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Advances in Engineering Software 32 (2001) 111–117

ADVANCES IN ENGINEERING SOFTWARE

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Applications of the complete multiple reciprocity method for solving the 1D Helmholtz equation of a semi-infinite domain

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Received 31 January 2000; revised 24 April 2000; accepted 1 June 2000

Abstract

In this paper, the complete multiple reciprocity method is adopted to solve the one-dimensional (1D) Helmholtz equation for the semi-infinite domain. In order to recover the information that is missing when the conventional multiple reciprocity method is used, an appropriate complex number in the zeroth order fundamental solution is added such that the kernels derived using this proposed method are fully equivalent to those derived using the complex-valued formulation. Two examples including the Dirichlet and Neumann boundary conditions are investigated to show the validity of the proposed method analytically and numerically. The numerical results show good agreement with the analytical solutions.

Keywords: Complete multiple reciprocity method; Helmholtz equation; Semi-infinite domain

1. Introduction

Solving partial differential equations for an infinite domain is practically important in engineering applications. For example, the earth has often been modeled as a semi-infinite domain in the field of earthquake engineering, and the sea has also been modeled as a semi-infinite domain in underwater acoustic problems. The conventional finite element method encounters difficulty with this kind of problem since discretization of an infinite domain is numerically impractical [1]. Use of a finite domain to approximate an infinite domain has been adopted [2] in finite element analysis. In this algorithm, a 'large enough' finite domain is selected to simulate the infinite domain. In this method, the boundary data of this finite domain is assumed to be negligible so that it will not affect the results inside this finite domain. However, how large this finite domain should be so that the boundary data can be neglected depends on the problem of concern. To overcome this difficulty, another approach, the so-called DtN method [3–6], has been employed. In the DtN method, appropriate boundary conditions are given on the boundary of this artificially selected finite domain, so that the dissipation of energy can be correctly modeled. For solving infinite domain problems, Wolf and Song [7,8] have recently developed a boundary element formulation based on the finite element scheme. They have used the interpolation function to approximate the physical quantities on the boundary and have derived an ordinary differential equation in the direction normal to the boundary. The method they developed needs neither the fundamental solution required in the traditional boundary element method nor discretization inside the infinite domain, which is needed in the finite element method (although the finite element method only discretizes a large enough artificial finite domain). For the infinite domain problem, it seems that the boundary element method is an appropriate choice since the fundamental solution employed automatically satisfies the radiation condition at infinity. The complex-valued boundary element formulation has been successfully used to solve the Helmholtz equation for the infinite domain [9–11]. However, fictitious eigenfrequencies, which depend on the integral formulation one uses, arise when this approach is adopted [12–14].

The multiple reciprocity method (MRM) has also been adopted to solve the vibration problem [15,16] for a finite structure, where the governing equation is the Helmholtz equation; however, the problem of spurious eigenvalues is encountered in this case. Chen and Wong [17] proposed combined use of the singular integral and hypersingular integral equations to filter out such spurious eigenvalues

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0965-9978/01/$ - see front matter
PII: S0965-9978(00)00082-X

for rod and cavity problems. Yeih et al. [18] recently found that not only the spurious eigenvalue problem, but also the possible indeterminacy of boundary eigenvectors may be encountered in the beam vibration problem. They proposed use of the singular value decomposition method to filter out the spurious eigenvalues and determine the boundary eigenvector at the same time. Another useful approach, the domain partition technique, has also been proposed by Chang et al. [19] to eliminate the spurious eigenvalues in the root searching stage whatever the singular or hypersingular equation is chosen for the dual integral formulations. Kamiya et al. [20] found that the kernels of the conventional MRM are no more than the real parts of the kernels in the complex-valued formulation by providing a two-dimensional (2D) example. Yeih et al. [21] proved that kernels in the conventional MRM are no more than the real parts of kernels in the complex-valued formulation. Furthermore, they proposed a complete MRM whose kernels are equivalent to the kernels of the complex-valued formulation by adding an appropriate complex number in the zeroth order fundamental solution. The existence of spurious eigenvalues in the conventional MRM stems from the fact that the kernels in the conventional MRM lose the information contributed by the imaginary part. This is known to be very important for the vibration problem, which can be understood when one solves the wave equation in the wave number domain for a finite region. However, applications of the MRM to solve the wave equation in the wave number domain for an infinite region have not been proposed to the authors' knowledge.

In this paper, the complete MRM is adopted to solve the Helmholtz equation for an infinite domain. The reason why the conventional MRM fails to find the solution for this kind of problem is explained. One-dimensional examples, including the Dirichlet and Neumann excitation problems, are given to show the validity of the current method. Numerical results match the analytical results very well. When the wave number is very large or the observation point is far away from the source, numerical error may become large depending on the calculation precision of the computer one uses. To obtain the far field behavior or response for a large wave number, the shifting technique is introduced to treat this kind of problem.

2. Review of the conventional MRM and complete MRM

The MRM has been proposed to solve the Helmholtz equation for a finite region. The basic idea of the MRM is to treat the Helmholtz equation as a Poisson equation. After Green's identity is introduced, a domain integral involving the 'external source' is treated using a recursive formula such that this domain integral can be transformed into a series of boundary integrals. Thus, the remaining domain integral term will approach zero when the number of terms in the series tends to infinity. This method has two merits: (1) the wave number is separated from the determinacy of the leading coefficient matrix such that we need not recompute the leading coefficient matrix when the wave number varies; (2) the leading coefficient matrix is real-valued. However, application of the conventional MRM to an infinite domain may encounter difficulty, and this will be explained in the following.

Consider the harmonic wave excitation problem in a semi-infinite medium; the governing equation is the Helmholtz equation and can be written as

$$\frac{\mathrm{d}^2u(x)}{\mathrm{d}x^2} + k^2u(x) = 0, \qquad 0 \le x \le \infty \tag{1}$$

where $u(x)$ is the axial displacement of the rod and k is the wave number.

The auxiliary system is considered as

$$\frac{\mathrm{d}^2U(x,s)}{\mathrm{d}x^2} = \delta(x-s), \qquad -\infty < x < \infty \tag{2}$$

where $U(x,s)$ is called the fundamental solution of the Laplace operator and is equal to $|x-s|/2$ in the conventional MRM, and $\delta(\cdot)$ is the Dirac delta function. Green's identity gives

$$\begin{aligned} u(s) &= \int_0^\infty \frac{\mathrm{d}^2U(x,s)}{\mathrm{d}x^2}u(x)\,\mathrm{d}x \\ &= \left[\frac{\mathrm{d}U(x,s)}{\mathrm{d}x}u(x) - U(x,s)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\right]_0^\infty \\ &\quad -k^2\int_0^\infty U(x,s)u(x)\,\mathrm{d}x, \end{aligned} \tag{3}$$

where a domain integral remains in the formulation. The domain integral can be transformed into a series of boundary integral terms by using the following procedures.

Define the zeroth order fundamental solution, $U^{(0)}(x,s) \equiv U(x,s)$, and use the recursive formulae

$$\nabla^2U^{(j+1)}(x,s) = U^{(j)}(x,s) \qquad \text{for } j \ge 0; \tag{4}$$

then Eq. (3) can be reduced to

$$\begin{aligned} u(s) = \Bigg[&\left(\sum_{j=0}^{N}(-k^2)^j\frac{\mathrm{d}U^{(j)}(x,s)}{\mathrm{d}x}\right)u(x) \\ &-\left(\sum_{j=0}^{N}(-k^2)^jU^{(j)}(x,s)\right)\frac{\mathrm{d}u(x)}{\mathrm{d}x}\Bigg]_0^\infty - R_{N+1}, \end{aligned} \tag{5}$$

where

$$R_{N+1}(s) \equiv \int_0^\infty U^{(N+1)}(x,s)u(x)\,\mathrm{d}x. \tag{6}$$

Let N approach infinity; Eq. (5) becomes

$$u(s) = [T_R(x,s)u(x) - U_R(x,s)t(x)]_0^\infty, \tag{7}$$

where

$$U_R(x,s) \equiv \lim_{N\to\infty} \sum_{j=0}^{N} (-k^2)^j U^{(j)}(x,s) = \frac{\sin k|x-s|}{2k}, \tag{8}$$

$$T_R(x,s) \equiv \lim_{N\to\infty} \sum_{j=0}^{N} (-k^2)^j \frac{\mathrm{d}U^{(j)}(x,s)}{\mathrm{d}x} = \begin{cases} \dfrac{\cos k(x-s)}{2} & x > s \\ \dfrac{-\cos k(s-x)}{2} & s > x \end{cases} \tag{9}$$

and

$$t(x) \equiv \frac{\mathrm{d}u(x)}{\mathrm{d}x}. \tag{10}$$

It is known that $u(x)$ should satisfy the radiation condition

$$\lim_{x\to\infty} [t(x) - \mathrm{i}ku(x)] = 0 \tag{11}$$

at infinity. Therefore, we obtain the following equation from Eq. (7):

$$\begin{aligned} u(s) &= \lim_{x\to\infty} [T_R(x,s)u(x) - U_R(x,s)t(x)] \\ &\quad - [T_R(0,s)u(0) - U_R(0,s)t(0)] \\ &= \lim_{x\to\infty} \left[\frac{\cos k(x-s)}{2} - \frac{\sin k(x-s)}{2k}(\mathrm{i}k) \right] u(x) \\ &\quad - [T_R(0,s)u(0) - U_R(0,s)t(0)] \\ &= \lim_{x\to\infty} \frac{\mathrm{e}^{-\mathrm{i}k(x-s)}}{2} u(x) - [T_R(0,s)u(0) - U_R(0,s)t(0)]. \end{aligned} \tag{12}$$

The first term in Eq. (12) cannot be zero and is undetermined; therefore, the conventional MRM cannot be used to solve the Helmholtz equation for an infinite domain. The reason for the failure stems from the fact that the kernels derived cannot satisfy the radiation condition at infinity. To recover this information that is lost when the conventional MRM is used, we have to introduce an appropriate complex constant into the zeroth order fundamental solution at the very beginning; this is the so-called complete MRM. Let us go back to Eq. (2) and choose

$$U_c^{(0)}(x,s) \equiv U_c(x,s) = \frac{|x-s|}{2} + \frac{1}{2\mathrm{i}k}. \tag{13}$$

The difference between the new zeroth-order fundamental solution in the complete MRM and the conventional zeroth-order fundamental solution in the conventional MRM is only a complex constant; however, this new zeroth-order fundamental solution still satisfies Eq. (2).

Following the above-mentioned process, we have

$$u(s) = \left[\left(\sum_{j=0}^{N} (-k^2)^j \frac{\mathrm{d}U_c^{(j)}(x,s)}{\mathrm{d}x} \right) u(x) - \left(\sum_{j=0}^{N} (-k^2)^j U_c^{(j)}(x,s) \right) t(x) \right]_0^\infty - R_{N+1}^*, \tag{14}$$

where

$$\nabla^2 U_c^{(j+1)}(x,s) = U_c^{(j)}(x,s) \qquad \text{for } j \geq 0$$

and

$$R_{N+1}^*(s) \equiv \int_0^\infty U_c^{(N+1)}(x,s)u(x)\,\mathrm{d}x.$$

By taking N to approach infinity, Eq. (14) can be rewritten as

$$u(s) = [T_c(x,s)u(x) - U_c(x,s)t(x)]_0^\infty, \tag{15}$$

where

$$U_c(x,s) \equiv \lim_{N\to\infty} \sum_{j=0}^{N} (-k^2)^j U_c^{(j)}(x,s) = \frac{\mathrm{e}^{\mathrm{i}k\|x-s\|}}{2\mathrm{i}k}, \tag{16}$$

$$T_c(x,s) \equiv \lim_{N\to\infty} \sum_{j=0}^{N} (-k^2)^j \frac{\mathrm{d}U_c^{(j)}(x,s)}{\mathrm{d}x} = \begin{cases} \dfrac{\mathrm{e}^{\mathrm{i}k(x-s)}}{2} & x > s \\ \dfrac{-\mathrm{e}^{\mathrm{i}k(s-x)}}{2} & s > x \end{cases}. \tag{17}$$

Since $u(x)$ needs to satisfy the radiation condition at infinity as Eq. (11) states, Eq. (15) becomes:
UT equation

$$\begin{aligned} u(s) &= \lim_{x\to\infty} [T_c(x,s)u(x) - U_c(x,s)t(x)] \\ &\quad - [T_c(0,s)u(0) - U_c(0,s)t(0)] \\ &= \lim_{x\to\infty} \left[\frac{\mathrm{e}^{\mathrm{i}k(x-s)}}{2} - \frac{\mathrm{e}^{\mathrm{i}k(x-s)}}{2\mathrm{i}k}(\mathrm{i}k) \right] u(x) \\ &\quad - [T_c(0,s)u(0) - U_c(0,s)t(0)] = \frac{\mathrm{e}^{\mathrm{i}ks}}{2\mathrm{i}k} t(0) + \frac{\mathrm{e}^{\mathrm{i}ks}}{2} u(0) \end{aligned} \tag{18}$$

It can be seen from Eq. (18) that the terms involving x approaching infinity just cancel each other out and result in no contribution at all since the kernel functions used satisfy the radiation condition at infinity. For the finite term approach, Eq. (14) becomes an approximate equation:

Table 1
Kernels used in the complete MRM method

Kernels	$x > s$	$x < s$
$U^{(0)}(x,s)$	$1/2(x-s)/1! + 1/2\mathrm{i}k$	$1/2(s-x)/1! + 1/2\mathrm{i}k$
$T^{(0)}(x,s)$	$1/2$	$-1/2$
$L^{(0)}(x,s)$	$-1/2$	$1/2$
$M^{(0)}(x,s)$	0	0
$U^{(j+1)}(x,s)$	$1/2(x-s)^{2j+3}/(2j+3)! + 1/2\mathrm{i}k(x-s)^{2j+2}/(2j+2)!$	$-1/2(x-s)^{2j+3}/(2j+3)! + 1/2\mathrm{i}k(x-s)^{2j+2}/(2j+2)!$
$T^{(j+1)}(x,s)$	$1/2(x-s)^{2j+2}/(2j+2)! + 1/2\mathrm{i}k(x-s)^{2j+1}/(2j+1)!$	$-1/2(x-s)^{2j+2}/(2j+2)! + 1/2\mathrm{i}k(x-s)^{2j+1}/(2j+1)!$
$L^{(j+1)}(x,s)$	$-1/2(x-s)^{2j+2}/(2j+2)! - 1/2\mathrm{i}k(x-s)^{2j+1}/(2j+1)!$	$1/2(x-s)^{2j+2}/(2j+2)! - 1/2\mathrm{i}k(x-s)^{2j+1}/(2j+1)!$
$M^{(j+1)}(x,s)$	$-1/2(x-s)^{2j+1}/(2j+1)! - 1/2\mathrm{i}k(x-s)^{2j}/(2j)!$	$1/2(x-s)^{2j+1}/(2j+1)! - 1/2\mathrm{i}k(x-s)^{2j}/(2j)!$

UT equation

$$u(s) \cong -\left[\left(\sum_{j=0}^{N}(-k^2)^j T_c^{(j)}(0,s)\right)u(0) - \left(\sum_{j=0}^{N}(-k^2)^j U_c^{(j)}(0,s)\right)t(0)\right], \tag{19}$$

where

$$T_c^{(j)}(x,s) \equiv \frac{\mathrm{d}U_c^{(j)}(x,s)}{\mathrm{d}x}.$$

When N approaches infinity, Eq. (19) becomes an exact equation. By taking the derivative of Eq. (19), the approximate equation for the secondary field, $t(x)$, can be obtained
LM equation

$$t(s) \cong -\left[\left(\sum_{j=0}^{N}(-k^2)^j M_c^{(j)}(0,s)\right)u(0) - \left(\sum_{j=0}^{N}(-k^2)^j L_c^{(j)}(0,s)\right)t(0)\right], \tag{20}$$

where

$$L_c^{(j)}(x,s) \equiv \frac{\mathrm{d}U_c^{(j)}(x,s)}{\mathrm{d}s}, \tag{21}$$

$$M_c^{(j)}(x,s) \equiv \frac{\mathrm{d}^2U_c^{(j)}(x,s)}{\mathrm{d}s\,\mathrm{d}x}. \tag{22}$$

Eqs. (19) and (20), thus, form the complete dual MRM. The kernel functions used in the complete dual MRM are listed in Table 1. If N approaches infinity, Eq. (20) becomes:
LM equation

$$t(s) = \frac{e^{\mathrm{i}ks}}{2}t(0) + \frac{\mathrm{i}ke^{\mathrm{i}ks}}{2}u(0). \tag{23}$$

For a detailed discussion of the complete MRM and the conventional MRM, the reader may refer to Ref. [21].

3. Demonstrated examples

Two examples, including the Dirichlet and Neumman excitation problems, are presented in the following.

Example 1: Dirichlet excitation
$(\mathrm{d}^2u(x)/\mathrm{d}x^2) + k^2u(x) = 0$ for $0 \le x \le \infty$, $u(0) = 1$ and $u(x)$ satisfies the radiation condition at infinity. The analytical solution can be easily obtained as $u(x) = e^{\mathrm{i}kx}$.

By using Eq. (18) and taking s to approach zero, we have

$$(1 - \tfrac{1}{2})u(0) = \frac{1}{2\mathrm{i}k}t(0). \tag{24}$$

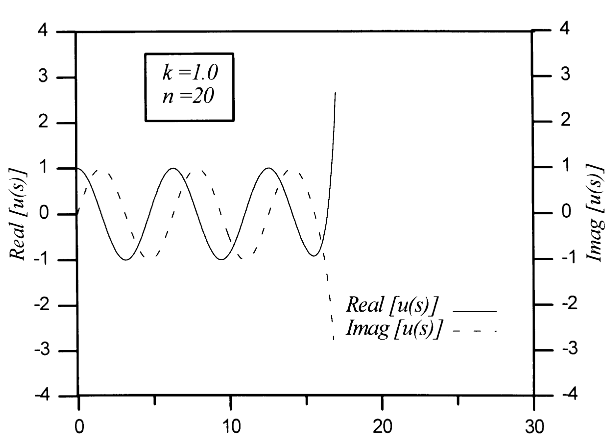

Fig. 1. Numerical divergence in the far field for the case of Dirichlet excitation using the complete *UT* MRM method.

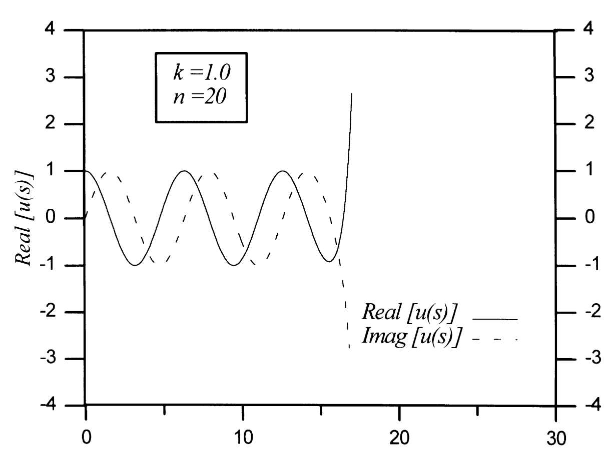

Fig. 2. Numerical divergence in the far field for the case of Dirichlet excitation using the complete *LM* MRM method.

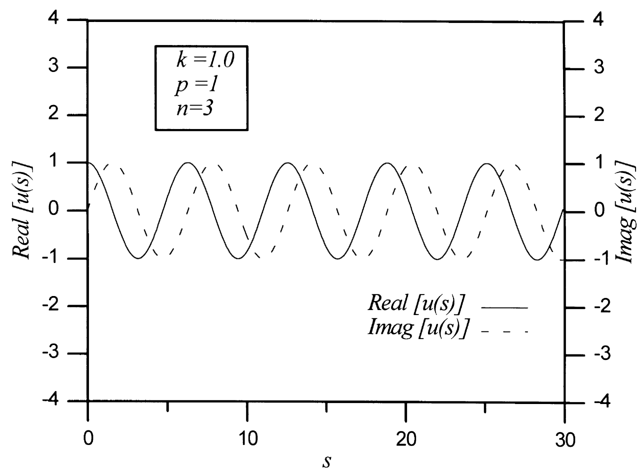

Fig. 3. Improvement of numerical divergence obtained by using the small marching step technique: the *UT* method.

Substituting the boundary condition into Eq. (24) yields

$$t(0) = \tfrac{1}{2}2\mathrm{i}k = \mathrm{i}k. \tag{25}$$

Substituting the boundary data into Eq. (18), we have

$$u(s) = \frac{\mathrm{e}^{\mathrm{i}ks}}{2\mathrm{i}k}t(0) + \frac{\mathrm{e}^{\mathrm{i}ks}}{2}u(0) = \frac{\mathrm{e}^{\mathrm{i}ks}}{2\mathrm{i}k}\mathrm{i}k + \frac{\mathrm{e}^{\mathrm{i}ks}}{2} = \mathrm{e}^{\mathrm{i}ks}. \tag{26}$$

This solution is exactly the same as the analytical solution. Alternatively, we can use Eq. (23) by taking s to approach zero and substituting the boundary condition; then the following equation can be obtained:

$$t(0) = \frac{1}{2}t(0) + \frac{\mathrm{i}k}{2}u(0) \Rightarrow t(0) = \mathrm{i}k. \tag{27}$$

The numerical solutions obtained using the *UT* and *LM* equations with the finite terms (Eqs. (19) and (20)) are shown in Figs. 1 and 2, respectively. In these two figures, the wave number is set to be 1. It can be found that the numerical solution becomes better as the number of series terms becomes larger. However, the numerical solution becomes very poor when the field point exceeds some range and locates in the far field. The reason stems from the numerical error of the computer when the trigonometric functions are expanded in a power series form [22]. To overcome this difficulty, we can set an artificial range p and then calculate the response for the field point s located between $(m-1)p$ and mp in the mth stage, where m is a positive number. If the range p is small enough, we can obtain accurate results. After obtaining the boundary data at $s = mp$, we can shift to the next range and recompute again. The algorithm is given below.

Step 1: Set $m = 1$.
Step 2: Solve $(\mathrm{d}^2u(x)/\mathrm{d}x^2) + k^2u(x) = 0$ for $0 \le x \le \infty$ with the boundary condition. By using Eq. (19) or Eq. (20) and letting s approach zero, we can solve for the unknown boundary data. (In this example, the unknown boundary data is $t(1)$.)
Step 3: Calculate the response of $u(s)$ in the range $s = (m-1)p$ to mp by substituting the boundary data into Eq. (19).
Step 4: Set $m = m + 1$ and use boundary data calculated in Step 3; then go to Step 3.

The above algorithm is called the small marching step technique since only a small region in considered each time. The numerical results of the *UT* and *LM* equations obtained using the shifting technique as shown in Figs. 3 and 4, respectively, show that the numerical errors can be eliminated as expected. It should be noted that Figs. 1 and 2 as well as Figs. 3 and 4 show the same error tendency due to the fact that there exists no discretization but only truncation error in the 1D case whether the *UT* or *LM* formulation is adopted. Furthermore, the number of series terms, which need to converge to the analytical solution, becomes larger when the shifting range becomes larger as shown in Fig. 5. When the wave number becomes very large, similar difficulties will be encountered. However, since the roles of the wave number, k, and the filed point position, s, are the same mathematically, for a large wave number, we can use the scaling technique, $\bar{k} \equiv k/k_0$ and $\bar{s} \equiv k_0 s$, where k_0 is an artificially chosen normalization parameter, to cope with these difficulties. Therefore, calculating the response for the wave number k at position s is now transformed into calculating the response for the wave number $\bar{k}$ at position $\bar{s}$, where $\bar{s}$ may be very large. Thus, we can obtain good results using the shifting technique as mentioned above. The responses of $s = 1$ versus the wave number are shown in Fig. 6. It is found that the scaling technique can effectively eliminate the numerical error.

Example 2: Neumann excitation

$(\mathrm{d}^2u(x)/\mathrm{d}x^2) + k^2u(x) = 0$ for $0 \le x \le \infty$, $t(0) = 1$ and $u(x)$ satisfies the radiation condition at infinity. The analytical solution is $u(x) = (-\mathrm{i}/k)\mathrm{e}^{\mathrm{i}kx}$.

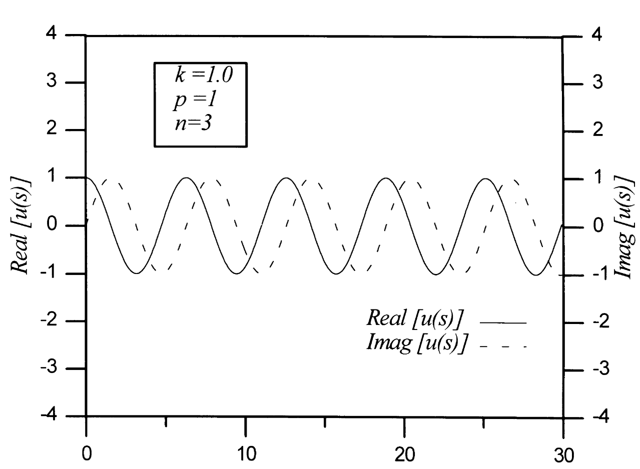

Fig. 4. Improvement of numerical divergence obtained by using the small marching step technique: the *LM* method.

By using Eq. (18) and taking s to approach zero, we can

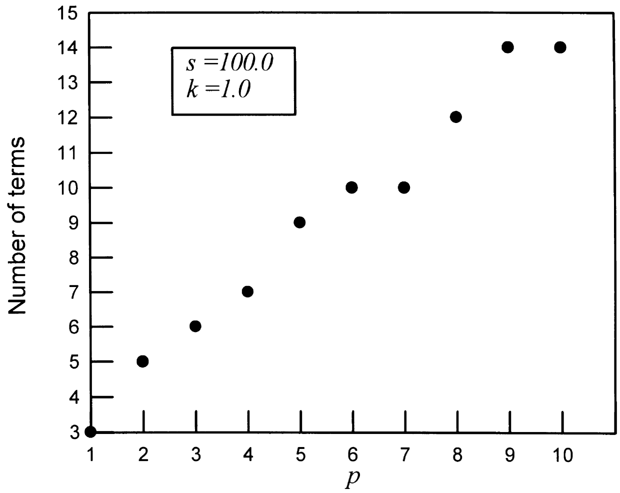

Fig. 5. Marching step size and needed convergent terms in series form.

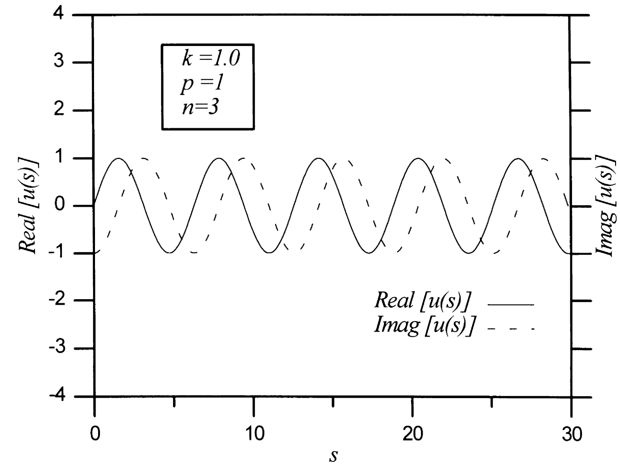

Fig. 7. Numerical results for the Neumann excitation obtained by using the complete *UT* MRM equation.

have the same equation as Eq. (24). Substituting the boundary condition into Eq. (24), the boundary unknown can be obtained

$$u(0) = \frac{-\mathrm{i}}{k}. \tag{28}$$

Substituting the boundary data into Eq. (18), we have

$$u(s) = \frac{\mathrm{e}^{\mathrm{i}ks}}{2\mathrm{i}k}t(0) + \frac{\mathrm{e}^{\mathrm{i}ks}}{2}u(0) = \frac{\mathrm{e}^{\mathrm{i}ks}}{2\mathrm{i}k}1 + \frac{\mathrm{e}^{\mathrm{i}ks}}{2}\frac{-\mathrm{i}}{k} = -\frac{\mathrm{i}}{k}\mathrm{e}^{\mathrm{i}ks}. \tag{29}$$

This is exactly the same as the analytical solution. For the *LM* equation, by taking s to approach zero and substituting the boundary condition into Eq. (23), the following equation can be obtained:

$$t(0) = \frac{1}{2}t(0) + \frac{\mathrm{i}k}{2}u(0) \Rightarrow u(0) = \frac{-\mathrm{i}}{k}. \tag{30}$$

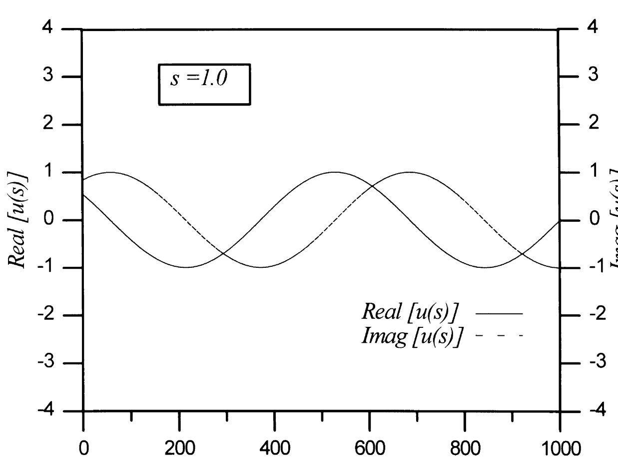

Fig. 6. Responses of a point versus wave number obtained by using the scaling technique.

Numerical solutions obtained using finite terms in the *UT* and *LM* equations (Eqs. (19) and (20)) are shown in Figs. 7 and 8, respectively. It should be noted that the shifting technique has already been adopted in these two figures.

Although the proposed approach only concerns with a 1D problem, however, similar treatments are expected in 2D or 3D cases. For a 2D case, the fundamental solution of the complete MRM series is written as

$$U_{\mathrm{c}} = \frac{\mathrm{i}}{4}[J_0(kr) - \mathrm{i}Y_0(kr)] = \sum_{j=0}^{\infty} A_j(-k^2)^j, \tag{31}$$

where

$$A_j = \frac{1}{4}\left[\frac{2}{\pi}F_j(\ln r - S_j) + \frac{2}{\pi}F_j\left(\gamma + \ln\frac{k}{2}\right) + \mathrm{i}F_j\right]$$

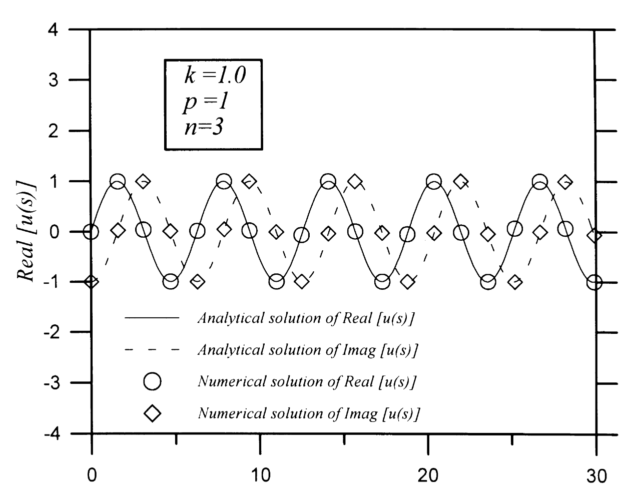

Fig. 8. Numerical results for the Neumann excitation obtained by using the complete *LM* MRM equation.

with

$$F_j = \frac{r^{2j}}{(j!)^2 4j}, \qquad S_j = \sum_{l=1}^{j} \frac{1}{l}, \qquad \gamma \cong \lim_{j \to \infty} \left(\sum_{l=1}^{j} \frac{1}{l} - \ln j \right).$$

Although this series converges for any argument, kr, theoretically it is not yet feasible in the numerical sense [22]. In order to calculate the far field response, we can write U_c in an asymptotic series as

$$U_c \sim \frac{e^{i(kr-(\pi/4))}}{\sqrt{kr}}. \tag{32}$$

In order to separate k and r for keeping the merit of MRM, Eq. (32) can be expanded into a power series again by the use of Taylor's series expansion simply. However, the convergency of such a power series mainly relies on the expansion radius. If the expansion radius is too large, numerical divergency occurs. Therefore, we have to choose a suitable expansion center to make the expansion radius small. Further, one question is open for discussion: can one derive Eq. (32) directly from the MRM algorithm?

When a 3D case is considered, the fundamental solution of the complete MRM series can be written as

$$U_c = \frac{-e^{ikr}}{4\pi r}. \tag{33}$$

Since Eq. (33) can be expanded into a power series as mentioned above, similar treatments are required.

4. Concluding remarks

In this paper, the complete MRM has been used successfully to solve the 1D Helmholtz equation for the semi-infinite domain. Use of the conventional MRM leads to difficulty in solving such problems since the kernels derived do not satisfy the radiation condition. To recover the information that is lost when the conventional MRM is used, we have proposed adding an appropriate complex constant to the zeroth-order fundamental solution so that the complete MRM can be constructed. Two examples, the Dirichlet and Neumann excitation problems, have been investigated to show the validity of the proposed approach. It has been shown that the shifting technique can be used successfully to cope with the numerical error in the far-field responses owing to the power series expansion of the trigonometric functions. Similarly, by using the scaling technique and then the shifting technique, we can effectively eliminate the error in the calculation of responses under a large wave number.

Acknowledgements

The second author gratefully thanks the National Science Council, ROC, for the financial support under grant number: NSC-88-2211-E019-009.

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