^2)^{3/2}}. \quad (4.15)
\]
For the elliptic problem (4.13)–(4.14), the main idea of Kansa’s method is to approximate the solution, \( u \), by \( \varphi \); i.e.,

\[
\hat{u}(x,y) = \sum_{j=1}^{n} c_j \varphi_j(x,y),
\]

(4.17)

where \( \{c_j\}_{j=1}^{n} \) are coefficients to be determined. Then, from (4.13) – (4.14), we have

\[
\sum_{j=1}^{n} c_j \left( \Delta \varphi_j + \alpha \frac{\partial \varphi_j}{\partial x} + \beta \frac{\partial \varphi_j}{\partial y} + \gamma \varphi_j \right) (x_i, y_i) = f(x_i, y_i), \quad i = 1, 2, \cdots, n_i,
\]

(4.18)

\[
\sum_{j=1}^{n} c_j B \varphi_j (x_i, y_i) = g(x_i, y_i), \quad i = n_i + 1, \cdots, n,
\]

(4.19)

which is an \( n \times n \) linear system for the unknowns \( \{c_j\}_{j=1}^{n} \). More specifically, from (4.15) – (4.16), (4.18) becomes

\[
\sum_{j=1}^{n} c_j \left( \frac{r_j^2 + 2c^2}{(r_j^2 + c^2)^{3/2}} + \frac{\alpha (x - x_j) + \beta (y - y_j)}{\sqrt{r_j^2 + c^2}} + \gamma \sqrt{r_j^2 + c^2} \right) = f(x_i, y_i),
\]

(4.20)

for \( i = 1, 2, \cdots, n_i \). Once \( \{c_j\}_{j=1}^{n} \) are known through solving (4.19) – (4.20), the approximation of \( u \) and its derivatives can be obtained. In general, Kansa’s method is very effective for the approximation of \( u \), but the approximation of the derivatives of \( u \) are normally not as accurate as we might expect.

4.3 Methodology

There are disadvantages to the above mentioned two-stage formulation in Section 4.2.1. First, closed form particular solutions, \( \Phi \), are available only for simple differential operators, \( L \). Second, for Helmholtz-type differential operators or more complicated differential operators, the availability of RBFs are often restricted to polyharmonic splines \([5, 21]\). As such, the accuracy of the solution of the partial differential equation is limited. Third, such approach can only be used to solve partial differential equations with
constant coefficients. In this section, we introduce a new formulation, without the need for the two-stage approach, to alleviate the difficulties mentioned above. For simplicity, we only consider the formulation in 2D cases. For 3D cases, a similar procedure can be applied.

We consider solving the elliptic partial differential equation (4.13) – (4.14). Since the general differential operators were variable coefficients, the fundamental solution is not available. Similar to the formulation of the DRBEM [22, 23], the Laplacian is kept on the left hand side and all the other terms with differential operators containing reaction and convection terms are moved to the right hand side and are treated as the forcing term. In this way, (4.13) is rearranged into Poisson’s equation; i.e.,

$$\Delta u(x, y) = F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}), \quad (x, y) \in \Omega,$$

(4.21)

where

$$F(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}) = -\alpha(x, y) \frac{\partial u}{\partial x} - \beta(x, y) \frac{\partial u}{\partial y} - \gamma(x, y) u + f(x, y).$$

(4.22)

First, we assume the solution can be directly approximated by the sum of the particular solution and homogeneous solution [3, 25]

$$u(x, y) \simeq \hat{u}(x, y) = \sum_{j=1}^{n} a_j \Phi(r_j) + \sum_{j=1}^{m} b_j G(\rho_j),$$

(4.23)

where $r_j$ and $\rho_j$ are defined in the same way as in (4.7) and (4.11) respectively. Due to the reformulation of (4.13) into (4.21), we choose the fundamental solution in (4.23) as $G(\rho) = \ln \rho$. We will soon illustrate how we choose $\Phi(r)$ as a basis function. Figure 4.1 shows the distribution of interior collocation points in $\Omega$, boundary collocation points on $\partial \Omega$, and source points on the fictitious boundary on $\Gamma$. Furthermore,

$$\frac{\partial u}{\partial x} \simeq \frac{\partial \hat{u}}{\partial x} = \sum_{j=1}^{n} a_j \frac{\partial \Phi}{\partial x}(r_j) + \sum_{j=1}^{m} b_j \frac{\partial G}{\partial x}(\rho_j),$$

(4.24)

$$\frac{\partial u}{\partial y} \simeq \frac{\partial \hat{u}}{\partial y} = \sum_{j=1}^{n} a_j \frac{\partial \Phi}{\partial y}(r_j) + \sum_{j=1}^{m} b_j \frac{\partial G}{\partial y}(\rho_j).$$

(4.25)

In (4.23), we apply two different basis functions with two different distance functions to approximate the solution of (4.13)–(4.14). Instead
of finding the particular solution and homogeneous solution separately, we intend to obtain the undetermined coefficients \( \{a_j\}_{j=1}^n \) and \( \{b_j\}_{j=1}^m \) simultaneously. In (4.21), we reformulated the given differential equation (4.13) into Poisson’s equation where the particular solutions for various basis functions are well-known. Using RBFs, \( \phi \), to approximate \( F \), we have

\[
F \left( x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} \right) = \sum_{j=1}^{n} a_j \phi(r_j). \tag{4.26}
\]

Furthermore, we note that \( \Delta G(\rho_j) = 0 \) for \( (x,y) \in \Omega \) and \( (x_j,y_j) \in \Gamma \). As a result, from (4.23), we have

\[
\Delta u \simeq \Delta \hat{u} = \sum_{j=1}^{n} a_j \Delta \phi(r_j) + \sum_{j=1}^{m} b_j \Delta G(\rho_j)
\]

\[
= \sum_{j=1}^{n} a_j \phi(r_j) \quad \text{in } \Omega, \tag{4.27}
\]
where \( \Delta \Phi(r) = \phi(r) \). Replacing the Laplacian in (4.21) using (4.27), we obtain
\[
\sum_{j=1}^{n} a_j \phi(r_j) = -\alpha(x,y) \frac{\partial \hat{u}}{\partial x} - \beta(x,y) \frac{\partial \hat{u}}{\partial y} - \gamma(x,y) \hat{u} + f(x,y), \quad \text{for} (x,y) \in \Omega.
\]
(4.28)

Using (4.23) – (4.25), the above expression (4.28) can be rearranged as follows
\[
\sum_{j=1}^{n} a_j \Psi(r_j) + \sum_{j=1}^{m} b_j \Theta(\rho_j) = f(x,y), \quad \text{for} (x,y) \in \Omega,
\]
(4.29)

where
\[
\Psi(r_j) = \phi(r_j) + \alpha(x,y) \frac{\partial \Phi(r_j)}{\partial x} + \beta(x,y) \frac{\partial \Phi(r_j)}{\partial y} + \gamma(x,y) \Phi(r_j),
\]
(4.30)

\[
\Theta(\rho_j) = \alpha(x,y) \frac{\partial G}{\partial x}(\rho_j) + \beta(x,y) \frac{\partial G}{\partial y}(\rho_j) + \gamma(x,y) G(\rho_j).
\]
(4.31)

The boundary condition in (4.14) becomes
\[
\sum_{j=1}^{n} a_j \mathcal{B} \Phi(r_j) + \sum_{j=1}^{m} b_j \mathcal{B} G(\rho_j) = g(x,y), \quad (x,y) \in \partial \Omega.
\]
(4.32)

It is clear that \( \Psi(r_j), \Theta(\rho_j), \) and \( \Phi(r_j) \) in (4.29) and (4.32) are completely known functions. In particular, \( \Psi(r_j) \) contains the radial basis function \( \phi \), the particular solution \( \Phi \), and the derivatives \( \partial \Phi/\partial x \), and \( \partial \Phi/\partial y \). Since the differential operator in (4.21) is the Laplacian, we have
\[
G(\rho) = \ln \rho, \quad \frac{\partial G(\rho)}{\partial x} = \frac{x}{\rho^2}, \quad \frac{\partial G(\rho)}{\partial y} = \frac{y}{\rho^2}.
\]

If we choose \( \phi = \sqrt{r^2 + c^2} \) (MQ), it is known that [11]
\[
\Phi(r) = \frac{1}{g} \left( 4c^2 + r^2 \right) \sqrt{r^2 + c^2} - \frac{c^3}{3} \ln \left( c + \sqrt{r^2 + c^2} \right),
\]
(4.33)

\[
\frac{\partial \Phi(r)}{\partial x} = \frac{x \left( c \sqrt{r^2 + c^2} + c^2 + r^2 \right)}{3 \left( c + \sqrt{r^2 + c^2} \right)} , \quad \frac{\partial \Phi(r)}{\partial y} = \frac{y \left( c \sqrt{r^2 + c^2} + c^2 + r^2 \right)}{3 \left( c + \sqrt{r^2 + c^2} \right)}.
\]
(4.34)
For the implementation, we choose $n$ interior points in $\Omega$, $m$ boundary points on $\partial \Omega$, and $m$ source points on the fictitious boundary $\Gamma$ (see Figure 4.1). By collocation, we have thus formulated a system of equations of order $(m+n) \times (m+n)$; i.e.,

$$
\begin{bmatrix}
\Psi(r_{ij}) & \Theta(\rho_{ij}) \\
B\Phi(r_{ij}) & B\Gamma(\rho_{ij})
\end{bmatrix}
\begin{bmatrix}
a \\ b
\end{bmatrix}
=
\begin{bmatrix}
f \\ g
\end{bmatrix},
$$

(4.35)

where $a = [a_1 \ a_2 \ \ldots \ a_n]^T$, $b = [b_1 \ b_2 \ \ldots \ b_m]^T$, $f = [f(x_1,y_1) \ f(x_2,y_2) \ \ldots \ f(x_n,y_n)]^T$, $g = [g(x_1,y_1) \ g(x_2,y_2) \ \ldots \ g(x_m,y_m)]^T$. Once $\{a_j\}_{i=1}^n$ and $\{b_j\}_{j=1}^m$ are determined by solving (4.35), the approximate solution of $u$ can be obtained from (4.23).

The major improvement of this proposed method is that we can extend the MFS to not only nonhomogeneous differential equations, but also to a large class of elliptic partial differential equations with variable coefficients. Contradicting the conventional wisdom that sophisticated particular solutions and fundamental solutions are required for the solution of partial differential equations, in this case all are expressed in simple terms. Furthermore, as we know that MQ converges exponentially for smooth functions, we have reason to expect rapid convergence in this solution process. As we shall see in the next section, high accuracy and rapid convergence were observed in our numerical tests. Another attractive feature of the current approach is its simplicity in implementation.

Note that $\Delta u$ in (4.27) can be approximated without taking second derivatives of $u$ with respect to $x$ and $y$. As such, the numerical accuracy and stability are preserved since taking the derivative numerically is not a stable process. This gives a great advantage over Kansa’s method [13]. On the other hand, the convection terms $\partial u/\partial x$ and $\partial u/\partial y$ are integrated naturally into the matrix system in (4.35) without additional treatment which is a significant advantage over the DRBEM [22, 23]. In other words, while the current approach resembles both Kansa’s method and the DRBEM, it is also different from them through the above attractive features.

Unlike the two-stage process of the MPS described in Section 4.2.1, which requires solving two smaller matrix systems for the particular solution and the homogeneous solution individually, the current approach requires only one system of equations to be solved. The selection of the shape parameter of MQ and fictitious boundary of the MFS are basically
4.4 Numerical Results

To demonstrate the effectiveness of the current approach, four numerical examples are presented in this section. The first three examples involve the modified Helmholtz equations with constant and variable coefficients. In the second example, we experimentally examine the convergence rate of the current approach. The fourth example is the convection-diffusion equation with variable coefficients, which is difficult using other standard numerical methods. In the first example, we examine the performance of different matrix solvers for the proposed method. All the numerical results are compared with both the analytical solution and the Kansa’s method. In addition, we propose a useful guide for determining the optimal shape parameter of MQ, $c$, by tracking the residual with the known boundary conditions. In the following examples, we provide a detailed comparison and the performance for the current method.

Through all the numerical tests in this section, we use the following formula to choose the location of the source points in the MFS:

$$x^s = x^b + \sigma (x^b - x^c),$$  \hspace{1cm} (4.36)

where $x^s, x^b$, and $x^c$ denote the source, boundary, and central nodes. The parameter $\sigma$ determines how far away the source points from the boundary.

For the evaluation of the numerical accuracy, we use the root mean square error (RMSE) and the root mean square error of the derivative with respect to $x$ (RMSEx), which are defined as follows:

$$RMSE = \sqrt{\frac{1}{nt} \sum_{j=1}^{nt} (\hat{u}_j - u_j)^2},$$  \hspace{1cm} (4.37)

$$RMSEx = \sqrt{\frac{1}{nt} \sum_{j=1}^{nt} \left( \frac{\partial \hat{u}_j}{\partial x} - \frac{\partial u_j}{\partial x} \right)^2},$$  \hspace{1cm} (4.38)

where $nt$ is the number of testing nodes chosen randomly in the domain. $\hat{u}_j$ denotes the approximate solution at the $j^{th}$ node. We shall not present the numerical results for RMSEy since they are very similar to RMSEx.
How to identify the optimal shape parameter of MQ is a challenge. For this purpose, we propose using the given boundary conditions to identify it since the boundary conditions are known and can be considered part of the analytical solution. Comparing the boundary conditions is similar to comparing the analytical solution. The definition of the residual of Dirichlet and Neumann boundary conditions are defined as follows:

\[
\text{Residual (Dirichlet)} = \frac{1}{nr} \sum_{j=1}^{nr} |\hat{u}_j - g_j|, \quad (4.39)
\]

\[
\text{Residual (Neumann)} = \frac{1}{nr} \sum_{j=1}^{nr} \left| \frac{\partial \hat{u}_j}{\partial n} - g_j \right|, \quad (4.40)
\]

where \(nr\) is the number of test nodes on the boundary, \(\partial \Omega\), and \(g_j\) is the given boundary condition at the \(j^{th}\) node.

**Example 4.4.1.** We consider the following modified Helmholtz equation

\[
(\Delta - \lambda^2)u = f(x, y), \quad (x, y) \in \Omega, \quad (4.41)
\]

\[
u = y \sin(\pi x) + x \cos(\pi y), \quad (x, y) \in \partial \Omega, \quad (4.42)
\]

where \(\lambda\) is a constant. The computational domain is a Cassini curve defined by the parametric equation

\[
\Omega = \{(x, y) \mid x = \rho \cos \theta, y = \rho \sin \theta, 0 \leq \theta \leq 2\pi\}, \quad (4.43)
\]

where

\[
\rho = \left( \cos(3\theta) + \sqrt{2 - \sin^2(3\theta)} \right)^{\frac{1}{3}}. \quad (4.44)
\]

The analytical solution is given by

\[
u(x, y) = y \sin(\pi x) + x \cos(\pi y). \quad (4.45)
\]

The computational domain and profile of the solution in the extended domain are shown in Figure 4.2. To obtain the results in Figures 4.3, 4.4, and 4.5, we chose \(n = 208\) evenly distributed interior points and \(m = 80\) equally distributed boundary points as shown in Figure 4.2, with \(\lambda = 1, \sigma = 10, nt = 230\) and \(nr = 60\). To show the effect of ill-conditioning on our newly formulated matrix system, we employed LU decomposition and
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Figure 4.2: Cassinian curve (left) and profile of solution in the extended domain (right).

Figure 4.3: Comparison of RMSE (left) and RMSEx (right) using different matrix solvers.

the truncated singular value decomposition (TSVD) [24]. The numerical results with respect to different shape parameters are presented in Figure 4.3.

To demonstrate that our method can be very effective even using a small number of interpolation nodes, we compared our numerical results with those obtained by Kansa’s method. The comparison of the numerical results is depicted in Figure 4.4. We observed the behaviors of the RMSE and RMSEx, both shown in this figure, with respect to different shape parameters. Note that Kansa’s method quickly becomes unstable while the
current method remains stable for a wide range of shape parameters. It should be noticed that the best performance of Kansa’s method seems to be occurring at random peaks. Hence, it is very difficult to identify the optimal shape parameter using Kansa’s method. On the other hand, our method remains stable and accurate for a wide range of shape parameters. We believe that the current method is clearly more effective and more stable than Kansa’s method in this case.

![Graph showing numerical results of RMSE and RMSEx by Kansa’s method and the MFS.](image1)

**Figure 4.4:** Numerical results of RMSE (left) and RMSEx (right) by Kansa’s method and the MFS.

![Graph showing profiles of RMSE and residual with respect to different shape parameters.](image2)

**Figure 4.5:** Profiles of RMSE and residual with respect to different shape parameters.

*The overall profiles of the residual and RMSE are shown in Figure 4.5.*
The profiles of RMSE and boundary residual clearly behave in the same way. This means that the acceptable range of shape parameter can be tracked down by calculating the residual along the boundary. We tried different combinations for the number of nodes and location of source points in the MFS. The results were quite similar. In Table 4.1, we list the best RMSE and RMSEx for both Kansa’s method and our method. Upon close observation of the RMSE, the best performances of each method seem to yield the same level of accuracy. However, the ranges of acceptable shape parameters for each method are quite different as we showed previously in Figure 4.4. Additionally, we noticed that there was a difference between the RMSEx of each method. The current method outperforms Kansa’s method in evaluating partial derivatives, which is critical for many physical problems. In Table 4.1, we choose $\lambda$ as 1, 10, and 100. The results are consistent for all these $\lambda$. The case for large $\lambda$ is important to solutions of diffusion and wave equations.

Example 4.4.2. The purpose of this example is to experimentally observe the convergent rate of the current approach. In this example, we consider the same equation as in the previous example with $\lambda = 1$. The domain is the unit square. We choose uniform grid points on the unit square to carry out the computation. In addition, we choose 200 random nodes for calculating RMSE and RMSEx. As in the last example, MQ have been used as basis functions. As seen in Figure 4.6, $\delta h = 1/n$ where $n$ is the number of mesh on each coordinate. The optimal shape parameter $c$ is selected for each $\delta h$. In this example we observe rapid convergence rate of the proposed method.

Example 4.4.3. In this example, we consider the nonhomogeneous modified Helmholtz equation with variable coefficient as follows:

$$
\Delta u - (x^2 y) u = f(x,y), \quad (x,y) \in \Omega,
$$

$$
u = \sin(y^2 + x) - \cos(y - x^2), \quad (x,y) \in \partial \Omega^D,
$$

$$
\frac{\partial u}{\partial n} = \left[ \nabla \left( \sin(y^2 + x) - \cos(y - x^2) \right) \right] \cdot n, \quad (x,y) \in \partial \Omega^N,
$$

where $n$ is the unit normal vector and $f(x,y)$ is a given function based on the analytical solution

$$
u(x,y) = \sin(y^2 + x) - \cos(y - x^2).$$
The MFS for PDEs with variable coefficients

Table 4.1: Comparison of best RMSE, RMSEx and corresponding shape parameter for Kansa’s method and the MFS.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>RMSE</th>
<th>RMSEx</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Kansa’s method</td>
<td>MFS</td>
</tr>
<tr>
<td>(\lambda = 1)</td>
<td>111 80</td>
<td>(2.54 \times 10^{-5})</td>
<td>(1.37 \times 10^{-6})</td>
</tr>
<tr>
<td></td>
<td>208 80</td>
<td>(2.05 \times 10^{-6})</td>
<td>(3.20 \times 10^{-7})</td>
</tr>
<tr>
<td></td>
<td>310 80</td>
<td>(3.97 \times 10^{-6})</td>
<td>(1.76 \times 10^{-7})</td>
</tr>
<tr>
<td>(\lambda = 10)</td>
<td>111 80</td>
<td>(4.56 \times 10^{-6})</td>
<td>(4.49 \times 10^{-6})</td>
</tr>
<tr>
<td></td>
<td>208 80</td>
<td>(3.16 \times 10^{-6})</td>
<td>(8.56 \times 10^{-7})</td>
</tr>
<tr>
<td></td>
<td>310 80</td>
<td>(1.05 \times 10^{-6})</td>
<td>(4.91 \times 10^{-7})</td>
</tr>
<tr>
<td>(\lambda = 100)</td>
<td>111 80</td>
<td>(2.90 \times 10^{-6})</td>
<td>(8.07 \times 10^{-6})</td>
</tr>
<tr>
<td></td>
<td>208 80</td>
<td>(3.74 \times 10^{-7})</td>
<td>(2.81 \times 10^{-7})</td>
</tr>
<tr>
<td></td>
<td>310 80</td>
<td>(1.26 \times 10^{-6})</td>
<td>(1.08 \times 10^{-7})</td>
</tr>
</tbody>
</table>

The computational domain is \(\Omega = \Omega_1 \cup \Omega_2\), which is formed by combining...
The MFS for PDEs with variable coefficients

Figure 4.6: Profile of convergent rate in Example 4.4.2.

Let us denote by \( \partial \Omega^D \) and \( \partial \Omega^N \) the boundaries on which Dirichlet and Neumann boundary conditions are given respectively. Let \( \partial \Omega = \partial \Omega^D \cup \partial \Omega^N \) and

\[
\begin{align*}
\partial \Omega^N &= \{(x,y) | 0 \leq x \leq 2, y = 0, \text{and } 0 \leq x \leq 1, y = 2\}, \quad (4.52) \\
\partial \Omega^D &= \partial \Omega \setminus \partial \Omega^N. \quad (4.53)
\end{align*}
\]

The Neumann and Dirichlet boundaries are shown in Figure 4.7 (left). The profile of the solution in the extended domain is shown in Figure 4.7 (right). To obtain the results in Figures 4.8 and 4.9, we chose \( m = 144 \) equally distributed points on the boundary, \( n = 300 \) quasi-random interior points, \( \sigma = 5 \), \( nr = 250 \), and the cut off singular value for TSVD as \( 10^{-11} \). Due to the complicated equation and mixed boundary conditions, Kansa’s method is very erratic with regard to the shape parameter of MQ. As shown in Figure 4.8, the best performance of Kansa’s method only appears in peaks and it is difficult to locate the optimal shape parameter. In contrast, the numerical results of the MFS are very stable and accurate for various shape parameters. Also, the range of suitable shape parameters is very wide for the MFS. Though the best RMSE for Kansa’s method is
better than the MFS, the instability of Kansa’s method with respect to the shape parameter of MQ could limit its application for more complicated problems.

In Figure 4.9, the same technique as the one used in the previous example was used to determine the optimal shape parameter. Since there are two types of boundary conditions, we calculate the residual for each boundary condition, separately. 21 and 63 nodes are used for calculating the residuals on boundaries with the Dirichlet and Neumann conditions.

We noticed that the overall profiles of the RMSE, the residual for the
Figure 4.9: Profiles of RMSE and residual with respect to different shape parameters.

Dirichlet condition, and the residual for the Neumann condition are very similar. Therefore, it is clear that the criterion for choosing the optimal shape parameter is valid. Both or one of the two boundary conditions can be used to determine the optimal shape parameter.

In Table 4.2, we listed the best results for Kansa’s method and the current method using 100, 200 and 300 interior nodes with 96, 120 and 144 boundary nodes. Both methods produce excellent results for RMSE and RMSEx using only a small number of nodes. Obtaining accurate approximate derivatives is an important issue. Although Kansa’s method is as accurate as the current method, we observed that an acceptable solution with Kansa’s method is only available within a small range of shape parameters. In contrast, the current method appears to be more robust and stable. The c values shown in Table 4.2 are the optimal shape parameters.

**Example 4.4.4.** In this example, we consider the convection-diffusion equation:

\[
\Delta u + \left( x^2 + y^2 \right) u + y \cos(y) \frac{\partial u}{\partial x} + \sinh(x) \frac{\partial u}{\partial y} = f(x,y), \quad (x,y) \in \Omega,
\]

\[
\frac{\partial u}{\partial x} = \sin(\pi x) \cosh(y) - \cos(\pi x) \sinh(y), \quad (x,y) \in \partial \Omega,
\]

where \( f(x,y) \) depends upon the analytical solution, which is given as:

\[
u(x,y) = \sin(\pi x) \cosh(y) - \cos(\pi x) \sinh(y). \]

(4.56)
Table 4.2: Comparison of best RMSE with corresponding optimal shape parameters and RMSEx for Kansa’s method and the MFS.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>RMSE</th>
<th>RMSEx</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Kansa’s method</td>
<td>Kansa’s method</td>
</tr>
<tr>
<td>100</td>
<td>96</td>
<td>$2.50 \times 10^{-3}$</td>
<td>$1.50 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(c = 1.03)$</td>
<td>$(c = 1.50)$</td>
</tr>
<tr>
<td>200</td>
<td>96</td>
<td>$2.82 \times 10^{-5}$</td>
<td>$8.68 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(c = 0.43)$</td>
<td>$(c = 1.44)$</td>
</tr>
<tr>
<td>300</td>
<td>96</td>
<td>$1.71 \times 10^{-5}$</td>
<td>$3.04 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(c = 0.68)$</td>
<td>$(c = 1.23)$</td>
</tr>
<tr>
<td>100</td>
<td>120</td>
<td>$3.10 \times 10^{-4}$</td>
<td>$1.50 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(c = 1.46)$</td>
<td>$(c = 1.50)$</td>
</tr>
<tr>
<td>200</td>
<td>120</td>
<td>$4.07 \times 10^{-6}$</td>
<td>$8.79 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(c = 0.74)$</td>
<td>$(c = 1.42)$</td>
</tr>
<tr>
<td>300</td>
<td>120</td>
<td>$1.48 \times 10^{-5}$</td>
<td>$2.82 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(c = 1.13)$</td>
<td>$(c = 1.24)$</td>
</tr>
<tr>
<td>100</td>
<td>144</td>
<td>$1.19 \times 10^{-4}$</td>
<td>$1.49 \times 10^{-3}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(c = 1.10)$</td>
<td>$(c = 1.49)$</td>
</tr>
<tr>
<td>200</td>
<td>144</td>
<td>$1.25 \times 10^{-5}$</td>
<td>$8.30 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(c = 0.80)$</td>
<td>$(c = 1.44)$</td>
</tr>
<tr>
<td>300</td>
<td>144</td>
<td>$6.12 \times 10^{-7}$</td>
<td>$2.86 \times 10^{-5}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$(c = 0.56)$</td>
<td>$(c = 1.24)$</td>
</tr>
</tbody>
</table>

The domain is defined by the following star shape parametric equation:

$$
\Omega = \{(x, y) \mid x = (1 + \cos^2(4\theta)) \cos \theta, y = (1 + \cos^2(4\theta)) \sin \theta, 0 \leq \theta < 2\pi\}.
$$

(4.57)

The domain and profile of the solution in the extended domain are depicted in Figure 4.10. To obtain the results in Figures 4.11 and 4.12, we chose 100 equally distributed boundary points, 317 evenly distributed interior points, $\sigma = 5, nt = 193, nr = 80$, and the truncated singular value of TSVD equal to $10^{-13}$. The results of RMSE and RMSEx for Kansa’s method and the MFS are shown in Figure 4.11. As in the previous examples, the optimal shape parameter for Kansa’s method is not easy to iden-
In contrast, the current method remains stable and converges rapidly as the shape parameter increases. In Figure 4.11, the range of acceptable shape parameters for the current method is quite large even though the governing equation is more complicated.

Similarly, we used the residual on the boundary to determine the optimal shape parameter. The results are depicted in Figure 4.12. As was the case with the results in the previous examples, the RMSE and residual behave in similar way. In spite of different governing equations in the current and previous examples, the proposed criterion for choosing the
In Table 4.3, we observed a similar pattern for RMSE and RMSE\textsubscript{x} as the previous two examples. In particular, the current method performs better than Kansa’s method for RMSE\textsubscript{x}. Again, we produce excellent solutions as well as derivatives using a small number of nodes, which indicates that the proposed method converges rapidly.

To alleviate the difficulty of finding the optimal shape parameter of MQ, we can always choose other radial basis functions such as \( r^{2k+1} \) where \( k \) is a nonnegative integer. In Tables 4.4 and 4.5, we compare the results using MQ and \( r^{2k+1}, k = 3, 4, 5 \). We observe that the accuracy obtained with \( r^{2k+1} \) is consistently one order lower than MQ. Without the difficulty of tracking the optimal shape parameter of MQ, it may be worth considering these types of basis functions in real applications. Other RBFs such as polyharmonic splines are also available. The closed form particular solution for these RBFs for the Laplacian are easy to derive. This will be the subject of a future investigation.

4.5 Concluding Remarks
In this paper, we proposed a one-stage approach to incorporate the MPS, the MFS, the DRBEM, and Kansa’s method into one matrix system for solving general elliptic partial differential equations with variable coefficients. The numerical tests reveal that we can obtain excellent results even
Table 4.3: Comparison of best RMSE, RMSEx and corresponding shape parameter for Kansa’s method and MFS.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>RMSE</th>
<th>RMSEx</th>
</tr>
</thead>
<tbody>
<tr>
<td>113</td>
<td>50</td>
<td>4.44 × 10⁻⁵</td>
<td>3.29 × 10⁻⁴</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c = 0.03)</td>
<td>(c = 2.85)</td>
</tr>
<tr>
<td>212</td>
<td>50</td>
<td>5.81 × 10⁻⁵</td>
<td>5.04 × 10⁻⁵</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c = 0.69)</td>
<td>(c = 2.35)</td>
</tr>
<tr>
<td>317</td>
<td>50</td>
<td>6.05 × 10⁻⁵</td>
<td>2.45 × 10⁻⁵</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c = 0.93)</td>
<td>(c = 1.73)</td>
</tr>
<tr>
<td>113</td>
<td>100</td>
<td>1.68 × 10⁻⁶</td>
<td>6.50 × 10⁻⁵</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c = 0.92)</td>
<td>(c = 2.82)</td>
</tr>
<tr>
<td>212</td>
<td>100</td>
<td>1.33 × 10⁻⁶</td>
<td>1.57 × 10⁻⁵</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c = 0.35)</td>
<td>(c = 2.36)</td>
</tr>
<tr>
<td>317</td>
<td>100</td>
<td>3.38 × 10⁻⁶</td>
<td>4.70 × 10⁻⁶</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c = 0.58)</td>
<td>(c = 1.78)</td>
</tr>
<tr>
<td>113</td>
<td>150</td>
<td>1.64 × 10⁻⁵</td>
<td>4.86 × 10⁻⁵</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c = 0.60)</td>
<td>(c = 2.80)</td>
</tr>
<tr>
<td>212</td>
<td>150</td>
<td>1.34 × 10⁻⁵</td>
<td>1.37 × 10⁻⁵</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c = 0.43)</td>
<td>(c = 2.30)</td>
</tr>
<tr>
<td>317</td>
<td>150</td>
<td>6.64 × 10⁻⁷</td>
<td>2.77 × 10⁻⁶</td>
</tr>
<tr>
<td></td>
<td></td>
<td>(c = 0.33)</td>
<td>(c = 1.54)</td>
</tr>
</tbody>
</table>

when using a small number of interpolation points and boundary points. This indicates that our proposed method converges rapidly relative to the number of interior interpolation points and boundary points. The advantages of the proposed method are summarized as follows:

1. The method can be used to solve general elliptic PDEs with variable
The MFS for PDEs with variable coefficients

Table 4.4: Comparison of RMSE by the proposed method using different RBFs.

<table>
<thead>
<tr>
<th>n</th>
<th>m</th>
<th>$r^7$</th>
<th>$r^9$</th>
<th>$r^{11}$</th>
<th>$\sqrt{r^2 + c^2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>113</td>
<td>50</td>
<td>$6.76 \times 10^{-3}$</td>
<td>$4.25 \times 10^{-3}$</td>
<td>$2.23 \times 10^{-3}$</td>
<td>$3.29 \times 10^{-4}$</td>
</tr>
<tr>
<td>212</td>
<td>50</td>
<td>$1.02 \times 10^{-3}$</td>
<td>$1.93 \times 10^{-3}$</td>
<td>$5.06 \times 10^{-4}$</td>
<td>$5.04 \times 10^{-5}$</td>
</tr>
<tr>
<td>317</td>
<td>50</td>
<td>$5.71 \times 10^{-4}$</td>
<td>$3.24 \times 10^{-3}$</td>
<td>$6.38 \times 10^{-4}$</td>
<td>$2.45 \times 10^{-5}$</td>
</tr>
<tr>
<td>113</td>
<td>100</td>
<td>$9.59 \times 10^{-4}$</td>
<td>$7.30 \times 10^{-4}$</td>
<td>$1.10 \times 10^{-3}$</td>
<td>$6.50 \times 10^{-5}$</td>
</tr>
<tr>
<td>212</td>
<td>100</td>
<td>$2.32 \times 10^{-4}$</td>
<td>$2.02 \times 10^{-4}$</td>
<td>$4.72 \times 10^{-4}$</td>
<td>$1.57 \times 10^{-5}$</td>
</tr>
<tr>
<td>317</td>
<td>100</td>
<td>$8.76 \times 10^{-5}$</td>
<td>$4.00 \times 10^{-5}$</td>
<td>$2.41 \times 10^{-5}$</td>
<td>$4.70 \times 10^{-6}$</td>
</tr>
<tr>
<td>113</td>
<td>150</td>
<td>$8.41 \times 10^{-4}$</td>
<td>$5.50 \times 10^{-4}$</td>
<td>$9.58 \times 10^{-4}$</td>
<td>$4.86 \times 10^{-5}$</td>
</tr>
<tr>
<td>212</td>
<td>150</td>
<td>$1.84 \times 10^{-4}$</td>
<td>$1.20 \times 10^{-4}$</td>
<td>$3.43 \times 10^{-4}$</td>
<td>$1.37 \times 10^{-5}$</td>
</tr>
<tr>
<td>317</td>
<td>150</td>
<td>$6.95 \times 10^{-5}$</td>
<td>$3.36 \times 10^{-5}$</td>
<td>$1.31 \times 10^{-5}$</td>
<td>$2.77 \times 10^{-6}$</td>
</tr>
</tbody>
</table>

coefficients.

2. $\Delta u$ can be efficiently evaluated without direct numerical differentiation.

3. The method is truly meshless and thus very easy to implement.

4. The numerical results are highly accurate and converge rapidly.

5. No complicated fundamental solution or particular solutions are required.

As was the case in the two-stage method, finding the optimal shape parameter, $c$, of MQ and the location of the fictitious boundary still pose challenges in implementation. In this paper, we identify an effective way to track the optimal shape parameter. The unified approach has great potential and is expected to make significant contributions to the future development of the MFS. The method is especially attractive for solving equations with nonconstant convection terms, which are very challenging when using other standard methods. Moreover, our proposed method has excellent
results in evaluating derivatives, which is crucial for solving many engineering problems.

We believe the current numerical technique has great potential for further application to science and engineering problems. Meanwhile, there are still outstanding theoretical issues to be resolved. The current numerical scheme can be directly extended to 3D, time-dependent, and nonlinear problems. These subjects are currently under investigation.

Acknowledgement

The first author acknowledges the support of NATO Collaborative Linkage Grant under reference ESP.CLG.982891. The authors would like to thank the referee for the constructive suggestions for improving the paper.

References

References


References


CHAPTER 5

On Efficient MFS Algorithms Using Complex Representations

A. Karageorghis\textsuperscript{1}, S. G. Mogilevskaya\textsuperscript{2}, and H. Stolarski\textsuperscript{3}

Abstract. We propose an efficient complex Method of Fundamental Solutions (MFS) algorithm for the solution of certain two–dimensional potential and linear elasticity problems. We consider the solution of such problems in rectangular domains subject to Dirichlet boundary conditions. Numerical experiments for both types of problems are presented.

5.1 Introduction

The Method of Fundamental Solutions (MFS) is a meshless boundary method which is applicable to certain elliptic boundary value problems. In the last three decades the MFS has been successfully applied to a large variety of physical problems, see for example the survey papers [2, 3, 4]. In recent years, the efficient implementation of the method to problems with rotational symmetry, has attracted a lot of interest, see for example [7, 10].

In this work, our goal is to exploit these algorithms in order to solve efficiently problems which do not possess rotational symmetry. As is well-documented, one fundamental difficulty in the application of the MFS is

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the choice of the pseudoboundary on which the sources are located. The theoretical predictions in the case of harmonic problems in a disk subject to Dirichlet boundary conditions \cite{11} indicate that the error converges (exponentially) to zero as the distance of the pseudoboundary from the boundary tends to infinity. However, this theoretical prediction is of little practical value as the ill-conditioning of the MFS matrices for large distances of the pseudoboundary from the boundary leads to inaccurate results. Also, when the pseudoboundary is located too close to the boundary the MFS approximation is inaccurate as we are essentially approximating an integral with a singular integrant using simple quadrature. The optimal choice of the location of the pseudoboundary thus remains an open problem. A practical way of determining this optimal location of the pseudoboundary was proposed in \cite{12}. In it, the problem is solved for a sequence of pseudoboundaries which are located at a distance $\varepsilon_\ell$, $\ell \in \mathbb{N}$ from the boundary. For each $\varepsilon_\ell$ the maximum error in the boundary conditions on a fixed set of points on the boundary is calculated. The optimal location of the pseudoboundary is the one for which the maximum error is minimal. The process can be rendered more efficient by using a bisection type method to locate the minimum, that is starting with a larger step and gradually reducing it. Still, in the case of rectangular domains, this algorithm requires the solution of a sequence of dense linear systems. In order to avoid this, we propose an algorithm in which most of the problems in the sequence are solved in circular domains, instead of the rectangular domain. In this way, we solve a sequence of problems in circular domains which are computationally much less expensive.

An alternative to the proposed approach is to use a conformal mapping technique which maps the domain under consideration onto the unit disk. The problem is then solved efficiently in the unit disk. This approach might, however, involve complicated evaluations of conformal mappings \cite{6}.

Although the examples presented in this work are for rectangular geometries these are easily applicable to other geometries.

An additional feature in this work is the use of the complex representation of the MFS. This has proved particularly suitable in the case of potential and elastostatics problems.
5.2 Potential Problems

Consider the solution of Laplace’s equation in the rectangle $\Omega = (a,b) \times (c,d)$ subject to the Dirichlet boundary condition $u = f = f_R + if_I$. We chose to consider complex boundary conditions (and hence complex solutions) to be consistent with the corresponding elasticity problems. By doing this we are essentially solving two harmonic problems simultaneously. The solution $u \in \mathbb{C}$ considered subsequently is approximated by the complex harmonic function

$$u_N(\mathbf{c}, \mathbf{Q}; P) = \sum_{j=1}^{N} c_j \log(z_{Q_j} - z_P)(\bar{z}_{Q_j} - \bar{z}_P), \quad P \in \overline{\Omega},$$

where $\mathbf{c} = (c_1, c_2, \ldots, c_N)^T \in \mathbb{C}^N$ and $\mathbf{Q}$ is a $2N$-vector containing the coordinates of the singularities (sources) $Q^\alpha_j$, $j = 1, \ldots, N$, which lie outside $\overline{\Omega}$. The singularities $Q^\alpha_j$ are fixed on the boundary $\partial \tilde{\Omega}$ of a disk $\tilde{\Omega}$ concentric to $\Omega$ and defined by $\tilde{\Omega} = \{ x \in \mathbb{R}^2 : |x| < R \}$. A set of collocation points $\{z_P\}_{i=1}^{N}$ is uniformly distributed on $\partial \Omega$ and the positions of the singularities are given by

$$z_{Q^\alpha_j} = \text{Re}^{\frac{j(\alpha - 1 - \alpha)}{N}} \pi, \quad j = 1, \ldots, N,$$

where $0 \leq \alpha < 1$.

The coefficients $\mathbf{c}$ are determined so that the boundary condition is satisfied at the boundary points $\{P_i\}_{i=1}^{N}$:

$$u_N(\mathbf{c}, \mathbf{Q}; P) = f(P_i), \quad i = 1, \ldots, N.$$  

This yields a complex linear system of the form

$$G \mathbf{c} = \mathbf{f},$$

for the coefficients $\mathbf{c}$, where the elements of the matrix $G$ are given by

$$G_{i,j} = \log(z_{Q_j} - z_{P_i})(\bar{z}_{Q_j} - \bar{z}_{P_i}), \quad i = 1, \ldots, N, \quad j = 1, \ldots, N.$$  

The matrix $G$ is dense with no particular structure and the system is solved by standard Gaussian elimination.

One fundamental question in the implementation of the MFS is the optimal positioning of the circle on which the singularities are to be placed.
In Dirichlet harmonic problems one may evaluate the maximum error on a fixed set of points on the boundary (not coinciding with the boundary points) for values of the radius $R$ on a certain interval. The maximum error on $\partial \Omega$ is also the maximum error in $\bar{\Omega}$ because the error in the MFS, being harmonic, satisfies the maximum principle. Therefore the value of $R$ which yields the minimum maximum error on the boundary determines the optimal circular pseudoboundary $\partial \tilde{\Omega}$ [12]. This task clearly involves the solution of a sequence of MFS discrete problems for a number of different $R$ which is potentially expensive.

In order to render this task computationally efficient we propose the following algorithm:

(i) We first use the MFS for the rectangular region with an arbitrary choice of the circular pseudoboundary $\partial \tilde{\Omega}$.

(ii) We compute the solution on a circle of radius $\rho < R$ which is concentric with the pseudoboundary $\partial \tilde{\Omega}$.

(iii) We solve a sequence of harmonic problems for the disk of radius $\rho$ for pseudoboundaries with varying radius $R$. For each problem we evaluate the maximum error on a fixed set of points on the (rectangular) boundary $\partial \Omega$.

(iv) The optimal value of $R$ is the one for which the maximum error is minimized.

(v) The original problem for the rectangular domain is solved once more for the optimal location of the pseudoboundary.

This algorithm leads to substantial savings in the computational cost, as in this case the consecutive matrices $G$ (for the disk) are circulant [1, 5] and the coefficients can be obtained efficiently using Fast Fourier Transforms (FFTs). In particular, from [10], the approximate solution is evaluated on set of collocation points $\{z_{S_i}\}_{i=1}^{N}$ where

$$z_{S_i} = \rho e^{\frac{2i(i-1)}{N} \pi}, \quad i = 1, \ldots, N.$$  \hspace{1cm} (5.6)

These now yield the boundary conditions for the problem on the disk. System (5.4) now becomes

$$E \mathbf{d} = \mathbf{g},$$  \hspace{1cm} (5.7)
where the matrix $E$ is circulant. System (5.7) is pre multiplied by the Fourier matrix $U$ ([10]) to yield

$$UEU^*U \hat{d} = U \hat{g} \quad (5.8)$$

or

$$D \hat{d} = \hat{g} \quad (5.9)$$

where

$$\hat{d} = U \hat{d} \quad \text{and} \quad \hat{g} = U \hat{g}. \quad (5.10)$$

The solution is thus clearly,

$$\hat{d}_i = \frac{\hat{g}_i}{d_i}, \quad i = 1, \cdots, N, \quad (5.11)$$

where $D = \text{diag}(d_i)_{i=1}^N$ is a diagonal matrix having as diagonal elements the eigenvalues of $E$. Having obtained $\hat{d}$, we can find $d$ from

$$d = U^* \hat{d}. \quad (5.12)$$

Because of the circulant structure of the matrices $E$, the cost of solving each system is $O(N \log N)$ operations.

In our numerical experiments, we calculated the maximum error on a set of 100 points on $\partial \Omega$, when solving the problem using the conventional MFS and when solving the proposed algorithm.

**Example 5.2.1** In Figures 5.1-5.3, we present the maximum error obtained solving the original problem and the maximum error obtained using the proposed (modified) algorithm versus the radius of the pseudoboundary $R$ for the problem with exact solution $u = z^6$, for the case when $\Omega = (-1, 1) \times (-1, 1)$. In Figure 5.1, we took $\rho = 1$, in Figure 5.2, we took $\rho = \sqrt{2}$ while in Figure 5.3, we took $\rho = 2$. We observe that in all three cases the general behaviour of the maximum boundary error is reproduced accurately by the proposed algorithm.

**Example 5.2.2** In Figures 5.4-5.6, we present the corresponding results for the problem with exact solution $u = z^6$, for the case when $\Omega = (-1, 1) \times (-1/2, 1/2)$. In Figures 5.4, 5.5, 5.6, we took $\rho = 1$, $\rho = \sqrt{2}$, $\rho = 2$, respectively. As in the previous case, the modified approach reproduces the behaviour of the maximum boundary error accurately.
Further, in Table 5.1, we present the CPU times for various numbers of degrees of freedom $N$ when the problem is solved for one hundred steps (i.e. the system is solved one hundred times), in the case the original problem is solved and in the case the proposed algorithm is used. These times include the evaluation of the maximum error on a set of 100 points on the boundary of the domain and were recorded on an IBM PC (Processor: Intel Pentium 4, 3.4 GHz). From the table one can observe the savings in CPU time when using the proposed algorithm.

### Table 5.1: CPU time (in seconds)

<table>
<thead>
<tr>
<th>$N$</th>
<th>Original problem (s)</th>
<th>Proposed algorithm (s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>80</td>
<td>0.56</td>
<td>0.53</td>
</tr>
<tr>
<td>160</td>
<td>2.91</td>
<td>2.26</td>
</tr>
<tr>
<td>320</td>
<td>18.81</td>
<td>9.55</td>
</tr>
<tr>
<td>640</td>
<td>138.79</td>
<td>41.95</td>
</tr>
</tbody>
</table>

5.3 Elastostatics Problems

We now consider the Cauchy–Navier equations of elasticity

$$\frac{(\lambda + \mu)}{\mu} u_{k,i} + \mu u_{i,kk} = 0, \quad i = 1, 2 \quad \text{in} \quad \Omega, \quad (5.13a)$$

where $\Omega$ is the rectangle $(a, b) \times (c, d)$, subject to the Dirichlet (displacement) boundary conditions

$$u_i = f_i, \quad i = 1, 2 \quad \text{on} \quad \partial \Omega. \quad (5.13b)$$

In equations (5.13a), the parameters $\lambda$ and $\mu$ are the Lamé elastic constants. These constants can be expressed in terms of Poisson’s ratio $\nu$ and Young’s modulus $E$ as $\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)}$ and $\mu = \frac{E}{2(1+\nu)}$. We introduce the complex displacement $u = u_1 + i u_2$ and the complex function $f = f_1 + i f_2$. Using Kelvin’s fundamental solutions [9], the displacement $u \in \mathbb{C}$ is approximated by the complex function [8]

$$u_N(c, Q, P) = \frac{1}{4\pi \mu (1+\kappa)} \left( \sum_{j=1}^{N} a_j \left( -\kappa \log(z_{Q_j} - z_P)(\bar{z}_{Q_j} - \bar{z}_P) + \frac{z_{Q_j} - z_P}{\bar{z}_{Q_j} - \bar{z}_P} \right) \right)$$
Figure 5.1: Maximum error versus $d$ for different values of $N$ for $\rho = 1$ in Example 2.1

\begin{align}
&+ \sum_{j=1}^{N} b_j \left( -\kappa i \log(z_{Q_j} - z_P)(\bar{z}_{Q_j} - \bar{z}_P) - i \frac{z_{Q_j} - z_P}{\bar{z}_{Q_j} - \bar{z}_P} \right) \right), \quad P \in \overline{\Omega},
\end{align}

where now $c = (c_1, c_2, \ldots, c_N)^T \in \mathbb{C}^N$, with $c_j = a_j + ib_j$, $j = 1, 2, \ldots, N$, and the singularities $Q$ are defined as before. Here $(a_j, b_j)$ is the point force loaded at the point $z_{Q_j}$ of an infinite elastic plane. In (5.14), $\kappa =$
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3 − 4ν. Approximation (5.14) can also be written as

\[ u_N(c, Q; P) = \frac{1}{4\pi\mu(1 + \kappa)} \left\{ \sum_{j=1}^{N} \left( -\kappa \log(z_{Q_j} - z_P)(\bar{z}_{Q_j} - \bar{z}_P) \right) c_j \right. \\
\left. + \sum_{j=1}^{N} \left( \frac{z_{Q_j} - \bar{z}_P}{z_{Q_j} - \bar{z}_P} \right) \bar{c}_j \right\} . \] (5.15)

The coefficients \( c \) are obtained from the system resulting from the collocation of the boundary conditions

\[ u_N(c, Q; P_\ell) = f(P_\ell), \quad \ell = 1, \ldots, N. \] (5.16)
Figure 5.3: Maximum error versus \( d \) for different values of \( N \) for \( \rho = 2 \) in Example 2.1.

We follow the algorithm described in the potential case. In contrast to the potential case, the resulting matrix in this case (for the circular domain) is not circulant. However, we follow the algorithm described in [7] in order to render it circulant. In particular, system (5.16) using expression (5.14) may be written as the \( 2N \times 2N \) real linear system \( A c = f \) or

\[
\begin{pmatrix}
A_{11} & A_{12} & A_{13} & \cdots & A_{1N} \\
A_{21} & A_{22} & A_{23} & \cdots & A_{2N} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
A_{N1} & A_{N2} & A_{N3} & \cdots & A_{NN}
\end{pmatrix}
\begin{pmatrix}
c_1 \\
c_2 \\
\vdots \\
c_N
\end{pmatrix}
= \begin{pmatrix}
f_1 \\
f_2 \\
\vdots \\
f_N
\end{pmatrix},
\]  
(5.17)
Figure 5.4: Maximum error versus $d$ for different values of $N$ for $\rho = 1$ in Example 2.2

where

$$A_{ij} = \begin{pmatrix} G_{11}(P_i, Q_j) & G_{12}(P_i, Q_j) \\ G_{21}(P_i, Q_j) & G_{22}(P_i, Q_j) \end{pmatrix}, \quad c_t = \begin{pmatrix} a_t \\ b_t \end{pmatrix}, \quad f_t = \begin{pmatrix} f_1(P_t) \\ f_2(P_t) \end{pmatrix},$$

and

$$G_{11}(P_i, Q_j) = \frac{1}{4\pi\mu(1 + \kappa)} \Re \left\{ -\kappa \log(z_{Q_j} - z_{P_i}) (\bar{z}_{Q_j} - \bar{z}_{P_i}) + \frac{z_{Q_j} - z_{P_i}}{\bar{z}_{Q_j} - \bar{z}_{P_i}} \right\},$$

$$G_{12}(P_i, Q_j) = \frac{1}{4\pi\mu(1 + \kappa)} \Im \left\{ -\kappa \log(z_{Q_j} - z_{P_i}) (\bar{z}_{Q_j} - \bar{z}_{P_i}) + \frac{z_{Q_j} - z_{P_i}}{\bar{z}_{Q_j} - \bar{z}_{P_i}} \right\}.$$
System (5.17) is the same as system (3.3) of [7], and can be rendered block circulant via an appropriate rotation. The block circulant system can then be solved efficiently using FFTs. The algorithm is described in detail in [7].
In our numerical experiments, as in the case of the Laplacian, we calculated the maximum error on a set of 100 points on $\partial \Omega$, when solving the original problem and when using the proposed (modified) algorithm.

**Example 5.3.1** In Figures 5.7-5.9, we present the maximum error obtained solving the original problem and the maximum error obtained using the modified problem versus the radius of the pseudo boundary $R_1$ for the problem with exact solution $u = e^{3z}$, for the case when $\Omega = (-1,1) \times (-1,1)$. In Figure 5.7, we took $\rho = 1$, in Figure 5.8, we took $\rho = \sqrt{2}$ while in Figure 5.9, we took $\rho = 2$. As in the corresponding harmonic case, the maximum boundary error is reproduced accurately by the modified
Example 5.3.2 In Figures 5.10-5.12, we present the corresponding results for the problem with exact solution \( u = e^{3z} \), for the case when \( \Omega = (-1,1) \times (-1/2,1/2) \). In Figures 5.10, 5.11, 5.12, we took \( \rho = 1 \), \( \rho = \sqrt{2} \), \( \rho = 2 \), respectively. As in the previous case, the modified approach reproduces the behaviour of the maximum boundary error accurately.

Figure 5.7: Maximum error versus \( d \) for different values of \( N \) for \( \rho = 1 \) in Example 3.1

5.4 Conclusions
In this work we attempted to exploit two features of the MFS. Firstly, we use complex representations of the fundamental solutions. The advantages
Figure 5.8: Maximum error versus $d$ for different values of $N$ for $\rho = \sqrt{2}$ in Example 3.1

of complex representations in both potential and elasticity theories are well documented. In particular, the complex representations of fundamental solutions, for example Melan’s and Flamant’s solutions are more compact than their real counterparts [8]. Also, in a future extension of the approach presented herein, one may exploit the properties of analytic functions (e.g., analytic continuation, series expansions, etc). The second feature we attempted to exploit in this work is the location of the optimal boundary in conjunction with the efficient solution of circulant systems. The algorithm proposed in this work is not restricted to rectangular domains but may be used for more general geometries. The choice of the radius of the circle of
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Figure 5.9: Maximum error versus $d$ for different values of $N$ for $\rho = 2$ in Example 3.1

testing nodes $\rho$ can influence the performance of the MFS. It was found that the closer the circle of testing nodes is to the original domain, the better the results. In future work we intend to use the advantages of complex representations to accurately calculate the boundary conditions on the circular domain.

Acknowledgements
The authors are grateful to the referee for his/her constructive comments.
Figure 5.10: Maximum error versus $d$ for different values of $N$ for $\rho = 1$ in Example 3.2

References


Figure 5.11: Maximum error versus $d$ for different values of $N$ for $\rho = \sqrt{2}$ in Example 3.2


Figure 5.12: Maximum error versus $d$ for different values of $N$ for $\rho = 2$ in Example 3.2


CHAPTER 6

The Method of Fundamental Solutions for Steady-State Nonlinear Heat Conduction

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Abstract. The steady-state heat conduction in heat conductors with temperature dependent thermal conductivity and mixed boundary conditions involving convection and radiation is investigated using the method of fundamental solutions. The locations of the singularities outside the solution domain are optimally determined using a nonlinear least-squares procedure. Numerical results for nonlinear materials are presented and discussed.

6.1 Introduction

In many heat transfer problems the assumption of constant thermal conductivity, i.e. that the heat conductors are homogeneous within the whole temperature variation interval, may lead to unacceptable errors in high-temperature environments or if large temperature differences are present. In the steady-state situation, the nonlinearity associated with the temperature dependence of the thermal conductivity can be removed by employing the Kirchhoff transformation, which replaces the original nonlinear partial differential equation in divergence form by the Laplace equation in the transformed space. Boundary conditions of the Dirichlet (first kind) or

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Neumann (second kind) types pose no problem for the transformation, but the Robin convective (third kind) boundary conditions become non-linear. Although this non-linearity is not strong, convergence problems may arise if radiative heat transfer (fourth kind) boundary conditions are also present, see [9]. Since all the non-linearities are transferred to the boundary conditions, the Kirchhoff transformation approach is very well-suited for applying the boundary element method (BEM), [5, 18], or the method of fundamental solutions (MFS), [19]. In the same manner these techniques can be extended to composite bodies through the subregion technique. In it, each region is dealt with separately and then the whole body is linked together by applying compatibility and equilibrium conditions along the interfaces between subregions.

Two-dimensional boundary value problems of heat conduction in non-linear single and composite materials have been the subject of several studies using the BEM, [1, 3, 4, 5, 18]. However, the implementation of the BEM becomes quite tedious, especially in three-dimensional irregular domains. Moreover, the evaluation of the gradient of the temperature solution on the boundary requires the use of finite differences or the evaluation of hypersingular integrals. In order to alleviate some of these difficulties, we propose the use of the MFS. The merits and drawbacks of the MFS over the BEM for solving elliptic boundary value problems are thoroughly discussed in [7, 10, 17, 19].

Prior to this study, the MFS was used for the solution of problems of heat conduction in linear single material with nonlinear boundary conditions, [19], and composite materials with linear boundary conditions [2]. It is the purpose of this paper to extend these analyses to nonlinear, both single and composite, materials with nonlinear boundary conditions.

6.2 Mathematical Formulation

Consider a bounded domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, with piecewise smooth boundary $\partial \Omega$, formed from two (or more) subregions $\Omega_1$ and $\Omega_2$ separated by the interfacial surface $\Gamma_{12} = \partial \Omega_1 \cap \partial \Omega_2$. The material of subregion $\Omega_1$ has a temperature dependent thermal conductivity $k_1 > 0$ and material of subregion $\Omega_2$ has a different thermal conductivity $k_2 > 0$. The governing
steady-state heat conduction equations are
\[ \nabla \cdot (k_i(T_i) \nabla T_i) = 0, \quad \text{in } \Omega_i, \quad i = 1, 2, \] (6.1)
where \( T_i \) is the temperature solution in domain \( \Omega_i, \quad i = 1, 2, \) and, for the sake of simplicity, we have assumed that there is no heat generation within \( \Omega \). The technique developed in this paper is valid for bodies \( \Omega \) consisting of an arbitrary finite number of subregions.

Boundary conditions of the mixed type can be prescribed at the external surface \( \partial \Omega \) of the composite body \( \Omega = \Omega_1 \cup \Omega_2 \) and they include (dropping for simplicity the region subscript \( i = 1, 2 \)):

- **Dirichlet boundary conditions** (prescribed temperature \( f \))
  \[ T = f, \quad \text{on } S_1. \] (6.2)

- **Neumann boundary conditions** (prescribed heat flux \( g \))
  \[ -k(T) \frac{\partial T}{\partial n} = g, \quad \text{on } S_2, \] (6.3)
  where \( n \) is the outward normal to the boundary \( \partial \Omega \).

- **Robin boundary conditions** (prescribed heat transfer coefficient \( h \))
  \[ -k(T) \frac{\partial T}{\partial n} = h(T - T_f) - q_3, \quad \text{on } S_3, \] (6.4)
  where \( T_f \) is the temperature of fluid exchanging heat with surface \( S_3 \), and \( q_3 \) is a given function.

- **Radiation boundary condition** (prescribed fourth-order power law)
  \[ -k(T) \frac{\partial T}{\partial n} = \sigma \varepsilon (T^4 - T_s^4) - q_4, \quad \text{on } S_4, \] (6.5)
  where \( q_4 \) is a given function, \( \sigma = 5.67051 \times 10^{-8} W/(m^2 K^4) \) is the Stefan-Boltzmann constant and \( \varepsilon \) is the radiation interchange factor (emissivity) between the irradiated boundary \( S_4 \) and the environment, having a temperature \( T_s \).

In (6.2)-(6.5) the boundary portions \( S_j, \quad j = 1, 4, \) which cover the
boundary \( \partial \Omega \), i.e. \( \partial \Omega = \bigcup_{j=1}^{4} S_j \), have no common parts, i.e. \( S_i \cap S_j = \emptyset, \ i \neq j \). Also, in the above boundary conditions the nonlinearity occurs mainly due to the heat radiation (6.5), although the method of solution can also allow nonlinearities to occur from a temperature dependent heat transfer coefficient \( h(T) \), or from a temperature dependent radiation interchange factor \( \epsilon(T) \).

In addition to the above boundary conditions (6.2)-(6.5), both imperfect and ideal contact conditions can occur at the interface \( \Gamma_{12} \), namely

- **Interface continuity**
  \[
  -k_1(T_1) \frac{\partial T_1}{\partial n^+} = k_2(T_2) \frac{\partial T_2}{\partial n^-}, \quad \text{on } \Gamma_{12}, \quad (6.6)
  \]
  where \( n^+ \) and \( n^- \) are the outward normals to the boundaries \( \partial \Omega_1 \cap \Gamma_{12} \) and \( \partial \Omega_2 \cap \Gamma_{12} \), respectively, i.e. since \( \Gamma_{12} = \partial \Omega_1 \cap \partial \Omega_2 \) we have \( n^+ = -n^- \).

- **Ideal contact (interface temperature continuity)**
  \[
  T_1 = T_2, \quad \text{on } S_5, \quad (6.7)
  \]
  and imperfect contact (interface temperature jump)
  \[
  T_1 = T_2 - R k_1(T_1) \frac{\partial T_1}{\partial n^+}, \quad \text{on } S_6, \quad (6.8)
  \]
  where \( R \) is the contact resistance. In (6.7) and (6.8) the interface portions \( S_5 \) and \( S_6 \) cover \( \Gamma_{12} \), i.e. \( S_5 \cup S_6 = \Gamma_{12} \), and have no common parts, i.e. \( S_5 \cap S_6 = \emptyset \).

### 6.2.1 Kirchhoff’s transformation

The governing nonlinear partial differential equations (6.1) can be easily transformed into the Laplace equation by employing the Kirchhoff transformation defined as

\[
\Psi_i = \psi_i(T_i) := \int_0^{T_i} \frac{k_i(y)}{k_0} dy, \quad i = 1, 2, \quad (6.9)
\]
where \( k_i(T) = k_0(1 + m_i(T_i)) \), \( k_0 \) are positive constants and \( m_i(T_i) > -1 \) are known functions.

Since \( k_i > 0 \), the inverse transformation to (6.9) exists and is given by

\[
T_i = \psi_i^{-1}(\Psi_i), \quad i = 1, 2. \tag{6.10}
\]

Under (6.9), problem (6.1)-(6.8) transforms into the equivalent form

\[
\nabla^2 \Psi_i = 0, \quad \text{in } \Omega_i, \quad i = 1, 2, \tag{6.11}
\]

subject to the boundary conditions (dropping for simplicity the subscript \( i = 1, 2 \))

\[
\Psi = \psi(f), \quad \text{on } S_1, \tag{6.12}
\]

\[
-k_0 \frac{\partial \Psi}{\partial n} = g, \quad \text{on } S_2, \tag{6.13}
\]

\[
-k_0 \frac{\partial \Psi}{\partial n} = h[\psi^{-1}(\Psi) - T_f] - q_3, \quad \text{on } S_3, \tag{6.14}
\]

\[
-k_0 \frac{\partial \Psi}{\partial n} = \sigma \varepsilon [\psi^{-1}(\Psi)^4 - T_s^4] - q_4, \quad \text{on } S_4, \tag{6.15}
\]

and the interface conditions

\[
-k_{0_1} \frac{\partial \Psi_1}{\partial n^+} = k_{0_2} \frac{\partial \Psi_2}{\partial n^-}, \quad \text{on } \Gamma_{12}, \tag{6.16}
\]

\[
\psi_1^{-1}(\Psi_1) = \psi_2^{-1}(\Psi_2), \quad \text{on } S_5, \tag{6.17}
\]

\[
\psi_1^{-1}(\Psi_1) = \psi_2^{-1}(\Psi_2) - R k_{0_1} \frac{\partial \Psi_1}{\partial n^-}, \quad \text{on } S_6. \tag{6.18}
\]

It can be seen that in the Kirchhoff space of the transform, the governing equation (6.11), the Dirichlet boundary condition (6.12), the Neumann boundary condition (6.13) and the flux continuity condition (6.16) are linear, whilst the convective boundary condition (6.14) and the interface temperature conditions (6.17) and (6.18) become nonlinear. The nonlinearity caused by the fourth power law radiation (6.5) is also present in the space of transform (6.15), and furthermore, the Kirchhoff transform is no longer continuous across the interface \( S_5 \), i.e. a jump in the transforms occurs there where \( \Psi_1 \neq \Psi_2 \).

The solvability of problem (6.1)-(6.8), or equivalently the transformed
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problem (6.11)-(6.18) depends on the form, e.g. smoothness, monotonicity, of the input data \( \Omega, \partial \Omega, \Gamma_{12}, k_i, f, g, T_f, T_s, h \) and \( \varepsilon \) and it may be established using classical boundary integral equation methods, see e.g. [15, 23].

Once \( \Psi \) has been determined, the temperature solution \( T \) may be readily obtained from equation (6.10), via (6.9).

\subsection*{6.3 The method of Fundamental Solutions (MFS)}

As the sources of nonlinearity are associated with the boundary conditions (6.14), (6.15), (6.17) and (6.18) only, the boundary value problem (6.11)-(6.18) for each subregion can be converted into a minimization problem, or equivalently an algebraic system of nonlinear equations, using the MFS.

From [6, 22], the MFS approximations for the solutions \( \Psi_1 \) and \( \Psi_2 \) of the Laplace equation (6.11) have the form

\begin{equation}
\Psi_i^N(c^i, x) = \sum_{k=1}^{N} c_i^k G_d(y_i^k, x), \quad x \in \Omega_i, \quad i = 1, 2, \quad (6.19)
\end{equation}

where \( N \) is the number of unknown singularities (sources) \( (y_i^k)_{k=1,N} \notin \Omega_i \), \( (c_i^k)_{k=1,N} \) are unknown real coefficients and \( G_d \) is a fundamental solution for the Laplace equation, given by

\begin{equation}
G_d(y, x) = \begin{cases} 
\ln |y - x|, & d = 2, \\
\frac{1}{|y - x|}, & d = 3.
\end{cases} \quad (6.20)
\end{equation}

The heat flux is obtained by differentiating (6.19) with respect to the outward normal \( n \).

In (6.19), the coordinates of the singularities may be either preassigned, or let free and determined as part of the solution [10]. Here, we adopt the former option, where the singularities are fixed, although their location is parameterized by a single unknown parameter, as described at the end of this section. Therefore, in equation (6.19) there are \( 2N \) unknowns, namely, the coefficients \( (c_i^k)_{k=1,N} \). These coefficients can be determined by collocating (imposing) the boundary and interface
conditions (6.12)-(6.18) at \( M_\partial \Omega \) distinct points on the boundary \( \partial \Omega_1 \setminus \Gamma_{12} \), \( M_\partial \Omega \) distinct points on the boundary \( \partial \Omega_2 \setminus \Gamma_{12} \), and \( M_T \) distinct points on the interface \( \Gamma_{12} \). We denote the boundary points on each of the four parts \( \{ S_\ell \}_{\ell=1}^4 \subset \partial \Omega \) in the following way:

- On \( S_1 \cap \partial \Omega_i \) we take \( (x_j^i)_{j=1}^{M_1} \)
- On \( S_2 \cap \partial \Omega_i \) we take \( (x_j^i)_{j=1}^{M_1+M_2} \)
- On \( S_3 \cap \partial \Omega_i \) we take \( (x_j^i)_{j=1}^{M_3} \)
- On \( S_4 \cap \partial \Omega_i \) we take \( (x_j^i)_{j=1}^{M_1+M_2+M_3+M_4} \)

On the interface \( \Gamma_{12} \) we take:

- On \( S_5 \) we take \( (x_j)_{j=1}^{M_5} \)

and

- On \( S_6 \) we take \( (x_j)_{j=1}^{M_5+1, M_5+M_6} \)

Clearly, here we have that \( M_\partial \Omega = M_1 + M_2 + M_3 + M_4 \) and \( M_T = M_5 + M_6 \).

We thus have a total number of \( 2N \) unknowns and a total of \( 2M_\partial \Omega + 2M_T \) conditions to be satisfied.

Let us denotes \( \Psi_N^{ij} := \Psi_N^{i}(e^i; x_j^i) \), \( \Psi_N^{1:j} := \Psi_N^{i}(e^i; x_j) \), and \( \Psi_N^{2:j} := \Psi_N^{2}(e^2; x_j) \). Substituting (6.19) into (6.12)-(6.18), we minimize the nonlinear least-squares objective function

\[
S(c^1, c^2) := \sum_{i=1}^{2} \sum_{j=1}^{M_1} \left[ \left( \Psi_N^{ij} - \psi_i(f(x_j^i)) \right)^2 + \sum_{j=M_1+1}^{M_1+M_2} \left[ -k_0 \frac{\partial \Psi_N^{ij}}{\partial n} - g(x_j^i) \right]^2 \right]
+ \sum_{j=M_1+M_2+1}^{M_1+M_2+M_3} \left[ -k_0 \frac{\partial \Psi_N^{ij}}{\partial n} - h(x_j^i) (\psi_i^{-1}(\Psi_N^{ij}) - T_f(x_j^i)) + q_3(x_j^i) \right]^2
+ \sum_{j=M_1+M_2+M_3+1}^{M_1+M_2+M_3+M_4+1} \left[ -k_0 \frac{\partial \Psi_N^{ij}}{\partial n} - \sigma \varepsilon(x_j^i) \left( \psi_i^{-1}(\Psi_N^{ij})^4 - T_s(x_j^i)^4 \right) + q_4(x_j^i) \right]^2
+ \sum_{j=1}^{M_5} \left[ \psi_1^{-1}(\Psi_N^{1:j}) - \psi_2^{-1}(\Psi_N^{2:j}) \right]^2 + \sum_{j=1}^{M_6} \left[ -k_0 \frac{\partial \Psi_N^{1:j}}{\partial n} - k_0 \frac{\partial \Psi_N^{2:j}}{\partial n} \right]^2
\]
The minimization of (6.21) is carried out using the MINPACK [14], routines lmdif or lmder which minimize the sum of the squares of nonlinear functions. In lmder the Jacobian is provided by the user, whilst in lmdif the Jacobian is calculated internally by forward finite differences. Preliminary investigations, [20, 21], found that lmder converged faster than lmdif.

Two pseudo-boundaries are taken as exterior similar deformations of the boundaries $\partial \Omega_i$ of the original domains $\Omega_i \subset \Omega'_i$, $i = 1, 2$. A number of $N$ singularities is placed on each pseudo-boundary. An important question in the implementation of the MFS is the positioning of these pseudo-boundaries. This point is addressed by extending the approach used in [26]. In particular, the pseudo-boundaries $\partial \Omega'_i$, $i = 1, 2$ are taken at a distance $\eta > 0$ from the boundaries $\partial \Omega_i$, $i = 1, 2$, respectively. In order to determine the optimal value of $\eta$, the minimization problem is solved for various values of $\eta_\ell = \eta_0 + \ell(\delta\eta)$, $\ell = 1, \ldots, L$, where $\delta\eta$ is a small positive increment. For each $\eta_\ell$, the maximum error in the boundary conditions at a selected set of uniformly spaced points on the boundary $\partial \Omega \cup \Gamma_{12}$ (different from the boundary collocation points) is calculated. The optimal value of $\eta$ is chosen the one for which the maximum error was minimized.

The MFS formulation described in this section may be viewed as a domain decomposition technique. Such approaches have been used, in conjunction with the MFS, in [2, 11].

**6.4 Numerical Results and Discussion**

In this section we present numerical results obtained from the application of the MFS described in the previous section for a single nonlinear material with radiative boundary condition. More examples including composite materials as well can be found in [20, 21].

We consider a single nonlinear heat conductor with the thermal conductivity given by

$$k(T) = k_0(1 + aT), \quad k_0 = 1 \text{W/mK}, \quad a \in \{0, 0.1, 0.3, 0.5\} \text{K}^{-1},$$

(6.22)
occupying part of an L-shaped cross-section of an industrial furnace, as depicted in Figure 6.1. We take nonlinear radiative boundary conditions throughout $S_4 = \partial \Omega$ with $T_s = 0$ and $\varepsilon = 10^8 / 5.67051$, such that the boundary condition (6.5) becomes

$$k(T) \frac{\partial T}{\partial n} + |T| T^3 = q_4, \quad \text{on} \quad S_4 = \partial \Omega. \quad (6.23)$$

In equation (6.23), the function $q_4$ is chosen in such a way that the exact (analytical) solution of the problem

$$\nabla \cdot (k(T) \nabla T) = 0, \quad \text{in} \quad \Omega \quad (6.24)$$

subject to (6.23), where $k(T)$ is given by (6.22), is given by

$$T(x) = -1 + \sqrt{1 - 2a \ln |x - x^*|} - 2a \ln |x - x^*|, \quad x \in \Omega. \quad (6.25)$$
where \( x^* = (0.3, -0.3) \notin \Omega \).

For linear material, i.e. \( a = 0 \), this problem was investigated using a Galerkin BEM in [13]. It should be noted that the nonlinearity of type \( T^4 \), which occurs frequently in heat transfer problems, does not satisfy the monotonicity assumption needed for an existence theory result, [12]; in fact, \( |T|T^3 \) was considered instead of \( T^4 \) in [23].

Employing the Kirchhoff transformation

\[
\Psi = \psi(T) := \int_0^T k(\xi) \, d\xi = T + \frac{aT^2}{2}, \quad (6.26)
\]

problem (6.23) - (6.24) becomes

\[
\nabla^2 \Psi = 0, \quad \text{in} \ \Omega, (6.27)
\]

\[
\left. \frac{\partial \Psi}{\partial n} + \left| \frac{1 + \sqrt{1 + 2a\Psi}}{a} \right| \left( \frac{1 + \sqrt{1 + 2a\Psi}}{a} \right)^3 = q_4, \quad \text{on} \ \partial \Omega, (6.28)
\]

The analytical solution of problem (6.27) and (6.28) is given by

\[
\Psi(x) = -\ln |x - x^*|, \quad x \in \Omega. \quad (6.29)
\]

To generate the data \( q_4 \) in (6.23), or (6.28), we use the derivatives

\[
\begin{align*}
\frac{\partial}{\partial x_1} (-\ln |x - x^*|) &= -\frac{x_1 - 0.3}{(x_1 - 0.3)^2 + (x_2 + 0.3)^2}, \\
\frac{\partial}{\partial x_2} (-\ln |x - x^*|) &= -\frac{x_2 + 0.3}{(x_1 - 0.3)^2 + (x_2 + 0.3)^2}.
\end{align*} \quad (6.30)
\]

We choose \( M \) uniformly distributed points \( (x_j)_{j=1}^M \) on the boundary \( \partial \Omega \), \( M/8 \) on each \( DE, EO, OA \) and \( AB \), and \( M/4 \) on each \( BC \) and \( CD \), and \( N \) uniformly distributed sources \( (y_j)_{j=1}^N \) on the pseudo-boundary \( \partial \Omega' \) which is taken at a distance \( \eta > 0 \) from \( \partial \Omega \). Then we minimize the functional

\[
S(c) = \sum_{j=1}^M \left\{ \frac{\partial \Psi_N(c;x_j)}{\partial n} + \left| \frac{1 + \sqrt{1 + 2a\Psi_N(c;x_j)}}{a} \right| \right. \\
\times \left. \left( \frac{1 + \sqrt{1 + 2a\Psi_N(c;x_j)}}{a} \right)^3 - q_4(x_j) \right\}^2, \quad (6.31)
\]
where

$$\Psi_N(c;x_j) = \sum_{k=1}^{N} c_k G_2(y_k;x_j), \quad j = 1,M.$$ 

Once $\Psi_N$ has been obtained accurately, the temperature $T$ can be obtained by inverting (6.26), i.e.

$$T = \frac{-1 + \sqrt{1 + 2a\Psi_N}}{a}.$$ (6.32)

We first solve the problem with $a = 0$. In this case $k(T) = 1$ and there is no need to employ transformation (6.9), the problem to be solved being linear and given by

$$\nabla^2 T = 0, \quad \text{in } \Omega,$$ (6.33)

$$\frac{\partial T}{\partial n} + |T|T^3 = q_4, \quad \text{on } \partial\Omega,$$ (6.34)

which has the analytical solution

$$T(x) = -\ln|x - x^*|, \quad x \in \bar{\Omega}.$$ (6.35)

In Figure 6.2, we present the boundary temperature along the perimeter of the L-shaped boundary, as a function of the arc-length $s \in [0,2)$, starting from the origin and oriented counter-clockwise in the cases $a = 0, 0.1, 0.3,$ and 0.5 obtained using the MFS with $M = N = 96$. From this figure very good agreement with the analytical solution (6.32) can be observed. The corresponding elongated plots are presented in Figure 6.3. In Figure 6.4, we present the maximum error calculated on a fixed set of points on the boundary (different than the boundary collocation points) versus the distance $\eta$ of the pseudoboundary from the boundary for various numbers of degrees of freedom $M = N = 32, 64$ and 96 in the case $a = 0.1$. As can be observed from this plot the accuracy improves with increasing the number of degrees of freedom. Also, the error decreases up to a certain value of $\eta$ after which it starts increasing again. This behaviour is typical of the MFS where the initial decrease of the error is consistent with the theoretical predictions [25], while the eventual increase of the error is due to the ill-conditioning of the MFS system [24]. The accuracy of the scheme was found to be independent of the values of $a$ considered.
Figure 6.2: The analytical (−) and the MFS numerical (+) boundary temperature along the perimeter of the L-shaped domain for \(a = 0, 0.1, 0.3, 0.5\).

6.5 Conclusions

In this paper, the application of the MFS to steady-state nonlinear heat conduction problems has been investigated. The method recasts the problem as a nonlinear minimization problem. The numerical obtained results are in good agreement with the available analytical solution showing high accuracy and stable convergence, and with the BEM results of [3, 5]. However, unlike the BEM, the MFS can be easily extended to three-dimensional nonlinear steady-state heat conduction problems. Moreover, if a heat source is present in equation (6.1), then one can apply a modification of the MFS, as described in [16]. In the numerical test performed in Section 4, the relations between the dependent variables \(\Psi\) and \(T\) before and after the Kirchhoff transformation were given explicitly by equations (6.26) and (6.32). However, if the thermal conductivity would be given at some discrete nodes instead of an explicit function then one can approxi-
mate $k(T)$ by a piecewise linear function using these nodes and then apply the Kirchhoff transformation over each linear portion, [1]. The proposed MFS technique can be implemented in a commercial code aimed at solving general convective, radiative, steady-state, nonlinear heat transfer in composite layered (ideal or non-ideal interface contact) heat conductors. In the case of transient heat conduction problems the proposed scheme would have to be changed in order to accommodate the transient term. One way of doing this is to use the Laplace transform to remove the time dependence and then to deal with the resulting inhomogeneous Helmholtz-type problem using the method of particular solutions. A comprehensive account of the details involved in this approach may be found in [8], see also [17].
Figure 6.4: Maximum error on the boundary versus $\eta$ for $M = N = 32, 64$ and 96 for $a = 0.1$.

Acknowledgements

The authors would like to thank the UK Royal Society for supporting this research. D. Lesnic would like to thank the University of Cyprus for the hospitality shown to him during his outgoing short research visit. Finally, the authors wish to thank the anonymous referee for his/her constructive comments and suggestions.

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CHAPTER 7

A Multi-Level Regularized Version of the Method of Fundamental Solutions

C. Gáspár

Abstract. A regularization technique of the method of fundamental solutions is presented. Instead of the fundamental solution of the original problem, the fundamental solution of a higher order auxiliary problem is used. Thus, the computational problems caused by the singularity of the original fundamental solution is avoided. The proposed method is equivalent to a boundary interpolation technique based on the fundamental solution of the applied higher order problem. The computational cost can be significantly reduced by directly solving the interpolation problem on quadtree-based cell system using standard multi-level tools. Thus, the use of large, dense and ill-conditioned matrices can also be avoided.

7.1 Introduction

The Method of Fundamental Solutions (MFS) is a truly meshless method, which has quickly become popular because of its simplicity and accuracy. It has been applied to many types of problems e.g. elliptic partial differential equations [3, 4, 5, 10, 12]; eigenvalue problems [1, 2]; inverse problems [11] etc. The method can be considered also an approximation of the indirect Boundary Element Method. In both cases, the fundamental solutions of the original partial differential equation is needed. For instance, in

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the case of the simple 2D Laplace equation
\[ \Delta U = 0, \quad (7.1) \]
the fundamental solution has the form:
\[ \Phi(x) = \frac{1}{2\pi} \log ||x||, \quad (7.2) \]
where \( ||.|| \) denotes the Euclidean norm. In the indirect BEM, the solution is sought in the boundary integral form:
\[ U(x) = \int_{\Gamma} \sigma(y) \cdot \Phi(x - y) d\Gamma_y \quad (7.3) \]
(with an a priori unknown function \( \sigma \)). The MFS defines the approximate solution as a finite sum:
\[ U(x) = \sum_{j=1}^{N} \alpha_j \cdot \Phi(x - \bar{x}_j), \quad (7.4) \]
where the points \( \bar{x}_1, \bar{x}_2, \ldots, \bar{x}_N \) (the source points) are located in the exterior of the domain of the original PDE \( \Omega \). Thus, the function \( U \) (both in (7.3) and (7.4)) automatically satisfies the PDE (7.1). The unknown function \( \sigma \) (and the coefficients \( \alpha_1, \alpha_2, \ldots, \alpha_N \), respectively) can be determined from the boundary conditions. For simplicity, assume that Dirichlet boundary condition is prescribed:
\[ U(x_k) = u_k \quad (k = 1, 2, \ldots, N), \quad (7.5) \]
where \( x_1, x_2, \ldots, x_N \) are located on the boundary \( \Gamma \) (collocation points).

After a proper boundary discretization, (7.3) leads to an algebraic system of equations similar to (7.4). (In this case, the points \( \bar{x}_j \) are located on the boundary \( \Gamma \).) Both approaches suffer from the same computational disadvantage: they lead to a linear algebraic system with large, full and often severely ill-conditioned matrix.

If the sources are located outside of the domain \( \Omega \) (like in (7.4), cf. [3, 4]), the situation is even worse as shown through the following simple example. Let \( \Omega \) be a circle with radius \( R \), centered at the origin. Then the solution of (7.1) can be expressed in the form:
\[ U(x) = \int_{\Gamma_0} \sigma(y) \cdot \Phi(x - y) d\Gamma_y, \quad (7.6) \]
where $\Gamma_0$ is a larger circle with radius $R_0 > R$. It is well known that the boundary function $u_0 := U|\Gamma_0$ belongs to the Sobolev space $H^{s_0}(\Gamma_0)$ provided that $\sigma \in H^{s_0-1}(\Gamma_0)$ (for arbitrary $s_0 \in \mathbb{R}$), where $U|\Gamma_0$ denotes the trace of $U$ taken on $\Gamma_0$. Expressing $u_0$ in terms of (boundary) Fourier series:

$$ u_0(t) = \sum_k \hat{u}_k e^{ikt}, $$

it is clear that the trace of $U$ taken at the original boundary $\Gamma$ is expressed as:

$$ u(t) := U|\Gamma = \sum_k \hat{u}_k \left( \frac{R}{R_0} \right)^k e^{ikt}. \quad (7.7) $$

Since $R < R_0$ and $u_0 \in H^{s_0}(\Gamma_0)$, the boundary function $u$ belongs to the Sobolev space $H^s(\Gamma_0)$ for any index $s \geq s_0$. That is, $u$ is extremely smooth. In other words, if $u$ is not extremely smooth, then $u_0$ as well as $\sigma$ become extremely irregular, which can cause serious numerical instability when implementing the solution of (7.6) (after a proper discretization e.g. applying the form (7.4)).

To illustrate the above phenomenon, consider the following two model problems. In both cases, let $\Omega$ be a circle centered at the origin, with radius $1/4$. Along the boundary $\Gamma$, Dirichlet boundary conditions are given, consistent with the exact solutions.

**Test 1:**

$$ U(x, y) := -x + 2y \quad (7.8) $$

**Test 2:**

$$ U(x, y) := \sqrt{\frac{4x + \sqrt{(4x)^2 + (4y + 1)^2}}{2}} - \sqrt{\frac{4x + \sqrt{(4x)^2 + (4y - 1)^2}}{2}} \quad (7.9) $$

Test 1 is a smooth function, while the function Test 2 has two singular points ($(0, 1/4)$ and $(0, -1/4)$), but still belongs to the Sobolev space $H^1(\Omega)$. (Note that the function is the real part of the complex-valued function $\sqrt{4z + i - \sqrt{4z - i}}$.) Table 7.1 summarizes the exactness of the approximate solutions measured in the discrete $L_2$-norm. Here $N$ denotes
the number of source and the collocation points and \( d \) denotes the distance of the source points from the boundary. The asterisk indicates that the approximate solution exhibits strong irregularity outside of \( \Omega \).

<table>
<thead>
<tr>
<th>( N ) ( \backslash ) ( d )</th>
<th>Test 1</th>
<th>Test 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>32</td>
<td>1.085</td>
<td>0.011</td>
</tr>
<tr>
<td>64</td>
<td>0.136</td>
<td>0.000</td>
</tr>
<tr>
<td>128</td>
<td>0.005</td>
<td>0.000</td>
</tr>
<tr>
<td>256</td>
<td>0.000</td>
<td>0.000</td>
</tr>
</tbody>
</table>

Table 7.1: MFS, relative \( L_2 \)-errors of the smooth and weakly singular test problems

As can be seen from Table 7.1, in case of non-smooth solution, the MFS produces very irregular approximate solutions; the numerical instability destroys the approximation if the sources are located too far from the boundary. This phenomenon is much more moderate when the solution is smooth; here the approximate solution is still quite exact in the interior of the domain while irregularities already appear in the vicinity of the sources. Figure 7.1 shows a typical example for this situation. Here the sources are located at a distance of 0.1 from the boundary. The number of sources was 256.

### 7.2 The Idea of the Regularized Method of Fundamental Solutions (RMFS)

As indicated in the previous section, the numerical difficulties of the MFS can be decreased by locating the source points in the vicinity of the boundary (though the accuracy also decreases in this case). However, the source and collocation points are not allowed to be identical because of the singularity of the fundamental solution \( \Phi \) at the origin.

To overcome this difficulty, introduce the fundamental solution of the fourth-order Laplace-Helmholtz operator \( \Delta(\Delta - c^2I) \), where \( I \) denotes the identity operator and \( c \) is a positive scaling constant which will be defined
Figure 7.1: MFS-solution of the smooth test problem. Irregularities appear later. This function has the form:

$$\Phi(x) = -\frac{1}{2\pi c^2}(K_0(c||x||) + \log(c||x||)),$$

(7.10)

where $K_0$ is the usual modified Bessel function of the third kind. This function is continuous at the origin, which is an immediate consequence of the well-known asymptotic expansion:

$$K_0(r) = (\log 2 - \log r - \gamma) + \frac{1}{4}(\log 2 - \log r + 1 - \gamma)r^2 +$$

$$+ \frac{1}{64}(\log 2 - \log r + \frac{3}{2} - \gamma)r^4 + \ldots$$

Here $\gamma$ denotes the Euler constant: $\gamma \approx 0.57721$. Moreover, since $K_0$ decreases rapidly, $\Phi$ is approximately harmonic far from the origin.

Replacing the fundamental solution of the Laplacian with the function defined in (7.10), from (7.4) we obtain the simplest form of the regularized
A multi-Level regularized version of the MFS

\[ U(x) = \sum_{j=1}^{N} \alpha_j \cdot \Phi(x - \bar{x}_j), \quad (7.11) \]

where the coefficients \( \alpha_j \) are determined by solving the linear system:

\[ \sum_{j=1}^{N} \alpha_j \cdot \Phi(x_k - \bar{x}_j) = u_k \quad (k = 1, 2, \ldots, N). \quad (7.12) \]

Again, \( x_1, \ldots, x_N \) are the collocation points on \( \Gamma \) and \( \bar{x}_1, \ldots, \bar{x}_N \) are the source points. Unlike the traditional MFS, the collocation and the source points are allowed to coincide. Thus, the condition number of the matrix in (7.12) can be kept at a more moderate level.

As illustrative examples, consider again the test problems presented in the previous section. We have applied the regularized method of fundamental solutions with the same scaling parameter \( c := 500 \). The results are summarized in Table 7.2. The numerical instability is now less than in case of MFS especially when the source and the collocation points coincide. On the other hand, the accuracy increases when the distance of the source points and the boundary grows. The goal is to achieve an acceptable compromise between the accuracy and instability.

<table>
<thead>
<tr>
<th>( N \setminus d )</th>
<th>Test 1</th>
<th>Test 2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N )</td>
<td>0</td>
<td>0.01</td>
</tr>
<tr>
<td>32</td>
<td>3.175</td>
<td>1.074</td>
</tr>
<tr>
<td>64</td>
<td>0.870</td>
<td>0.131</td>
</tr>
<tr>
<td>128</td>
<td>0.115</td>
<td>0.002</td>
</tr>
<tr>
<td>256</td>
<td>0.341</td>
<td>0.002</td>
</tr>
</tbody>
</table>

Table 7.2: Regularized MFS, relative \( L_2 \)-errors of the smooth and weakly singular test problems

7.3 Error Estimations

As shown in the previous section, the RMFS can be considered a method in which the solution of the original Laplace equation is approximated by
A multi-Level regularized version of the MFS

a special solution of the fourth-order Laplace-Helmholtz-equation. In the following, we give some error estimations for this type of approximations.

Let \( \Omega \subset \mathbb{R}^2 \) be a bounded, simply connected domain with a smooth boundary \( \Gamma \). Let \( U^* \) be the (unique) solution of the Dirichlet problem

\[
\Delta U^* = 0, \quad U^*|_{\Gamma} = u, \tag{7.13}
\]

where \( u \in H^{1/2}(\Gamma) \), and let \( U \) be the (unique) solution of the fourth-order Dirichlet problem

\[
\Delta(\Delta - c^2 I)U = 0, \quad U|_{\Gamma} = u, \quad \frac{\partial U}{\partial \mathbf{n}}|_{\Gamma} = v, \tag{7.14}
\]

with an arbitrary function \( v \in H^{-1/2}(\Gamma) \).

**Theorem 1:** There exists a constant \( C \geq 0 \), independent of \( u, v \) and \( c \) such that the following inequality holds:

\[
||U - U^*||^2_{L^2(\Omega)} \leq \frac{C}{c^2} \left( ||u||^2_{H^{1/2}(\Gamma)} + ||v||^2_{H^{-1/2}(\Gamma)} \right).
\]

**Proof:** It can be assumed that \( \Omega \) is the half-stripe \((0, L) \times (0, +\infty)\) (the general case can be converted to this case by an proper coordinate transform), and the appearing boundary functions as well as the functions \( U^* \) and \( U \) are \( L \)-periodic with respect to the first variable. Expressing the function \( u \) in terms of complex Fourier series:

\[
u(x) = \sum_k \alpha_k e^{i\kappa x} \quad \text{(where } \kappa := \frac{2\pi k}{L}),
\]

the solution of (7.13) has the form:

\[
U^*(x, y) = \sum_k \alpha_k e^{-|\kappa| y} e^{i\kappa x}. \tag{7.15}
\]

Similarly, if \( v \) is expressed in terms of Fourier series:

\[
v(x) = \sum_k \beta_k e^{i\kappa x}, \tag{7.16}
\]

then it is easy to check that the solution of (7.14) is as follows:

\[
U(x, y) = \sum_k \left( A_k e^{-|\kappa| y} + B_k e^{-\sqrt{\kappa^2 + c^2} y} \right) e^{i\kappa x}. \tag{7.17}
\]

(Note that \( \alpha_0 = 0 \) and \( \beta_0 = 0 \), i.e. the summation is performed for the nonzero indices only.) The coefficients \( A_k, B_k \) are determined by the boundary conditions in (7.14): \( U|_{\Gamma}(x) = U(x, 0) = \sum_k (A_k + B_k) e^{i\kappa x} = \sum_k \alpha_k e^{i\kappa x} \), whence

\[
A_k + B_k = \alpha_k.
\]
Moreover,
\[
\frac{\partial U}{\partial n}(x) = -\frac{\partial U}{\partial y}(x,0) = \sum_k (|\kappa| A_k + \sqrt{\kappa^2 + c^2} B_k) e^{i\kappa x} = \sum_k \beta_k e^{i\kappa x},
\]
which implies that
\[|\kappa| A_k + \sqrt{\kappa^2 + c^2} B_k = \beta_k.\]

From the above two equations, \(A_k\) and \(B_k\) can be computed:
\[
A_k = \frac{\sqrt{\kappa^2 + c^2} \alpha_k - \beta_k}{\sqrt{\kappa^2 + c^2} - |\kappa|}, \quad B_k = -\frac{|\kappa| \alpha_k + \beta_k}{\sqrt{\kappa^2 + c^2} - |\kappa|}.
\]

The difference of the approximate and exact solution:
\[
U(x,y) - U^*(x,y) = \sum_k \left( (A_k - \alpha_k) e^{-|\kappa| y} + B_k e^{-\sqrt{\kappa^2 + c^2} y} \right) e^{i\kappa x}
\]
Applying Parseval’s theorem and the fact that \(A_k - \alpha_k = -B_k\), we obtain:
\[
\int_0^L |U(x,y) - U^*(x,y)|^2 \, dx = L \sum_k |B_k|^2 \left( e^{-2|\kappa| y} - 2e^{-|\kappa| y} e^{-\sqrt{\kappa^2 + c^2} y} + e^{-2\sqrt{\kappa^2 + c^2} y} \right)
\]
Integrating from 0 to +\(\infty\) with respect to \(y\):
\[
||U - U^*||^2_{L^2(\Omega)} = L \sum_k |B_k|^2 \left( \frac{1}{2|\kappa|} - \frac{2}{|\kappa| + \sqrt{\kappa^2 + c^2}} + \frac{1}{2\sqrt{\kappa^2 + c^2}} \right).
\]
Substituting the expression of \(B_k\) into the right-hand side and applying the elementary inequality \(|z + w|^2 \leq 2|z|^2 + 2|w|^2\), we have:
\[
||U - U^*||^2_{L^2(\Omega)} \leq 2L \sum_k \frac{|\kappa|^2 |\alpha_k|^2 + |\beta_k|^2}{(\sqrt{\kappa^2 + c^2} - |\kappa|)^2} \left( \frac{1}{2|\kappa|} - \frac{2}{|\kappa| + \sqrt{\kappa^2 + c^2}} + \frac{1}{2\sqrt{\kappa^2 + c^2}} \right)
\]
\[
= 2L \sum_k \left( |\kappa| \cdot |\alpha_k|^2 + |\beta_k|^2 \right) \left( \frac{|\kappa|^2}{2|\kappa|} - \frac{2}{|\kappa| + \sqrt{\kappa^2 + c^2}} + \frac{1}{2\sqrt{\kappa^2 + c^2}} \right) \left( \frac{1}{\sqrt{\kappa^2 + c^2} - |\kappa|^2} \right).
\]
After some algebraic manipulations, the expression in the square brackets can be significantly simplified:

\[
\|U - U^{*}\|_{L_{2}(\Omega)}^2 \\
\leq 2L \cdot \sum_{k} \left( \frac{|\kappa| \cdot |\alpha_k|^2 + |\beta_k|^2}{|\kappa|} \right) \cdot \frac{1}{2\sqrt{\kappa^2 + c^2} \cdot (\sqrt{\kappa^2 + c^2} + |\kappa|)}
\]

\[
\leq \frac{L}{c^2} \cdot \sum_{k} \left( \frac{2\pi}{L} |k| \cdot |\alpha_k|^2 + \frac{L}{2\pi} |\beta_k|^2 \right),
\]

whence the theorem follows.

The theorem implies that if the scaling constant \( c \) is large enough, then the solution of the (second-order) problem (7.13) can be approximated by that of the fourth-order problem (7.14) with arbitrary boundary function \( v \).

In practice, the solution of the fourth-order problem (7.14) is approximated by the RMFS-form (7.11). If the source and the collocation points coincide (which is allowed, since the function \( \Phi \) is continuous everywhere), (7.11) provides a boundary type interpolation formula based on the radial basis function \( \Phi \):

\[
U(x) = \sum_{k=1}^{N} \alpha_k \Phi(x - x_j), \quad \Phi(x) = -\frac{1}{2\pi c} (K_0(c|x|) + \log(c||x||)).
\]

Thus, the RMFS can be considered also a special (boundary) RBF-interpolation method.

The key issue is the proper choice of the scaling parameter \( c \) in the formula (7.18) (see [9]). If it is too low e.g. \( c = 0 \) (biharmonic interpolation), then the boundary condition is approximated fairly well (see [7], [8]), but the function defined by (7.18) can be considered harmonic only far from the boundary. If it is too high, then the function (7.18) is 'almost' harmonic, however, the approximation of the boundary condition is poor (numerical singularities appear at the source points). Both cases result in large errors in the approximate solution. This phenomenon is illustrates in Table 7.3. Here the smooth test function (7.8) is approximated by the RMFS-formula (7.18) using 32 boundary points.

As a quasi-optimal compromise, the definition \( c \approx \frac{1}{h} \) (where \( h \) denotes the characteristic distance of the boundary points) makes it possible to
A multi-Level regularized version of the MFS

Table 7.3: Regularized MFS, relative $L_2$-errors of the smooth test problem

<table>
<thead>
<tr>
<th>$c$</th>
<th>1</th>
<th>100</th>
<th>140</th>
<th>200</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative error (%)</td>
<td>7.5637</td>
<td>1.0375</td>
<td>0.4398</td>
<td>1.1869</td>
<td>3.7053</td>
<td>5.5052</td>
</tr>
</tbody>
</table>

keep both the error of the approximation of the boundary condition and the error estimated by Theorem 1 at an acceptably low level.

### 7.4 Improvement by Off-boundary Sources

As can be seen from the results summarized in Table 7.2, the exactness of the RMFS-approximation can be improved by locating the sources outside of the domain (off-boundary sources). If the sources are kept in the vicinity of the boundary, the condition number of the discrete problem does not grow significantly. Now we give an error estimation for such cases showing that a significant amount of improvement can be achieved.

Assume that the original domain $\Omega$ is contained in a larger domain $\Omega_\delta$, the boundary of which (denoted by $\Gamma_\delta$) lies at the distance $\delta$ from the original boundary $\Gamma$. Consider the same model problem (7.13) than in the previous section. Now denote by $U$ the solution of the following fourth-order problem:

$$\Delta(\Delta - c^2 I) U = 0, \quad U|_\Gamma = u, \quad \frac{\partial U}{\partial n}|_{\Gamma_\delta} = v,$$  \hspace{1cm} (7.19)

where $U$ is defined in the larger domain $\Gamma_\delta$. Note that (7.19) is not a usual boundary value problem since the Dirichlet and Neumann data are attached to different boundaries. For simplicity, assume that both $\Omega$ and $\Omega_\delta$ are half-stripes: $\Omega = (0,L) \times (\delta, +\infty)$, $\Omega_\delta = (0,L) \times (0, +\infty)$. Assume also that the boundary data $u$ and $v$ are expressed in terms of Fourier series:

$$u(x) = \sum_k \alpha_k e^{ikx}, \quad v(x) = \sum_k \beta_k e^{ikx},$$  \hspace{1cm} (7.20)

where $\kappa = 2\pi k/L$.

**Theorem 2:** There exists a constant $C \geq 0$, independent of $u$, $v$, $c$ and $\delta$
such that the following inequality holds:

$$||U - U^*||_{L^2(\Omega)}^2 \leq \frac{C}{c^2} \left( \sum_k |\alpha_k|^2 \cdot |k| \cdot e^{-2(\sqrt{\kappa^2 + c^2} - |\kappa|)\delta} + \frac{|\beta_k|^2}{|k|} \cdot e^{-2\sqrt{\kappa^2 + c^2}\delta} \right).$$

**Proof:** The technique is quite similar to the proof of Theorem 1, therefore we only outline the proof. Now the exact solution $U^*$ has the form:

$$U^*(x, y) = \sum_k \alpha_k e^{-|\kappa|(y-\delta)} e^{i\kappa x},$$

while the solution of (7.19) is expressed as

$$U(x, y) = \sum_k \left( A_k e^{-|\kappa|y} + B_k e^{-\sqrt{\kappa^2 + c^2}y} \right) e^{i\kappa x}.$$

The coefficients $A_k, B_k$ are determined by the boundary conditions in the following way:

$$A_k = \frac{\sqrt{\kappa^2 + c^2} \alpha_k - e^{-\sqrt{\kappa^2 + c^2} \delta} \beta_k}{\sqrt{\kappa^2 + c^2}e^{-|\kappa|\delta} - |\kappa|e^{-\sqrt{\kappa^2 + c^2}\delta}},$$

$$B_k = \frac{-|\kappa| \alpha_k + e^{-|\kappa|\delta} \beta_k}{\sqrt{\kappa^2 + c^2}e^{-|\kappa|\delta} - |\kappa|e^{-\sqrt{\kappa^2 + c^2}\delta}}.$$

Computing the quantity $||U - U^*||_{L^2(\Omega)}^2$ via the same steps as in the proof of Theorem 1, we obtain:

$$||U - U^*||_{L^2(\Omega)}^2 \leq \frac{L}{c^2} \sum_k \left( \frac{(\sqrt{\kappa^2 + c^2} - |\kappa|) e^{-|\kappa|\delta}}{\sqrt{\kappa^2 + c^2}e^{-|\kappa|\delta} - |\kappa|e^{-\sqrt{\kappa^2 + c^2}\delta}} \right)^2 \times \left[ e^{-2(\sqrt{\kappa^2 + c^2} - |\kappa|)\delta} |\alpha_k|^2 + e^{-2\sqrt{\kappa^2 + c^2}\delta} \frac{1}{|\kappa|} \cdot |\beta_k|^2 \right].$$

Now the theorem is a simple consequence of the elementary inequality:

$$\frac{(\sqrt{\kappa^2 + c^2} - |\kappa|) e^{-|\kappa|\delta}}{\sqrt{\kappa^2 + c^2}e^{-|\kappa|\delta} - |\kappa|e^{-\sqrt{\kappa^2 + c^2}\delta}} \leq 1.$$

Theorem 2 implies that each Fourier coefficient of the error are reduced
A multi-Level regularized version of the MFS

at least by a factor $\frac{1}{2} e^{-c\delta}$ for the low-frequency components (i.e. for the indices $k$ for which $\kappa \leq c$). This may be a significant improvement if the solution of the original problem is smooth enough i.e. the high-frequency components of $u$ are small enough.

When discretizing (7.19), a possible way is to directly apply the RMFS-technique i.e. (7.11) and (7.12). Another possibility is to convert the problem (7.19) to a sequence of fourth-order Dirichlet problems, each of which defined in the same domain $\Omega_\delta$:

$$\Delta(\Delta - c^2 I)U = 0, \quad U|_{\Gamma_\delta} = \tilde{u}, \quad \frac{\partial U}{\partial n}|_{\Gamma_\delta} = v,$$

(7.21)

where the Dirichlet data $\tilde{u}$ is defined by the following iteration:

$$\tilde{u}^{\text{improved}} := \tilde{u} - \omega \cdot (U|_\Gamma - u),$$

(7.22)

where $\omega > 0$ is a (small) iteration parameter. Roughly speaking, the iteration enforces the original boundary condition $u$ along the original boundary $\Gamma$.

One can easily see that the above iteration is convergent for any sufficiently small positive parameter $\omega$. Assume for simplicity again that $\Omega = (0, L) \times (\delta, +\infty)$, $\Omega_\delta = (0, L) \times (0, +\infty)$. Expressing $\tilde{u}$ in terms of Fourier series:

$$\tilde{u}(x) = \sum_k \tilde{\alpha}_k e^{i\kappa x},$$

the solution of (7.21) is as follows:

$$U(x, y) = \sum_k(A_k e^{-|\kappa| y} + B_k e^{-\sqrt{\kappa^2 + c^2} y}) e^{i\kappa x},$$

where $A_k, B_k$ are determined by (cf. the proof of Theorem 1):

$$A_k = \frac{\sqrt{\kappa^2 + c^2} \tilde{\alpha}_k - \beta_k}{\sqrt{\kappa^2 + c^2} - |\kappa|}, \quad B_k = \frac{-|\kappa| \tilde{\alpha}_k + \beta_k}{\sqrt{\kappa^2 + c^2} - |\kappa|}.$$

Based on these expressions, the iteration (7.22) can be split into separate
iterations for the Fourier coefficients $\tilde{\alpha}_k$:

$$\tilde{\alpha}_k^{\text{improved}} = \tilde{\alpha}_k - \omega \cdot \left( A_k e^{-|\kappa| \delta} + B_k e^{-\sqrt{\kappa^2 + c^2} \delta} - \alpha_k \right)$$

$$= \left( 1 - \omega \cdot \frac{\sqrt{\kappa^2 + c^2} e^{-|\kappa| \delta} - |\kappa| e^{-\sqrt{\kappa^2 + c^2} \delta}}{\sqrt{\kappa^2 + c^2} - |\kappa|} \right) \tilde{\alpha}_k$$

$$+ \omega \cdot \frac{e^{-|\kappa| \delta} - e^{-\sqrt{\kappa^2 + c^2} \delta}}{\sqrt{\kappa^2 + c^2} - |\kappa|} \tilde{\beta}_k + \omega \alpha_k$$

Since obviously

$$\sqrt{\kappa^2 + c^2} e^{-|\kappa| \delta} > |\kappa| e^{-\sqrt{\kappa^2 + c^2} \delta},$$

the absolute value of the factor multiplying $\tilde{u}_k$ is less than 1 for any sufficiently small positive parameter $\omega$. This implies that every Fourier coefficient $\tilde{\alpha}_k$ is convergent (the speed of convergence becomes low for the high-frequency components). For the limit function $\tilde{u}$, it is obvious that the corresponding solution $U$ satisfies the equality $U|_{\Gamma} = u$, i.e. the iteration results in the solution of the problem (7.19).

The above iteration plays no role if the problem (7.19) is solved (approximately) by a direct RMFS-formulation. However, it can be usefully applied when a Direct Multi-Elliptic Interpolation approach is used as presented in the next section.

### 7.5 Combination with the Direct Multi-Elliptic Interpolation

Though the RMFS leads to a linear system which is far less ill-conditioned than the original MFS, the matrix of this system is still large and dense. Therefore the computational cost remains high. Using e.g. the traditional Gaussian elimination, the number of the arithmetic operations is proportional to the third power of $N$, which is inadmissibly high if $N$ is large. The computational cost can be significantly reduced by applying the idea of the Direct Multi-Elliptic Interpolation (see [7], [8], [9]). In this approach, instead of solving the RMFS-equations (7.12), the approximate solution is defined by directly solving the fourth-order Laplace-Helmholtz-equation:

$$\Delta (\Delta - c^2 I) U = 0 \quad \text{in} \quad \Omega_0 \setminus \{x_1, x_2, \ldots, x_N\}$$

(7.24)
supplied with the interpolation condition

\[ U(x_k) = u_k \quad (k = 1, 2, \ldots, N). \quad (7.25) \]

Here \( x_1, x_2, \ldots, x_N \in \Gamma \) are the collocation points \( \Omega_0 \supset \Omega \) is a larger domain. Along the boundary \( \Gamma_0 := \partial \Omega_0, \) any usual boundary condition can be imposed e.g.:

\[ U|_{\Gamma_0} = 0, \quad \frac{\partial U}{\partial n}|_{\Gamma_0} = 0. \quad (7.26) \]

Based on variational tools, it has been proved (see [7]) that the problem (7.24)-(7.26) has a unique solution in the Sobolev space \( H^2_0(\Omega_0) \) in spite of the pointwise interpolation conditions (7.25), which destroys the well-posedness for second-order elliptic equations but not for fourth-order ones.

The connection between the Direct Multi-Elliptic Interpolation and the Method of Radial Basis Functions is given by the representation theorem of the Direct Multi-Elliptic Interpolation (see [7]), which states that the interpolation function defined by (7.24)-(7.26) is uniquely represented in the form:

\[ U(x) = w(x) + \sum_{j=1}^{N} \beta_j \Phi(x - x_j), \]

where \( w \) is a function which satisfies the Laplace-Helmholtz-equation everywhere (including also the interpolation points \( x_1, \ldots, x_n \)) and \( \Phi \) is the fundamental solution of the Laplace-Helmholtz-operation defined by (7.10).

To compute the interpolation function \( U \) defined by (7.24)-(7.26), a multi-level technique can be recommended on a quadtree cell system generated by the interpolation points \( x_1, \ldots, x_N \). This cell system exhibits local refinements at the interpolation points, while the total number of cells remains moderate (\( O(N) \)). Using quite natural finite volume schemes and multigrid tools (see [6]), the overall computational cost of the solution of the interpolation (7.24)-(7.26) can be reduced to \( O(N) \). It should also be pointed out that the numerical treatment of large, dense and ill-conditioned matrices is completely avoided.

To illustrate the method, consider again the smooth and non-smooth test functions (7.8) and (7.9). The interpolation problem (7.24)-(7.26) was solved on a square \( \Omega_0 \) with sidelength 4. Starting from this square, a quadtree cellsystem was generated with 8 levels of subdivision. The subdivision process was controlled by the interpolation points, which were
A multi-Level regularized version of the MFS equi-spatially distributed along $\Gamma$. The scaling constant in (7.24) was set to 500. Table 7.5 shows the relative $L_2$-errors with different numbers of boundary points. The results are quite similar to the results summarized in Table 7.2. However, the use of dense matrices is now avoided and no numerical instability appeared. Figure 7.2 shows the approximate solutions on the finest cell system.

Finally we show that (in the case of smooth solutions) the accuracy can be improved again by using off-boundary points similarly to the technique in Section 7.4. Let $x_1, x_2, \ldots, x_N \in \Gamma$ be boundary interpolation points again and let $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N$ be located at a small positive distance from $x_1, x_2, \ldots, x_N$ in the outward direction, i.e. $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N$ are located on the boundary $\Gamma_\delta$ of a somewhat larger domain $\Omega_\delta \supset \Omega$. The problem (7.24)-(7.25) is now converted to a sequence of boundary interpolation problem

$$
\Delta (\Delta - \epsilon^2 I) U^{(n)} = 0 \quad \text{in} \quad \Omega_0 \setminus \{\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N\}
$$

$$
U^{(n)}(\tilde{x}_k) = \tilde{u}_k^{(n)} \quad (k = 1, 2, \ldots, N),
$$

where the interpolation conditions taken at the points $\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_N$ are defined recursively:

$$
\tilde{u}_k^{(n)} := \tilde{u}_k^{(n-1)} - \omega \cdot \left( U^{(n-1)}(x_k) - u_k \right) \quad (k = 1, 2, \ldots, N) \quad (7.27)
$$

Here $\omega$ is a sufficiently small positive iteration parameter and $u_1, u_2, \ldots, u_N$ are the original interpolation conditions taken at the points $x_1, x_2, \ldots, x_N$. Roughly speaking, the iteration (7.27) enforces the original boundary condition along the original boundary.

As an example, consider the smooth test problem (7.8). We applied 64 equally spaced boundary points. The distance between the boundary and the corresponding off-boundary points was 0.03. Table 7.5 summarizes

<table>
<thead>
<tr>
<th>$N$</th>
<th>32</th>
<th>64</th>
<th>128</th>
<th>256</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative error (%) Test 1</td>
<td>9.733</td>
<td>3.218</td>
<td>0.685</td>
<td>0.299</td>
</tr>
<tr>
<td>Relative error (%) Test 2</td>
<td>10.137</td>
<td>3.452</td>
<td>0.802</td>
<td>0.306</td>
</tr>
</tbody>
</table>

Table 7.4: Regularized MFS via Direct Multi-Elliptic Interpolation. Relative $L_2$-errors of the smooth and weakly singular test problems on the finest cell system.
the relative $L_2$-errors of the approximate solutions computed on a quadtree cell system with different scaling constants. The relative errors without using off-boundary points can also be seen. The maximal level of subdivision was 8. Observe that the errors become significantly smaller when off-boundary points were introduced. It can also be seen that the method is not very sensitive to the optimal choice of the scaling constant. Figure 7.3 shows the approximate solution at the value $c = 1000$. It should be pointed out that in the vicinity of the off-boundary points, the approximate solution is irregular (it exhibits numerical singularities), however, inside the original domain, it remains smooth and of good accuracy.
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<table>
<thead>
<tr>
<th>c</th>
<th>1</th>
<th>100</th>
<th>200</th>
<th>500</th>
<th>1000</th>
</tr>
</thead>
<tbody>
<tr>
<td>Relative error (%)</td>
<td>27.729</td>
<td>2.691</td>
<td>0.258</td>
<td>3.218</td>
<td>4.218</td>
</tr>
<tr>
<td>Relative error (%), $\delta = 0.03$</td>
<td>28.766</td>
<td>0.175</td>
<td>0.004</td>
<td>0.004</td>
<td>0.004</td>
</tr>
</tbody>
</table>

Table 7.5: Regularized MFS via Direct Multi-Elliptic Interpolation without and with off-boundary points. Smooth test, relative $L_2$-errors

Figure 7.3: Regularized MFS via Direct Multi-Elliptic Interpolation, using off-boundary points. Smooth test, approximate solution

7.6 Summary and Conclusions

The main idea of the presented regularization technique of the Method of Fundamental Solutions is to replace the original fundamental solution with the fundamental solution of a higher-order partial differential operator in the MFS-formulation. In the case of the Laplace equation, this leads to the use of the fundamental solution of the fourth-order Laplace-Helmholtz operator with a carefully chosen scaling parameter. The new fundamental solution is continuous everywhere (including also the origin), which allows to locate the source points to the boundary. This reduces the com-
putational difficulties arising necessarily in the traditional version of the MFS if the sources are far from the boundary.

The method can be considered also a boundary Laplace-Helmholtz type interpolation method. It can be implemented in an economic way by solving the Laplace-Helmholtz equation in a multi-level way using quadtrees and quadtree-based finite volume schemes. This approach makes it possible to completely avoid large dense and often severely ill-conditioned matrices.

If the solution of the original problem is sufficiently smooth, the exactness can be improved by using off-boundary sources which remain, however, in the vicinity of the boundary. In this case, the RMFS-procedure can be converted to a sequence of Dirichlet problems for the Laplace-Helmholtz equation. This can be easily incorporated in the multi-level solution algorithm, so that the computational cost remains relatively low.

Finally it should be pointed out that the approach can be easily generalized also to Neumann or mixed type boundary conditions. However, the local density of the boundary points should be more or less uniform on the boundary in order that a proper scaling parameter can be defined which is adequate in the whole domain.

References


CHAPTER 8

The Method of Fundamental Solutions for Solving Axi-symmetric Isothermal Gas Flow in Porous Medium

Anita Uscilowska

Abstract. This paper presents numerical solution to a problem of the transient flow of gas within an three-dimensional porous medium. The considered problem is treated as axisymmetric one, therefore the dimension of the problem is reduced to 2D problem. A method of fundamental solution for space variables and finite difference method for time variable are employed to obtain a solution of the non-linear partial differential equation describing the flow of gas. Picard iteration is used for treating nonlinearity. The inhomogeneous term is expressed by radial basis functions and polynomials at each iteration step.

Nomenclature
\( \phi \) - porosity
\( \mu \) - viscosity
\( k \) - permeability, which is related to hydrodynamic conductivity coefficient
\( K \) by: \( k = \frac{\mu}{\rho g} K \)
\( \rho \) - mass density of the fluid
\( g \) - gravity acceleration
\( p \) - pressure
\( q \) - superficial fluid velocity
\( T \) - temperature

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The MFS for axi-symmetric gas flow in porous medium

\( r, z \) - geometry variables
\( \theta \) - angle variable
\( a, b, c \) - geometry parameters
\( t \) - time
\( R_g \) - individual gas constant
\( \tau \) - dimensionless time parameter
\( R, Z \) - dimensionless geometry variables
\( D, E \) - dimensionless geometry parameters

8.1 Introduction

The method of fundamental solutions (MFS) can be understood as a method in which differential equation is satisfied exactly whereas the boundary condition are satisfy in approximate way. For case of non-linear equations the set of functions that fulfill exactly these equation is usually unknown. Then method of fundamental solution cannot be use in straight way for non-linear Boundary Value Problems (BVPs). But, there are procedures of implementing the MFS for solving the nonlinear problems. First case when it can be used is BVPs with linear equation but with non-linear boundary conditions. Examples of such applications of this method are given in papers [5, 10]. Second case known in literature is BVP with non-linear Poisson equation in the form [1, 2, 3, 8, 9]:

\[
\nabla^2 u = f(x, y, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial y})
\]

where \( u \) is unknown function, and \( f \) is known function in which same arguments are unknown. In paper [3] the non-linear thermal explosions problem was solved by method of fundamental solutions. The radial basis functions were used for interpolation of right hand side on Picard iteration method was used to read non-linearity. In paper [1] method called as “particular solution Trefftz method” was used. Really its is extension and improvement of ideas proposed in paper [3]. Another version of boundary element method for solution of non-linear Poisson equation was presented in paper [9]. A two-dimensional Poisson equation with spatially varying conductivity is solved by homotopy analysis method. Also steady state heat conduction problem with temperature dependent conductivity was
The MFS for axi-symmetric gas flow in porous medium

considered in paper [8]. Combination of method fundamental solutions with Picard iteration was used for non-linear Poisson equation. Evolutionary algorithm was applied for optimal determination of method parameters. More complicated application of the MFS was presented in paper [2] where the method of operator splitting with method of fundamental solution was used for transient non-linear Poisson problems. These problems are widely encountered in the modelling many physical phenomena and governing differential equation has a form:

$$\frac{du}{dt} = \nabla^2 u + f(u)$$  \hspace{1cm} (8.2)

where \( t \) is time. The purpose of the present paper is application of the MFS to a problem of the transient flow of gas within a three-dimensional porous medium. The considered problem is axisymmetric. Unsteady gas flow through semi-infinite porous medium was considered in paper [7]. In such case problem is described by ordinary differential equation. In case of finite porous region the governing equation for pressure of gas as unknown has a form similar to (8.2). In our proposition (as in [11]) the method of fundamental solution for space variables and finite difference method for time variable are employed to obtain a solution of the non-linear partial differential equation describing the flow of gas. Picard iteration is used for treating nonlinearity. The inhomogeneous term is expressed by radial basis functions at each iteration step. The fundamental solution for axisymmetric Poisson problem is known in literature [6]. Particular solutions of axisymmetric Poisson problem for chosen radial basis functions and polynomials are proposed in paper [4].

8.2 Problem Description

Considered region of the porous medium with flowing fluid is presented on Figure 8.1. The porous medium is filled with gas under uniform pressure. The edges of considered reservoir are insulated, except one piece of edge which is opened. Pressure outside the reservoir is lower than pressure in porous medium.

The reservoir is cylindrical porous medium with radius \( a \) and length \( b \). The open edge is a circle of radius \( c \). Due to axisymmetry of the problem the rectangular plane region is of the considered and presented in Figure 8.2.
For investigation gas flow in porous medium we introduce the following assumptions:

- The flow of gas follows Darcy’s law;
- The only phase flowing is a gas of constant composition and viscosity
- The gas is perfect and gas flow is isothermal
- The permeability of the porous medium is constant and uniform
- Gravitational forces are neglected.

8.3 Motion Equations

Darcy Law is filtration equation for fluid flow in porous media and in 3-D case has form

\[ \mathbf{q} = -\frac{k}{\mu} \nabla p \]  (8.3)
The MFS for axi-symmetric gas flow in porous medium

which in polar coordinates is

\[ \mathbf{q} = -\frac{k}{\mu} \left( \frac{\partial p}{\partial r} e_r + \frac{\partial p}{\partial \theta} e_\theta + \frac{\partial p}{\partial z} e_z \right) \]  

(8.4)

and for axisymmetric case has form

\[ \mathbf{q} = -\frac{k}{\mu} \left( \frac{\partial p}{\partial r} e_r + \frac{\partial p}{\partial z} e_z \right) \]  

(8.5)

The continuity equation for porous media is

\[ \text{div} (\rho \mathbf{q}) = -\frac{\partial}{\partial t} (\varphi \rho) \]  

(8.6)
The MFS for axi-symmetric gas flow in porous medium

which for 3D case in cylindrical coordinates is

\[ \frac{1}{r} \frac{\partial}{\partial r} (rpq_r) + \frac{1}{r} \frac{\partial}{\partial \theta} (\rho q_\theta) + \frac{\partial}{\partial z} (\rho q_z) = -\frac{\partial}{\partial t} (\varphi \rho) \] (8.7)

and for axisymmetric case

\[ \frac{1}{r} \frac{\partial}{\partial r} (rpq_r) + \frac{\partial}{\partial z} (\rho q_z) = -\frac{\partial}{\partial t} (\varphi \rho) \] (8.8)

The gas equation for isothermal phenomena

\[ \rho = \frac{p}{R_g T} \] (8.9)

and \( T \) - temperature is constant.

Applying the eq. (8.5) and eq. (8.9) to the eq. (8.8) gives

\[ \frac{\partial}{\partial x} \left( \frac{p}{RT} \frac{k}{\mu} \frac{\partial p}{\partial x} \right) + \frac{\partial}{\partial y} \left( \frac{p}{RT} \frac{k}{\mu} \frac{\partial p}{\partial y} \right) = \frac{\mu}{k} \frac{\partial}{\partial t} \left( \varphi \frac{p}{RT} \right) \] (8.10)

Rearranging the eq. (8.10) yields to the equation

\[ \frac{1}{r} \frac{\partial}{\partial r} \left( pr \frac{\partial p}{\partial r} \right) + \frac{\partial}{\partial z} \left( p \frac{\partial p}{\partial z} \right) = \frac{\mu}{k} \frac{\partial}{\partial t} (\varphi p) \] (8.11)

The initial condition says that there is uniform pressure in porous medium:

\[ p(r,z,t) = p_0 \] (8.12)

for \( t = 0, 0 < r < a \) and \( 0 < z < b \).

The boundary condition at open edge \( \{(r,z) \mid (0 < r < c) \cap (z = b)\} \) is:

\[ p(r,z,t) = p_1 < p_0 \] (8.13)

for \( 0 < t < \infty \)

For the other reservoir edges

\[ \{(r,z) \mid ((0 < r < a) \cap (z = 0)) \cup ((c < r < a) \cap (z = b)) \cup ((r = a) \cap (0 < z < b))\} \]

insulated boundary condition is

\[ \frac{\partial p}{\partial n} = 0 \] (8.14)
and for \{ (r, z) | (r = 0) \cap (0 < z < b) \} the symmetry condition is applied
\[
\frac{\partial p}{\partial n} = 0
\] (8.15)
for \( 0 < t < \infty \).

The dimensionless variables are introduced
\[
R = \frac{r}{a}, \ Z = \frac{z}{a}, \ E = \frac{b}{a}, \ D = \frac{c}{a}, \ P = \frac{p}{p_0}, \ P_1 = \frac{p_1}{p_0}, \ \tau = \frac{kp_0}{\varphi \mu a^2 t}
\] (8.16)
Therefore the equation (8.11) has the dimensionless form
\[
\frac{1}{R} \frac{\partial}{\partial R} \left( PR \frac{\partial P}{\partial R} \right) + \frac{\partial}{\partial Z} \left( P \frac{\partial P}{\partial Z} \right) = \frac{\partial P}{\partial \tau}
\] (8.17)
and the initial condition is
\[
P(R, Z, \tau) = 1
\] (8.18)
for \( \tau = 0, 0 < R < 1, 0 < Z < E \). The boundary conditions are the boundary condition for the open edge
\[
P(R, Z, \tau) = P_1 < 1
\] (8.19)
for \( 0 < R < D, Z = E \) and insulating and symmetry condition is
\[
\frac{\partial P}{\partial n} = 0
\] (8.20)
for the boundary
\[
\{(R, Z) | ((0 < R < 1) \cap (Z = 0)) \cup ((D < R < 1) \cap (Z = E)) \cup ((R = 0) \cap (0 < Z < E)) \cup ((R = 1) \cap (0 < Z < E))\}
\]
8.4 Algorithm for Solving Initial-boundary Problems
Assuming that time derivative term can be expanded using finite difference
\[
\frac{\partial P}{\partial \tau} = \frac{P^{(n+1)} - P^{(n)}}{\Delta \tau}
\] (8.21)
for \( n = 0, 1, 2, \ldots, \) eq. (8.17) can be approximated as

\[
\frac{1}{R} \frac{\partial P^{(n+1)}}{\partial R} + \frac{\partial^2 P^{(n+1)}}{\partial R^2} + \frac{\partial^2 P^{(n+1)}}{\partial Z^2} - \frac{P^{(n+1)} - P^{(n)}}{P^{(n+1)} \Delta \tau} = - \frac{1}{P^{(n+1)}} \left\{ \left( \frac{\partial P^{(n+1)}}{\partial R} \right)^2 + \left( \frac{\partial P^{(n+1)}}{\partial Z} \right)^2 \right\}
\]

(8.22)

with initial condition

\[
P^{(0)}(R, Z, \tau) = 1
\]

(8.23)

for \( \tau = 0, 0 < R < 1, 0 < Z < E \) and boundary conditions

\[
P^{(n+1)} = P^{(n+1)}_1 < 1
\]

(8.24)

for \( 0 < R < D, Z = E \)

\[
\frac{\partial P^{(n+1)}}{\partial n} = 0
\]

(8.25)

for the boundary

\[
\{(R, Z) \mid ((0 < R < 1) \cap (Z = 0)) \cup ((D < R < 1) \cap (Z = E)) \\
\cup ((R = 1) \cap (0 < Z < E)) \cup ((R = 0) \cap (0 < Z < E))\}
\]

where \( P^{(n)} \) is dimensionless pressure at the \( n \)-th time step, \( P^{(n+1)} \) is this pressure in the next time step.

The calculation of pressure in every time step is based also on eq. (8.22), which is transformed into Poisson equation:

\[
\frac{1}{R} \frac{\partial P^{(n+1)}}{\partial R} + \frac{\partial^2 P^{(n+1)}}{\partial R^2} + \frac{\partial^2 P^{(n+1)}}{\partial Z^2} = - \frac{1}{P^{(n+1)}} - \frac{1}{P^{(n+1)} \Delta \tau} \left\{ \left( \frac{\partial P^{(n+1)}}{\partial R} \right)^2 + \left( \frac{\partial P^{(n+1)}}{\partial Z} \right)^2 \right\}
\]

(8.26)

with initial and boundary conditions (8.24), (8.25). for \( n = 0, 1, 2, \ldots \). The equation is strongly non-linear with respect to \( P^{(n+1)} \), therefore, it
The MFS for axi-symmetric gas flow in porous medium is solved in an iterative fashion:

\[
\frac{1}{R} \frac{\partial P(n+1, i+1)}{\partial R} - \frac{\partial^2 P(n+1, i+1)}{\partial R^2} - \frac{\partial^2 P(n+1, i+1)}{\partial Z^2} + \frac{\partial P(n+1, i+1)}{\partial Z} = \frac{1}{\Delta \tau} - \frac{P(n)}{P(n+1, i+1)} \left( \left( \frac{\partial P(n+1, i)}{\partial R} \right)^2 + \left( \frac{\partial P(n+1, i)}{\partial Z} \right)^2 \right)
\]

(8.27)

with boundary conditions (8.24), (8.25), where \( P(n+1, i) \) is the \( i \)-th iteration result at \((n+1)\)-th time step. We introduce an initial condition for iterative procedure e.g. trial equation in Laplace form, which is modified version of eq. (8.27):

\[
\frac{1}{R} \frac{\partial P^{n+1, 1}}{\partial R} - \frac{\partial^2 P^{n+1, 1}}{\partial R^2} - \frac{\partial^2 P^{n+1, 1}}{\partial Z^2} = 0
\]

(8.28)

with trial boundary conditions

\[
P^{n+1, 1} = P_1 < 1
\]

(8.29)

for \( 0 < R < D, Z = E \) and

\[
\frac{\partial P^{n+1, 1}}{\partial n} = 0
\]

(8.30)

for the boundary

\[
\{(R, Z) | ((0 < R < 1) \cap (Z = 0)) \cup ((D < R < 1) \cap (Z = E)) \}
\]

\[
\cup \{(R = 0) \cap (0 < Z < E)\} \cup \{(R = 1) \cap (0 < Z < E)\}
\]

One extra boundary condition is added

\[
P^{n+1, 1} = P(n)
\]

(8.31)

for \( \{(R, Z) | (0 < R < 1) \cap (Z = 0)\} \) to combine the previous time step pressure distribution with the solution at the next time step.

Equation (8.28) is solved by the fundamental solution method, including the appropriate boundary conditions into calculation.

Solutions at second and next iteration steps are found by Trefftz method, based on the eq. (8.27) with its boundary conditions. Therefore, in one time step we obtain the sequence of solutions: \( P^{n+1, 1}, P^{n+1, 2}, \ldots \).
The iterative process is terminated when difference between solutions of two successive iteration steps is quite small, less than a chosen small parameter. We introduce \( m \), which points the iteration step number, at which solution is taken as the solution at \( n \)-th time step, noticed as \( P^{(n+1,m)} = P^{(n+1)} \).

### 8.5 The MFS for Solving Boundary Value Problems

Partial differential inhomogeneous equation

\[
Lu = f(R, Z) \tag{8.32}
\]

is considered on the region \( \Omega \).

Operator \( L \) is a partial differential operator, which includes Laplace operator.

The boundary condition has the general form

\[
Bu = g(R, Z) \tag{8.33}
\]

where \( B \) is an operator imposed as boundary conditions, such Dirichlet, Neumann, and Robin.

Let us denote \( \{P_{k,m} = (R_i, Z_i)\}_{i=1}^N \) to be \( N \) collocations points in \( \Omega \cup \partial \Omega \) of which \( \{(R_i, Z_i)\}_{i=1}^{Nl} \) are interior points; \( \{(R_i, Z_i)\}_{i=Nl+1}^{N} \) are boundary points and \( N_b = N - Nl \) is a number of boundary points.

The right-hand side function \( f \) is approximated as Radial Basis Functions (RBFs) as

\[
f_N(R, Z) = \sum_{j=1}^{N} a_j \varphi_j(R, Z) + \sum_{k=0}^{l_1} \sum_{m=0}^{l_2} b_{k,m}p_{k,m}(R, Z) \tag{8.34}
\]

where \( \varphi_j(R, Z) = \varphi(R, Z, R_j, Z_j) : R^d \to R^+ \) is a RBF, \( \{p_{k,m}\}_{k=0}^{l_1} \) is the complete basis for \( d \)-variate polynomials of degree \( \leq (k - 1)(m - 1) \). The coefficients \( \{a_j\}, \{b_{k,m}\} \) can be found by solving the system

\[
\sum_{j=1}^{N} a_j \varphi_j(R_i, Z_i) + \sum_{k=0}^{l_1} \sum_{m=0}^{l_2} b_{k,m}p_{k,m}(R_i, Z_i) = f(R_i, Z_i) \tag{8.35}
\]

for \( 1 \leq i \leq N, \)

\[
\sum_{k=0}^{l_1} \sum_{m=0}^{l_2} a_j p_{k,m}(R_j, Z_j) = 0 \tag{8.36}
\]
for $1 \leq j \leq N$, where $\{R_i, Z_i\}_{i=1}^{N}$ are the collocation points on $\Omega \cup \partial \Omega$.

The approximate particular solutions $u_p$ can be obtained using the coefficients $\{a_j\}$ and $\{b_{k,m}\}$ by

$$u_p(R, Z) = \sum_{j=1}^{N} a_j \phi_j(R, Z) + \sum_{k=1}^{l_1} \sum_{m=1}^{l_2} b_{k,m} \psi_{k,m}(R, Z)$$

(8.37)

where

$$L \phi_j = \varphi_j(R, Z)$$

(8.38)

$$L \psi_{k,m} = p_{k,m}(R, Z)$$

(8.39)

for $1 \leq j \leq N$, $0 \leq k \leq l_1$, $0 \leq m \leq l_2$. Solution of differential equation (8.32) now can be given as

$$u = u_p + v$$

(8.40)

where $v$ is solution of boundary value problem in the form

$$Lv = 0 \text{ in } \Omega$$

(8.41)

$$Bv = g(R, Z) - Bu_p \text{ on } \partial \Omega$$

(8.42)

The method of fundamental solution is used to solve problem presented above, what means that

$$v(R, Z) = \sum_{j=1}^{N_i} c_j f_{S,j}(R, Z)$$

(8.43)

where $f_S$ is the fundamental solution function and

$$f_{S,j}(R, Z) = f_S(\| (R - R_{i,j}^s, Z - Z_{i,j}^s) \|)$$

denotes the fundamental solution function determined for the $j$-th source point. Points $\{R_{i,j}^s, Z_{i,j}^s\}_{j=1}^{N_i}$ are so called source points, and $N_i$ is number of source points on artificial boundary outside the considered region $\Omega$.

Putting (8.43) into boundary condition (8.42):

$$\sum_{j=1}^{N_i} c_j B f_{S,j}(R_i, Z_i) = g(R_i, Z_i) - Bu_p(R_i, Z_i) \text{ for } 1 \leq i \leq N_b$$

(8.44)

coefficients $c_j$ are obtained. The solution of the boundary problem (8.32) and (8.33) is calculated by equation (8.40).
8.5.1 Numerical implementation

In Poisson equation differential operator is the Laplace operator:

\[ L = \nabla^2 \]  \hspace{1cm} (8.45)

The radial basis function for axisymmetric problem is

\[ \varphi_j(R, Z) = 4E(k) \sqrt{\alpha^2 + \beta^2} \]  \hspace{1cm} (8.46)

and proper particular solution is

\[ \phi(R, Z) = \frac{1}{9} \left( \alpha + \beta \right)^{1/2} \left( k^2 - 1 \right) \left( K(k) + (4 - 2k^2) E(k) \right) \]  \hspace{1cm} (8.47)

where

\[ \alpha = R^2 + R^2_j + (Z - Z_j)^2, \quad \beta = 2RR_j, \quad k^2 = \frac{2\beta}{\alpha + \beta}, \]

and \( E(k), K(k) \) are the complete elliptic integrals of first and second kind. For polynomials given in the form

\[ p_{k,m}(R, Z) = R^k Z^m, \]  \hspace{1cm} (8.48)

the proper particular solutions are

\[ \psi_{k,m}(R, Z) = \sum_{j=0}^{m/2} \frac{(-1)^j \cdot m!}{(m - 2j)!} \left( \frac{k!!}{(k + 2j + 2)!!} \right)^2 R^{k+2j+2} Z^{m-2j} \]  \hspace{1cm} (8.49)

The fundamental solution for Poisson equation is function:

\[ f_s,j(R, Z) = \frac{4K(k)}{\sqrt{\alpha^2 + \beta^2}} \]  \hspace{1cm} (8.50)

where

\[ \alpha = R^2 + R^2_j + (Z - Z_j)^2, \quad \beta = 2RR_j, \quad k^2 = \frac{2\beta}{\alpha + \beta} \]

and \( K(k) \) is the complete elliptic integral of second kind.
8.6 Numerical Results

The example solved by the combination of the methods described above is presented. The considered region is a square with boundary of length equal to 1. The pressure inside the region is equal to 1 and outside pressure has value 0.5. For first time step, $d\tau = 1.0$ the first iteration result is presented in Figure 8.3.

It shows results obtained for auxiliary boundary value problem (8.28 - 8.30). The second iteration is calculated for Poisson equation given by (8.27) with boundary conditions (8.24), (8.25), where $n = 0, i = 1$. The right-hand side function is approximated by radial basis functions and polynomials. The relative error of the approximation shown in Figure 8.4 is acceptably low, to continue iterations procedure.

Finally, solution obtained for second time step is presented in Figure 8.5. Next, two time steps results at Figure 8.6 and 8.7 gas flow out from reservoir. The results are compliant with expected ones.

Values of the pressure inside the considered region are very close to value of the pressure at the open edge. The next time result shows that the pressure field is almost uniform (with tolerance $10^{-3}$) and equal to the outside pressure.
8.7 Conclusions

In our paper the MFS is used for solving time dependent phenomena. The finite difference method is implemented to approximate differential with respect to time parameter. The technique for obtaining solution of inhomogeneous nonlinear Poisson equation is applied. The problem given in implicit form is solved using the iterative technique. Obtained result, which
Figure 8.6: Pressure distribution at third time step.

Figure 8.7: The solution at fourth time step.

is gas outflow from reservoir - porous medium, agrees with expected ones.

References


CHAPTER 9

The Method of Fundamental Solutions for Low Reynolds Number Flows with Moving Rigid Body

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Abstract. The Eulerian-Lagrangian method of fundamental solutions (ELMFS) is proposed to solve the unsteady nonlinear Navier-Stokes equations of low Reynolds number flows with a moving rigid body by combining the Eulerian-Lagrangian method (ELM) and the method of fundamental solutions (MFS). The concept of ELM is used to deal with the nonlinear advection term, while the MFS based on the unsteady Stokeslets is applied to solve the converted characteristic Navier-Stokes equations. The advantages of the MFS, free from mesh generation and numerical integration, are very suitable and effective for direct handling the moving rigid body problems in comparison with other mesh-dependent schemes. By properly placing the source and field points in the time-space domain, the time derivative term can be solved together with the spatial variables without using the techniques of Laplace transform or finite difference discretization or eigenfunction expansion separation. The numerical solutions of the Navier-Stokes equations can then be obtained by tracing the unsteady Stokes results along the characteristics through the ELM scheme. The benchmark lid-driven cavity flows with Re=10 and Re=50 are first validated by comparing with other numerical methods. The proposed method is further applied to the Navier-Stokes equations for low Reynolds number flows with a moving rigid cylinder. The numerical results show very good agreement with the immersed-boundary finite element method (IBFEM) and therefore demonstrate the feasibility and perspective of the present ELMFS algorithm to deal with the nonlinear Navier-Stokes equations with and without moving domains even very coarse nodes are used.

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9.1. Introduction

The numerical solutions by the MFS are represented as a combination of a series of fundamental solutions of the differential equations with different source intensities. The source intensities of the fundamental solutions can be obtained by collocating the known boundary and initial data. Since the meshless MFS is free from mesh generation and numerical integration, this numerical scheme is easily applied to some linear steady or transient problems with forward or inverse calculations only using very coarse node points [1-11]. In the references of [12-14], various literature surveys addressed early applications of the MFS to some simple linear elliptical or parabolic partial differential equations such as Laplace, Helmholtz, modified Helmholtz, biharmonic, and diffusion operators. Few investigations are undertaken for the unsteady and nonlinear partial differential equations by using the MFS due to the difficulty of obtaining the fundamental solutions of the system. Nevertheless the time-dependent and nonlinear problems are always important subjects in many applications for engineering and the sciences. The MFS can be adopted to solve linear diffusion problems either by the modified Helmholtz fundamental solution [11] through the finite difference discretization of time derivative, Laplace transform of time domain, or eigenfunction expansion method of time separation [15]. Recently, an interesting concept of time-space unification by the extended MFS has been introduced to solve the linear diffusion equation [16-17] without using the finite difference scheme or Laplace transform or eigenfunction expansion when dealing with the time domain. A more important achievement of the concept of time-space unification is that the extended MFS can be used to solve the nonlinear or, more precisely, the quasilinear differential operators such as the advection-diffusion or Burgers or Navier-Stokes equations. The MFS used to solve the linear diffusion equation can be further modified and extended by adopting the ELMFS scheme to obtain the solutions of the advection-diffusion equations [18], the Burgers equations [19], as well as the Navier-Stokes equations.

The MFS has been well developed for solving the linear scalar field problems [12-14] since it was initially proposed in 1964 by Kupradze and Aleksidze [1]. However for the vector field problems such as the Burgers,
Stokes, and Navier-Stokes equations there is very limited literature available on computational fluid dynamics (CFD), especially for the time-dependent and nonlinear problems. Simple MFS or extended ELMFS were employed in the simulation of vector Burgers equations [19-21]. When the nonlinear term is involved and simple MFS is adopted, we have to make use of some kind of parameter transformations to remove the nonlinearity such as the Cole-Hopf transformation in the advection term and Kirchhoff transformation in the diffusion term [20]. The studies of Burgers equations will provide a preliminary and valuable background to simulate the more difficult Navier-Stokes equations. The Stokes equations are linear forms of the Navier-Stokes equations and will serve as a first approximation for investigating the notorious Navier-Stokes equations with low Reynolds number flows. The studies of steady forward and inverse Stokes flows in the interior or exterior problems have also been well developed recently [22-29]. Tsai et al. works are highly recommended for the unsteady Stokes flows [30-31]. Due to the infancy stage and the unsteady and nonlinear vector system, the MFS applied to the Navier-Stokes equations still remains a challenging task. Young and his collaborators are the pioneers among the recent researchers on the applications of the meshless MFS or ELMFS to the Navier-Stokes equations [32-34].

There are two or more general MFS approaches for solving the Navier-Stokes equations in the developed processes. Tsai [32] used the conventional MFS concept to combine the MFS and the method of particular solutions (MPS), such as the dual reciprocity method (DRM) based on the modified Helmholtz fundamental solutions to solve the Navier-Stokes equations through the velocity-vorticity formulation. In contrast to Tsai’s approach, Fan [33] and Chen [34] directly used the ELMFS to simulate the Navier-Stokes equations based on the unsteady Stokeslets through time-space unification concept, to avoid the nonhomogeneous source terms due to the discretization of time dimension. The Navier-Stokes equations are then converted to the characteristic Navier-Stokes equations in the Lagrangian coordinates by the ELM and then solved by the characteristic MFS based on the unsteady Stokeslets. To model the transport mechanism by the ELM, the advection part is solved by the Lagrangian method. It can be computed
independently at each time step by the method of characteristics applied to a grid fixed domain. Once the solutions of the unsteady Stokes problems are obtained by the MFS, the numerical results of the Navier-Stokes equations can be acquired by retrieving from the unsteady Stokes solutions through the method of characteristics. Since the physical behaviors will follow the unsteady Stokes equations along the characteristics, the back-tracking scheme can be used to transfer the solutions of the unsteady Stokes equations to the solutions of the Navier-Stokes equations [33-34].

In this chapter, the MFS with ELM scheme can be further extended directly and effectively to simulate the dynamics of a moving rigid body in viscous fluids. It is much simpler and more convenient to use the present meshless ELMFS algorithm for solving the dynamics of a moving rigid body in a viscous fluid in comparison to the treatments of the same problems by mesh-dependent methods. For illustrations the arbitrary Lagrangian-Eulerian method (ALE) [35], Cartesian grid method [36], or immersed boundary method (IBM) [37] are among the most popular approaches adopted to deal with the moving boundary. When the ELMFS is used during the transient process, the field points and source points can be easily modified at each instant and collocated again with the moving rigid body without further difficulty. In order to test the proposed method, the benchmark lid-driven cavity flow will be validated first. Then the Navier-Stokes equations with moving cylinder in a viscous fluid will be obtained and compared with numerical results by the immersed-boundary FEM. The chapter contents are organized as follows: Section 9.2 introduces the governing equations and the unsteady Stokeslets. Section 9.3 presents the numerical methods, and Section 9.4 shows the numerical results and comparisons with other numerical methods. Final discussions and conclusions are drawn in section 9.5.

9.2. Governing Equations

The governing equations for mass and momentum conservation of an incompressible fluid are the Navier-Stokes equations, expressed as following.

\[ \nabla \cdot \vec{u} = 0 \quad (9.1) \]
\[ \rho \left( \frac{\partial \tilde{u}}{\partial t} + \tilde{u} \cdot \nabla \right) \tilde{u} = -\nabla p + \mu \nabla^2 \tilde{u} \quad (9.2) \]

where \( \rho \) is the density, \( \mu \) is the dynamic viscosity, \( p \) is the pressure, and \( \tilde{u} = (u, v, w) \) is the velocity vector in \( \tilde{x} = (x, y, z) \) directions.

Using the ELM [33-34], we can write the Navier-Stokes equations as the following characteristic form.

\[ \nabla \cdot \tilde{u} = 0 \quad (9.3) \]

\[ \rho \frac{D\tilde{u}}{Dt} = -\nabla p + \mu \nabla^2 \tilde{u} \quad (9.4) \]

where \( \frac{D}{Dt} = \frac{\partial}{\partial t} + (\tilde{u} \cdot \nabla) \) is the material derivative. Equations (9.3) and (9.4) represent the Navier-Stokes equations using the Lagrangian approach [38].

The linear Stokes equations can be obtained from the nonlinear Navier-Stokes equations (9.1) and by neglecting the nonlinear terms in (9.2) equation. The fundamental solutions for the unsteady Stokes operators are called the unsteady Stokeslets, which represent the flow fields due to concentrated point forces.

The governing equations of the 2D unsteady Stokeslets are

\[ \frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0 \quad (9.5a) \]

\[ \rho \frac{\partial \tilde{u}}{\partial t} = -\frac{\partial \tilde{p}}{\partial x} + \mu \nabla^2 \tilde{u} + \alpha_1 \delta(\tilde{x}) \delta(t) \quad (9.5b) \]

\[ \rho \frac{\partial \tilde{v}}{\partial t} = -\frac{\partial \tilde{p}}{\partial y} + \mu \nabla^2 \tilde{v} + \alpha_2 \delta(\tilde{x}) \delta(t) \quad (9.5c) \]

where \( \delta() \) is the well-known Dirac-delta function, and \( \alpha_1, \alpha_2 \) are the source intensities of the unsteady Stokeslets in \( x \) and \( y \) directions.
respectively. Without loss of generality, we locate the source at the origin of time-space coordinates in our derivations.

Properly applying the integral transforms, we are able to obtain the 2D unsteady Stokeslets as follows [39].

\[
\begin{bmatrix}
\mathbf{u}^* \\
\mathbf{v}^* \\
\mathbf{p}^* \\
\omega_2^*
\end{bmatrix} = \frac{1}{2\pi \rho} \begin{bmatrix}
A_{11} & A_{12} \\
A_{21} & A_{22} \\
A_{31} & A_{32} \\
A_{41} & A_{42}
\end{bmatrix} \begin{bmatrix}
\alpha_1 \\
\alpha_2
\end{bmatrix}
\]

(9.6)

with

\[
A_{11} = \frac{x^2 - y^2}{r^4} + \frac{1}{r^4 \nu t} e^{-\frac{r^2}{4\nu t}} \left( \frac{y^2 + x^2 y^2}{2} + tv \left( y^2 - x^2 \right) \right)
\]

(9.7a)

\[
A_{12} = A_{21} = \frac{2xy}{r^4} - \frac{xy}{r^4 \nu t} e^{-\frac{r^2}{4\nu t}} \left( \frac{r^2}{2} + 2tv \right)
\]

(9.7b)

\[
A_{22} = \frac{-x^2 + y^2}{r^4} + \frac{1}{r^4 \nu t} e^{-\frac{r^2}{4\nu t}} \left( \frac{x^4 + x^2 y^2}{2} + tv \left( x^2 - y^2 \right) \right)
\]

(9.7c)

\[
A_{31} = \frac{\rho x}{r^2} \delta(t)
\]

(9.7d)

\[
A_{32} = \frac{\rho y}{r^2} \delta(t)
\]

(9.7e)

\[
A_{41} = \frac{y}{4(tv)^2} e^{-\frac{r^2}{4\nu t}}
\]

(9.7f)
\[ A_{42} = \frac{-x}{4(\nu)} e^{-\frac{r^2}{4\nu}} \tag{9.7g} \]

where \( r = \|\vec{x}\| \) is the Euclidean norm of \( \vec{x} = (x,y) \), and \( \nu \) is the kinematic viscosity. For dealing with the nondimensional Stokeslets in solving the nondimensional Navier-Stokes equations, we simply set \( \nu = 1/\text{Re} \) and \( \rho = 1 \). Re is defined as the dimensionless Reynolds number.

### 9.3. ELMFS formulations

In the spirit of the MFS, the solutions are assumed to be a combination of the unsteady Stokeslets at different time-space source points. Thus the 2D numerical solutions of the unsteady Stokes equations will be represented in the following forms:

\[
\begin{align*}
    u(\vec{x}, t) &= \sum_{j=1}^{N} \alpha_{1j} A_{11}(\vec{x}, t; \vec{\xi}_j, \tau_j) + \sum_{j=1}^{N} \alpha_{2j} A_{12}(\vec{x}, t; \vec{\xi}_j, \tau_j) \quad (9.8a) \\
    v(\vec{x}, t) &= \sum_{j=1}^{N} \alpha_{1j} A_{21}(\vec{x}, t; \vec{\xi}_j, \tau_j) + \sum_{j=1}^{N} \alpha_{2j} A_{22}(\vec{x}, t; \vec{\xi}_j, \tau_j) \quad (9.8b) \\
    \omega_z(\vec{x}, t) &= \sum_{j=1}^{N} \alpha_{1j} A_{41}(\vec{x}, t; \vec{\xi}_j, \tau_j) + \sum_{j=1}^{N} \alpha_{2j} A_{42}(\vec{x}, t; \vec{\xi}_j, \tau_j) \quad (9.8c)
\end{align*}
\]

where \((\vec{x}, t)\) represents the time-space location of the field points, \((\vec{\xi}, \tau)\) gives the time-space location of the source points, \(N\) is the number of source points, and \(\alpha_{1j}\) as well as \(\alpha_{2j}\) are the undetermined coefficients or the intensities of sources.
Typical distributions of the field and the source points are illustrated in Fig. 9.1a in which the field points are placed at the boundary portion at \( t = (n + 1) \Delta t \) and in the interior domain at \( t = n \Delta t \). The source points are placed in the same positions but at different time levels. We are able to form a linear matrix system through collocating these field and source points by satisfying the initial and boundary conditions and using equations (9.8) as follows:

\[
[A] \{\alpha\} = \{b\} \tag{9.9}
\]

The matrix \( \{b\} \) is the combination of known initial and boundary data. The components of \([A]\) are obtained from the unsteady Stokeslets as listed above. After inverting the matrix system, the coefficients \( \{\alpha\} \) can be found; the function values inside the time-space box can then be obtained by equations (9.8). Once the solutions of unsteady Stokes equations have been obtained, the results of the Navier-Stokes equations can be computed by back-tracking technique along the characteristics. In 2005, Fan [33] and Hu [40] introduced the processes of the ELM to solve the unsteady and nonlinear partial differential equations such as Burgers and Navier-Stokes equations. According to the principle of the ELM, from the definition of the characteristic, the convective velocity can be expressed as:

\[\lambda < \Omega (n+\Delta t) \]

\[\Omega (n+\Delta t) \]

\[\lambda < \Omega (n-\Delta t) \]

\[\lambda < \Omega (n) \Delta t \]

\[\lambda < \Omega \]

\[\lambda < \Omega (n+1) \Delta t \]

\[\lambda < \Omega (n) \Delta t \]

\[\lambda < \Omega \]

\[\lambda < \Omega (n+1) \Delta t \]

\[\lambda < \Omega (n) \Delta t \]

\[\lambda < \Omega \]
\[ u = \frac{dx}{dt} \] (9.10a)

\[ v = \frac{dy}{dt} \] (9.10b)

Fig. 9.2. Schematic diagram and boundary conditions of a square cavity

\[ x^n = x^{n+1} - u^n \Delta t \] (9.11a)

\[ y^n = y^{n+1} - v^n \Delta t \] (9.11b)

In Fig. 8-1b, the line AB is a line of characteristic. If the velocity at point A in the \((n+1)\Delta t\) level is solved, then the position of point B in the \(n\Delta t\) level can be traced back by using equations (9.11a) and (9.11b). When the position of point B is obtained, the solutions of the Navier-Stokes equations at point A are replaced by the unsteady Stokes results at point C. Since the physical behaviors will follow the unsteady Stokes equations along the characteristics, the back-tracking scheme can be used to transfer the ready solutions of the Stokes equations to the desired solutions of the Navier-Stokes equations.

9.4. Numerical results

The proposed numerical scheme based on the ELMFS is validated by comparing the results for the benchmark lid-driven cavity flow with Re=10
and Re=50. Further, the phenomena of Navier-Stokes flow with a moving cylinder will also be simulated to demonstrate the advantages of the ELMFS to handle the moving domain. The numerical results of both the lid-driven cavity and moving cylinder problems will be compared with the solutions of the conventional FEM and immersed-boundary FEM.

### 9.4.1. Lid-driven cavity flow

First, the benchmark square cavity flow has been solved to test the capability of the present meshless numerical scheme. Figure 9.2 represents the schematic diagram of a square cavity whose top lid is assumed to move with a constant velocity of one unit in the x-direction. The top lid brings to rise a recirculation flow pattern inside the cavity with Re=10 and Re=50 as shown by the velocity vector in Fig. 9.3. A non-symmetric velocity vector distribution is observed in this figure for Re=50 by the effect of inertial force. Figure 9.4 depicts the u-velocity along x=0.5 and v-velocity along y=0.5 with Re=10 and Re=50 (1000 points, dt=0.0001, \( \lambda = 150 \)) which compares well with FEM results. The reasonable numerical results can be obtained by the proposed meshless ELMFS method.

### 9.4.2. Navier-Stokes problem with a moving cylinder
The flow with a moving rigid body or irregular domain is an important topic in computational fluid dynamics (CFD). We consider a stationary circular cavity flow driven by a moving cylinder to validate the capability of the developed model for moving rigid body simulations. As shown in Fig. 9-5 there is a two-dimensional incompressible flow of one circular cylinder with radius \( R_2 = 0.25 \) confined in a circular cavity of radius \( R_1 = 1.0 \). The center of the inner cylinder is initially located at \((x_0, y_0) = (0.5, 0.0)\) and then moves counterclockwise with respect to the center of cavity with the angular velocity \( \omega = 2 \). In conventional treatment for mesh-dependent numerical methods, the body-fitted curvilinear and unstructured mesh approaches are widely used for simulating the flows with irregular geometry as illustrated in Fig. 9.6a. Although these schemes have been adopted, the moving structure problem is still a big challenge for scientific computation. Recently the ideas of ALE [35] or IBM [37] have been introduced and applied to solve the moving boundary or moving structure successfully.

In this section the new concept of time-space unification with ELMFS is used to solve the moving rigid body problems. The field points inside the computational domain at \((n)\Delta t\) level are uniformly distributed. Only the source and field points at \((n)\Delta t\) and \((n+1)\Delta t\) levels on the moving rigid body will be updated, modified, and collocated again at each instant (Fig. 9.6b). According to the ELMFS process, the numerical results at \((n+1)\Delta t\) level will be obtained. The moving rigid body phenomena in the fluid can be described clearly and solved directly by the ELMFS. The cylinder is regarded as a rigid body and moves on a circular trajectory without rotating, where the position of the cylinder center is expressed as:

\[
\begin{align*}
  x_c &= 0.5 \cos(\omega t) \\
  y_c &= 0.5 \sin(\omega t)
\end{align*}
\]  \hspace{1cm} (9-12)

The FEM with unstructured mesh is displayed in Fig. 9.6a. The numerical results of FEM with IBM are chosen for comparison. In opposition to the mesh-dependent method, only the field points are collocated on the boundary and initial data by the ELMFS (see Fig. 9.6b). The comparisons of \(u\)-velocity and \(v\)-velocity obtained by immersed-
boundary FEM with time evolution for $Re=0.1$ at point $(0.0, 0.0)$ and $(0.9, 0.0)$ are shown in Figs. 9.7 and 9.8 (600 points, $dt=0.0001, \lambda = 210$). Then, we compute the moving rigid body for at least three periods and record the velocity at point $(0.9, 0.0)$. Figure 9.9 indicates the period of the numerical results (about $\pi$). It also demonstrates the stability of the proposed method.

---

**Fig. 9.4.** (a) $u$-velocity profile along $x=0.5$ (b) $v$-velocity profile along $y=0.5$ for a square (c) $u$-velocity profile along $x=0.5$ (d) $v$-velocity profile along $y=0$ cavity for $Re=10$ and $Re=50$ (1000 points, $dt=0.0001, \lambda = 150$)
The MFS for Low Reynolds Number Flows with Moving Rigid Body

Fig. 9-5. Schematic diagram and boundary conditions of a circular cavity with moving rigid body

Fig. 9.6. (a) FEM for unstructured mesh with immersed boundary method
(b) MFS for field points in time-space domain
Fig. 9.7. Comparison for moving rigid body problem with time evolution of (a) u-velocity (b) v-velocity at point \( (0,0,0) \) for \( Re=0.1 \) (600 points, \( dt=0.0001 \), \( \lambda = 210 \)).
Fig. 9.8. Comparison for moving rigid body problem with time evolution of (a) $u$-velocity (b) $v$-velocity at point $(0.9,0,0)$ for Re=0.1 (600 points, dt=0.00001, $\lambda = 210$)
Fig. 9.9. (a) $u$-velocity (b) $v$-velocity with time evolution at point $(0.9, 0.0)$ about three period $(t=0~10)$ (600 points, $dt=0.0001$, $\lambda = 210$)
Numerical results of u-velocity and v-velocity distributions which are computed by immersed-boundary FEM and ELMFS for Re=0.1 are depicted in Figs. 9.10 and 9.11 at t=0.01 and t=0.5 respectively. The ELMFS results show good agreement with immersed-boundary FEM for moving rigid body problems. From these figures the small vorticity is observed to occur near the moving cylinder. The velocity vector and streamlines at t=0.01 and t=0.5 are displayed in Figs. 9.12 and 9.13. The flow motions are driven by the moving cylinder which is displayed by the velocity vector. The variations of streamlines around the moving cylinder clearly demonstrate the physics phenomena underlying the Navier-Stokes equations with moving rigid body problems. The streamlines exhibit the small vortex and eddy along with the moving cylinder with time evolution as vividly shown in Fig. 9.14.

9.5. Conclusions

The Navier-Stokes equations with the moving rigid body are solved by using the ELMFS based on the unsteady Stokeslets and time-space unification without using any kind of time transformation, time discretization or time separation techniques. The Navier-Stokes equations are converted to the characteristic Navier-Stokes equations in the Lagrangian coordinates by the ELM and solved by the MFS based on the unsteady Stokeslets. The advantages of the ELMFS, free from the mesh generation and numerical integration, can be used clearly and directly to describe and solve the moving rigid body problem. The numerical solutions of MFS are expressed as a combination of unsteady Stokeslets with different source intensities which are determined by collocating the known initial and boundary data. Then the results of Navier-Stokes equations can be computed by the back-tracking scheme along the characteristics. The proposed meshless method shows good agreement with the immersed-boundary FEM. Thus it has been demonstrated that the ELMFS is an effective tool in dealing with the Navier-Stokes equations with moving rigid body even very coarse nodes are used.
Fig. 9.10. Numerical results distributions of u-velocity (a) FEM (b) MFS at t=0.01 for moving rigid body problem (Re=0.1) (600 points, dt=0.0001, $\lambda = 210$)
Fig. 9.11. Numerical results distributions of v-velocity (a) FEM (b) MFS at t=0.5 for moving rigid body problem (Re=0.1) (600 points, dt=0.0001, \( \lambda = 210 \) )
Fig. 9.12. (a) velocity vector (b) streamlines at $t=0.01$ for flow system with moving rigid body

$\text{Re}=0.1$ (600 points, $dt=0.0001, \lambda = 210$)
Fig. 9.13. (a) velocity vector (b) streamlines at $t=1.6$ for flow system with moving rigid body

(Re=0.1) (600 points, $dt=0.0001$, $\lambda = 210$)
Fig. 9.14. Streamlines for flow system with moving rigid body at (a) $t=0.01$ (b) $t=0.5$ (c) $t=1.6$ (d) $t=3.0$ ($Re=0.1$) (600 points, $dt=0.0001$, $\lambda = 210$)

Acknowledgments

We would like to thank the National Science Council of Taiwan for its financial support under the Grant No. 95-2221-E-002-406 and No.95-2221-E-002-262. We also thank referees’ constructive comments and suggestions.
References


The MFS for Low Reynolds Number Flows with Moving Rigid Body


CHAPTER 10

The Method of Fundamental Solutions for Fully Developed Laminar Flow of Power-Law Non-Newtonian Fluids in Ducts of Arbitrary Cross-Section

Jan Adam Kolodziej\textsuperscript{1} and Tomasz Klekiel\textsuperscript{2}

Abstract  Chapter is concerned with the application of the method of fundamental solutions and radial basis functions for solving fully developed laminar flow of non-Newtonian fluids, inside ducts with constant, but arbitrary cross-section. An iterative solution of non-linear governing equation by Picard iteration is proposed. The solution for Newtonian fluid as first approximation of solution of non-linear boundary value problem is chosen. In each iterative step, the inhomogeneous harmonic problem is solved by method of fundamental solutions. The boundary condition is fulfilled by boundary collocation method in least square sense in each iterative step. The inhomogeneous term is interpolated by radial basis functions. Numerical results for flow of power-law fluid in circular and rectangular ducts are presented to compare the obtained results with previous solutions.

10.1. Introduction

Fluid flow in ducts of irregular cross-section appears in many industrial applications. Many authors have analyzed the motion of non-Newtonian fluids and particularly the power-law in ducts. Usually the purpose of such analysis is the pressure drop/flow rate equation, which is quite useful for practical engineering needs. Existing papers that examine estimating the

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pressure drop/flow rate equation for non-Newtonian fluids in ducts can be divided into two categories: theoretical and empirical. In the first case, the equation of motion has to be solved first in order to obtain the velocity distribution and then integration is performed to determine the pressure drop/flow rate equation. In the second case, some empirical equations are proposed.

The theoretical studies in the literature usually consider the fully developed flow in a duct of regular cross-section (circular, rectangular, elliptic, and triangular). In the first studies, Schechter [17] applied a variational method to obtain the velocity profile and corresponding pressure drop/flow rate equation for power-law fluid in a square duct. Wheeler and Wissler [20] solved the same problem by finite-difference method. Finite element analysis for flow of power-law fluid in rectangular ducts was considered in papers [13, 15, 19]. The comparison of Galerkin FEM and boundary-fitted co-ordinate transformation method is given in paper [14]. The generalized integral transformation method was presented in paper [11]. This method can be applied for rectangular ducts, only.

All papers mentioned above used mesh methods, which means that they require mesh in calculations. In last two decades, there was considerable interest in developing mesh-free methods for the solution of boundary value problems encountered in applied mechanics [12]. Today, there are many versions of mesh-free methods, including some versions of Trefftz methods as boundary methods. Trefftz method can be understood as a method in which differential equation is fulfilled exactly whereas the boundary condition is fulfilled approximately. There are few possibilities in approximate fulfilment of the boundary condition in frame of Trefftz method. Among others one of them is the boundary collocation method (BCM) [10], where the boundary conditions are fulfilled in collocation manner.

Two different sets of functions, which satisfy exactly differential equations, are used in frame of BCM: T-complete Herrera functions [6] and fundamental solutions of governing equation [5]. In the second case the method is known as the method of fundamental solutions (MFS). Basically, Trefftz method can be used when functions which fulfill governing differential equation are known. Then Trefftz method and in frame of its MFS cannot be use in straight way for non-linear Boundary Value Problems (BVPs) which appear at investigation of non-Newtonian flow.
However, it does not mean that Trefftz method can not be used in any way for non-linear BVPs. Firstly it can be used in BVPs with linear equation but with non-linear boundary conditions. Examples of such applications of this method are given in papers [7, 16]. The second case known in literature is BVP with non-linear Poisson equation [1, 2, 4, 8]. In paper [4] the non-linear thermal explosions problem was solved by method of fundamental solutions. The radial basis functions were used for interpolation of right hand side and Picard iteration method was used to tread non-linearity. In paper [2] method called “particular solution Trefftz method” was used. In fact it is extension and improvement of ideas proposed in paper [4]. Another version of Trefftz method for solution of non-linear Poisson equation was presented in paper [1]. For non-linear thermal conductivity problem by Kirchoff transformation the nonlinearity exists only in boundary conditions. The non-linear algebraic equation was solved by stabilized continuation method. Kita at al. [8] considered steady state heat conduction problems for functionally gradient materials. In order to overcome the difficulty with non-linear Poisson equation, the combination scheme of the Trefftz method with the computing point analysis method was presented. Also steady state heat conduction problem with temperature dependent conductivity was considered in paper [9]. Combination of the MFS with Picard iteration was used for non-linear Poisson equation.

The governing equation of considered non-Newtonian flow problem may be formulated as nonlinear equation where on the left hand side the Laplace operator has occurred. On the right hand side of equation is the non-linear function which can be assumed as known at each iteration step. The problem described by such form of equation may be solved by iterations using the MFS.

The purpose of our paper is the application of method fundamental solution and radial basis functions for solving fully developed laminar flow of non-Newtonian fluids inside ducts with constant but arbitrary cross-section. An iterative solution of non-linear governing equation by Picard iteration is proposed. At first approximation of solution non-linear boundary value problem the solution for Newtonian fluid is chosen. In each iterative step the inhomogeneous harmonic problem is solved by the MFS. The boundary condition is fulfilled by the BCM in least square sense in each iterative step. The inhomogeneous term is interpolated by radial basis functions (RBF).
Numerical results for flow of power-law fluid in circular and rectangular ducts are presented to compare the obtained results with previous solutions.

10.2. Statement of the Problem

Consider a steady, fully developed, laminar, isothermal flow of incompressible viscous non-Newtonian fluids driven by a constant pressure gradient in a duct of arbitrary cross-section. The duct’s configuration and coordinate system are shown in Fig. 10.1. The cross-section area of the duct remains constant in the $z$-direction. We assume that the velocity is purely axial and $u$ is the only non-vanishing velocity component. As a consequence of assumption related with flow, the equation of continuity is satisfied automatically. The equation of motion, on the other hand, is reduced to a single partial differential equation of the form

\[
\frac{\partial}{\partial x} \left( \eta(\gamma) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \eta(\gamma) \frac{\partial u}{\partial y} \right) = \frac{dp}{dz}
\]  

where $u$ is velocity component in $z$ direction, $p$ is pressure, and $\eta(\gamma)$ is viscosity function. The power-law model is used in this work to describe the non-Newtonian behavior of the fluid and consequently the viscosity function is given by

\[
\eta(\gamma) = K \cdot \gamma^{m-1}
\]

where $K$ is consistency factor, $m$ is the power-law index, and

\[
\gamma = \sqrt{\left( \frac{\partial u}{\partial x} \right)^2 + \left( \frac{\partial u}{\partial y} \right)^2}
\]

The constant pressure gradient in Eq. (10.1) can be given approximately by

\[
\frac{dp}{dz} = \frac{p_2 - p_1}{L} = - \frac{p_1 - p_2}{L}
\]
where $p_1$ is the pressure at the beginning of the duct, $p_2$ is the pressure at the end of the duct, and $L$ is the length of the duct.

Now, the Eq. (10.1) can be written as:

$$\frac{\partial}{\partial x} \left( \eta(y) \frac{\partial u}{\partial x} \right) + \frac{\partial}{\partial y} \left( \eta(y) \frac{\partial u}{\partial y} \right) = -\frac{(p_1 - p_2)}{L} \tag{10.5}$$

There is a non-linear differential equation for which the boundary condition is defined as:

$$u = 0 \quad \text{on the walls of the duct} \quad \tag{10.6}$$

The dimensionless variables are formulated as:

$$X = \frac{x}{a}, \quad Y = \frac{y}{a}, \quad \nu = \frac{\mu L u}{a^2 (p_1 - p_2)}, \quad E(y) = \frac{\eta(y)}{\mu_0} \tag{10.7}$$

where $\mu_0$ is reference viscosity, $a$ is characteristic dimension of cross section of duct.
On the base Eq. (10.5) and Eq. (10.7) the governing equation has form:

\[
\frac{\partial}{\partial X} \left[ E(\chi) \frac{\partial V}{\partial X} \right] + \frac{\partial}{\partial Y} \left[ E(\chi) \frac{\partial V}{\partial Y} \right] = -1
\]  

(10.8)

where

\[
\chi = \frac{\gamma}{\frac{a(p_1 - p_2)}{\mu_0 L}} = \sqrt{\left( \frac{\partial V}{\partial X} \right)^2 + \left( \frac{\partial V}{\partial Y} \right)^2}
\]  

(10.9)

The variable \( \chi \) is also dimensionless because \( \gamma \) as well as \( \frac{a(p_1 - p_2)}{\mu_0 L} \) have the same units.

The Eq. (10.8) may be rewritten as:

\[
\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} = -\frac{1}{E(\chi)} \sqrt{E(\chi)} \left( \frac{\partial \chi}{\partial X} \frac{\partial V}{\partial X} + \frac{\partial \chi}{\partial Y} \frac{\partial V}{\partial Y} \right)
\]  

(10.10)

In Eq. (10.10) the shape of the function \( E(\chi) \) depends on model of non-Newtonian fluid. In this paper the power-law model was used. On the basis of Eq. (10.2) the function \( E(\chi) \) has form:

\[
E(\chi) = \frac{\eta(\gamma)}{\mu_0} = K \frac{\alpha^{-1} \chi^{-m^{-1}}}{\mu_0} = B_1 \chi^{-m^{-1}}
\]  

(10.11)

where

\[
\alpha = \frac{a(p_1 - p_2)}{\mu_0 L}, \quad B_1 = \frac{K}{\mu_0} \alpha^{-1}
\]

The parameter \( B_1 \) is dimensionless.

The Eq. (10.10) by using Eq. (10.11) becomes

\[
\frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} = -\frac{1}{B_1 \chi^{-m^{-1}}} \left( \frac{m-1}{a^2 + b^2} \phi \right)
\]  

(10.12)
where

\[ a = \frac{\partial V}{\partial X}, \quad b = \frac{\partial V}{\partial Y}, \quad \Phi = a^2 \frac{\partial^2 V}{\partial X^2} + 2ab \frac{\partial^2 V}{\partial X \partial Y} + b^2 \frac{\partial^2 V}{\partial Y^2} \] (10.13)

The Eq. (10.12) is solved with boundary condition

\[ V = 0 \quad \text{on} \quad \Gamma \] (10.14)

where \( \Gamma \) is the dimensionless contour of cross-section of duct.

10.3. Solution Methodology

The non-linear governing partial differential Eq. (10.12) describes fluid flow along a straight duct. The solution of this problem was obtained by the MFS, the RBF, and Picard iteration method. These three methods were used according to following scheme:

**Step 1**

For \( m=1 \) the fluid is Newtonian and the Eq. (10.12) is reduced to the form:

\[ \frac{\partial^2 V}{\partial X^2} + \frac{\partial^2 V}{\partial Y^2} = -1 \] (10.15)

For this case on the base of MFS the approximated solution is received as:

\[ V(X,Y) = -\frac{X^2 + Y^2}{4} + \sum_{j=1}^{N} c_j \ln \sqrt{(X - X_j)^2 + (Y - Y_j)^2} \] (10.16)
where $c_j$ coefficients is obtained from boundary collocation method. During calculations some number of collocation and sources points is chosen. The Fig. 10.2 presents the place of source, collocation and interpolation points.

**Step 2**
The right hand of Eq. (10.12) is interpolated using RBF. As the RBF function we choose function called multiquadrics which is defined as:

$$\phi(r_j) = \sqrt{r_j^2 + c^2}$$

(10.17)

In this case the value of shape parameter $c$ was assumed as 0.001. The interpolation function has following form:

$$f(X,Y) = \sum_{i=1}^{MI} a_i \phi(r_i) + \sum_{k=1}^{I} b_k p_k(X,Y)$$

(10.18)

where $MI$ is the number of interpolation points in consideration area and on the boundary, the function $p_k(X,Y)$ is a polynomial function [9] given in Table 1, and the coefficients $a_i$ and $b_k$ are calculated in result of interpolation.
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Fig. 10.2. The location of source, collocation and interpolation points for any profile of cross section of duct

Step 3
In current iteration the solution is defined as:

\[
V(X,Y) = \sum_{i=1}^{M} a_i \psi(t_i) + \sum_{k=1}^{I} b_k \varphi_k(X,Y) + \sum_{j=1}^{N} c_j \ln \sqrt{(X - X_j)^2 + (Y - Y_j)^2}
\]  

(10.19)

where

\[
\psi(r) = -\frac{1}{3} c^3 \ln\left( c \sqrt{c^2 + r^2} + c \right) + \frac{1}{9} \left( 4c^2 + r^2 \right) \sqrt{c^2 + r^2}
\]

and \(q_k\) are given in Table 1. The coefficients \(c_j\) are calculated by boundary collocation method.

Step 4
If, for two next solutions, the difference of maximum values of velocity is smaller than some value the algorithm is finished. If this condition is not fulfilled the solutions are continued and algorithm goes to Step 2.

Table 10.1.

<table>
<thead>
<tr>
<th>(k)</th>
<th>(p_k(x,y))</th>
<th>(q_k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>(\frac{1}{4}x^2 + \frac{1}{4}y^2)</td>
</tr>
<tr>
<td>2</td>
<td>(x)</td>
<td>(\frac{1}{8}x^3 + \frac{1}{8}xy^2)</td>
</tr>
</tbody>
</table>
10.4. The Considered Ducts

10.4.1. The Circular Duct

At first, the problem of the fluid flow along circular duct was solved. On Fig. 10.3 the cross section with boundary condition is shown. This problem has exact solution in the form:

\[
V(r) = \frac{1}{R} \int_0^R V(r) r dr
\]  

This problem was also solved by other authors. They have used the average velocity which may be calculated by [3]:

\[
\bar{w} = \frac{2}{r^2} \int_0^R V(r) r dr
\]

The coordinate axis is located along symmetry axis of duct. On the boundary, value of velocity has to be zero. For this case the interpolation points are generated on rings which are located centrally to coordinates’ axis and the radius of these rings \( \rho \) is solved by equation:
\[ \rho = i \cdot \frac{R}{n}, \]  
\text{(10.22)}

where \( i \) - is a ring number, \( R \) – radius of consideration area, \( n \) – number of rings. The number of rings is selected arbitrary. Each next ring from 0 to \( R \) includes one more point. For example when \( n=10 \) the interpolation points are located such as Fig. 10.3.

![Fig. 10.3. The location of collocation, source and interpolation points for circular duct where: X - are the interpolation points, O - are the source points, □- are the collocation points.](image)

The radius for source points is defined by \( S \) parameter where this radius is calculated as sum of radius \( R \) and parameter \( S \).

\textbf{10.4.2. The Square and Rectangular Duct}

For rectangular duct the dimensions are proposed as \( 2a, 2b \) (see Fig. 10.4), for which the aspect ratio parameter was defined as: \( \alpha = \frac{b}{a} \), but for square duct the \( \alpha \) parameter is equal 1.
The average flow velocity is given as [11, 15]:

\[ w = \frac{\int_{-a}^{a} \int_{-b}^{b} V(x, y) dx dy}{\int_{-a}^{a} \int_{-b}^{b} dx dy} = \frac{1}{4ab} \int_{-a}^{a} \int_{-b}^{b} V(x, y) dx dy \]  

(10.23)

In a lot of publications [3, 11, 15, 20] the parameter called friction factor Reynolds number, is used to compare several solutions. For rectangular duct for which dimensions are defined as \( a \times b \), the friction factor Reynolds number parameter may be written as:

\[ f \cdot Re = \frac{(2/w)^m}{(\alpha + 1)^m} \]  

(10.24)

where \( f \) is friction factor, \( Re = \frac{\rho \cdot w^{2-m} \cdot D_h^m}{K} \) is Reynolds number, \( \rho \) is density of fluid.

The source points in MFS are generated on the contour which is similar to consideration area. The Fig. 10.8 presents location for several points. The distance between contour of source points and the boundary of profile is defined by shape parameter \( S \).
10.5. Results of Numerical Experiments

The method presented in section 10.3 was programmed in Pascal language. The solution was realized on the computer AMD Athlon 1700+. The time cost of particular solution depended on the number of iteration $IT$. The results obtained for circular duct were presented in Table 1. It was solved for a few values of $m$ parameter (power-law index) such as 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0. However, for $m = 0.2$ the result has too much error. Table 10.2 include following data: $m$ – power-law index, $IT$ - the number of iterations, $\delta$ - error defined as: $(V_{m}(0) - V(0,0)) / V_{m}(0)$, $\delta_{int}$ - interpolation error, $\delta_{bnd}$ - error on the boundary, $V(0,0)$ is value of velocity at center point from approximate solution, $V_{m}(0)$ is value of velocity in center point for exact solution. If the $m$ parameter was smaller, the calculated process required more iteration $IT$. The boundary normal error [9] haven’t been large for all cases. These solutions were calculated for the number of collocation points $M$=15, the number of source points $MS$=10, the number of the rings for interpolation points $n$=10, distance between a contours source and collocation points $S$=2, the shape parameter for RBF $c$=0.001.

The proposed method was compared with exact solution and solution which was obtained by authors [11, 15]. The results for exact solution were shown in Table 10.2. A difference between exact and approximated solution is small for $m$ near 1, but when it is decreased the error is great. Fig. 10.7. presents a normal error along $y$ coordinate.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$IT$</th>
<th>$\delta$</th>
<th>$\delta_{int}$</th>
<th>$\delta_{bnd}$</th>
<th>$w$</th>
<th>$V(0)/w$</th>
<th>$V(0)$</th>
<th>$F(0,0)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.00000</td>
<td>0.12500</td>
<td>2.00000</td>
<td>0.25000</td>
<td>0.25000</td>
</tr>
<tr>
<td>0.9</td>
<td>6</td>
<td>0.00845</td>
<td>0.05531</td>
<td>0.00001</td>
<td>0.11212</td>
<td>1.93935</td>
<td>0.21743</td>
<td>0.21929</td>
</tr>
<tr>
<td>0.8</td>
<td>10</td>
<td>0.01486</td>
<td>0.04373</td>
<td>0.00002</td>
<td>0.09800</td>
<td>1.87848</td>
<td>0.18409</td>
<td>0.18687</td>
</tr>
<tr>
<td>0.7</td>
<td>14</td>
<td>0.02233</td>
<td>0.05060</td>
<td>0.00003</td>
<td>0.08249</td>
<td>1.81294</td>
<td>0.14955</td>
<td>0.15297</td>
</tr>
<tr>
<td>0.6</td>
<td>20</td>
<td>0.02693</td>
<td>0.07041</td>
<td>0.00004</td>
<td>0.06589</td>
<td>1.74438</td>
<td>0.11494</td>
<td>0.11812</td>
</tr>
</tbody>
</table>
On Fig. 10.5 the profiles of velocity along $R$ was presented. Fig. 10.6. shows the behavior of the velocity ratio $v(r)/w$ along $R$. These results are presented for several values of $m$ parameter.
The second problem concerns the square duct which was solved by proposed method. The problem was calculated for following values of \( m \) parameter: 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1.0 and the parameters of MFS and RBF were values as: the number of collocation points \( M = 15 \), the number of source points \( M_S = 10 \), the number of interpolation points \( n = 100 \), distance between contours source and collocation points \( S = 0.1 \), shape parameter for RBF \( c = 0.001 \). The Fig. 10.9 presents location of points. The source points are located similarly to the boundary.
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Fig. 10.7. The normal error $\delta$ along y coordinate for several values of $m$.

Fig. 10.8. The location of collocation, source and interpolation points for square duct where: × - are the interpolation points, ○ - are the source points, □ - are the collocation points.

Table 10.3. The results for square duct. Where: $m$ – power-law index, $IT$ - the number of iterations, $\delta_{int}$ - normal interpolation error, $\delta_{bnd}$ - normal error on the boundary, $w$ - average flow velocity, $u(0,0)/w$ - velocity ratio, $u(0,0)$ – obtained solution in center point, $f_{Reg}$ – friction factor Reynolds number.

<table>
<thead>
<tr>
<th>$M$</th>
<th>$IT$</th>
<th>$\delta_{int}$</th>
<th>$\delta_{bnd}$</th>
<th>$W$</th>
<th>$u(0,0)/w$</th>
<th>$u(0,0)$</th>
<th>$f_{Reg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2</td>
<td>0.00000</td>
<td>0.00004</td>
<td>0.03514</td>
<td>2.09625</td>
<td>0.07367</td>
<td>14.22701</td>
</tr>
<tr>
<td>0.9</td>
<td>5</td>
<td>0.06852</td>
<td>0.00004</td>
<td>0.02990</td>
<td>2.05326</td>
<td>0.06140</td>
<td>11.77177</td>
</tr>
</tbody>
</table>
The Fig. 10.9 presents profiles of velocity ratio $u(0,y)/w$ along $y$ coordinate for several values of $m$ parameter. For $m=1$ the friction factor Reynolds number parameter is exactly the same as presenting in [11, 15]. If the $m$ parameter is lessened the difference between these values is greater. The values of $f_{Reg}$ (from Table 10.3) are different in comparison with other solutions, but they are not too great.

The third example concerns the fluid flow in rectangular duct where $\alpha$ parameter equals 0.5. The problem has been solved for various values of $m$ parameter. The Table 10.4 presents results obtained for the number of collocation points $M=25$, the number of source points $MS=15$, the number of interpolation points $n=100$, distance between contours source and collocation points $S=0.5$ and the shape parameter for RBF $c=0.001$. Like previous example for $m=1$ the $f_{Reg}$ is exactly the same as presenting in literature [11, 15].

Table 10.4. The results for rectangular duct

<table>
<thead>
<tr>
<th>$m$</th>
<th>$IT$</th>
<th>$\delta_{int}$</th>
<th>$\delta_{bnd}$</th>
<th>$w$</th>
<th>$V(0,0)/w$</th>
<th>$u(0,0)$</th>
<th>$f_{Reg}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>0.0005</td>
<td>0.02563</td>
<td>2.01038</td>
<td>0.05152</td>
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10.6. Conclusions

In this paper the new method of solution of problem of fluid flow in duct about various shapes was presented. As first, the circular duct has been considered. For this example the proposed method has been compared with the exact solution. This comparison shows that proposed method gives results which are near to exact so we made conjectures that application of fundamental solution method and radial basis functions are suitable to solve a Navier-Stokes equation for other shapes of duct. For square and rectangular configuration of duct obtained results in many cases are the same as in other publications. The obtained solutions characterized a small error on the boundary. It means that this calculated system of several methods has been correctly built.

The proposed iteration technique called Picard iteration method has been used to solution a nonlinear equation. This method is characterized by weakly convergence but for this application this problem disappeared.
References


CHAPTER 11

Shear Deformable Plate Analysis by the Method of Fundamental Solutions: Static and Dynamic Cases

P.H. Wen¹, C.S. Chen² and Y.W. Liu³

Abstract. The aim of this study is to investigate the application of the method of fundamental solutions (MFS) to the shear deformable (Reissner/Mindlin’s theories) plate problem under either static or dynamic loads. The fundamental solutions for Reissner/Mindlin plate theory are derived in the Laplace transform domain. It has been shown that the new fundamental solutions exhibit the same singularities as in the static case and can be used directly in the method of fundamental solutions. Deflection, moment and shear boundary conditions are satisfied at collocation points on the boundary by applying concentrated forces at source points outside the domain. The physical variables in the time domain can be obtained by Durbin’s inversion method. Numerical examples are presented to demonstrate the accuracy of the method of fundamental solutions and comparisons are made with exact solutions and approximate solutions obtained using the finite element method (ABAQUS). In addition, stability and convergence are discussed for the static problem. The proposed MFS is found to be simple to implement and gives satisfactory results for a shear deformable plate under static and dynamic loads.

11.1. Introduction

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Numerical methods such as the finite element method and the boundary element method are well-developed in plate bending structure analysis. It is well-known that many commercial software such as DYNA3D, ABAQUS/Explicit have been developed and are available to solve large scale engineering problems (shell element). To reduce model's dimension, in 1968, the application of boundary integral equation method to the plate bending problem was presented by Jaswon and Maiti [1] for the classical theory. Later, indirect boundary integral equation solutions of Kirchhoff plate bending problems were presented by Altiero and Sikarskie [2] and Tottenham [3]. For the shear deformable Reissner/Mindlin plates, the boundary integral equation method was reported by Vander Weeën [4] for static problems. Later, the direct boundary element formulation for Reissner/Mindlin's plate bending by Wen et al [5] derived fundamental solutions for displacement and traction in the Laplace transform domain under dynamic loads. In their work, the behaviour of three elasticity waves, i.e. slow flexural, fast flexural and thickness shear waves, were studied.

In recent years, there has been growing interest in meshless methods for the numerical solution of partial differential equations, see Atluri [6]. The method of fundamental solutions is regarded as a mesh free method since the MFS does not require an elaborate discretisation of the boundary, and the integrations over the boundary and domain are avoided and the solution in the interior of the domain is evaluated without extra treatments. The main idea in the MFS consists of approximating the solution of the problem by a linear combination of fundamental solutions with respect to source points which are located outside the domain. This method was originally introduced by Kupradze et al in 1962 [7] and was successfully applied for solving a wide range of boundary value problems when the fundamental solution is available for partial differential equations. The application of the method of fundamental solutions to Kirchhoff plates with Winker’s foundation can be found in Wen 1984 [8]. A review of the MFS for two- and three-dimensional elasticity was given by in Marin and Lesnic [9] and for the diffusion equation by Chen et al [10].

The aim of the current study is to extend the MFS to the solution of shear deformable plates under static and dynamic loads by using the Laplace transform method (static problem is considered as a particular case of the
Fundamental solutions of deflection/rotation and moment/shear force are derived in the Laplace transform domain for concentrated moments and shear force. A set of linear algebraic equations with unknown coefficients (the intensity of the point force located outside of the domain) is solved by satisfying the boundary conditions at specified boundary collocation points. Validation of this method is given by comparison with either available analytical solutions or approximate solution obtained using the finite element method (DYNA3D).

11.2. Fundamental Solution for Reissner/Mindlin Plate in Laplace Domain

With small motion assumption, the governing equations for deflection and rotations for the Reissner/Mindlin plate can be expressed as follows, see [11]

\[
\frac{D}{2} \left[ (1-\nu)\nabla^2 w_\alpha + (1+\nu)w_{\beta,\beta} \right] - \kappa h (w_{3,\alpha} + w_\alpha) = \frac{\rho h^3}{12} \frac{\partial^2 w_\alpha}{\partial t^2}
\]

\[
\kappa h (\nabla^2 w_3 + w_{\alpha,\alpha}) + q(t) = \rho h \frac{\partial^2 w_3}{\partial t^2}
\]

where \(q(t)\) is the pressure load on the plate and \(w_\alpha\) (see Figure 11.1) denote rotations in the \(x_\alpha\) direction and \(w_3\) is the out-of-plane deflection, where Greek indices vary from 1 to 2. The moment \(M_{\alpha\beta}\) and shear force \(Q_\alpha\) may be expressed in terms of displacements as

Fig 11.1. Sign convention of displacement and internal forces.
Shear Deformable Plate Analysis: Static and Dynamic Cases

\[ M_{\alpha\beta} = \frac{1-\nu}{2} D \left( w_{\alpha,\beta} + w_{\beta,\alpha} + \frac{2\nu}{1-\nu} w_{\gamma,\gamma} \delta_{\alpha\beta} \right) \]  \hspace{1cm} (11.2)

\[ Q_\alpha = \kappa^2 \mu h (w_\alpha + w_{\gamma,\gamma}) \]

in which \( \delta_{\alpha\beta} \) denotes the Kronecker delta function, \( \kappa \) the shear coefficient \((= \pi^2/12 \text{ for Mindlin theory and } = 5/6 \text{ for Reissner theory})\), \( D = Eh^3/12(1-\nu^2) \) is the bending stiffness of the plate, \( \mu = E/2(1+\nu) \) and \( h \) is the thickness of the plated, \( E \) and \( \nu \) are elastic constants. Consider the Laplace transform of a function \( f(x,t) \)

\[ L[f(x,t)] = \tilde{f}(x, p) = \int_0^\infty f(x, t)e^{-pt} dt \]  \hspace{1cm} (11.3)

where \( p \) is the parameter of the Laplace transform. Applying the Laplace transform to the governing equations (11.1) yields

\[ 6D \left[ (1-\nu) \nabla^2 \tilde{w}_\alpha + (1+\nu) \tilde{w}_{\beta,\beta} \right] - 12\kappa \mu h (\tilde{w}_{3,\alpha} + \tilde{w}_\alpha) = \rho h^3 p^2 \tilde{w}_\alpha \]  \hspace{1cm} (11.4)

\[ \kappa \mu h (\nabla^2 \tilde{w}_3 + \tilde{w}_{\alpha,\alpha}) + \tilde{q} = \rho hp^2 \tilde{w}_3 \]

It is apparent that the rotation and the deflection can be written in terms of the three displacement potentials \( \phi_k \) \((k = 1,2,3)\) as

\[ \tilde{w}_1 = (a_2 - 1) \phi_{1,1} + (\alpha_1 - 1) \phi_{2,1} + \phi_{3,1} \]

\[ \tilde{w}_2 = (a_2 - 1) \phi_{1,2} + (\alpha_1 - 1) \phi_{2,2} + \phi_{3,2} \]

\[ \tilde{w}_3 = \phi_1 + \phi_2 \]  \hspace{1cm} (11.5)

where

\[ a_\beta = \frac{2}{1-\nu} \left( \frac{\alpha_\beta}{\alpha_3} \right)^2, \quad \beta = 1,2 \]  \hspace{1cm} (11.6)

in which
Shear Deformable Plate Analysis: Static and Dynamic Cases

\[ \alpha_1^2 = \frac{1}{2} \left( \frac{p}{\omega_0} \right)^2 \left\{ \frac{1}{S} + \frac{1}{R} + \left[ \left( \frac{1}{S} - \frac{1}{R} \right)^2 - \frac{4}{SR} \left( \frac{\omega_0}{p} \right)^2 \right]^{1/2} \right\} \]

\[ \alpha_2^2 = \frac{1}{2} \left( \frac{p}{\omega_0} \right)^2 \left\{ \frac{1}{S} + \frac{1}{R} - \left[ \left( \frac{1}{S} - \frac{1}{R} \right)^2 - \frac{4}{SR} \left( \frac{\omega_0}{p} \right)^2 \right]^{1/2} \right\} \]

(11.7)

\[ \alpha_3^2 = \left( \frac{\pi}{h} \right)^2 \left[ \left( \frac{p}{\omega_0} \right)^2 + 1 \right] \]

where \( S = \frac{12D}{\pi^2 \mu h}, R = \frac{h^2}{12}, \omega_0 \) denotes the cut-off frequency \( \pi c_2 / h \) and \( c_2 = \sqrt{\mu / \rho} \) is the velocity of the shear wave. Substituting equation (11.5) into the governing equation (11.4) gives the following Helmholtz-type differential equations for the displacement potentials

\[ (\nabla^2 + \alpha_k^2) \phi_k = 0 \quad k = 1, 2, 3. \]

(11.8)

For the symmetric problem with respect to the \( x_1 \)-axis, the general solution of above equations can be obtained (see Wen et al [5]) using a Fourier transform technique as

\[ \phi_1 = \frac{2}{\pi} \int_0^\infty A_1(s) e^{-\beta_1 x_1} \cos(sx_1) ds, \]

\[ \phi_2 = \frac{2}{\pi} \int_0^\infty A_2(s) e^{-\beta_2 x_1} \cos(sx_1) ds \]

(11.9)

\[ \phi_3 = \frac{2}{\pi} \int_0^\infty A_3(s) e^{-\beta_3 x_1} \cos(sx_1) ds \quad x_2 \geq 0 \]

where \( \beta_k^2 = s^2 + \alpha_k^2 \) and \( A_k(s) \) are unknown coefficients to be determined by the boundary conditions for each concentrated force. It is easy to determine the rotation and the deflection as follows
Shear Deformable Plate Analysis: Static and Dynamic Cases

\[
\tilde{w}_1 = \frac{2}{\pi} \int_0^\infty \left[ s(1-a_2) A_1(s) e^{-\beta_1 s_2} + s(1-a_1) A_2(s) e^{-\beta_2 s_2} - \beta_3 A_3(s) e^{-\beta_3 s_2} \right] \sin(s x_1) ds
\]

\[
\tilde{w}_1 = \frac{2}{\pi} \int_0^\infty \left[ (1-a_2) A_1(s) e^{-\beta_1 s_2} + (1-a_1) A_2(s) e^{-\beta_2 s_2} - s A_3(s) e^{-\beta_3 s_2} \right] \cos(s x_1) ds
\]

\[
\tilde{w}_1 = \frac{2}{\pi} \int_0^\infty \left[ A_1(s) e^{-\beta_1 s_2} + A_2(s) e^{-\beta_2 s_2} \right] \cos(s x_1) ds \quad x_2 \geq 0
\]  

(11.10)

The moment \( \tilde{M}_{\alpha\beta} \) and shear stress \( \tilde{Q}_\alpha \) in the transformed domain can be obtained as

\[
\tilde{M}_{11} = \frac{2}{\pi} \int_0^\infty \left[ (1-a_2)(s^2 - \nu \beta_1^2) A_1(s) e^{-\beta_1 s_2} + (1-a_1)(s^2 - \nu \beta_2^2) A_2(s) e^{-\beta_2 s_2} \right. \\
- s \beta_3 (1-\nu) A_3(s) e^{-\beta_3 s_2} \left. \right] \sin(s x_1) ds
\]

\[
\tilde{M}_{22} = \frac{2}{\pi} \int_0^\infty \left[ (1-a_2)(s^2 - \beta_1^2) A_1(s) e^{-\beta_1 s_2} + (1-a_1)(s^2 - \beta_2^2) A_2(s) e^{-\beta_2 s_2} \right. \\
+ s \beta_3 (1-\nu) A_3(s) e^{-\beta_3 s_2} \left. \right] \cos(s x_1) ds
\]

\[
\tilde{M}_{12} = -\frac{2}{\pi} \int_0^\infty \left[ 2s \beta_1 (1-a_2) A_1(s) e^{-\beta_1 s_2} + 2s \beta_2 (1-a_1) A_2(s) e^{-\beta_2 s_2} \right. \\
- (\beta_3^2 + s^2) A_3(s) e^{-\beta_3 s_2} \left. \right] \cos(s x_1) ds
\]  

(11.11)

and

\[
\tilde{Q}_1 = -\frac{2k_{\Delta} \beta}{\pi} \int_0^\infty \left[ s a_1 A_1(s) e^{-\beta_1 s_2} + s a_1 A_2(s) e^{-\beta_2 s_2} + \beta_3 A_3(s) e^{-\beta_3 s_2} \right] \sin(s x_1) ds
\]

\[
\tilde{Q}_2 = -\frac{2k_{\Delta} \beta}{\pi} \int_0^\infty \left[ \beta_1 a_2 A_1(s) e^{-\beta_1 s_2} + \beta_2 a_1 A_2(s) e^{-\beta_2 s_2} + s A_3(s) e^{-\beta_3 s_2} \right] \cos(s x_1) ds
\]  

(11.12)
For a concentrated moment acting at the origin, the boundary conditions along axis $x_1$ are represented as

$$\bar{M}_{22} = -\frac{1}{4} \delta(x_1), \quad \bar{w}_i = \bar{w}_j = 0. \tag{11.13}$$

In the Laplace transform domain, the coefficients can be determined from the above equations

$$\begin{align*}
(1 - a_2)(\nu^2 - \beta_i^2)A_i(s) + (1 - a_1)(\nu^2 - \beta_2^2)A_2(s) + s\beta_1(1 - \nu)A_3(s) &= -\frac{1}{4} \\
\nu(1 - a_2)A_i(s) + s(1 - a_1)A_2(s) - \beta_2 A_i(s) &= 0 \\
A_i(s) + A_2(s) &= 0
\end{align*} \tag{11.14}$$

Therefore, we have

$$A_i(s) = -\frac{1}{4(\alpha_2 - \alpha_1)} \quad A_2(s) = \frac{1}{4(\alpha_2 - \alpha_1)} \quad A_3(s) = \frac{1}{(1 - \nu)(\alpha_2 - \alpha_1)} \tag{11.15}$$

In the same way, we can obtain two other sets of coefficients for a concentrated moment and a concentrated shear force acting at origin. Finally the fundamental solutions for the rotations and the deflection $\bar{U}_{ik}$ in the Laplace transform domain can be arranged as [5]

$$\bar{U}_{\alpha\beta}(x, p) = -\frac{1}{2\pi D}(f_{\alpha\beta} + g_{\alpha\beta}), \quad \bar{U}_{\alpha3}(x, p) = \frac{1}{2\pi D} h_{\alpha} \tag{11.16}$$

and

$$\begin{align*}
\bar{U}_{3\alpha}(x, p) &= \frac{(1 - a_1)(1 - a_2)(1 - \nu)}{4\pi \kappa \mu h} h_{\alpha} \\
\bar{U}_{33}(x, p) &= \frac{1}{2\pi \kappa \mu} \left[(1 - a_1)K_0(z_1) - (1 - a_2)K_0(z_2)\right] \tag{11.17}
\end{align*}$$
in which the subscript \( i \) denotes the direction of displacement and \( k \) the direction of point forces, \( K_0(z_i) \) represents the zero order Bessel function, \( z_j = \alpha_j r \) and functions

\[
f = \frac{1}{\alpha_2^2 - \alpha_1^2} \left[ (1 - a_2)K_0(z_1) - (1 - a_1)K_0(z_2) + (a_2 - a_1)\alpha_2 K_0(z_3) \right]
\]

\[
g = \frac{2}{1 - \nu} K_0(\alpha_3), \quad h = \frac{1}{\alpha_2^2 - \alpha_1^2} \left[ K_0(z_1) - K_0(z_2) \right]
\]

Taking into account the properties of Bessel functions, the displacement fundamental solutions can be written as

\[
\bar{U}_{\alpha} = \frac{1}{2\pi D(\alpha_2^2 - \alpha_1^2)} \left[ (1 - a_2)\alpha_2 K_1(z_1) - (1 - a_1)\alpha_2 K_1(z_2) + (a_2 - a_1)\alpha_2 K_1(z_3) \right]
\]

\[
(2r_{\alpha\beta} r_{\beta\delta} - \delta_{\alpha\delta})/r + \left[ (1 - a_2)\alpha_2^2 K_0(z_1) - (1 - a_1)\alpha_2^2 K_0(z_2) + (a_2 - a_1)\alpha_2^2 K_0(z_3) \right] \delta_{\alpha\beta}
\]

\[
\bar{U}_{\alpha 3} = \frac{1}{2\pi D(\alpha_2^2 - \alpha_1^2)} \left[ \alpha_2 K_1(z_1) - \alpha_2 K_1(z_2) \right] \delta_{\alpha 3}
\]

\[
\bar{U}_{3\alpha} = \frac{(1 - a_2)}{2\pi \kappa \mu (a_2 - a_1)} \left[ \alpha_2 K_1(z_1) - \alpha_2 K_1(z_2) \right] \delta_{3\alpha}
\]

\[
\bar{U}_{33} = \frac{(1 - a_2)}{2\pi \kappa \mu (a_2 - a_1)} \left[ (1 - a_1)K_1(z_1) - (1 - a_2)K_1(z_2) \right]
\]

and the traction fundamental solutions from equation (11.2)
where $\hat{T}_{\alpha\beta} = \begin{pmatrix} 1 - \nu \end{pmatrix} \left[ -\frac{1}{4\pi(a_2^* - a_1^*)} \right] \left[ -\frac{1}{(a_2 - a_1)\alpha_1^* (2K_1(z_i) + z_i K_0(z_i)) - (1 - a_2)\alpha_2^* (2K_1(z_i) + z_i K_0(z_i))] \times \right. \\
\left. \frac{2}{r_2^2} (4r_{\alpha\beta} r_{\alpha\beta}^* - r_{\alpha\beta} n_\beta + r_{\alpha\beta} n_\alpha + r_{\alpha\beta} \delta_{\alpha\beta}) - ((1 - a_2)\alpha_1^* K_1(z_i) - \\
(1 - a_2)\alpha_2^* K_1(z_i) + (a_2 - a_1)\alpha_3^* K_1(z_i))r_{\alpha\beta} n_\beta + (a_2 - a_1)\alpha_3^* K_1(z_i) (r_{\alpha\beta} n_\alpha + r_{\alpha\beta} \delta_{\alpha\beta}) \right] \\
+ \frac{2\nu}{1 - \nu} ((1 - a_2)\alpha_1^* K_1(z_i) + (1 - a_2)\alpha_2^* K_1(z_i)) r_{\alpha\beta} n_\beta \right]

\hat{T}_{\alpha\alpha} = \frac{\kappa^2 \mu h}{2\pi(a_2^* - a_1^*)} \left[ (a_2 \alpha_1 K_1(z_i) - a_2 \alpha_2 K_1(z_i)) - (a_2 - a_1)\alpha_1 K_1(z_i)) \times \right. \\
\left. \frac{1}{r} (2r_{\alpha\beta} n_\alpha - n_\alpha) + (a_2 \alpha_1^* K_0(z_i) - a_1 \alpha_2^* K_0(z_i)) - \\
(a_2 - a_1)\alpha_3^* K_0(z_i) + (a_2 - a_1)\alpha_2^* K_0(z_i) n_\alpha \right]

(11.21)

\hat{T}_{33} = \frac{1}{2\pi(a_2 - a_1)} \left[ (1 - a_1)\alpha_1 K_1(z_i) - (1 - a_2)\alpha_2 K_1(z_i) - \\
(1 - a_1)(a_1 \alpha_1 K_1(z_i) - \alpha_2 K_1(z_i)) r_{\alpha\beta} \right]

(11.22)

in which $n_\beta$ denotes the component of the outward normal vector to the boundary of plate and $r_{\alpha\beta} = r_{\alpha\beta} n_\alpha$.

11.3. The MFS in the Laplace Transform Domain

In the MFS we place the source points outside the physical domain in order to avoid the singularities of the fundamental solution. By using the principle of superposition for linear elasticity, Wen [5] applied the MFS to a Kirchhoff plate resting on the Winkle foundation. As the superposition
principle is still valid in the transformed domain, the approximate solution for the displacement (deflection and rotations) and resultants of moment and shear force in the Laplace domain on the boundary collocation point can be expressed as

\[
\begin{align*}
\tilde{w}_i(P) &= \tilde{w}_i^*(P) + \sum_{n=1}^{N} \tilde{U}_{ik}(P,Q)c_k^P(Q) \\
\tilde{P}_i(P) &= \tilde{P}_i^*(P) + \sum_{n=1}^{N} \tilde{T}_{ik}(P,Q)c_k^P(Q) & P \in \Gamma, & i = 1,2,3
\end{align*}
\]

(11.23)

where \(\tilde{P}_a = \tilde{M}_{ap}\eta_{\beta}\), \(\tilde{P}_3 = \tilde{Q}_{p}\eta_{\beta}\), \(\tilde{w}_i^*(P)\) and \(\tilde{P}_i^*(P)\) present particular solutions of the governing equations, and \(P\) and \(Q\) denote collocation and source points respectively, see Figure 11.2, and \(c_k^P\) denotes the density of concentrated forces at the source point \(n\). If a uniform load or a linearly distributed load is considered in the domain, i.e.

\[
q(P,t) = q_0(t) + q_1(t)x_1 + q_2(t)x_2
\]

(11.24)

where \(q_0(t), q_1(t)\) and \(q_2(t)\) are time-dependent functions only, in the Laplace transform domain, we have.

![Fig.11.2.](image-url)
Therefore, particular solutions can be easily obtained from the governing equations in (11.4) as

\[
\tilde{w}_1^* = -\frac{\kappa^2 \mu h \bar{q}_1}{\kappa^2 \mu h + \rho h^3 p^2}, \quad \tilde{w}_2^* = -\frac{\kappa^2 \mu h \bar{q}_2}{\kappa^2 \mu h + \rho h^3 p^2}, \quad \tilde{w}_3^* = \frac{1}{\rho p h^2} (\bar{q}_1(t) + \bar{q}_2(t)x_1 + \bar{q}_2(t)x_2)
\]

(11.26)

and particular solutions for the resultants of the moments and shear force are obtained as

\[
\tilde{M}_{11}^* = \tilde{M}_{12}^* = \tilde{M}_{22}^* = 0,
\]

\[
\tilde{Q}_1^* = \kappa^2 \mu \tilde{q}_1 \left(1 - \frac{\kappa^2 \mu h}{\kappa^2 \mu h + \rho h^3 p^2}\right),
\]

\[
\tilde{Q}_2^* = \kappa^2 \mu \tilde{q}_2 \left(1 - \frac{\kappa^2 \mu h}{\kappa^2 \mu h + \rho h^3 p^2}\right).
\]

(11.27)

From the boundary conditions of the plate, we have the following linear algebraic equations:

\[
\tilde{w}_i(P) = \sum_{a=1}^{N} \bar{U}_{ia}(P,Q)c_{ia}^*(Q) = \bar{w}_i(P) - \tilde{w}_i^*(P) \quad \text{for } P \in \Gamma_u
\]

(11.28)

\[
\tilde{p}_i(P) = \sum_{a=1}^{N} \bar{T}_{ia}(P,Q)c_{ia}^*(Q) = \bar{p}_i(P) - \tilde{p}_i^*(P) \quad \text{for } P \in \Gamma_{\sigma}
\]

where \(\bar{w}_i\) and \(\bar{p}_i\) are specified displacement and traction boundary values in the Laplace domain respectively, \(\Gamma_u\) and \(\Gamma_{\sigma}\) present displacement and traction boundaries, which \(\Gamma = \Gamma_u \cup \Gamma_{\sigma}\). Thus the MFS can be applied by solving the set of linear equations in (11.28) for both the static problem and the dynamic problem in the Laplace domain. The total number of linear equations is \(3(N_{\Gamma_u} + N_{\Gamma_{\sigma}})\) for the Reissner/Mindlin plate bending and therefore \(3(N_{\Gamma_u} + N_{\Gamma_{\sigma}})\) densities of concentrated forces at the source points can be obtained by solving equations (11.28). Although the MFS
system (11.28) is ill-conditioned, convergent solutions can be obtained by selecting the distance between collocation and source points properly. The accuracy and stability have been examined for both static and dynamic problems in the numerical examples.

The transformed values are calculated for these transform parameters and the physical values in the time domain must be obtained by an inverse transformation technique. Here, the method given by Durbin [12] is used. A demonstration of the Durbin’s inverse method was made by Wen et al [13] for elastodynamic crack problems with an investigation of elasticity wave propagation. The calculation formula used is

\[
\frac{2e^{\eta p}}{T} \left[ -\frac{1}{2} \tilde{f}(\eta) + \sum_{k=0}^{L} \Re \{ \tilde{f}(\eta + 2k\pi i / T) e^{2k\pi i / T} \} \right]
\]  

(11.29)

where \(\tilde{f}(s_k)\) stands for the transformed variables in the Laplace space for the parameters \(s_k = \eta + 2k\pi i / T\) \((i = \sqrt{-1})\) and \(L\) is the number of sample. The selection of the two free parameters \(\eta\) and \(\tau\) affects the accuracy only slightly. In addition, in the following numerical calculations, we only consider Mindlin’s theory plate.

### 11.4. Numerical examples

#### 11.4.1. A Cantilever Plate Subjected to Static Loads Along the Edge

In this example, a square cantilever plate of length \(2a\) and thickness \(h\) subjected to a uniformly distributed static moment \(M_0\) and a static shear force \(Q_0\), as shown in Figure 11.3, is analysed. For the static problem, the fundamental solutions derived by Vander Weeën [4] can be obtained by letting \(p=0\) directly in the fundamental solutions in the Laplace domain. The aim of this example is to demonstrate the accuracy and stability of the MFS for a shear deformable plate under static loads. Collocation points and source points are distributed uniformly along each edge of plate as shown in Figure 11.2, where \(\Delta\) presents the gap between collocation and source points and \(d\) is the distance between collocation points. In order to compare our solution
with the exact solution for thin plate theory, we take Poisson’s ratio $\nu = 0$, the thickness of plate $h/a = 0.1$ and the number of collocation/source points $N = 124$. The normalised deflections $w_3^0 / w_3^0$ in the middle of the right edge (A) are shown in Figure 11.4. The exact solutions for the deflection $w_3^0 = M_d l^2 / 2EI$ for the bending load and $w_3^0 = Q_0 l^3 / 3EI$ for the shear load, where $l = 2a$ and $I = h^3 / 12$. Apparently, the relative error is less than 5% for the moment load and 7% for the shear load when the gap $\Delta/d = 1$ and the accuracy improves significantly and the method stable if $\Delta/d > 2$.

![Fig.11.3. A square cantilever plate subjected to static uniform distributed loads along the edge, and coordinate system.](image)

Comparison has been made between the MFS and the finite element method (ABAQUS). In this case, Poisson’s ratio $\nu = 0.33$, the thickness of the plate $h/a = 0.1$ and the gap $\Delta/d = 1$. Only a uniformly distributed shear force $Q_0$ loaded at the edge of plate is considered. The normalised deflections $w_3^0 (A) / w_3^0$ along the $x_1$-axis are shown in Figure 11.5 for different numbers of collocation points on the boundary, where again $w_3^0 = Q_0 l^3 / 3EI$, $l = 2a$ and $I = h^3 / 12$. It is clear that the relative error reduces significantly when the number of collocation points increases. In this case, the maximum
deflection is less than that given by beam theory \((\nu = 0)\) as the effect of Poisson’s ratio.

![Graph](image1.png)

**Fig. 11.4.** Stability and convergence investigations for the distance between collocation and source points.

![Graph](image2.png)

**Fig. 11.5.** Deflection of plate on symmetry line when Poisson ratio \(\nu = 0.33\) for different numbers of collocation points and results obtained by FEM.
11.4.2. Cantilever Plate Subjected to Shear Force Suddenly Applied at the End

The same square cantilever plate as in example 11.4.1 subjected to a uniformly distributed shear force \( Q_0 H(t) \) is analysed in this example, where \( H(t) \) is the Heaviside step function of time. Also, we take Poisson's ratio \( \nu = 0.33 \), the thickness \( h = 0.1a \) and the gap \( \Delta/d = 1 \). In this example, we select the two free parameters to be \( \eta = 5/t_0 \) and \( T/t_0 = 20 \) respectively, where \( t_0 = a/c_2 \) is the unit of time and the number of samples in the Laplace domain is \( L = 100 \). The dynamic normalised moments \( M_{ij}/Q_0a \) and normalised shear forces \( Q_i(t)/Q_0 \) at the internal points B and C vs. normalised time \( c_2t/a \) are presented in Figures 11.6(a-b) and 11.7(a-b). The results obtained by the FEM (DYNA3D) with 101×101 nodes and 100×100 shell elements are also plotted in these figures for comparison purposes. Apparently, before the thickness shear flexural waves arrive at these points, the moments and shear forces remain almost equal to zero, as expected. To obtain solutions for a large time period, i.e. \( c_2t/a > 16 \), we need to select large number for the free parameter \( T \) in the Durbin formulation. The accuracy of the MFS is shown to be satisfactory for dynamic Reissner/Mindlin plate analysis.

11.4.3. Simply Supported Plate Subjected to Uniform Load Suddenly Applied in Domain

A simply supported square plate subjected to a uniformly distributed pressure \( q_0 H(t) \) in the domain \( \Omega \) is considered as shown in Figure 11.8. Apparently the particular solution for this constant distribution of loads is \( \hat{w}_1^* = 0, \hat{w}_2^* = 0, \hat{w}_3^* = q_0/\rho h p^3 \) and the resultants of moment and shear forces
Fig. 11.6. Component of moment $M_{11}(t)$ at different locations: (a) moment at B; (b) moment at C.
Fig. 11.7. Component of shear force $Q_1(t)$ at different locations: (a) shear force at B; (b) shear force at C.

are zero. Poisson's ratio is taken to be 0.33, the distance $\Delta/d = 3$ and the two free parameters are taken to be $\eta = 5/t_0$ and $T/t_0 = 100$. The number of samples in the Laplace domain is $L=100$ the same as in example 11.4.2. The
Fig. 11.8. Simply supported square subjected to uniform pressure \( q_0 \) load, and coordinate system.

Fig. 11.9. Normalized deflection \( w_3(B)/[q_0a^4/D] \) at the centre of plate under uniform pressure for different thickness of plate \( h \).

Normalised deflections \( w_3(t)/[q_0a^4/D] \) in the centre B of the plate vs. the normalised time \( c_2t/a \) are plotted in Figure 11.9 for two selections of thickness \( h = 0.1a \) and \( h = 0.4a \) respectively. In this figure, the dashed lines present the position of the static uniform pressure \( q_0 \) obtained by Panc [14], i.e. \( w_3^0/[q_0a^4/D] = 0.0702 \) and 0.0915 for \( h = 0.1a \) and \( h = 0.4a \).
respectively. We can see that the plate starts to vibrate about these static positions of static load with different frequencies for each thickness and the maximum altitudes of vibration are almost twice those of the static value.

11.5. Conclusions

In this paper, the method of fundamental solutions has been developed for shear deformable (Reissner/Mindlin theories) plate problems in both the static and the dynamic case. Fundamental solutions in the Laplace transform domain were derived using displacement potential functions. The accuracy and stability of the MFS was examined for the static case and the optimal distance between collocation and source points was studied. Excellent agreements with the finite element method for both static and dynamic problems were achieved. As a mesh free technique, the MFS has the advantages of mesh free methods and has demonstrated three major features in our computations: simplicity, accuracy and efficiency. We can conclude with the following observations: (1) The MFS is available for shear deformable plate for static and dynamic problems; (2) Compared with the FEM and the BEM, the MFS is more flexible and simpler to program; (3) For a proper selection of the distance between the collocation and source points, the accuracy of the FMS is the same as the accuracy of the boundary element method when using the same number of boundary nodes (collocation points); (4) A possible disadvantages of the method is the fact that it is applicable only if a fundamental solution of the governing equation is available. Finally, the optimal source point distribution is still a problem to be investigated.

References


CHAPTER 12

Calculation on Stress Concentration of Dull Elliptical Holes by the MFS Using Equally Dispersed Point Loads

Wataru Fujisaki 1

Abstract. In this paper, the accuracy by the method of fundamental solutions with two improvements of discretization was investigated. Maximum stresses of a circle and an elliptical hole with internal pressure were calculated. In the former part of this paper, the theory of the method of fundamental solution is explained for the elastic problems. To verify the usefulness of the MFS program, the value of point load and its shape are investigated. According to the result, in a case of a circular hole, the exact values are obtained with a wide range of the calculation conditions. The more elliptical hole becomes sharp, the more calculation error increases in case that the source points are located by equal center angle (this is called no bias). On the other hand, by using biased position, better accuracy can be obtained. In the latter part of this paper, in order to obtain more good accuracy, the program was improved using the equally dispersed point loads from a point load. For an elliptical hole, it is found that the stable range of accuracy becomes wider by using the improvements. In addition, the check points to obtain better accuracy are investigated using the equally dispersed point loads with biased position.

12.1. Introduction

Recently, valuable application to material strength using the method of fundamental solutions has been reported [1,2]. Accuracy of stress in some cases is excellent compared with the finite element method. However, in

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other cases, an error of stress increases depending on point dispositions and some other parameters [3]. In this paper, the effects of the point dispositions and the dispersion loads to the accuracy of maximum stress on elliptical-hole problems were investigated using the method of fundamental solutions in infinite plate.

12.2. The Method of Fundamental Solutions

Figure 12.1 shows one example of a circular hole model in an infinite plane. We consider a bounded domain $\Omega \subset \mathbb{R}^2$ with boundary $\Gamma$. A surface of source is defined exterior to $\Omega$. Boundary points are defined as $\Gamma(i) = (x_i', x_i^2)$, and source points as $S(j) = (x_j', x_j^2)$ and domain points as $\Omega(k) = (x_k', x_k^2)$ where $\Omega = \Gamma \cup \Omega$.

We consider a linear elastic problem with body forces, governed by the following Navier equation.

$$
G\mu_{i,nn} + \frac{G}{(1-2\nu)} u_{m,ji} + b_j = 0
$$

(12.1)

where $G=E/(1+\nu)$ is the shear modulus, $E$ is the Young’s modulus, $\nu$ is the Poisson’s ratio, $u_{i,n}(x)$ are displacements in the directions $x_n$ and $b_j$ are body forces where $m, l = 1, 2$. Equation (12.1) is subjected to boundary conditions in displacements, $u_{i,n}(x') = \overline{u}_n$ on $\Gamma_{nm}$ and in surface tractions $p_{i,m}(x') = \overline{p}_m$ on $\Gamma_{mn}$.

The surface tractions are given by $p_{i,m}(x) = \sigma_{mn}(x)n_i(x)$, where $n_i(x)$ are the direction cosines of the outward normal at the boundary points $\Gamma$ and $\sigma_{mn}(x)$ is the stress tensor[2]. The overline signifies known values. The fundamental solutions for displacements and surface tractions respectively are

$$
u_{i,m}(x) = \frac{1}{8\pi(1-\nu)G} \left\{ (3-4\nu)\ln(r'(x)) \delta_{mn} - r'(x)_{,m} r'(x)_{,n} \right\}$$

(12.2)

$$
p_{i,m}(x) = \frac{-1}{4\pi(1-\nu)r'(x)} \left\{ \left[ (1-2\nu)\delta_{mn} + 2r'(x)_{,m} r'(x)_{,n} \right] \frac{\partial r'(x)}{\partial n} - (1-2\nu) r'(x)_{,m} r'(x)_{,n} - r'(x)_{,m} n_i(x) \right\}$$

(12.3)

where $r'(x) = \left[ (x'_i - x_i)^2 + (x'_j - x_j)^2 \right]^{1/2}$, $r_x' = \partial r'/\partial x_n$, and $\delta_{mn}$ is the Kronecker delta. Equations (12.2) and (12.3) are written for the plane strain
Calculation on stress concentration of dull elliptical holes
by the MFS using equally dispersed point loads

problem. For plane stress problem, $\nu$ is replaced with
$\nu' = \nu/(1 + \nu)$.

$x^j = (x^j_1, x^j_2) \in S, j = 1, N.$

We consider a point $x^j \in \Gamma_m$, on which a
boundary condition $u_m = \overline{u}_m$ is
prescribed. In the homogeneous problem, we can obtain the known value of
the variable by the sum of a series of fundamental solutions $u^*_m$ and $u'^*_m$ as
given in Eq. (12.2) with unknown source loads $f(j), j = 1, 2N$.

$$
\sum_{j=1}^{N} \left[ u^*_m (x^j) \cdot f(j) + u'^*_m (x^j) \cdot f(j + N) \right] = \overline{u}_m
$$

(12.4)

In Eq. (12.4), $N$ is the number of points on $S$. There is one source point
$x^j \in S$ for each collocation point $x^j \in \Gamma$. Similarly, for another point $x^j \in \Gamma_m$,
having a boundary condition $p_m = \overline{p}_m$, following equation (12.5) is expressed.

$$
\sum_{j=1}^{N} \left[ p^*_m (x^j) \cdot f(j) + p'^*_m (x^j) \cdot f(j + N) \right] = \overline{p}_m
$$

(12.5)

where $p^*_m$ and $p'^*_m$ are fundamental solutions for surface tractions.

Once values of $f$ are known, we can obtain unknown values of $p_m$ on the
boundary $\Gamma_m$ using

$$
p_m (x^j) = \sum_{j=1}^{N} \left[ p^*_m (x^j) \cdot f(j) + p'^*_m (x^j) \cdot f(j + N) \right]
$$

(12.6)

Redefining $x^j$ as a point in $\Gamma \cup \Omega$, we can find values of $u_m$ for points both on
boundary $\Gamma_m$ and on the domain $\Omega$ by the following equation.

$$
u_m (x^j) = \sum_{j=1}^{N} \left[ u^*_m (x^j) \cdot f(j) + u'^*_m (x^j) \cdot f(j + N) \right]
$$

(12.7)

Stresses can be calculated both for boundary and internal points using

$$
\begin{align*}
\sigma_{11} (x^j) &= \sum_{j=1}^{N} \left[ \sigma^*_{11} (x^j) \cdot f(j) + \sigma'^*_{11} (x^j) \cdot f(j + N) \right] \\
\sigma_{12} (x^j) &= \sum_{j=1}^{N} \left[ \sigma^*_{12} (x^j) \cdot f(j) + \sigma'^*_{12} (x^j) \cdot f(j + N) \right]
\end{align*}
$$

(12.8)
Calculation on stress concentration of dull elliptical holes by the MFS using equally dispersed point loads

\[ \sigma_{22}(x') = \sum_{j=1}^{N} \left[ \sigma_{221}^{i*}(x') \cdot f(j) + \sigma_{222}^{i*}(x') \cdot f(j + N) \right] \]

where

\[ \sigma_{22}^{i*}(x) = \frac{-1}{4\pi(1-\nu)^2}\left\{ \left(1-2\nu\right)r'(x_m)\delta_{m} + r'(x_n)\delta_{mn} - r'(x_n)\delta_{mn} + 2r'(x_n)r'(x_m) \right\} \]

Fig. 12.1. Domain, boundary and source

Fig. 12.2 Model of elliptical hole

12.3. Infinite Plate Including Elliptical Hole with Internal Pressure

The accuracy of the maximum value (\( \sigma_{\text{max}} \)) of stress \( \sigma_{22} \) at a point \( i=1 \) in Fig.12.2 is investigated with points disposition using a bias ratio \( B \). We consider the coordinates of the elliptical hole \( (a, b, \theta) \) as shown in Fig.12.2, if \( \theta_i \) is equally divided (namely \( B = 1.0 \)), the angle \( \theta_i \) can be expressed as \( \theta_i = (i-1)(n-1)\times\pi/2 \). On the other hand, if \( B \neq 1.0 \), then the angle \( \theta_i \) is expressed as

\[ \theta_i = \frac{1-B}{1-B^*} \cdot \frac{\pi}{2} \]  \( (12.9) \)
Calculation on stress concentration of dull elliptical holes by the MFS using equally dispersed point loads

\[ \theta_i = \sum_{j=2}^{i} \theta_2 \cdot B^{j-2} \quad (i = 2, n) \quad (12.10) \]

\[
\begin{array}{c|c|c|c|c|c|c|c}
 B & 1.0 & 1.02 \\
\hline
 \theta_i & & & & & & & \\
\hline
 0 & & & & & & & \\
\end{array}
\]

Fig.12.3. Examples of \( \theta_i \) depending on bias ratio \( B \)

Model points and source points are divided by the bias ratio. Examples \((n=17; \ B = 1.0 \text{ and } 1.02)\) are shown in Fig.12.3. When \( B > 1 \), the points are located to the outer side of the hole \((x_1 = \pm a)\), and \( B < 1 \), the points are moved to the center of the hole \((x_1 = 0)\). Ratios of the source distance to the model distance \( r_s/r_m \) are changed from 0.1 to 0.8.

12.4. Effect of Bias Disposition

The features of the calculated source loads with the elliptical ratio \( b/a = 1.0 \sim 0.7 \) are shown in Fig.12.4. \(|\text{Error}|/(\%)\) is the relative error between calculated value \( \sigma_{\text{max,c}} \) and the rigid value \( \sigma_{\text{max,r}} \). With decreasing \( b/a \), the maximum value of the source load \( |f_{\text{max}}| \) reaches an extremely high ordered value (from 5.16 to 6.33E+10). Also, it is observed that the patterns of the source loads are almost consisted of similar forms of the several local sources.

Table 12.1 shows the calculation results of maximum stress \( \sigma_{\text{max,c}} \) and the relative error (\%) depending on \( B \) and \( r_s/r_m \). If the value is underlined, the error is under 1%, and if the value is shown bold, the error is under 0.1%. 

\begin{align*}
|\text{Error}|/(%) & \leq 1
go to top
|\text{Error}|/(%) & \leq 0.1
\end{align*}
Calculation on stress concentration of dull elliptical holes by the MFS using equally dispersed point loads

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Table 12.1. Accuracy of $\sigma_{\max,c}$ in case of point load with bias

( Upper value: $\sigma_{\max,c}$, Lower value: Error %, *: see Fig. 12.4 )

(a) $b/a = 1.0$

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Calculation on stress concentration of dull elliptical holes by the MFS using equally dispersed point loads

\[ \sigma_{\text{max},r} = 1.8571 \]

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<td>1.8404</td>
<td>1.8916</td>
<td>1.8560</td>
<td>1.8418</td>
</tr>
<tr>
<td>1.00</td>
<td>1.8697</td>
<td>1.8517</td>
<td>1.8536</td>
<td>1.8166</td>
</tr>
</tbody>
</table>

Fig.12.4. Features of calculated source loads \((N=64, \, r_s/r_m=0.5)\)

In case of \(b/a=1.0\) (a circle hole), high accuracy is obtained in very wide range of \(B\) and \(r_s/r_m\). With decreasing \(b/a\) (an elliptical hole), the range of good accuracy becomes small due to the rapid increase of \(f_{\text{max}}\). In case of
Calculation on stress concentration of dull elliptical holes
by the MFS using equally dispersed point loads

For a deep elliptical hole \( (b/a<0.7) \), to obtain an accurate \( \sigma_{\text{max}} \) in a wide range, more points \( (N>64) \) are needed.

From Table12.1, we can see good accuracy is obtained with \( r_s/r_m=0.5 \) in any case. In general, the effects of \( r_s/r_m \) are supposed to be following. When the distance between source and model is small \( (r_s/r_m=1) \), the maximum value of source load is small and has a strong effect to the neighbor model points. On the other hand, when the distance is not that far \( (r_s/r_m=0.5) \), the value of source load gradually becomes huge in order to keep the effect to the model points. In the end, if it is extremely far \( (r_s/r_m=0.1) \), the source loads locate densely in a small area. In such situation, these source loads could not make the satisfaction of various boundary conditions.

12.5. Effect of Equally Dispersed Point Loads (EDPL)

To obtain good accuracy, one new improvement is attempted. In Fig.12.5, the improvement called “Equally Dispersed Point Load” is shown. If the position of point load is near the model boundary, the influence to the boundary is not the same as the real model. Therefore, the boundary condition at the middle point between the model points cannot be satisfied enough. To avoid this situation, the conventional point load will be divided into 2 to 10 equally dispersed point loads. When division number \( N_{ed} \) increases, it works as a line load. The advantage of this improvement is that the freedom number does not increase. To apply this improvement, \( p_{1m}(x), p_{2m}(x) \) in equation (12.4) are modified.

The satisfaction status of the boundary condition is shown in Fig.12.6. In case of purely point load (see Fig.12.6(a)), when the parameter \( r_s/r_m \) is under 0.9, the boundary condition \( p=1 \) is well satisfied. But if \( r_s/r_m=0.95 \), an error of \( p \) at the middle points increases sharply. On the other hand, in case of EDPL (Fig.12.6(b)), the accuracy of \( p \) in case of \( r_s/r_m=0.95 \) is remarkable, and the maximum error is below 0.005%.

Effects of division number \( N_{ed} \) to the accuracy are discussed with different \( r_s/r_m \). Figure 12.7(a) shows the results in case of \( r_s/r_m=0.7 \) which is some far distance from the boundary. In this case, after \( N_{ed}=4 \), the error decreases rapidly but still remains in some degree because of the sharp shape of the elliptical hole \( (b/a=0.6) \). Far from the model boundary, the error mainly
Calculation on stress concentration of dull elliptical holes by the MFS using equally dispersed point loads

arises from a matrix illness due to an extremely high value of the source load $f_{\text{max}}$. In this case, there is no apparent difference between a purely point load and EDPL.

<table>
<thead>
<tr>
<th>Point Load</th>
<th>EDPL</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ned=1</td>
<td></td>
</tr>
<tr>
<td>Ned=2</td>
<td></td>
</tr>
<tr>
<td>Ned=3</td>
<td></td>
</tr>
<tr>
<td>Ned=10</td>
<td></td>
</tr>
</tbody>
</table>

Fig.12.5. Point load and equally dispersed point loads

On the other side, when the source is near the source, EDPL is effective enough. Fig.12.7(b) shows the example of the case with $r_s/r_m=0.85$. In this case, after $N_{\text{ed}}=2$, $\sigma_{\text{max}}$ is obtained with good accuracy. From these figures, the source points are better located around $r_s/r_m=0.85$.

The relation between $b/a$ and error of $\sigma_{\text{max}}$ without bias in case of $r_s/r_m=0.9$ is shown in Fig.12.8. In case of point load, accuracy of $\sigma_{\text{max}}$ is not stable. On the other hand, using EDPL, very good accuracy (maximum error < 0.04 %) has been achieved.

When the source loads locate near the boundary ($r_s/r_m > 0.7$), the stable ranges of $\sigma_{\text{max}}$ between point load and EDPL are shown in Fig.12.9. In case of point load (Fig. 12.9(a)), with increasing $b/a$, the error is becoming worse. If EDPL is applied (Fig. 12.9(b)), better accuracy is obtained. The good condition to obtain reasonable accuracy locates $0.8 < r_s/r_m < 0.95$ for a dull elliptical hole ($b/a > 0.6$). In addition, when we use a single bias ratio $B_s$ for source points, the stable range becomes wider, shown in Fig.12.10. In this case, the model points are located with no bias and the proper $B_s$ is 1.02.
In order to obtain the wider stable range for a sharp elliptical hole \((b/a < 0.6)\), it seems further improvement is needed to determine the best position of each source point and each model point.

### 12.6. Conclusions

The characteristics of the source loads and two improvements to obtain good accuracy on dull elliptical-hole problems were examined. The following main results are obtained.

1. In case of a circular-hole problem, the pattern of the source loads with a good accuracy shows a perfect radial shape. With decreasing \(b/a\), similar forms of the several source loads are repeated in each local part of the elliptical arc. In addition, the maximum value of the source load \(|f_{\text{max}}|\) increases rapidly.

2. Good accuracy of \(\sigma_{\text{max}}\) can be obtained using a point load and the bias ratio \(B\) in some range and the stable range of \(r_s/r_m\) exists around 0.5, even if \(b/a\) is decreasing.

3. By using the equally dispersed point load (EDPL) and the single bias ratio \(B_s\), a stable with wider range of high accuracy (<0.1%) can be obtained in case of a dull elliptical shape \((b/a > 0.6)\).

4. The proper check points to obtain good accuracy are following;
   1) boundary condition on middle points between model points
   2) the shape of source load
   3) the maximum value of the source loads
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Fig. 12.6. Accuracy of boundary condition at middle point
Calculation on stress concentration of dull elliptical holes by the MFS using equally dispersed point loads

Fig. 12.7. Effect of division number $N_{ed}$ of EDPL ($b/a=0.6$, $B=1.0$)

Fig. 12.8. Effect of equally dispersed point loads
Calculation on stress concentration of dull elliptical holes by the MFS using equally dispersed point loads

<table>
<thead>
<tr>
<th>Point Load</th>
<th>EDPL (Ned=10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\frac{K_a}{\Gamma_m}$</td>
<td>$\frac{K_a}{\Gamma_m}$</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9</td>
</tr>
<tr>
<td>0.95</td>
<td>0.95</td>
</tr>
</tbody>
</table>

Fig. 12.9. Stable range of accuracy by equally distributed point loads. $N=64$ points, $B=1.0$ (No Bias)

<table>
<thead>
<tr>
<th>$b/a$</th>
<th>$\frac{K_a}{\Gamma_m}$</th>
<th>$\frac{K_a}{\Gamma_m}$</th>
<th>Error of $\sigma_{max}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.6</td>
<td>0.6</td>
<td>&lt; 0.1%</td>
</tr>
<tr>
<td>0.7</td>
<td>0.7</td>
<td>0.7</td>
<td>&lt; 1.0%</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8</td>
<td>0.8</td>
<td>&lt; 10%</td>
</tr>
<tr>
<td>0.9</td>
<td>0.9</td>
<td>0.9</td>
<td>&lt; 100%</td>
</tr>
</tbody>
</table>

(a) $B_s=1.01$  (b) $B_s=1.02$  

Fig. 12.10. Stable range of accuracy by equally distributed point loads with single bias $B_s$
Calculation on stress concentration of dull elliptical holes by the MFS using equally dispersed point loads

References


CHAPTER 13

Dipole plate bending formulation for the method of fundamental solutions

S.W. Mohareb¹, Y.F. Rashed²,³, *, A.Y. Akl²

Abstract. In this work a new method of fundamental solutions (MFS) is developed using dipoles for Reissner’s plate theory. At first the traditional technique of using monopole sources, is reviewed, then, the dipole formulation is derived. New kernels for the dipole formulation are derived and given in explicit form. Three examples are tested including circular, square and thin rectangular domains to demonstrate the validity and the accuracy of the developed formulation.

13.1. Introduction

The Method of Fundamental Solutions (MFS) is a mesh-less technique used to solve Boundary Value Problems (BVPs). The idea is to represent solution via superposition of fictitious source intensities multiplied by appropriate fundamental solutions. The method is developed first in the work of Kupradze & Aleksițe [1], and by Antes [2] for the Reissner’s plate bending theory [3]. The dipole formulation is previously developed for 2D potential and elasticity problems in the work of Fam & Rashed [6, 7]. This work is concerned with the development of dipole a MFS formulation for Reissner’s plate bending theory. A comparison of the proposed

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formulation against the traditional one is carried out in the numerical examples.

13.2. The Traditional Monopole Formulation

Consider an arbitrary plate of domain $\Omega$ with boundary $\Gamma$, shown in Fig. (13.1). In the traditional MFS, the governing partial differential equation for Reissner’s plate theory [3]-[5] is transformed into a superposition operation using appropriate fundamental solutions multiplied by fictitious source intensities as follows [2]:

$$u_j(x_i) = \sum_{k=1}^{n} U^*_j(\xi_k, x_i) \phi_i(\xi_k), \quad (13.1)$$

$$t_j(x_i) = \sum_{k=1}^{n} T^*_j(\xi_k, x_i) \phi_i(\xi_k). \quad (13.2)$$

where $u_j(x_i)$, $t_j(x_i)$ are the displacement and traction values in the $j$ direction at a field point $x_i$, respectively; $\phi_i(\xi_k)$ is the intensity of a source in the $i$ direction at a source point $\xi_k$; $U^*_j(\xi_k, x_i)$ and $T^*_j(\xi_k, x_i)$ represent the displacement and traction two-point fundamental solution kernels given by Vander Weeën [8], respectively. Applying a collocation operation over the predefined values at the boundary nodes using Eqs. (13.1), (13.2) the unknown fictitious source intensities can be calculated. Then using Eqs. (13.1), (13.2) the displacement and traction can be calculated at any internal or boundary point. The stress-resultants can be calculated using the following superposition operations:
Dipole plate bending formulation for the method of fundamental solutions

Fig. 13.1: A typical MFS definition.

\begin{align*}
M_{\alpha\beta}(x_i) &= \sum_{k=1}^{n} M_{\alpha\beta}^*(\xi_k, x_i) \varphi_i(\xi_k), \\
Q_{1\beta}(x_i) &= \sum_{k=1}^{n} Q_{1\beta}^*(\xi_k, x_i) \varphi_i(\xi_k),
\end{align*}

where \(M_{\alpha\beta}(x_i)\) and \(Q_{1\beta}(x_i)\) are the values of the bending and shear stress-resultants at the field point \(x_i\), respectively; the generalized stress-resultant fundamental solutions \(M_{\alpha\beta}^*(\xi_k, x_i)\) and \(Q_{1\beta}^*(\xi_k, x_i)\) are given in [8].

13.3. The Proposed Dipole Formulation

The dipole formulation is derived by assuming that every monopole (source) used in Sec. (13.2) is replaced by two monopoles at \((\xi_k, \xi_k')\), as shown in Fig. (13.2), with two different intensities offset by distance \(h\) in the normal direction of the fictitious surface at the point \(\xi_k\). Equation (13.1) can be then modified as follows:

\begin{align*}
u_j(x_i) &= \sum_{k=1}^{n} U_{\alpha}^*(\xi_k, x_i) \varphi_i(\xi_k) + \sum_{k=1}^{n} U_{\beta}^*(\xi_k, x_i) \varphi_i(\xi_k).
\end{align*}
we know that:
\[ U_y^r(\xi_k, x_i) = U_y^r(r(\xi_k, x_i)), \]  
(13.6)
\[ U_y^r(\xi_k, x_i) = U_y^r(r(\xi_k, x_i)). \]  
(13.7)

Using the geometrical relationship shown in Fig. (13.2) between the two poles, it is clear that:
\[ r^* = r + h. \]  
(13.8)

So Eq. (13.7) can be modified as follows:
\[ U_y^r(\xi_k, x_i) = U_y^r(r(\xi_k, x_i) + h). \]  
(13.9)

Now Eq. (13.5) can be rewritten as follows:
\[ u_i(x_j) = \sum_{k=1}^{n} U_y^r(r(\xi_k, x_i))\phi_i(\xi_k) + \sum_{k=1}^{n} U_y^r(r(\xi_k, x_i) + h)\phi_i(\xi_k). \]  
(13.10)

Taking \( \phi_i(\xi_k) = -\phi_i(\xi_k) \), Eq. (13.10) becomes:
Dipole plate bending formulation for the method of fundamental solutions

\[ u_j(x_i) = \sum_{k=1}^{n} \left( U_{ij}^* \left( r(\xi_k, x_i) \right) - U_{ij}^* \left( r(\xi_k, x_i) + h \right) \right) \varphi_i(\xi_k). \]  

(13.11)

Multiplying Eq.(13.11) by \( \frac{h}{h} \) to give:

\[ u_j(x_i) = \sum_{k=1}^{n} \left( \frac{U_{ij}^* \left( r(\xi_k, x_i) \right) - U_{ij}^* \left( r(\xi_k, x_i) + h \right)}{h} \right) h \varphi_i(\xi_k). \]  

(13.12)

The term \( \frac{U_{ij}^* \left( r(\xi_k, x_i) \right) - U_{ij}^* \left( r(\xi_k, x_i) + h \right)}{h} \) is the difference quotient of the function \( U_{ij}^* \left( r(\xi_k, x_i) \right) \) with respect to the direction of \( h \). Taking the limit as \( h \) tends to zero, and assuming:

\[ \psi_i(\xi_k) = h \varphi_i(\xi_k), \]  

(13.13)

in which \( \psi_i(\xi_k) \) is the intensity of the dipole, then Eq (13.12) can be written as follows:

\[ u_j(x_i) = \sum_{k=1}^{n} \lim_{h \to 0} \left( \frac{U_{ij}^* \left( r(\xi_k, x_i) \right) - U_{ij}^* \left( r(\xi_k, x_i) + h \right)}{h} \right) \psi_i(\xi_k). \]  

(13.14)

Equation (13.14) represents the differentiation of the functional \( U_{ij}^* \left( r(\xi_k, x_i) \right) \) with respect to the normal direction at the point \( \xi_k \) as follows:

\[ u_j(x_i) = \sum_{k=1}^{n} U_{ij}^* \left( r(\xi_k, x_i) \right) \rho_{ij}(\xi_k) \psi_i(\xi_k). \]  

(13.15)

or

\[ u_j(x_i) = \sum_{k=1}^{n} U_{ij}^* \left( \xi_k, x_i \right) \rho(\xi_k) \psi_i(\xi_k), \]  

(13.16)
Dipole plate bending formulation for the method of fundamental solutions

where the notation \( (\cdot, \rho(\xi_k)) \) indicates that the differentiation operation is performed with respect to the spatial coordinates of the source point \( \xi_k \).

Setting:
\[
n_{\rho}(\xi_k)\psi_{i}(\xi_k) = \psi_{i\rho}(\xi_k), \tag{13.17}
\]

the final form of the new superposition operation for dipoles is:
\[
u(x_i) = \sum_{k=1}^{n} U^{*}_{ij}(\xi_k, x_i)\psi_{i\rho}(\xi_k)
\tag{13.18}
\]
or
\[
u(x_i) = \sum_{k=1}^{n} U^{*d}_{ij}(\xi_k, x_i)\psi_{i\rho}(\xi_k).
\tag{13.19}
\]

where the superscript \( (d) \) refers to the new two-point fundamental solution kernels used in the case of dipoles. These new two-point kernels are the derivatives of the corresponding traditional kernels with respect to the two spatial coordinates of the source point. The explicit form of the new two-point kernels used in Eq. (13.19) can be obtained as follows:
\[
U^{*d}_{a\rho}(\xi_k, x_i) = \frac{1}{4D\pi(1-\nu)r} [(4A + 1 - \nu)(r_{\rho}\delta_{a\rho} + r_{\beta}\delta_{a\beta} + r_{\alpha}\delta_{\rho\beta}) + 2r_{\alpha}\delta_{\rho\beta}], \tag{13.20}
\]
\[
U^{*d}_{a3\rho}(\xi_k, x_i) = -U^{*d}_{3a\rho}(\xi_k, x_i) = \frac{1}{8\pi D} \left(2\ln(z) - 1\right)\delta_{a\rho} + 2r_{\alpha}r_{\rho}, \tag{13.21}
\]
\[
U^{*d}_{33\rho}(\xi_k, x_i) = \frac{rr_{\rho}}{8\pi D(1-\nu)} \left[ (1-\nu)(2\ln(z) - 1) - \frac{8\ln^2(z)}{z^2} \right]. \tag{13.22}
\]

It has to be noted that in Eq. (13.19) there are two dummy indices, \( \rho \) which varies from 1 to 2 and \( i \) which varies from 1 to 3. Such indices are representing two summation operations as follows:
u_j(x_i) = \sum_{k=1}^{n} \sum_{\rho=1}^{2} \sum_{i=1}^{3} U_{ijp}^{*d}(\xi_k, x_i) \psi_{\rho p}(\xi_k).
(13.23)

Similarly, the traction superposition operation in Eq. (13.2) can be written as follows:

\begin{align*}
t_j(x_i) &= \sum_{k=1}^{n} T_{ijp}^{*d}(\xi_k, x_i) \psi_{\rho p}(\xi_k),
(13.24)
\end{align*}

where

\begin{align*}
T_{ijp}^{*d}(\xi_k, x_i) &= T_{ijp}^{*}(\xi_k, x_i).
(13.25)
\end{align*}

and such new kernel can be obtained as follows:

\begin{align*}
T_{ijp}^{*d}(\xi_k, x_i) &= \frac{1}{4\pi r^2} [(-1 - 4A - \nu)\delta_{\rho \psi}n_\alpha]

+ (-1 - 4A + \nu - 2K_1(z)z)(n_\rho \delta_{\alpha p} + \delta_{\alpha \psi} n_\rho)

+ (2 + 16A - 2\nu + 4K_1(z)z)(r_{\rho \psi} r_{\alpha p} + r_\rho r_{\alpha \psi} + r_\gamma r_{\alpha \psi})

+ (2 + 16A + 2\nu + 4K_1(z)z)r_{\rho \psi} r_\alpha

+ (2 + 16A - 2\nu + 8K_1(z)z + 2K_3(z)z^2)(r_{\alpha r} r_{\psi \alpha} + r_{\alpha r} r_{\psi \alpha} + r_{\alpha \beta} r_{\psi \alpha})

+ (-8 - 96A + 8\nu - 32K_1(z)z - 4K_3(z)z^2)r_{\rho r} r_{\psi r} r_{\alpha r},
\end{align*}

(13.26)

\begin{align*}
T_{ijp}^{*d}(\xi_k, x_i) &= \frac{1}{4\pi r^2} [-(1 + \nu)r_{\rho \psi} n_\alpha - (1 - \nu)r_{\alpha \psi} n_\rho - (1 - \nu)r_{\alpha \psi} \delta_{\rho \psi}]

+ 2(1 - \nu)r_{\rho r} r_{\psi r} r_{\alpha r},
\end{align*}

(13.27)
Dipole plate bending formulation for the method of fundamental solutions

\[ T^{*d}_{\theta\xi} (\xi_k, x_i) = \frac{\lambda^2}{2\pi r} \left[ -(A + K_1(z)z)r_\rho n_\theta - A(r_\rho n_\rho + r_\alpha \delta_{\theta\rho}) + (4A + K_1(z)z)r_\rho r_\rho' r_\alpha' \right], \]

(13.28)

\[ T^{*d}_{33\rho} (\xi_k, x_i) = \frac{-n_\rho + 2r_\rho' r_\alpha'}{2m^2}. \]

(13.29)

The stress-resultant dipole superposition representation can be written as follows:

\[ M_{\alpha\beta} (x_i) = \sum_{k=1}^{n} M^{*d}_{i\alpha\beta\rho} (\xi_k, x_i) \psi_{i\rho} (\xi_k), \]

(13.30)

\[ Q_{3\beta} (x_i) = \sum_{k=1}^{n} Q^{*d}_{3\beta\rho} (\xi_k, x_i) \psi_{i\rho} (\xi_k), \]

(13.31)

where

\[ M^{*d}_{i\alpha\beta\rho} (\xi_k, x_i) = M^{*d}_{i\alpha\beta,\rho} (\xi_k, x_i) \]

(13.32)

and

\[ Q^{*d}_{3\beta\rho} (\xi_k, x_i) = Q^{*d}_{3\beta,\rho} (\xi_k, x_i) \]

(13.33)

and the relevant new kernels can be obtained as follows:
The dipole formulation can be implemented numerically for cases of using only dipoles or mixed with the monopoles. In this work, only pure dipoles are used. Two steps are performed in order to carry out the solution numerically:

\[
M_{\theta_0 \beta_0}^{d} (\xi_k, x_i) = \frac{1}{4\pi r^2} \left[ (-1 - 4A - \nu)\delta_{\theta_0} \delta_{\alpha_0} + 
\right.
\left. (-1 - 4A + \nu - 2K_1(z)z)(\delta_{\theta_0} \delta_{\alpha_0} + \delta_{\theta_0} \delta_{\beta_0})
\right.
\left. + (2 + 16A - 2\nu + 4K_1(z)z)(r_{\alpha, r_{\rho}} \delta_{\theta_0} + r_{\rho, r_{\delta}} \delta_{\alpha_0} + r_{\alpha, r_{\delta}} \delta_{\beta_0})
\right.
\left. + (2 + 16A - 2\nu + 4K_1(z)z)r_{\rho, r_{\rho}} \delta_{\alpha_0} \delta_{\beta_0}
\right.
\left. + (2 + 16A - 2\nu + 8K_1(z)z + 2K_0(z)z^2)(r_{\alpha, r_{\rho}} \delta_{\theta_0} + r_{\beta, r_{\rho}} \delta_{\theta_0})
\right.
\left. + (-8 - 96A + 8\nu - 32K_1(z)z - 4K_0(z)z^2)r_{\rho, r_{\rho}} r_{\beta, r_{\beta}} \right],
\]

(13.34)

\[
M_{\theta_0 \beta_0}^{d} (\xi_k, x_i) = \frac{1}{4\pi r} \left[ -(1 + \nu)r_{\alpha, r_{\beta}} \delta_{\alpha_0} \delta_{\beta_0} - (1 - \nu)r_{\alpha, r_{\rho}} \delta_{\alpha_0} - (1 - \nu)r_{\beta, r_{\rho}} \delta_{\beta_0}
\right.
\left. + 2(1 - \nu)r_{\alpha, r_{\beta}} r_{\rho, r_{\rho}} \right],
\]

(13.35)

\[
Q_{\theta_0 \beta_0}^{d} (\xi_k, x_i) = \frac{\lambda^2}{2\pi r} \left[ -(A + K_1(z)z)r_{\alpha, r_{\rho}} \delta_{\alpha_0} \delta_{\beta_0} + A(r_{\rho, r_{\rho}} \delta_{\alpha_0} \delta_{\beta_0})
\right.
\left. + (4A + K_1(z)z)r_{\rho, r_{\rho}} r_{\beta, r_{\beta}} \right],
\]

(13.36)

\[
Q_{33}^{d} (\xi_k, x_i) = \frac{-\delta_{\alpha_0} \delta_{\beta_0} + 2r_{\beta, r_{\rho}} r_{\rho, r_{\rho}}}{2\pi r^2}.
\]

(13.37)

**13.4. Dipole Numerical Implementation**

The dipole formulation can be implemented numerically for cases of using only dipoles or mixed with the monopoles. In this work, only pure dipoles are used. Two steps are performed in order to carry out the solution numerically:
13.4.1. Obtaining the Source Intensities

Referring to Eqs. (13.19) and (13.24), these equations can be expanded as follows:

\[ u_1(x_i) = [U_{111}^{sd}(\xi_1, x_i)\psi_{11}(\xi_1) + U_{211}^{sd}(\xi_1, x_i)\psi_{21}(\xi_1) + U_{311}^{sd}(\xi_1, x_i)\psi_{31}(\xi_1) + \]
\[ U_{112}^{sd}(\xi_1, x_i)\psi_{12}(\xi_1) + U_{212}^{sd}(\xi_1, x_i)\psi_{22}(\xi_1) + U_{312}^{sd}(\xi_1, x_i)\psi_{32}(\xi_1) + \]
\[ \cdots + U_{312}^{sd}(\xi_n, x_i)\psi_{32}(\xi_n) \].

(13.38)

\[ t_1(x_i) = [T_{111}^{sd}(\xi_1, x_i)\psi_{11}(\xi_1) + T_{211}^{sd}(\xi_1, x_i)\psi_{21}(\xi_1) + T_{311}^{sd}(\xi_1, x_i)\psi_{31}(\xi_1) + \]
\[ T_{112}^{sd}(\xi_1, x_i)\psi_{12}(\xi_1) + T_{212}^{sd}(\xi_1, x_i)\psi_{22}(\xi_1) + T_{312}^{sd}(\xi_1, x_i)\psi_{32}(\xi_1) + \]
\[ \cdots + T_{312}^{sd}(\xi_n, x_i)\psi_{32}(\xi_n) \].

(13.39)

Expressing the boundary conditions defined at boundary nodes using Eq. (13.38) for displacement and Eq. (13.39) for traction boundary conditions; the following system of equations can be formed:

\[
\begin{pmatrix}
U_{111}^{sd}(\xi_1, x_1) & U_{111}^{sd}(\xi_1, x_2) & U_{211}^{sd}(\xi_1, x_1) & U_{211}^{sd}(\xi_1, x_2) & \cdots & U_{311}^{sd}(\xi_1, x_1) & U_{311}^{sd}(\xi_1, x_2) \\
U_{112}^{sd}(\xi_1, x_1) & U_{212}^{sd}(\xi_1, x_2) & U_{312}^{sd}(\xi_1, x_1) & U_{312}^{sd}(\xi_1, x_2) & \cdots & U_{312}^{sd}(\xi_1, x_1) & U_{312}^{sd}(\xi_1, x_2) \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
U_{112}^{sd}(\xi_1, x_m) & \cdots & \cdots & U_{312}^{sd}(\xi_1, x_m)
\end{pmatrix}
\begin{pmatrix}
\psi_{11}(\xi_1) \\
\psi_{21}(\xi_1) \\
\vdots \\
\psi_{32}(\xi_n)
\end{pmatrix}
= \begin{pmatrix}
u_1(x_1) \\
u_2(x_1) \\
\vdots \\
u_3(x_m)
\end{pmatrix}
\]

(13.40)

where \( m \) is the number of the known value boundary nodes; \( n \) is the number of dipole sources. The system of equations in (13.40) can be solved as a square or rectangular linear system of equation. In this paper square system
of equations is employed. Solving the system in (13.40), the intensities of the dipoles $\psi_i(\xi)$ can be obtained.

13.4.2. Calculation of the Solution at Interior Points

Similar to monopoles in sec. (13.2), once the dipole intensities are known, applying Eqs. (13.19) and (13.24) the displacement and traction can be calculated. Stress resultants can be also calculated by applying Eqs. (13.30) and (13.31) with the known dipole intensities.

13.5. Body forces

In this work the Particular Integral (PI) technique is adopted [2], in which the involved variables are divided into two parts:

1) A particular solution part (the domain of the problem is assumed to be embedded in an infinite domain which is only loaded with the domain loads ignoring all the predefined boundary conditions).

2) A complementary solution part (the domain of the problem is assumed to be embedded in an infinite domain containing all the problem predefined complementary properties without the domain loading).

The particular part is solved using the analytical particular solutions, and the complementary problem is solved regularly as mentioned before using the MFS, then the solution is obtained by superposition of the particular and complementary parts as follows:

$$ (\bullet) = (\bullet^p) + (\bullet^c) $$

where $(\bullet)$ can represent $u_i(x)$, $t_i(x)$ and $M_{\alpha\beta}(x)$ & $Q_{3a}(x)$; the superscripts $p$ and $c$ distinguish between the particular solution case and the complementary solution case.

13.6. Numerical Measures

Two measures are used for the numerical and the overall accuracy:
1) The numerical accuracy is measured using the 2-norm condition number (CND) \([9]\), which is the ratio of the largest singular value to the smallest one.

2) The overall accuracy \((U_{err})\) is measured using the root mean square (r.m.s.) value of the absolute error vector of the vertical displacement property for some randomly chosen points. This measure can be calculated for the group of \(n\) points \((x_i)\) as follows:

\[
U_{err} = \sqrt{\frac{\sum_{j=1}^{n} (u_{MFS}^{x_i}(x_j) - u_{Comp}^{x_i}(x_j))^2}{n}}, \quad (13.42)
\]

where \(u_{MFS}^{x_i}(x_j)\) is the deflection calculated using the MFS for the group of points \(x_j\); \(u_{Comp}^{x_i}(x_j)\) is the deflection calculated using any comparative method for the same group of points \(x_j\), where the chosen comparative method will be mentioned in each example in the following section. In this work, dipoles are placed along at equidistant surface in the normal direction of the boundary nodes (see Fig. (13.3)). It was found that by such location the best accuracy is obtained. The coordinates of the sources forming the used equidistant fictitious source surface are calculated as follows:

![Fig. 13.3: The equidistant surface generation.](image-url)
\[ x_\alpha \varphi = x_\alpha + d\alpha \times n_\alpha , \quad (13.43) \]

where \( x_\alpha \varphi \) represents the coordinates of the source point, \( x_\alpha \) represents the coordinates of the boundary node having a normal direction component in the \( \alpha \) direction equal to \( n_\alpha \).

### 13.7. Numerical Examples

Three examples are tested. The first example is a circular domain having a radius = 4 with a simply supported edge as shown in Fig. (13.4). The boundary conditions are represented using 50 boundary nodes. The second example is a cantilever square domain with dimensions 4 × 4 as shown in Fig. (13.5). The boundary conditions are represented using 32 boundary nodes.

The comparative method used in these two examples is the direct boundary element method (BEM) [10]. The third example is a rectangular thin beam domain with dimensions 0.25 × 10 as shown in Fig. (13.6), and the boundary conditions are represented using 44 boundary nodes. The comparative method for this example is the stiffness matrix method [11]. All the three examples have thickness of 0.2 and Poisson’s ratio = 0.2, the Young’s modulus is 10000.0, and loaded with a uniform domain loading of 1.0. In Figs. (13.4), (13.5) and (13.6) the dotted line named as “LINE 1” demonstrates the place of the chosen points used to calculate \( U_{err} \) (recall sec. (13.6) Eq. (13.42)). It should be noted that all the defined variables of the problems are treated as dimensionless values.
Fig. 13.4: The considered circular domain used in Example 1.

Fig. 13.5: The square domain used in Example 2.

Fig. 13.6: The beam used in Example 3.
Dipole plate bending formulation for the method of fundamental solutions

Fig. 13.7a: Condition number for different source locations in Example 1.

Fig. 13.7b: Condition number for different source locations in Example 2.
Fig. 13.7c: Condition number for different source locations in Example 3.

Figures (13.7a), (13.7b) and (13.7c) demonstrate the numerical behavior for the monopoles against the dipoles. It can be seen that the condition number for monopoles started with low values and gradually increases with the increase of the sources distant, until it reach machine precision. Then the curve trend started to oscillate. On the other hand, the dipoles condition number starts in oscillative trend very early, reflecting an early numerical instability. Such behavior demonstrates that regardless the fact that the distances between sources are larger, in case of dipoles (which has a positive effect), the numerical stability in the monopole case is better, and this could be is due to the fact that dipole kernels are derivatives of the monopole kernels.
Dipole plate bending formulation for the method of fundamental solutions

Fig. 13.8a: The overall accuracy curves for both mono & dipoles in Example 1.

Fig. 13.8b: The overall accuracy curves for both mono & dipoles in Example 2.
Fig. 13.8c: The overall accuracy curves for both mono & dipoles in Example 3.

Table (13.1.): Best overall accuracy for both monopole and dipoles.

<table>
<thead>
<tr>
<th>Example</th>
<th>Uerr di</th>
<th>Uerr mono</th>
<th>Uerr mono/Uerr di</th>
</tr>
</thead>
<tbody>
<tr>
<td>Example 1</td>
<td>3.89E-05</td>
<td>4.52E-05</td>
<td>1.162654867</td>
</tr>
<tr>
<td>dis</td>
<td>2.3</td>
<td>9.6</td>
<td></td>
</tr>
<tr>
<td>Example 2</td>
<td>0.037635651</td>
<td>0.018986905</td>
<td>0.504492533</td>
</tr>
<tr>
<td>dis</td>
<td>0.3</td>
<td>3</td>
<td></td>
</tr>
<tr>
<td>Example 3</td>
<td>0.0061464</td>
<td>0.086535</td>
<td>14.07897306</td>
</tr>
<tr>
<td>dis</td>
<td>1.6</td>
<td>2.1</td>
<td></td>
</tr>
</tbody>
</table>

Figures (13.8a), (13.8b) and (13.8c) demonstrate the overall accuracy using the dipole against the monopole sources. The accuracy peaks (points of global minimum in $U_{err}$ curve; see Figs. (13.8a), (13.8b) and (13.8c))
obtained are given in Table (13.1). It should be noted that for regular or non-slender domains, as in examples 1 & 2, the overall accuracy had no significant change between the dipole and monopole. Also, it is noticed that the dipole accuracy peaks were reached at smaller distances than those of the monopoles. This is because the numerical instability occurred in dipoles faster than in monopoles (as mentioned before). However, the thin structure in Example 3 demonstrated a significant improvement for the overall accuracy, in the case of using dipoles. It can be seen from Table (13.1) that despite the high numerical instability for dipoles they produce 14 times better in accuracy than that of the monopoles.

13.8. Conclusions

A dipole formulation for MFS is presented. All kernels required for the proposed formulation were derived and given in explicit form. The numerical implementation of such formulation was also discussed. Moreover the proposed formulation was tested against the monopole formulation using 3 numerical examples involving 3 different domains (circle, square and thin rectangle). We believe that: one should use the dipole sources in solving thin structures, which demonstrates a significant improvement in the accuracy. For regular domains, the monopole sources are recommended because of their numerical stability. A mixed approach could also be used, in order to increase the numerical stability, as opposed to using only dipoles, which will be considered in the future work.

References


Dipole plate bending formulation for the method of fundamental solutions


CHAPTER 14

Performance of the MFS in the Computation of Seismic Motion and Rotation

Luís Godinho¹, António Tadeu¹, Paulo Amado Mendes ¹

Abstract. The Method of Fundamental Solutions (MFS) is a powerful tool that may be used to solve wave propagation problems. The applicability of the MFS for the computation of the rotational motion in an elastic medium is addressed. The analyzed scenarios refer to a half-space with a topographical feature (a valley). For case of the half-space, fundamental solutions are known, and thus the MFS must only be used to describe the valley. Although these solutions are accurate, they are computationally demanding, and can lead to long computational times. Given this limitation, two different approaches have been analyzed in this work. The first makes use of the mentioned solutions both for the incident field and for the virtual sources required for the MFS representation. In a second approach, only the incident field is computed using these fundamental solutions, while the topography and the half-space surface are described by full-space fundamental solutions. It is important to note that the second approach allows a significant decrease in the computation times required for the analyzed problems.

14.1. Introduction

Site effects influence the nature of strong ground motion, and may cause marked amplification phenomena [1]. Surface topography is one of the factors that are known to produce these so-called site effects, and there is significant evidence of the ground motion amplification it generates. Most of the previous works in this field deal mainly with amplification and translational motion caused by the seismic loads, while the rotational

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component of motion has been mostly neglected. It is common in earthquake engineering to focus more on the translational motion, overlooking important rotations around horizontal (tilting) and vertical (spinning) axes, that occur during earthquakes. It is, however, known that the rotational movements can have a significant influence on the behaviour of structures, and that they may even influence the way displacements are registered. As an example, Grazier [2] studied the influence of rotational motion on the translational measurements obtained by pendular accelerometers, and observed that the pendulum is sensitive not only to translational movements but also to tilt. From his observations, he concluded that in the vicinity of active faults it would be desirable to measure the rotational components of motion in combination with translational motion in the vicinity of active faults.

In the past, several authors addressed the problem of rotational motion in theoretical works, such as Bouchon and Aki [3], Lee and Trifunac [4] and Castellani and Boffi [5]. In those works the authors concluded that the rotational ground motion generated in the near-field or by surface waves can be important.

A possible approach to assess the importance of the rotational motion is to make use of numerical and mathematical models that simulate the seismic phenomena. Li et al. [6] proposed a mathematical model that allows the computation of time-histories of rocking motion and torsional rotation corresponding to a set of three recorded orthogonal translational components. The proposed model is based on a representation of soil impedance and of the contribution of body waves.

The most common numerical methods, namely finite elements, finite differences, boundary elements and meshless techniques, have been used in the simulation of seismic wave propagation. Once again, different approaches have been used to simulate the propagation of seismic waves in the presence of specific topographical deformations, but mainly in what concerns motion amplification. Pedersen et al. [7] used an indirect Boundary Element Method (BEM) formulation to calculate the three-dimensional seismic response of two dimensional topographies, under the influence of plane waves, using Green’s functions for an infinite space. Later, Tadeu et al. [8] used a direct BEM approach to compute the 3D reflected field.
generated in the presence of smooth topographic deformations, making use of the Green’s functions for an unbounded space.

When the geometry of the problem consists of half-spaces or layered media containing inclusions, it is possible to use fundamental solutions that take into account the layered structure of the propagation media, without requiring its discretization. Tadeu et al. [9] proposed one such set of functions that allow simulating the cases of half-spaces and solid layers, subjected to 2.5D loads. The functions proposed are defined as summations of the effects of plane waves with different inclinations with respect to the horizontal axis.

In recent years, a different class of numerical techniques has been increasingly used, namely the so-called “meshless techniques”. A few examples of these techniques are: the method of fundamental solutions (MFS) [10, 11], the radial basis functions (RBF) collocation method [12], and the Meshless Local Petrov-Galerkin method [13]. The MFS seems to be particularly effective in the study of wave propagation phenomena, overcoming some of the mathematical complexity of the BEM, and providing accurate solutions. Godinho et al. [14] studied the performance of the MFS for simulating the propagation of acoustic waves in a fluid domain with an inclusion, concluding that the method can be very efficient, even surpassing the performance of the BEM for this type of problem. Later, Godinho et al. [15] applied the MFS to study elastic and acoustic wave propagation around thin structures using a domain decomposition technique.

In the present paper, the authors make use of the MFS to study the translational and rotational motion caused by seismic waves in the presence of simple valley topographies. First, the formulation of the MFS will be presented, followed by a brief description of the half-space fundamental solutions used here. An alternative formulation of the method will be proposed, allowing the reduction of the computation times needed to solve each problem. After this, the performance of both approaches will be assessed for a simple geometry, by computing the errors and by evaluating the computational times taken by each proposed approach.

14.2. The Method of Fundamental Solutions
The solution of the wave equation in two-dimensions can be obtained using different numerical schemes. In the present paper, the authors have used the MFS, which approximates the solution in terms of a linear combination of fundamental solutions of the governing equation. The approximate solution \( \hat{\mathbf{u}} \) is then given by

\[
\hat{\mathbf{u}} = \left\{ \begin{array}{l}
\hat{u}_x = \sum_{k=1}^{N_{inc}} \left[ a_{xk} G_{y} (x,x_k,y,y_k) + a_{yk} G_{x} (x,x_k,y,y_k) \right] + u_{x}^{inc} \\
\hat{u}_y = \sum_{k=1}^{N_{inc}} \left[ a_{yj} G_{y} (x,x,y_k,y_k) + a_{yj} G_{y} (x,x,y_k,y_k) \right] + u_{y}^{inc}
\end{array} \right. \tag{14.1}
\]

where \( \{(x_k,y_k)\}_{k=1}^N \) are \( N \) distinct source points on a fictitious boundary, \( G_{ij} (x,x_i,y,y_i) \) \( (i,j = 1, 2) \) are the displacements in the direction \( j \) at \( (x,y) \), caused by a unit point force applied at \( (x_k,y_k) \), and \( u_{x}^{inc} \) and \( u_{y}^{inc} \) represent a possible incident field generated by a source inside the domain. The fictitious sources are located along a fictitious boundary \( \Gamma' \) placed outside the domain to avoid singularities.

Once the source points have been chosen, the coefficients \( \{a_{ij}\}_{k=1}^N \) can be determined by satisfying the boundary conditions along the physical boundary \( \Gamma \). Two different cases are considered:

1) the number of source points \( N \) on \( \Gamma' \) and the number of collocation points \( M \) on boundary \( \Gamma \) are equal, and the resulting linear equation systems are solved using a direct solver;

2) the number of source points \( N \) on \( \Gamma' \) is half the number of collocation points \( M \) on boundary \( \Gamma \), and the resulting equation systems are solved using a least squares solver (the solver ZGELSS from the LAPACK set of numerical subroutines has been used, solving the equation systems by a singular-value decomposition method).
In the present work, the propagation domain consists of a half-space with a geometric discontinuity (a semi-circular valley) near its surface, as depicted in Figure 14.1, illuminated by a linear source located at \((x_0, y_0)\). In the same figure, the physical (\(\Gamma\)) and virtual (\(\Gamma'\)) boundaries are also represented. For this geometry, it is adequate to use the fundamental solutions known for half-spaces, presented by Tadeu et al. [9]. Following the conclusions of previous works [15], the source points are placed along the boundary \(\Gamma'\) at a constant distance to the boundary of \(0.2 \times R\), where \(R\) is the radius of the geometric discontinuity.

![Fig. 14.1. Schematic representation of the two-dimensional configuration.](image)

Using the solutions mentioned above, the discretization of the horizontal surface of the half-space may be avoided if fundamental solutions that take its presence into account are used. For a half-space medium, the total wavefield can be expressed by taking into account the incident field generated by the source (source terms), and the terms generated at the surface (surface terms). The source terms can be written making use of the equations proposed by Tadeu and Kausel [16] for 2.5D loads, while the surface terms can be represented by a set of one dilatational and two shear potentials, with unknown amplitude values. Both the source and the surface terms are expressed as continuous integrals of the effects of plane waves. These integrals can then be discretized into summations of discrete terms, assuming the existence of an infinite number of virtual sources placed along
the $x$ direction at equal intervals, $L_x$. The distance separating them is large enough to prevent the virtual loads from contaminating the response. For loads acting along the $x$ and $y$ directions, and considering the contribution of $2K + 1$ terms, the potentials that define the surface terms are:

**Load acting along $x$:**

$$
\phi^x_0 = E_a \sum_{n=-K}^{n=K} \left( \frac{k_n}{\nu_n} E_{\phi0} A^x_n \right) E_d; \quad \psi^x_0 = -E_d \sum_{n=0}^{n=Q} \left( E_{\psi0} B^x_n \right) E_d; \quad (14.2)
$$

**Load acting along $y$:**

$$
\phi^y_0 = E_a \sum_{n=-K}^{n=K} \left( E_{\phi0} A^y_n \right) E_d; \quad \psi^y_0 = E_d \sum_{n=0}^{n=Q} \left( \frac{k_n}{\gamma_n} E_{\psi0} B^y_n \right) E_d; \quad (14.3)
$$

with $E_a = \frac{1}{2\rho \omega^2 L_x}$, $E_{\psi0} = e^{-i\nu \gamma}$, $E_{\phi0} = e^{-i\nu \alpha}$, $E_d = e^{-ik \gamma}$, $\nu_n = \sqrt{k^2 - k^2_n}$ ($\text{Im} \nu_n \leq 0$), $\gamma_n = \sqrt{k^2 - k^2_n}$ ($\text{Im} \gamma_n \leq 0$), $k_p = \omega / \alpha$, $k_s = \omega / \beta$, $i = \sqrt{-1}$, and with $k = 2\pi n / L_x$ being the horizontal wavenumber along $x$.

Using this methodology, and imposing the necessary boundary conditions (null tangential and normal stresses at the surface), systems of equations can be established for each value of $k$ that allow the unknown amplitude factors to be calculated.

The rotational motion caused by the dynamic loads may be defined by half the difference between the cross-derivatives of the horizontal and vertical motion, $\Theta = \frac{1}{2} \left( \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} \right)$. This rotational motion can be derived analytically for a half-space, and the corresponding expressions will be

$$
\Theta_x = -\frac{1}{2} E_a \sum_{n=-K}^{n=K} E_d E_{\phi0} \left( B^x_n (\gamma_n^2 + k_n^2) \right) + \Theta_x^{\text{full}} \quad (14.4)
$$

$$
\Theta_y = \frac{1}{2} E_d \sum_{n=-K}^{n=K} E_{\psi0} \left( B^y_n (\gamma_n^2 + k_n^2) \right) + \Theta_y^{\text{full}},
$$

where $\Theta_x$ and $\Theta_y$ are the rotations produced by horizontal and vertical loads, respectively, and $\Theta_x^{\text{full}}$ and $\Theta_y^{\text{full}}$ represent the corresponding rotational motion caused by the load in an unbounded domain.
14.3. Alternative Simplified Approach

Although the described fundamental solutions are accurate and easy to implement, they may be computationally demanding, with the involved summations requiring long computational times to attain convergence, especially when the sources or the receivers are located close to the surface of the half-space. For this reason, it can be interesting to test alternative formulations, which may result in a significant improvement in the performance of the method.

One such alternative can be the use of a simplified version of the above formulation, neglecting only the contribution of the surface terms for the specific case of the virtual sources, while using the full half-space solution for the real source located at \((x_0, y_0)\). Using only the full-space solution for the virtual sources, the very long computational times required when the source is very close to the surface will be avoided, and the computational times are expected to decrease significantly. On the other hand, maintaining the complete solution for the incident field, the original surface waves generated at the half-space surface are still taken into account, and this helps the method to maintain its high accuracy. However, to model the geometry using only full-space solutions implies that some additional sources and collocation points have to be considered along the surface of the half-space, near the geometric discontinuity. The distance to be discretized to each side of the discontinuity should be large enough so that all relevant contributions from surface reflections are taken into account. In this work, the authors have considered a distance \(R\) to each side of the discontinuity along which the additional sources and collocation points should be placed. The final system can be observed in Figure 14.2.
Performance of the MFS in the computation of seismic motion and rotations

Fig. 14.2. Location of the sources for the alternative model.

For this simplified approach, the displacements at any point inside the domain may be computed as

\[
\begin{align*}
\hat{u}_x &= \sum_{k=1}^{N} \left[ a_{xk} G_{xx}^{\text{full}}(x,x_k,y,y_k) + a_{yk} G_{xy}^{\text{full}}(x,x_k,y,y_k) \right] + u_x^{\text{inc}}, \\
\hat{u}_y &= \sum_{k=1}^{N} \left[ a_{yj} G_{yj}^{\text{full}}(x,x_j,y,y_j) + a_{yj} G_{yx}^{\text{full}}(x,x_j,y,y_j) \right] + u_y^{\text{inc}},
\end{align*}
\]

(14.5)

where \(u_x^{\text{inc}}\) and \(u_y^{\text{inc}}\) represent the incident field generated by the source in a half-space medium, while \(G_{ij}^{\text{full}}(x,x_j,y,y_j)\) are the displacements along \(j\) generated by a load that acts along \(i\), in an unbounded space. The rotational motion can be computed by taking the necessary derivatives of these functions (\(\Theta = \frac{1}{2} \left( \frac{\partial \hat{u}_y}{\partial y} - \frac{\partial \hat{u}_x}{\partial x} \right) \)).

14.4. Performance of the MFS

To assess the performance of the MFS formulations in the calculation of seismic motion (both translational and rotational), the proposed approaches have been used to calculate the displacements and rotations generated in the
presence of a simple geometry, consisting of a half-space with a semi-circular valley of radius 1000 m. For this purpose, the propagation domain was assumed to have the same properties identified in the previous sections, being excited by linear vertical loads placed at $x=0$ m and $y=-5000$ m. The results were computed over a sequence of 120 receivers located at the surface of the valley structure. In all the simulations, the virtual sources were placed over a virtual boundary located 200 m away from the real boundary. The problem geometry is illustrated in Figure 14.3. Since no analytical solution is known for this type of configuration, a reference solution was calculated to serve as a basis for all comparisons. For this case, and after studying the convergence of the MFS in this specific case, this reference solution was calculated using a very high number of collocation points and virtual sources (400). These reference solutions are depicted in Figure 14.4.

The first set of results refers to the case in which the problem is modeled using the MFS with half-space fundamental solutions, and with a number of virtual sources equal to the number of collocation points ($N=M$). For this case, different numbers of sources/collocation points were tested, namely $N=30$, $N=60$ and $N=120$. The results were computed for excitation frequencies of 1 Hz, 2 Hz and 5 Hz.

Fig. 14.3. Schematic representation of the problem’s geometry (dimensions in meters).
Figure 14.5 shows the relative differences to the reference solution in what concerns vertical displacements and rotations, when the source is excited. For this case, it is clear that both the displacements and the rotations computed by the model converge to the reference solutions as the number of collocation points is increased. It is also clearly noticeable that the relative differences computed for the displacement responses are lower than those obtained for the rotational motion. This can be explained by the fact that for the computation of rotational motion it becomes necessary to compute the derivatives of the displacement, subsequently decreasing the precision in the numerical calculations. Although the global behavior of the model is similar for the three frequencies analyzed, it can be seen that the relative differences tend to increase as higher frequencies are considered.

Fig. 14.4. Reference solutions for vertical displacements and rotations.
Fig. 14.5. Relative errors computed using the MFS with half-space fundamental solutions and a direct solver ($M=N$).

Figure 14.6 illustrates the relative differences to the reference solution computed when full-space fundamental solutions are used to describe the semi-circular valley. It is important to note that, for this case, the number of collocation points is higher, since it becomes necessary to simulate part of the half-space boundary by means of placing additional virtual sources at a fixed distance to the real boundary. The displacements and the rotations computed using this model reveal a lower accuracy, which results in higher
relative differences to the reference solution, especially for the rotational motion. Even for the lower frequencies, these relative differences are very high, reaching values of 0.1. It also becomes evident that the results are not converging to the reference solution as the number of collocation points increases, revealing numerical instability of the linear equation systems when higher numbers of collocation points are used.

A second approach proposed in this study prescribes the use of a least-squares solver, with a higher number of collocation points than virtual sources. In the examples presented, the number of collocation points is always twice the number of virtual sources. Figure 14.7 illustrates the relative differences to the reference solution computed using half-space fundamental solutions with this least-squares approach. The results computed using this model seem to approach very well to the reference solution, with lower relative differences both in what concerns displacements and rotations. The results converge well to the solution, with these differences clearly decreasing as the number of collocation points increases. It is even possible to note that the computed relative differences are, for some cases, lower than those shown if Figure 14.4, computed using a standard direct solver. Interestingly, the relative differences computed for the rotational motion are higher near the center of the valley structure. This can be explained by the fact that, at this part of the system, these motions are very small, reaching a null value at the precise center of the valley.

The least-squares approach has also been used in conjunction with the model that makes use of full-space fundamental solutions to simulate the surface. The corresponding results are shown in Figure 14.8. Comparing these results with those shown in Figure 14.6, it becomes evident that this approach is much more stable and that it converges to the solution, with the computed relative differences becoming lower as more collocation points are used. Although the results are still not as good as those shown in Figure 14.5 or Figure 14.7, it now seems that the error is lower than before, and that the model gives fairly accurate results, and that it may be very useful from an engineering point of view.
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Fig. 14.6. Relative errors computed using the MFS with full-space fundamental solutions for the virtual sources and a direct solver ($M=N$).
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Fig. 14.7. Relative errors computed using the MFS with half-space fundamental solutions for the virtual sources and a least-squares solver ($M=2N$).
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Fig. 14.8. Relative errors computed using the MFS with full-space fundamental solutions for the virtual sources and a least-squares solver ($M=2N$).

Besides the accuracy of the models, their efficiency was also tested in what concerns computational times required for each set of calculations. Table 14.1 presents the corresponding computation times required by each model to compute the response at the 120 receivers, when the source is excited, and for a single frequency (5 Hz), and when the same number of collocation points is used to model the semi-circular part of the system. The values depicted in this table reveal that the model that makes use of a direct solver with half-space fundamental solutions requires much higher computational times. In fact, observing these values and taking into account...
the plots presented before, it is possible to conclude that similar accuracy can be obtained with half the computation time making use of a least-squares approach, with half the number of virtual sources. One important aspect that should be noted in this table is that, although their global accuracy is lower, the models that use full-space solutions to describe the surface require substantially less computational times. This observation indicates the usefulness of this formulation in engineering studies.

<table>
<thead>
<tr>
<th></th>
<th>Direct solver with $N \times N$ system matrix</th>
<th>Least-squares approach with $N \times N/2$ system matrix</th>
</tr>
</thead>
<tbody>
<tr>
<td>Half-space fundamental solutions ($N=120$)</td>
<td>474.4 s</td>
<td>228.9 s</td>
</tr>
<tr>
<td>Full-space fundamental solutions ($N=200$)</td>
<td>3.9 s</td>
<td>4.0 s</td>
</tr>
</tbody>
</table>

### 14.5. Conclusions

Different approaches were presented that allow the computation of displacements and tilting motion in 2D problems with a half-space containing a valley structure. Those approaches make use of the Method of Fundamental Solutions, and allow an efficient analysis of physical systems subject to the incidence of seismic waves. The presented methods and their accuracy were tested, enabling the authors to conclude that the best results could be obtained by making use of fundamental solutions that account for the presence of the half-space surface, and using virtual sources only for describing the valley geometry. The presented results also indicated that using full-space fundamental solutions for the virtual sources, together with half-space solutions for the real sources, yields acceptable results, while taking considerably less computational time.
References


CHAPTER 15

A Modified Method of Fundamental Solutions for Potential Flow Problems

Božidar Šarler¹

Abstract. This chapter describes an application of the recently proposed Modified Method of Fundamental Solutions (MMFS) to the potential flow problems. The solution in two dimensional Cartesian coordinates is represented in terms of the fundamental solution of the Laplace equation together with the first order polynomial augmentation. The collocation is used for determination of the expansion coefficients. This novel method does not require fictitious boundary as the conventional Method of Fundamental Solutions (MFS). The source and collocation points thus coincide on the physical boundary of the system. The desingularised value of the fundamental solution in case of the coincidence of the collocation and source points is determined directly as the average value of the fundamental solution on the boundary in the vicinity of the source point. The respective values of the derivatives of the fundamental solution in the coordinate directions, as required in potential flow calculations, are calculated indirectly from the considerations of the constant potential field. The normal on the boundary is calculated by parametrisation of its length and use of the cubic radial basis functions with the second order polynomial augmentation. The components of the normal are calculated in an analytical way. A numerical example of potential flow around two dimensional circular region is shown. The results with the MMFS are compared with the results of the classical MFS and the analytical solution. It is shown that the MMFS gives better accuracy of the velocity components as compared with the classical MFS.

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Modified MFS for Potential Flow Problems

15.1 Introduction

The MFS is a numerical technique that falls in the class of methods generally called boundary methods. The other well known representative of these methods is the Boundary Element Method (BEM) [1]. Both methods are best applicable in situations where a fundamental solution to the partial differential equation in question is known. In such cases, the dimensionality of the discretization is reduced. BEM for example requires polygonisation of the boundary surfaces in general 3D cases, and boundary curves in general 2D cases. This method requires solution of the complicated regular, weakly singular, strongly singular, and hypersingular integrals over boundary segments which is usually a cumbersome and non-trivial task. The MFS has certain advantages over BEM, that are mostly visible in the fact that pointisation of the boundary is needed only, that completely avoids any integral evaluations, and makes no principal difference in coding between the 2D and the 3D cases. A comprehensive survey of the MFS and related methods for elliptic boundary value problems can be found in [2,3,4]. Some important developments of the MFS, that might be put into connection with the Laplace equation, focused in the present paper, are as follows. The method has been used for harmonic problems with linear [5] and non-linear boundary conditions [6], free boundaries [7,8], multi-domains [9], for heat conduction in isotropic and anisotropic bimaterials [10], and for axisymmetric problems [11]. The method has been expanded for material non-linearities and all technically relevant boundary conditions in a systematic way and applied to thermal design of hollow bricks [12]. The later paper represents one of the rare industrial applications of the MFS. In the present paper, the potential flow problem, previously solved by the least squares version of MFS [13] and collocation version of MFS [8] is solved by the MMFS. This novel method, which essentially represents a sort of blend between BEM and MFS has been originally developed by [14] by using collocation with double layer Laplace equation fundamental solution. The method has been further extended to single-layer Laplace equation
fundamental solution in [15]. The main drawback of the MFS represent positions of the source points that need to be positioned outside the boundary. In case they are too close to the boundary, the solution is not accurate. In case they are too far away from the boundary, the discretisation matrix becomes ill conditioned. The novel MMFS overcomes this difficulty by allowing the source point positions to coincide with the collocation points on the physical boundary. A desingularisation technique thus has to be employed in order to be able to allow bounded values in the discretisation matrix. The desingularisation has been derived through the properties of the double layer potential in [14] and through the indirect BEM formulation in [15]. In the present paper, the desingularisation is extended to the calculation of the desingularised values through the direct BEM approach, as well as to calculation of the desingularised value of the partial (not normal) derivatives on the boundary, which was not the case in previous two cited MMFS pioneering papers by the D.L. Young’s group.

15.2. Governing Equations

Consider a connected two-dimensional domain $\Omega$ with boundary $\Gamma$. The domain is filled by a fluid that undergoes potential flow. The boundary is divided into the part $\Gamma^E$ that represents external boundaries of the system and into the part $\Gamma^I$, that represents internal boundary of the system i.e. $\Gamma = \Gamma^E \cup \Gamma^I$. The potential $\Phi$ is governed by the following boundary value problem: Laplace equation

$$\nabla^2 \Phi = 0,$$

(15.1)

and boundary conditions of the Dirichlet and Neumann type, located at the Dirichlet $\Gamma^D$ and Neumann $\Gamma^N$ parts of the boundary $\Gamma$, i.e. $\Gamma = \Gamma^D \cup \Gamma^N$

$$\Phi(p) = \Phi^D(p), \quad p \in \Gamma^D,$$

(15.2)

$$\frac{\partial \Phi}{\partial n} (p) = \Phi^N(p), \quad p \in \Gamma^N,$$

(15.3)
with \( \mathbf{p} \) standing for the position vector and \( \mathbf{n}_N \) for the outward normal on the boundary \( \Gamma \). \( \Phi^D \) and \( \Phi^N \) represent Dirichlet and Neumann boundary condition forcing functions. Let us introduce a two dimensional Cartesian coordinate system with ortho-normal base vectors \( \mathbf{i}_x \) and \( \mathbf{i}_y \) and coordinates \( p_x \) and \( p_y \), i.e. \( \mathbf{p} = p_x \mathbf{i}_x + p_y \mathbf{i}_y \). The potential field velocity components are calculated from the potential \( \Phi \) as

\[
v_{\xi}(\mathbf{p}) = \frac{\partial \Phi}{\partial p_{\xi}}(\mathbf{p});\quad \xi = x, y.
\] (15.4)
It is the purpose of this paper to determine the steady state potential flow components as a function of the posed geometry, governing equation and boundary conditions.

15.3. Solution Procedure

15.3.1 Solution of the Potential Flow

The common points of the MFS and MMFS for solution of the potential flow field are elaborated first. The differences are elaborated afterwards. The solution of the potential $\Phi$ is represented by the $N_\Gamma + 3$ global approximation functions $\psi^*_n(p)$ and their coefficients $\alpha^*_n$:

$$\Phi(p) \approx \sum_{n=1}^{N_\Gamma + 3} \psi^*_n(p) \alpha^*_n. \quad (15.5)$$

The global approximation functions have the property

$$\nabla^2 \psi^*_n(p) = \begin{cases} 0; & p \neq p_n; \quad n = 1, 2, \ldots, N_\Gamma, \\ \delta(p_n); & p = p_n, \end{cases} \quad (15.6)$$

i.e., they are fundamental solutions of the Laplace operator. $\delta$ denotes the Kronecker symbol. For two dimensional problems in Cartesian coordinates, the fundamental solution equals to

$$\psi^*_n(p) = \frac{1}{2\pi} \log \frac{r^*_n}{r_n}; \quad r^*_n = r_n \cdot r_n, \quad (15.7)$$

for $n = 1, 2, \ldots, N_\Gamma$. The following three augmented functions (that also represent solution of the Laplace equation) can be additionally optionally introduced into the global approximation functions set

$$\psi^*_{N_\Gamma + 1}(p) = 1, \quad (15.8)$$

$$\psi^*_{N_\Gamma + 2}(p) = p_x - p_{0x}, \quad (15.9)$$
where \( r^* \) denotes the reference radius and \( p_{0x} \) and \( p_{0y} \) represent the mean coordinates of the \( \Gamma \cup \Omega \). The augmentation functions have been introduced in order to be able to exactly satisfy possible constant or linear potential fields. Such global approximation functions set has been already successfully used in the calculation of the bubble shape in the potential flow (Šarler, 2006). Let us introduce the boundary condition indicators in order to be able to represent the boundary collocation equations in a compact form. The Dirichlet \( \chi^D \) and Neumann \( \chi^N \) type of boundary conditions indicators are

\[
\chi^D(p) = \begin{cases} 
1; & p \in \Gamma^D \\
0; & p \notin \Gamma^D 
\end{cases},
\]

(15.11)

\[
\chi^N(p) = \begin{cases} 
1; & p \in \Gamma^N \\
0; & p \notin \Gamma^N 
\end{cases}.
\]

(15.12)

The coefficients are calculated from a system of \( N_\Gamma + 3 \) algebraic equations

\[
\sum_{n=1}^{N_\Gamma+3} \psi_{n}^* \alpha_n^* = b_j, \quad j = 1, 2, ..., N_\Gamma + 3.
\]

(15.13)

The first \( N_\Gamma \) equations are obtained through collocation of Equation (5) in collocating points \( p_j; j = 1, 2, ..., N_\Gamma \) for \( n = 1, 2, ..., N_\Gamma + 3 \)

\[
\psi_{n}^* = \chi^D(p_j)\psi_{n}^*(p_j) + \chi^N(p_j)\frac{\partial \psi_n^*}{\partial n_t}(p_j),
\]

(15.14)

\[
b_j = \chi^D(p_j)\Omega^D(p_j) + \chi^N(p_j)\frac{\partial \Omega^N}{\partial n}(p_j),
\]

(15.15)

and the remaining three equations \( j = N_\Gamma + 1, N_\Gamma + 2, N_\Gamma + 3 \) for \( n = 1, 2, ..., N_\Gamma + 3 \) are obtained through constraints

\[
\psi_{n}^* = \psi_{n}^*(p_j),
\]

(15.16)

\[
b_j = 0,
\]

(15.17)
where the definition holds
\[ \psi_n^*(p_j) = 0; \quad j = N_f + 1, N_f + 2, N_f + 3. \]  
(15.18)

The coefficients \( \alpha_n^* \) can be expressed through inversion of the system (13), which gives
\[ \alpha_n^* = \sum_{j=1}^{N_r} \Psi_{nj}^{-1} \left[ \chi^O(p_j) \Phi^O(p_j) + \chi^N(p_j) \Phi^N(p_j) \right]; \]
\[ n = 1, 2, \ldots, N_f + 3 \]  
(15.19)

The velocity field components are calculated as
\[ v_\xi(p) = \sum_{j=1}^{N_r+3} \frac{\partial \psi_n^*}{\partial p_\xi}(p) \alpha_n^*; \quad \xi = x, y. \]  
(15.20)

The explicit form of the partial derivatives of the fundamental solution are
\[ \frac{\partial}{\partial p_\xi} f_n^*(p) = -\frac{1}{2\pi} \frac{P_\xi - S_{n\xi}}{(p_x - s_{n\xi})^2 + (p_y - s_{n\eta})^2}; \quad \xi = x, y, \]  
(15.21)
\[ \frac{\partial}{\partial p_\xi} f_{N_r+1}^*(p) = 0; \quad \xi = x, y, \]  
(15.22)
\[ \frac{\partial}{\partial p_x} f_{N_r+2}^*(p) = 1, \quad \frac{\partial}{\partial p_y} f_{N_r+2}^*(p) = 0, \]  
(15.23)
\[ \frac{\partial}{\partial p_x} f_{N_r+3}^*(p) = 0, \quad \frac{\partial}{\partial p_y} f_{N_r+3}^*(p) = 1. \]  
(15.24)

### 15.3.2 Classical Method of Fundamental Solutions

The fundamental solution source points are located outside physical boundary, i.e. \( p_j \neq s_j \) and \( s_j \notin \Omega \) in the classical MFS. One can consider that they form an artificial boundary. The proper location of the source points is
not a trivial task. It can be observed that the accuracy improves with the increasing distance from the physical boundary up to some extent. However, the collocation matrices become increasingly ill conditioned with increased distance from the boundary.

15.3.3 Modified Method of Fundamental Solutions

The key point of the modified method of fundamental solutions represents desingularisation of the value of the fundamental solution, because in this case the source and the collocation points coincide, i.e. $p_j = s_j$. The desingularisation value can be directly set as an average value of the fundamental solution over a portion of the boundary. This can be formulated as

$$
f_j^* (p_j) = \frac{1}{\ell_j} \int_{p_{j-1}}^{p_j} f_j^* \left[p(\Gamma)\right] d\Gamma + \frac{1}{\ell_{j+1}} \int_{p_j}^{p_{j+1}} f_j^* \left[p(\Gamma)\right] d\Gamma. \quad (15.25)
$$

The average value of the singularity at the boundary can be calculated in a closed form [16] from the direct BEM arsenal of analytical expressions

$$
f_j^* (p_j) = \frac{1}{4\pi} \left(1 + \log \frac{2r^*}{\ell_j}\right) + \frac{1}{4\pi} \left(1 + \log \frac{2r^*}{\ell_{j+1}}\right), \quad (15.26)
$$

where $\ell_j$ represents the Euclidean distance between points $p_{j-1}$ and $p_j$ on the boundary. (For details see Section 3.4). The derivatives of the fundamental solution can be calculated in the following indirect way. Let us assume a pure Dirichlet problem with all the boundary values set to a constant $\Phi^D (p) = c; p \in \Gamma$. We obtain in this case

$$
\Phi (p_j) = c = \sum_{n=1}^{N_c} \psi_n^* (p_j) \alpha_n^c, \quad (15.27)
$$
\[
\frac{\partial}{\partial \xi} \Phi(p_j) = \sum_{n=1}^{N-1} \frac{\partial}{\partial \xi} f^*_n(p_j) \alpha^*_n = 0; \quad \xi = x, y.
\]

The desingularised value of the partial derivative can be calculated as

\[
\frac{\partial}{\partial \xi} f^*_j(p_j) = -\frac{1}{\alpha^*_j} \sum_{n=1}^{N} \frac{\partial}{\partial \xi} f^*_n(p_j) \alpha^*_n; \quad \xi = x, y.
\]

The desingularised value of the normal derivative can be calculated from the desingularised values of the partial derivatives as

\[
\frac{\partial}{\partial n} f^*(p_j) = \frac{\partial}{\partial p_x} f^*(p_j) n_x(p_j) + \frac{\partial}{\partial p_y} f^*(p_j) n_y(p_j).
\]

### 15.3.4 Calculation of the Normal on the Boundary

The internal and external boundaries are given by a vector of points \( p_k, k = 1, 2, ..., N^\beta_B, B = I, E \). The length \( \ell_k \) of the contour between the boundary points \( p_k \) and \( p_{k-1} \) is parametrised by the simple Euclidean distance

\[
\ell_k = \left[ (p_{kx} - p_{k-1x})^2 + (p_{ky} - p_{k-1y})^2 \right]^{1/2},
\]

with the cyclic index conditions \( k - 1 = N^\beta_I, k = 1, k + 1 = 1; k = N^\beta_E \). The total Euclidean length \( \ell_I \) of the boundary contour equals to

\[
\ell_I = \sum_{k=1}^{N^\beta_I} \ell_k.
\]

The position of the boundary contour between the boundary points can be estimated by the meshless approximation with the contour parameter \( \ell \)
\[
p_{\xi}(\ell) = \sum_{k=1}^{N_{\Gamma}^p+1} \psi_k(\ell) \alpha_k^\xi \; ; \; \xi = x, y.
\]
\[(15.33)\]

The cubic splines
\[
\psi_k(\ell) = \left| \ell - \ell_k \right|^3 \; ; \; k = 1, 2, ..., N_{\Gamma}^p,
\]
\[(15.34)\]

with the augmented functions
\[
\psi_{N_{\Gamma}^p+1}(\ell) = 1, \]
\[(15.35)\]
\[
\psi_{N_{\Gamma}^p+2}(\ell) = \ell, \]
\[(15.36)\]
\[
\psi_{N_{\Gamma}^p+3}(\ell) = \ell^2,
\]
\[(15.37)\]

are used for the global approximation. The following three compatibility conditions are needed
\[
\sum_{k=1}^{N_{\Gamma}^p+3} \psi_k(0) \alpha_k^\xi = \sum_{k=1}^{N_{\Gamma}^p+3} \psi_k(\ell_{\Gamma}) \alpha_k^\xi \; ; \; \xi = x, y,
\]
\[(15.38)\]
\[
\sum_{k=1}^{N_{\Gamma}^p+3} \frac{d}{d\ell} \psi_k(0) \alpha_k^\xi = \sum_{k=1}^{N_{\Gamma}^p+3} \frac{d}{d\ell} \psi_k(\ell_{\Gamma}) \alpha_k^\xi \; ; \; \xi = x, y,
\]
\[(15.39)\]
\[
\sum_{k=1}^{N_{\Gamma}^p+3} \frac{d^2}{d\ell^2} \psi_k(0) \alpha_k^\xi = \sum_{k=1}^{N_{\Gamma}^p+3} \frac{d^2}{d\ell^2} \psi_k(\ell_{\Gamma}) \alpha_k^\xi \; ; \; \xi = x, y,
\]
\[(15.40)\]
in order to ensure the continuity and the smoothness of the first and the second derivatives required in calculation of the bubble normal and curvature. The coefficients are calculated from a system of \(N_{\Gamma}^p + 3\) algebraic equations
\[
\sum_{n=1}^{N_{\Gamma}^p+3} \Psi_{kn} \alpha_k^\xi = b_k^\xi \; ; \; n = 1, 2, ..., N_{\Gamma}^p + 3, \; \xi = x, y.
\]
\[(15.41)\]
The first $N^B_\Gamma$ equations are obtained through collocation of Eq. (34) for $x$ and $y$ directions in collocation points $\ell_k; k = 1, 2, \ldots, N^B_\Gamma$ for $n = 1, 2, \ldots, N^B_\Gamma + 3$, distributed over $\Gamma^B$

$$\Psi^{x}_{kn} = \Psi_n(\ell_k); \xi = x, y,$$
$$b_k^x = p_{k\xi}; \xi = x, y.$$  (15.42)

The remaining 3 equations are obtained through the compatibility conditions

$$\Psi^{x}_{(N^B_\Gamma+1)n} = \Psi_n(\ell_1) - \Psi_n(0); \xi = x, y,$$
$$b_k^x = 0; k = N^B_\Gamma + 1, \xi = x, y,$$  (15.44)
$$\Psi^{x}_{(N^B_\Gamma+2)n} = \frac{d\Psi_n(\ell_1)}{d\ell} - \frac{d\Psi_n(0)}{d\ell}; \xi = x, y,$$
$$b_k^x = 0; k = N^B_\Gamma + 2, \xi = x, y,$$  (15.46)
$$\Psi^{x}_{(N^B_\Gamma+3)n} = \frac{d^2\Psi_n(\ell_1)}{d\ell^2} - \frac{d^2\Psi_n(0)}{d\ell^2}; \xi = x, y,$$
$$b_k^x = 0; k = N^B_\Gamma + 3.$$  (15.48)

The coefficients $\alpha^x_\xi$ and $\alpha^y_\xi$ can be expressed through inversion of the related two systems (15.41)

$$\alpha^x_\xi = \sum_{k=1}^{N^B_\Gamma} \Psi^{x}_{nk} p_k^x; \xi = x, y.$$  (15.50)

The components of the normal on the boundary can be explicitly calculated as

$$n_{rx} = \frac{dp_x}{d\ell} \left[ \left( \frac{dp_x}{d\ell} \right)^2 + \left( \frac{dp_y}{d\ell} \right)^2 \right]^{-1/2},$$
$$n_{ry} = \frac{dp_y}{d\ell} \left[ \left( \frac{dp_x}{d\ell} \right)^2 + \left( \frac{dp_y}{d\ell} \right)^2 \right]^{-1/2}. $$  (15.51)

15.4. Numerical Examples
Potential flow around a circle is considered for a numerical example. The flow is confined to a square (exterior) region $\Gamma^E$, $p_x^* \leq p_x \leq p_x^+$, $p_y^- \leq p_y \leq p_y^+$ with $p_x^* = -p_x^- = p_0$, $p_y^* = -p_y^- = p_0$. The Dirichlet boundary conditions are defined at the square boundaries as

$$\Phi^D(p_x, p_y) = v_0 p_y; \quad p_x = p_x^\pm, \quad p_y = p_y^\pm.$$  \hspace{1cm} (15.53)

The potential field, defined from the boundary conditions (44) gives the following solution for the velocity field

$$v_{0x} = 0,$$ \hspace{1cm} (15.54)

$$v_{0y} = v_0.$$ \hspace{1cm} (15.55)

A circular hole $\Gamma'$ (internal boundary) with the radius $r_0$, centered around point $p_c$ with coordinates $p_{cx} = \left( p_x^* + p_x^+ \right)/2$, $p_{cy} = \left( p_y^* + p_y^+ \right)/2$ with the Neumann boundary conditions

$$\Phi^N[p_x(\Gamma'), p_y(\Gamma')] = 0; \quad \nu \in \Gamma'$$  \hspace{1cm} (15.56)

is inserted into the square. The solution of the potential field is for $r_0 \ll p_0$ equal to

$$\Phi_{ana} = v_{0y} (p_y - p_{cy}) \left[ 1 + \frac{r_0^2}{(p_x - p_{cx})^2 + (p_y - p_{cy})^2} \right],$$  \hspace{1cm} (15.57)

with $v_{0y}$ defined from equation (15.55). The respective analytical solution for the velocity field is

$$v_{ana,x} = -v_{0x} \frac{2v_{0y}r_0^2 (p_x - p_{cx})(p_y - p_{cy})}{(p_x - p_{cx})^2 + (p_y - p_{cy})^2},$$  \hspace{1cm} (15.58)

$$v_{ana,y} = v_{0x} \left[ 1 + \frac{r_0^2}{(p_x - p_{cx})^2 + (p_y - p_{cy})^2} \right] - \frac{2v_{0y}r_0^2 (p_y - p_{cy})^2}{(p_x - p_{cx})^2 + (p_y - p_{cy})^2}.$$  \hspace{1cm} (15.59)
We set $r_0 = 0.1\text{m}$ and $p_0 = 0.5\text{m}$ for defining the geometry. The square sides are virtually divided into 50 equal length segments and the collocation points are put at each of the segment centers. The total number of discretisation points on the external boundary is set to $N^E = 200$. The circle is discretised by five different discretisations $N^I = 8, 16, 32, 64, 128$. In case of the MFS, the source points are moved for 5 nodal distances in the direction of the outward normal on the external boundary. In case of the internal boundary, the source point boundaries are put on the artificial boundary with radius $r_{0x} < r_0$. The distances between the source points on this boundary and the distances between the source points and corresponding collocation points are assumed to be the same, i.e.

$$r_0 - r_{0x} = 2\pi r_{0x} / N^I.$$  (15.60)

From this equation, the radius of the circle on which the source points are set, is calculated as

$$r_{0x} = \frac{r_0}{1 + \frac{2\pi}{N^I}}.$$  (15.61)

In the MMFS, the source points are coincident with the boundary points. The Root Mean Square (RMS) error of the MFS and MMFS solutions are defined as

$$\Phi_{\text{rms}} = \sum_{n=1}^{N^I} \left\{ \frac{1}{N^I} \left[ \Phi(p_n) - \Phi_{\text{ana}}(p_n) \right]^2 \right\}^{1/2},$$  (15.62)

$$v_{\xi \text{rms}} = \sum_{n=1}^{N^I} \left\{ \frac{1}{N^I} \left[ v_{\xi}(p_n) - v_{\xi \text{ana}}(p_n) \right]^2 \right\}^{1/2} ; \xi = x, y,$$  (15.63)

$$v_{\text{rms}} = \left( v_{x \text{rms}}^2 + v_{y \text{rms}}^2 \right)^{1/2}.$$  (15.64)

The RMS errors of the potential, velocity components, and absolute value of velocity are in case of MFS given in Table 1 as a function of the discretisation density $N^I$. One can observe monotone convergence of the results with finer discretisation. Similar results are given in Table 2 for MMFS. The calculated potential is in the case of MFS calculated more
accurately as in the case of MMFS for \( N' = 8,16 \). The velocity components and absolute value of velocity are better predicted by MMFS than in MFS in all attempted discretisations.

Fig. 3a: Potential around circle. MFS. Note the artificial boundary. The normals are calculated through Eqs. (15.51,15.52).
Fig. 3b: Potential around circle. MMFS. Note the absence of artificial boundary. The normals are calculated through Eqs. (15.51,15.52).
Fig. 4a: Calculated potential as a function of the square height at the square centerline. MFS. The jump in the derivative coincides with the artificial boundary position.

Fig. 4b: Calculated potential as a function of the square height at the square centerline. MMFS. The jump in the derivative coincides with the physical boundary position.
Fig. 5a: Calculated potential as a function of the square width at the square centerline. MFS.

Fig. 5b: Calculated potential as a function of the square width at the square centerline. MMFS.
Fig. 6: Calculated potential on the circle. MMFS.

Fig. 7: Calculated x component of the velocity on the circle. MMFS.
Fig. 8: Calculated y component of the velocity on the circle. MMFS.

Fig. 9: Calculated absolute value of the velocity on the circle. MMFS.
Tab. 15.1: RMS error of the MFS as a function of the circle discretisation.

<table>
<thead>
<tr>
<th>$N'$</th>
<th>$\Phi_{\text{rms}}$</th>
<th>$v_{\text{rms}}$</th>
<th>$v_{\text{y rms}}$</th>
<th>$v_{\text{rms}}$</th>
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<td>6.9785379E-02</td>
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Tab. 15.2: RMS error of the MMFS as a function of the circle discretisation.

<table>
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<th>$N'$</th>
<th>$\Phi_{\text{rms}}$</th>
<th>$v_{\text{rms}}$</th>
<th>$v_{\text{y rms}}$</th>
<th>$v_{\text{rms}}$</th>
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<td>1.0623918E-02</td>
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<tr>
<td>128</td>
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<td>9.0229781E-03</td>
<td>1.0427632E-02</td>
</tr>
</tbody>
</table>

15.5. Conclusions

The MFS represents a numerical technique best applicable in situations where the fundamental solution of the partial differential equation is known. In recent years, the MFS has proved to be an effective alternative to the boundary element methods in specific problems. Due to its advantages with respect to the simplicity of formulation and the fact that the distribution of the calculation nodes is truly meshless, the method is an ideal candidate for the moving and free boundary problems. Its main drawback represents the “artificial boundary issue”. This issue has been in this work overcome.
through the MMFS concept. The desingularisation has been in the present work made in a direct BEM sense. The desingularisation of the spatial derivatives has been made in an indirect way through the constant potential field concept. Both approaches differ from the previous two pioneering works on the subject [14,15]. In addition, this paper extends the MMFS to potential flow situations. The components of the flow field are calculated more precisely with the MMFS as with the MFS at all discretisations used. The presented developments can be straightforwardly upgraded to axisymmetric problems [11] by inclusion of the axisymmetric fundamental solution. The flow physics can be extended to Navier-Stokes flow by the strategy, proposed in [17] which used the dual reciprocity with radial basis functions. The axisymmetric radial basis functions, such as thin plate splines [18] and multiquadrics [19] can be used for this purpose in axisymmetry.

Acknowledgement

This paper forms a part of the project J2-0099: Multiscale Modelling and Simulation of Liquid-Solid Systems, sponsored by the Slovenian Research Agency. The financial support is kindly acknowledged.

References


Modified MFS for Potential Flow Problems
Abstract. In order to analyze microstrip patch antennas of arbitrary shape the method of fundamental solutions is proposed. This method consists mainly in approximating the solution of a problem by a linear combination of fundamental solutions with respect to source points located outside the domain. Results obtained with the method of fundamental solutions combined with formulas for taking into account the fringing fields and the losses at the cavity are presented and compared with results obtained from experiments and from other results available in the literature.

16.1. Introduction

Microstrip antennas, or printed antennas, basically consist of a metal patch constructed on a dielectric substrate, see Fig.16.1, usually employing the same sort of lithographic patterning used to fabricate printed circuit boards.

The popularity of microstrip antennas stems from the following:
- they are relatively inexpensive to manufacture and design because of the simple 2-dimensional physical geometry;

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3 DECivil/ICIST, Instituto Superior Técnico, TULisbon, Av. Rovisco Pais, 1049-001 Lisboa, Portugal (e-mail: vitor@civil.ist.utl.pt).
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they can be adapted to a variety of geometries;
they are mechanically rugged.

For a microstrip patch, printed on a thin substrate, lying in the xy plane, and fed by a z directed current, the electric field $E$ in the substrate volume, between the patch and the ground plane, can be assumed to have only the z-direction component.

This is the basic assumption of the resonant cavity model [1] which is one of the more commonly used models for the analysis of microstrip antennas.

In this model, and from the assumption that there is only the z-component, the behaviour of the microstrip antenna may be described by the Helmholtz equation subject to Neumann boundary conditions as follows:

\[
\begin{align*}
\left(\nabla^2 + k^2\right) E_z &= j\omega \mu J_z \quad \text{in } \Omega, \\
\frac{\partial E_z}{\partial n} &= 0 \quad \text{in } \partial \Omega.
\end{align*}
\]

where $k$ is the wave number, $\omega$ is the angular frequency, $\mu$ is the magnetic permeability of the substrate, $n$ is the outward normal on the boundary of the patch and the input current $J_z$ can be modelled as $J_z = I\delta(r - r_0)$ for the case of a coaxial feed.

The main difficulty with this equation is that analytical solutions exist only for antennas of certain regular shapes. For antennas of arbitrary
shape we have to find a numerical solution, for which several methods have been proposed:

- analytical or quasi-analytical methods, such as the transmission line model and the cavity model, not suitable for general design;
- full-wave complex methods, such as finite differences, finite elements and methods of moments, see [2]-[3]. These methods are applicable to arbitrary configuration and are usually very accurate.

In this work we propose the use of the Method of Fundamental Solutions for solving the problem of microstrip antennas. The Method of Fundamental Solutions (which has already been applied to other Helmholtz problems, see [7], [11]) is somehow in between the two indicated classes of methods: it is simple to implement, can be used in the analysis of microstrip patch antennas of arbitrary geometry and provides reasonably accurate results.

Parameters of interest for the analysis of antennas are obtained and compared (in terms of accuracy and computational cost) with results available in the literature and obtained by other numerical methods.

### 16.2. The Method of Fundamental Solutions

The main idea of the method of fundamental solutions (MFS) is to approximate the solution of a problem, given as a partial differential equation with boundary conditions, by a linear combination of fundamental solutions with sources located outside the domain. For the Helmholtz equation:

\[
\left( \nabla^2 + k^2 \right) E_z = j \omega \mu \delta(r - r_0) .
\]  

A particular solution (not satisfying the boundary conditions of (16.1)) is:

\[
E_z = \frac{\omega \mu l}{4} H_0^1(k|r - r_0|) .
\]

Equation (1) can be written as:
\[
\left\{ \begin{array}{l}
L u = j \omega \mu I \delta(r - r_0) \quad \text{in } \Omega, \\
\frac{\partial u}{\partial n} = q = 0 \quad \text{in } \partial \Omega.
\end{array} \right. \tag{16.4}
\]

where the linear operator \( L \) is \( L = \nabla^2 + k^2 \).

So the general solution to (16.4) can be written as:
\[
u = \hat{u} + \bar{u}, \tag{16.5}
\]

with
\[
\bar{u} = \frac{\omega \mu l}{4} H_0^1(k|r - r_0|). \tag{16.6}
\]

Now, in order to find the general solution to (16.4), it is necessary to find \( \hat{u} \).

The conditions that \( \hat{u} \) must satisfy are:

Since \( Lu = j \omega \mu \delta(r - r_0) \) and \( L\bar{u} = j \omega \mu \delta(r - r_0) \) then:
\[
L\hat{u} = (\nabla^2 + k^2)\hat{u} = 0 \tag{16.7}
\]

Since \( q = \bar{q} + \hat{q} = 0 \) then:
\[
\hat{q} = -\bar{q}, \tag{16.8}
\]

where
\[
\bar{q} = \frac{\hat{n}}{\partial n} = -\frac{\omega \mu k l}{4} H_1^{(1)}(k|r - r_0|) \left( \frac{r - r_0}{|r - r_0|} \cdot \vec{n} \right). \tag{16.9}
\]

It can be observed that if we consider \( N \) points outside \( \Omega \), which will be called source points, then the functions:
\[
u_j = \frac{j}{4} H_0^{(3)}(k|r - r_j|), \tag{16.10}
\]

and their corresponding normal derivatives:
\[
q_j = -\frac{jk}{4} H_1^{(1)}(k|r - r_j|) \left( \frac{r - r_j}{|r - r_j|} \cdot \vec{n} \right), \tag{16.11}
\]

satisfy (16.4).

For regular shapes, it is well known (in the MFS community) that the source points can be located almost at any curve outside \( \Omega \).
Let us now write $\hat{u}$ in terms of $u_j$ as:

$$\hat{u} = \sum_{j=1}^{N} c_j u_j,$$

Then:

$$\hat{q} = \sum_{j=1}^{N} c_j q_j,$$

and the condition $\hat{q} = -\bar{q}$ can be written as:

$$\hat{q} = \sum_{j=1}^{N} c_j q_j = -\vec{q} = \frac{\omega_{ijkl}}{4} H^{(1)}_{i}(k|r-r_0)\left(\frac{r-r_0}{r_0} \cdot \vec{n}\right).$$

By applying the last equation to M points in the boundary:

$$\hat{q}(r_i) = \sum_{j=1}^{N} c_j q_j(r_i) = \frac{\omega_{ijkl}}{4} H^{(0)}_{i}(k|r_i-r_0)\left(\frac{r_i-r_0}{r_i-r_0} \cdot \vec{n}_i\right)$$

$$\vdots$$

$$\hat{q}(r_M) = \sum_{j=1}^{N} c_j q_j(r_M) = \frac{\omega_{ijkl}}{4} H^{(0)}_{i}(k|r_M-r_0)\left(\frac{r_M-r_0}{r_M-r_0} \cdot \vec{n}_M\right),$$

a linear system of equations is obtained as follows:
By solving (16.16), the unknowns of the problem, that is the $c_j$ coefficients, are obtained. From these it is possible to compute $\hat{u}$ after which the complete solution of (16.4), representing the electric field under the patch, is obtained.

16.3. Determination of Input Parameters

In the case of regular antennas the effect of the fringing field is taken into consideration by means of empirical formulas. These formulas allow for the calculation of the effective dimensions [4] of the antenna. For the case of irregular patches effective dimensions can be approximated by using the formulas for rectangular patches [5].

The input impedance of the antenna can be computed as:

$$Z_i = -tE_z(r_0)/I,$$

where $E_z(r_0)$ is the electric field in the feeding point evaluated with a modified value of $k$, $k_{\text{eff}}$ which is calculated by determining the radiation loss, dielectric loss and copper loss [6] and $t$ is the thickness of the substrate. The resonance frequency can be found as the frequency for which the maximum of the real part of the input impedance occurs. This can be easily determined computationally due to the fast convergence of the Method of Fundamental Solutions, by constructing a function that returns the impedance having as input parameter the frequency and having as constants the geometry of the patch and other constant parameters.
In a case without losses, it can be possible to calculate the resonance frequency simply by studying the eigenvalues of the system matrix in equation (16.16), see [7].

16.4. Examples and Results

In these examples the method of fundamental solutions was used to find the resonance frequencies of several antennas of different shapes and parameters. The results obtained by applying the Method of Fundamental Solutions together with formulas for the effective dimensions of the antennas are shown to be in agreement with other results available in the literature. It is important to notice that for all the presented cases there are analytical solutions for the electric field. The results for the electrical field under the patch for the case when 40 points are taken at the boundary exhibit errors of less than 5% from the analytical solution and this error decreases to less than 2% by doubling the number of points in the boundary. For this level of accuracy the proposed method outperforms, in terms of computational efficiency, most other methods.

In Figure 16.2, the simulated electric field (taking 80 points in the boundary) and its error as compared to the analytical solution for the rectangular antenna (case 1 in table 16.2), are shown:
Figure 16.2. Analytical solution electrical field distribution under the patch and percentage of error between the simulated and the analytical results.

The fact that the maximum error which occurs at some internal points is less than 1.5% of the value of $E_z$ at those points reveals the high accuracy this method can achieve.

*Circular patch antenna*
Figure 16.3. Circular patch antenna.

Table 16.1 presents the predictions of the resonant frequency obtained with the current method as compared to experimentally obtained ones and those available in the literature obtained with an enhanced cavity model [8] and with the method of moments [9].

<table>
<thead>
<tr>
<th>Case</th>
<th>$\varepsilon_r$</th>
<th>$\tan\delta$</th>
<th>$d$ (cm)</th>
<th>$d / \lambda_0$</th>
<th>$r$ (cm)</th>
<th>$r_o$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.2</td>
<td>0.0023</td>
<td>0.254</td>
<td>0.04</td>
<td>0.9920</td>
<td>0.2160</td>
</tr>
<tr>
<td>2</td>
<td>2.33</td>
<td>0.0013</td>
<td>0.079</td>
<td>0.01</td>
<td>1.4845</td>
<td>0.4150</td>
</tr>
<tr>
<td>3</td>
<td>2.20</td>
<td>0.0009</td>
<td>0.079</td>
<td>0.02</td>
<td>0.7502</td>
<td>0.2268</td>
</tr>
</tbody>
</table>

Table 16.1. Comparison of resonance frequency for the circular patch antenna.
Rectangular patch

Figure 16.4. Rectangular patch antenna.

Table 16.2 presents the same comparisons as in table 16.1 but now for the rectangular patch antenna.

<table>
<thead>
<tr>
<th>Case</th>
<th>$\varepsilon_r$</th>
<th>$\tan \delta$</th>
<th>$d$ (cm)</th>
<th>$d / \lambda_0$</th>
<th>$L$ (cm)</th>
<th>$W$ (cm)</th>
<th>$F$ (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>10.2</td>
<td>0.0023</td>
<td>0.127</td>
<td>0.01</td>
<td>2.00</td>
<td>3.00</td>
<td>0.65</td>
</tr>
<tr>
<td>2</td>
<td>10.2</td>
<td>0.0023</td>
<td>0.254</td>
<td>0.02</td>
<td>1.90</td>
<td>3.00</td>
<td>0.65</td>
</tr>
<tr>
<td>3</td>
<td>2.22</td>
<td>0.0009</td>
<td>0.079</td>
<td>0.01</td>
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<td>4.00</td>
<td>0.40</td>
</tr>
<tr>
<td>4</td>
<td>2.22</td>
<td>0.0009</td>
<td>0.079</td>
<td>0.02</td>
<td>1.25</td>
<td>2.00</td>
<td>0.20</td>
</tr>
</tbody>
</table>
Table 16.2. Comparison of resonance frequency for the rectangular patch antenna.

<table>
<thead>
<tr>
<th></th>
<th></th>
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<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>2.26</td>
<td>2.23</td>
<td>2.28</td>
<td>2.27</td>
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<tr>
<td>2</td>
<td>2.24</td>
<td>2.22</td>
<td>2.29</td>
<td>2.31</td>
</tr>
<tr>
<td>3</td>
<td>3.94</td>
<td>3.90</td>
<td>3.89</td>
<td>3.96</td>
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<tr>
<td>4</td>
<td>7.65</td>
<td>7.53</td>
<td>7.61</td>
<td>7.79</td>
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</table>

Equilateral triangular patch antenna

Figure 16.5. Equilateral triangular patch antenna.

Table 16.3 compares, for an equilateral triangular patch antenna, the results obtained with this method as compared to the measured ones and results obtained with the method of moments [10].

<table>
<thead>
<tr>
<th>Case</th>
<th>$\varepsilon_r$</th>
<th>$a$ (cm)</th>
<th>$h$ (cm)</th>
<th>$d$ (cm)</th>
</tr>
</thead>
</table>


Table 16.3. Comparison of resonance frequency for the equilateral triangular patch antenna.

Resonance resistance for the rectangular patch antenna

For the case of the rectangular patch antenna the present results (in terms of the resonance resistance) are compared to those obtained with an enhanced cavity model [8] and those obtained with the method of moments by [9]. These results show a very good accuracy for this method in the calculation of the resonance resistance.

Table 16.4. Comparison of input impedance for the circular patch antenna.

Experimental results and comparison with the MFS

Two of the previous discussed antennas where simulated with the program ensemble, constructed and experimentally analyzed with an Agilent Technologies Technologies E8361A network analyzer.
Figure 16.5. Constructed patch antennas.
Figure 16.6. Comparison of measured return loss with results expected from MFS and Ensemble for the circular antenna.

Figure 16.7. Comparison of measured return loss with results expected from MFS and Ensemble for the rectangular antenna.
The antennas were constructed in Duroid 5880 substrate with $\varepsilon_r = 2.22$, $h = 0.787\text{mm}$, $\tan\delta = 0.0009$. The dimensions of the rectangular antenna were: $W = 20\text{mm}, L = 12.5\text{mm}, F = 2\text{mm}$. And those of the circular antenna were: $r = 7.502\text{mm}, r_0 = 2.268\text{mm}$.

The results, similar to those obtained from Ensemble, reveal, as seen in the previous two figures, reasonably high accuracy. The differences in the results of the resonance frequency, less than 6%, are slightly lower for the circular case than for the rectangular case.

### 16.5. Conclusions

The method of the fundamental solution works as a very simple, fast and accurate method for the computation of the electric field under the patch of a microstrip antenna. The results obtained show that the present method, with the use of effective dimensions for the patch, can lead to quite accurate values for the input parameters of the antenna. The high convergence rate of this method compared to other methods previously used allows for an easy determination of those parameters simply by finding, computationally, the maximum of a single-variable function (and this is a simple task).

### Acknowledgments

This work was supported by the Technical University of Lisbon, by Instituto de Telecomunicações (where the experiments were carried out) and by the Alfa ELBENET EU funded project. We would also like to thank Carlos Brito and António Almeida, from IT/IST, for their help in the antennas fabrication and testing.

### References


