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Two regularization methods for the Cauchy problems of the Helmholtz equation ☆

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ABSTRACT

In this paper, the Cauchy problems for the Helmholtz equation are investigated. We propose two regularization methods to solve them. Convergence estimates are presented under an *a-priori* bounded assumption for the exact solution. Finally, the numerical examples show that the proposed numerical methods work effectively.

1. Introduction

The Helmholtz equation arises in many areas, especially in practical physical applications, such as acoustic, wave propagation and scattering, vibration of the structure, electromagnetic scattering and so on, see [1–4]. The direct problems, i.e. Dirichlet, Neumann or mixed boundary value problems for the Helmholtz equation, have been studied extensively in the past century. However, in some practical problems, the boundary data on the whole boundary can not be obtained. We only know the noisy data on a part of the boundary or at some interior points of the concerned domain, which will lead to some inverse problems. The Cauchy problem for the Helmholtz equation is an inverse problem and is severely ill-posed. That means the solution does not depend continuously on the given Cauchy data and any small perturbation in the given data may cause large change to the solution [5–7]. The Cauchy problem for the Helmholtz equation has been studied extensively in recent years, we refer the reader to [8–14,4,15–18]. Several numerical methods have been proposed to solve this problem, such as the boundary element method (BEM) [14], the conjugate gradient method [4,16], the Landweber method [15], the singular value decomposition (SVD) method [16], the method of fundamental solutions (MFS) [13,17,18], the boundary knot method [10,12], the plane wave method [11] and so on. In the literatures [11,10,12–14,4,15–18], the authors verified that their proposed regularization methods are stable and effective from their numerical results, but they did not give a mathematical analysis for the convergence and stability. In paper [19], the truncated Fourier transform technique was used to solve the Cauchy problem for the Helmholtz equation in a strip infinite region. In this paper, we use two regularization methods, a modified Tikhonov regularization method which was also used to solve the inverse heat conduction problem [20], and a truncation method which was also used to solve the inverse heat conduction problem [21], the source identification problem [22] and the Cauchy problem for the Laplace equation [23], to construct the stable approximate solutions for the Cauchy problems of the Helmholtz equation in a rectangular domain. We not only present the convergence and stability estimates

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but also do the numerical implementation by using the two different regularization methods. The numerical results show that the proposed methods are stable and effective.

The paper is organized as follows. In Section 2, we present the formulation of the Cauchy problems for the Helmholtz equation. In Section 3, we give the modified Tikhonov regularization method and obtain the convergence estimates. In Section 4, we use a truncation method to construct a stable approximation solution and give the convergence estimates. In Section 5, numerical examples are given to test the effectiveness of the proposed methods. Finally, conclusions are given in Section 6.

2. Mathematical problems

Firstly, we consider the following Cauchy problem for the Helmholtz equation with Dirichlet boundary conditions at $x = 0$ and $x = \pi$:

$$\Delta u(x,y) + k^2 u(x,y) = 0, \quad 0 < x < \pi, \quad 0 < y < 1, \tag{2.1}$$
$$u(x,0) = \varphi(x), \quad 0 \leqslant x \leqslant \pi, \tag{2.2}$$
$$u_y(x,0) = 0, \quad 0 \leqslant x \leqslant \pi, \tag{2.3}$$
$$u(0,y) = u(\pi,y) = 0, \quad 0 \leqslant y \leqslant 1, \tag{2.4}$$

where constant $k > 0$ is the wave number.

Assume that the exact data $\varphi \in L^2(0,\pi)$ and measured data $\varphi_\delta \in L^2(0,\pi)$ satisfy

$$\|\varphi_\delta - \varphi\| \leqslant \delta, \tag{2.5}$$

where $\|\cdot\|$ denotes the L^2-norm and $\delta > 0$ is a noise level. Further assume that $u(\cdot,y) \in L^2(0,\pi)$ and there exists a constant $E > 0$, such that the following *a-priori* bound exists,

$$\left\| \frac{\partial^p u(\cdot,1)}{\partial y^p} \right\| \leqslant E, \tag{2.6}$$

where $p \geqslant 1$ is an integer.

Separation of variables leads to a solution of problem (2.1)–(2.4) as follows,

$$u(x,y) = \begin{cases} \sum_{n=1}^{\infty} \cosh\left(\sqrt{n^2 - k^2}y\right)(\varphi, X_n)X_n, \quad 0 < k < 1, \\ \sum_{n=1}^{[k]} \cos\left(\sqrt{k^2 - n^2}y\right)(\varphi, X_n)X_n \\ + \sum_{n=[k]+1}^{\infty} \cosh\left(\sqrt{n^2 - k^2}y\right)(\varphi, X_n)X_n, \quad k \geqslant 1, \end{cases} \tag{2.7}$$

where $(\cdot,\cdot)$ denotes the inner product in $L^2(0,\pi)$, $X_n = X_n(x) := \sqrt{\frac{2}{\pi}}\sin(nx)$ and $[\cdot]$ denotes the integer part of a real number. Note that $\{X_n\}_{n=1}^{\infty}$ is the orthonormal basis of $L^2(0,\pi)$.

From the following statements we can show that problem (2.1)–(2.4) is unstable, that means the solution does not depend continuously on the given Cauchy data. Choose $u_n(x,y) = \frac{\sin(nx)}{n^2k^2}\cosh(\sqrt{n^2 - k^2}y)$ as the exact solution for problem (2.1)–(2.4) with initial data $\varphi_n(x) = \frac{\sin(nx)}{n^2k^2}$, here positive integers $n > k$. We find that $\sup_{x\in[0,\pi]}|\varphi_n(x)| \to 0$ as $n \to \infty$, but $\sup_{x\in[0,\pi]}|u_n(x,y)| \to \infty$ as $n \to \infty$ for fixed $y > 0$. Hence, we cannot use the classical numerical methods to solve Cauchy problem (2.1)–(2.4) and it needs some regularization techniques [24,6,25].

Secondly, we consider the Cauchy problem for the Helmholtz equation with Neumann data at $x = 0$ and $x = \pi$ instead of Dirichlet conditions (2.4), i.e.

$$u_x(0,y) = u_x(\pi,y) = 0, \quad 0 \leqslant y \leqslant 1. \tag{2.8}$$

By the method of separation of variables, it is easy to derive the solution of problem (2.1)–(2.4) and (2.8) is as follows,

$$u(x,y) = \sum_{n=0}^{[k]} \cos\left(\sqrt{k^2 - n^2}y\right)(\varphi, Z_n)Z_n + \sum_{n=[k]+1}^{\infty} \cosh\left(\sqrt{n^2 - k^2}y\right)(\varphi, Z_n)Z_n, \quad k > 0, \tag{2.9}$$

where

$$Z_0 = \sqrt{\frac{1}{\pi}} \quad \text{and} \quad Z_n = Z_n(x) := \sqrt{\frac{2}{\pi}}\cos(nx) \quad (n \geqslant 1), \tag{2.10}$$

and $[\cdot]$ denotes the integer part of a real number. Note that $\{Z_n\}_{n=0}^{\infty}$ is the orthonormal basis of $L^2(0,\pi)$.

Thirdly, we consider the Cauchy problem for the Helmholtz equation with Robin conditions at $x = 0$ and $x = \pi$ instead of Dirichlet conditions (2.4), i.e.

$$-u_x(0,y) + au(0,y) = 0, \quad 0 \leqslant y \leqslant 1. \tag{2.11}$$

$$u_x(\pi,y) + bu(\pi,y) = 0, \quad 0 \leqslant y \leqslant 1. \tag{2.12}$$

where constants $a > 0$ and $b > 0$.

Separation of variables leads to the solution of problem (2.1)–(2.3), (2.11) and (2.12) is as follows,

$$u(x,y) = \begin{cases} \sum_{n=1}^{\infty} \cosh\left(\sqrt{\beta_n^2 - k^2}y\right)(\varphi, T_n)T_n, & 0 < k < \beta_1, \\ \sum_{n=1}^{s} \cos\left(\sqrt{k^2 - \beta_n^2}y\right)(\varphi, T_n)T_n & \\ + \sum_{n=s+1}^{\infty} \cosh\left(\sqrt{\beta_n^2 - k^2}y\right)(\varphi, T_n)T_n, & k \geqslant \beta_1, \end{cases} \tag{2.13}$$

where β_n satisfies $0 < \beta_1 < \beta_2 < \cdots < \beta_n < \cdots$, $\lim_{n\to\infty}\beta_n = \infty$, and $\tan(\beta_n\pi) = \frac{(a+b)\beta_n}{\beta_n^2 - ab}$ and $n - 1 < \beta_n < n$, see [26],

$$T_n = T_n(x) = \frac{R_n(x)}{\sqrt{\int_0^\pi R_n^2(\xi)d\xi}}, \tag{2.14}$$

$$R_n(x) = \cos(\beta_n x) + \frac{a}{\beta_n}\sin(\beta_n x), \quad n = 1, 2\cdots, \tag{2.15}$$

and

$$s = \max\{n | \beta_n \leqslant k\}. \tag{2.16}$$

Note that $\{T_n\}_{n=1}^{\infty}$ is the orthonormal basis of $L^2(0,\pi)$.

Due to the ill-posedness of problem (2.1)–(2.4), (2.8), (2.11), and 2.12, the formula (2.9) and (2.13) cannot be used to obtain a stable approximation when replace the exact data φ with a measured data φ_δ. Hence, some regularized techniques are also needed [24,6,25].

In the following two sections, we will propose two regularization methods to solve problems (2.1)–(2.4), (2.8), (2.1)–(2.3), (2.11) and (2.12), respectively, and give the convergence estimates for $0 < y \leqslant 1$.

3. A modified Tikhonov regularization method

In this section, we will present a modified Tikhonov regularization method and give the convergence estimates for $0 < y \leqslant 1$ under an *a-priori* assumption for the exact solution.

In the following, we will first give a stable approximation of the solution u given by (2.7).

For $0 < k < 1$, according to the idea in [27], define an operator $K(\cdot) : L^2(0,\pi) \mapsto L^2(0,\pi)$ for $0 < y \leqslant 1$, such that problem (2.1)–(2.4) can be formulated as the following operator equation,

$$K(y)u(x,y) = \varphi(x), \quad 0 < y \leqslant 1. \tag{3.1}$$

By the first equation in (2.7), we can obtain

$$(u(x,y), X_n) = (\varphi, X_n)\cosh\left(\sqrt{n^2 - k^2}y\right). \tag{3.2}$$

By (3.1) and (3.2), we have

$$K(y)u(x,y) = \sum_{n=1}^{\infty}(\varphi, X_n)X_n = \sum_{n=1}^{\infty}\frac{(u(x,y), X_n)}{\cosh\left(\sqrt{n^2 - k^2}y\right)}X_n. \tag{3.3}$$

Then,

$$K(y)X_n = \frac{1}{\cosh\left(\sqrt{n^2 - k^2}y\right)}X_n, \tag{3.4}$$

here K is a linear, self-adjoint and compact operator [27].

In order to obtain a stable approximation of the solution u given by the first equation in (2.7), we seek a regularized approximation u_α^δ by solving the following minimization problem,

$$\min F_\alpha(u) := \|Ku(\cdot,y) - \varphi_\delta\|^2 + \alpha\|u(\cdot,y)\|^2, \quad \text{for} \quad u(\cdot,y) \in L^2(0,\pi). \tag{3.5}$$

By Theorem 2.11 of Chapter 2 in [25], we know that the functional F_α has a unique minimizer u_α^δ and this minimum u_α^δ is the unique solution of the following normal equation,

$$\alpha u_\alpha^\delta + K^* K u_\alpha^\delta = K^* \varphi_\delta, \tag{3.6}$$

where K^* is the adjoint operator of K. From (3.6) and the spectral theory [25], note that $K^* = K$ [27], we can derive that

$$u_\alpha^\delta(x,y) = (\alpha I + K^2)^{-1} K \varphi_\delta = \sum_{n=1}^{\infty} \frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{n^2-k^2}y\right)} (\varphi_\delta, X_n) X_n. \tag{3.7}$$

For $k \geqslant 1$, we define a regularized solution for the exact solution u as follows,

$$u_\alpha^\delta(x,y) = \sum_{n=1}^{[k]} \cos\left(\sqrt{k^2-n^2}y\right)(\varphi_\delta, X_n)X_n + \sum_{n=[k]+1}^{\infty} \frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{n^2-k^2}y\right)} (\varphi_\delta, X_n) X_n. \tag{3.8}$$

The regularized solutions (3.7) and (3.8) can be interpreted as using the modified kernel $\frac{\cosh(\sqrt{n^2-k^2}y)}{1+\alpha\cosh^2(\sqrt{n^2-k^2}y)}$ to replace the kernel $\cosh(\sqrt{n^2-k^2}y)$ for $n \geqslant k$. In this paper, we replace the modified kernel $\frac{\cosh(\sqrt{n^2-k^2}y)}{1+\alpha\cosh^2(\sqrt{n^2-k^2}y)}$ with a much simpler kernel $\frac{\cosh(\sqrt{n^2-k^2}y)}{1+\alpha\cosh^2(\sqrt{n^2-k^2})}$ and obtain a modified regularization solution $u_{*\alpha}^\delta$ of problem (2.1)–(2.4) as follows,

$$u_{*\alpha}^\delta(x,y) = \begin{cases} \sum_{n=1}^{\infty} \frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)} (\varphi_\delta, X_n) X_n, & 0 < k < 1, \\ \sum_{n=1}^{[k]} \cos\left(\sqrt{k^2-n^2}y\right)(\varphi_\delta, X_n)X_n & \\ + \sum_{n=[k]+1}^{\infty} \frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)} (\varphi_\delta, X_n) X_n, & k \geqslant 1. \end{cases} \tag{3.9}$$

Secondly, for the Cauchy problem of the Helmholtz equation with Neumann conditions at $x = 0$ and $x = \pi$, we define a modified regularization solution $u_\alpha^{*\delta}$ of problem (2.1)–(2.3), and (2.8) as follows,

$$\begin{aligned} u_\alpha^{*\delta}(x,y) = &\sum_{n=0}^{[k]} \cos\left(\sqrt{k^2-n^2}y\right)(\varphi_\delta, Z_n)Z_n \\ &+ \sum_{n=[k]+1}^{\infty} \frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)} (\varphi_\delta, Z_n) Z_n, \quad k > 0. \end{aligned} \tag{3.10}$$

Thirdly, for the Cauchy problem of the Helmholtz equation with Robin conditions at $x = 0$ and $x = \pi$, we define a modified regularization solution $u_{\alpha*}^\delta$ of problem (2.1)–(2.3), (2.11) and (2.12) as follows,

$$u_{\alpha*}^\delta(x,y) = \begin{cases} \sum_{n=1}^{\infty} \frac{\cosh\left(\sqrt{\beta_n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{\beta_n^2-k^2}\right)} (\varphi_\delta, T_n) T_n, & 0 < k < \beta_1, \\ \sum_{n=1}^{s} \cos\left(\sqrt{k^2-\beta_n^2}y\right)(\varphi_\delta, T_n)T_n & \\ + \sum_{n=s+1}^{\infty} \frac{\cosh\left(\sqrt{\beta_n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{\beta_n^2-k^2}\right)} (\varphi_\delta, T_n) T_n, & k \geqslant \beta_1, \end{cases} \tag{3.11}$$

where s is given by (2.16).

In the following theorem, we give a convergence estimate between the regularied solution $u_{*\alpha}^\delta$ given by (3.9) and the exact solution u given by (2.7).

Theorem 3.1. *Let u be the exact solution of problem (2.1)–(2.4) given by (2.7), and let $u_{*\alpha}^\delta$ be the regularized approximation solution given by (3.9). Assume that the measured data φ_δ fulfills (2.5) and the exact solution u at $y = 1$ satisfies condition (2.6).*

If the regularization parameter α is chosen as

$$\alpha = \frac{\delta}{E}, \tag{3.12}$$

then for fixed $0 < y \leqslant 1$, the following convergence estimates hold

$$\|u_{*\alpha}^{\delta}(\cdot,y)-u(\cdot,y)\| \leqslant \begin{cases} 2^{y}E^{\frac{y}{2}}\delta^{1-\frac{y}{2}}+\frac{2}{1-e^{-2\sqrt{1-k^2}}} \\ \max\left\{2Ee^{-\sqrt{1-k^2}(1-y)}\left(\ln\sqrt[3]{\frac{E}{\delta}}\right)^{-p},\delta^{\frac{2}{3}-\frac{y}{3}}E^{\frac{1}{3}+\frac{y}{3}}(1-k^2)^{-\frac{p}{2}}\right\}, \quad 0<k<1, \\ \delta+2^{y}E^{\frac{y}{2}}\delta^{1-\frac{y}{2}}+\frac{2}{1-e^{-2L}} \\ \max\left\{2Ee^{-L(1-y)}\left(\ln\sqrt[3]{\frac{E}{\delta}}\right)^{-p},\delta^{\frac{2}{3}-\frac{y}{3}}E^{\frac{1}{3}+\frac{y}{3}}L^{-p}\right\}, \quad k\geqslant 1, \end{cases} \tag{3.13}$$

where

$$L=\sqrt{([k]+1)^2-k^2}, \quad k\geqslant 1, \tag{3.14}$$

in which $[\cdot]$ *denotes the integer part of a real number.*

Proof. Note that condition (2.5) gives

$$\sum_{n=1}^{\infty}|(\varphi_\delta-\varphi,X_n)|^2\leqslant\delta^2. \tag{3.15}$$

For the case $0<k<1$. By the first equation in (2.7), note that condition (2.6) is equivalent to

$$\left\|\frac{\partial^p u(\cdot,1)}{\partial y^p}\right\|^2=\begin{cases}\sum_{n=1}^{\infty}|(\varphi,X_n)|^2(n^2-k^2)^p\sinh^2\left(\sqrt{n^2-k^2}\right)\leqslant E^2, \quad p \quad \text{is} \quad \text{odd},\\ \sum_{n=1}^{\infty}|(\varphi,X_n)|^2(n^2-k^2)^p\cosh^2\left(\sqrt{n^2-k^2}\right)\leqslant E^2, \quad p \quad \text{is} \quad \text{even}.\end{cases} \tag{3.16}$$

By the first equation in (2.7),(3.9) and (3.15), we have

$$\begin{aligned}\|u_{*\alpha}^{\delta}(\cdot,y)-u(\cdot,y)\| &\leqslant \left(\sum_{n=1}^{\infty}\frac{\cosh^2\left(\sqrt{n^2-k^2}y\right)}{\left(1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)\right)^2}|(\varphi_\delta-\varphi,X_n)|^2\right)^{\frac{1}{2}} \\ &\quad+\left(\sum_{n=1}^{\infty}\left(\frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)}-\cosh\left(\sqrt{n^2-k^2}y\right)\right)^2|(\varphi,X_n)|^2\right)^{\frac{1}{2}} \\ &\leqslant \delta\sup_{n\geqslant 1}\frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)} \\ &\quad+\left(\sum_{n=1}^{\infty}\frac{\alpha^2\cosh^2\left(\sqrt{n^2-k^2}y\right)\cosh^4\left(\sqrt{n^2-k^2}\right)}{\left(1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)\right)^2}|(\varphi,X_n)|^2\right)^{\frac{1}{2}} \\ &:=\delta\sup_{n\geqslant 1}A1(n)+A2(n),\end{aligned} \tag{3.17}$$

Let $\xi=\sqrt{n^2-k^2}$, note that

$$\frac{1}{2}e^{\xi y}\leqslant\cosh(\xi y)\leqslant e^{\xi y} \quad \text{for} \quad y>0, \tag{3.18}$$

and

$$\xi\geqslant\sqrt{1-k^2}, \tag{3.19}$$

then

$$A1(n)=\frac{\cosh\xi y}{1+\alpha\cosh^2\xi}\leqslant\frac{e^{\xi y}}{1+\frac{\alpha}{4}e^{2\xi}}:=a(\xi).$$

It is easy to obtain that

$$a_{\max}(\xi)=a\left(\frac{1}{2}\ln\frac{4y}{(2-y)\alpha}\right)=\frac{2-y}{2}\left(\frac{y}{2-y}\right)^{\frac{y}{2}}2^{y}\alpha^{-\frac{y}{2}}\leqslant 2^{y}\alpha^{-\frac{y}{2}}.$$

Hence,

$$\delta \sup_{n\geqslant 1} A1(n) \leqslant 2^y \delta \alpha^{-\frac{y}{2}}. \tag{3.20}$$

Now we estimate $A2(n)$. When p is even, by the second equation in (3.16), we have

$$A2(n) \leqslant E\sup_{n\geqslant 1}\left|\frac{\alpha\cosh\left(\sqrt{n^2-k^2}y\right)\cosh\left(\sqrt{n^2-k^2}\right)}{\left(1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)\right)(n^2-k^2)^{\frac{p}{2}}}\right| = E\sup_{n\geqslant 1}\left|\frac{\alpha\cosh(\xi y)\cosh\xi}{1+\alpha\cosh^2\xi}\xi^{-p}\right| := E\sup_{n\geqslant 1} C(n).$$

Case I: For large values of n with $\xi = \sqrt{n^2-k^2} \geqslant \ln\frac{1}{\sqrt[3]{\alpha}}$, by (3.18) and (3.19), we have

$$C(n) \leqslant \frac{\cosh(\xi y)}{\cosh\xi}\xi^{-p} \leqslant 2e^{-\xi(1-y)}\xi^{-p} \leqslant 2e^{-\sqrt{1-k^2}(1-y)}\left(\ln\frac{1}{\sqrt[3]{\alpha}}\right)^{-p}.$$

Case II: For $\xi = \sqrt{n^2-k^2} < \ln\frac{1}{\sqrt[3]{\alpha}}$, by (3.18) and (3.19), we have

$$C(n) \leqslant \alpha e^{\xi(1+y)}\xi^{-p} \leqslant \alpha^{\frac{2}{3}-\frac{y}{3}}(1-k^2)^{-\frac{p}{2}}.$$

Then when p is even, we obtain that

$$A2(n) \leqslant E\max\left\{2e^{-\sqrt{1-k^2}(1-y)}\left(\ln\frac{1}{\sqrt[3]{\alpha}}\right)^{-p}, \alpha^{\frac{2}{3}-\frac{y}{3}}(1-k^2)^{-\frac{p}{2}}\right\}. \tag{3.21}$$

When p is odd, by the first equation in (3.16), we have

$$A2(n) \leqslant E\sup_{n\geqslant 1}\left|\frac{\alpha\cosh\left(\sqrt{n^2-k^2}y\right)\cosh^2\left(\sqrt{n^2-k^2}\right)}{\left(1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)\right)\sinh\left(\sqrt{n^2-k^2}\right)}(n^2-k^2)^{-\frac{p}{2}}\right| = E\sup_{n\geqslant 1}\left|\frac{\alpha\cosh(\xi y)\cosh^2\xi}{(1+\alpha\cosh^2\xi)\sinh\xi}\xi^{-p}\right| := E\sup_{n\geqslant 1} D(n),$$

In the following, we estimate $D(n)$. Similar to the procedure of estimating $C(n)$, by (3.19), note that $\frac{\cosh\xi}{\sinh\xi} = \frac{e^{\xi}+e^{-\xi}}{e^{\xi}-e^{-\xi}} \leqslant \frac{2}{1-e^{-2\xi}}$, we have

$$D(n) \leqslant \frac{2}{1-e^{-2\xi}}\left|\frac{\alpha\cosh(\xi y)\cosh\xi}{1+\alpha\cosh^2\xi}\xi^{-p}\right| \leqslant \frac{2}{1-e^{-2\sqrt{1-k^2}}}\left|\frac{\alpha\cosh(\xi y)\cosh\xi}{1+\alpha\cosh^2\xi}\xi^{-p}\right| = \frac{2}{1-e^{-2\sqrt{1-k^2}}}C(n).$$

Consequently, when p is odd, by Case I and Case II, we can obtain that

$$A2(n) \leqslant \frac{2E}{1-e^{-2\sqrt{1-k^2}}}\max\left\{2e^{-\sqrt{1-k^2}(1-y)}\left(\ln\frac{1}{\sqrt[3]{\alpha}}\right)^{-p}, \alpha^{\frac{2}{3}-\frac{y}{3}}(1-k^2)^{-\frac{p}{2}}\right\}. \tag{3.22}$$

Hence, for $0 < k < 1$ and fixed $0 < y \leqslant 1$, by (3.17), (3.20), (3.21) and (3.22), we have

$$\|u_{*\alpha}^{\delta}(\cdot,y) - u(\cdot,y)\| \leqslant 2^y\delta\alpha^{-\frac{y}{2}} + \frac{2E}{1-e^{-2\sqrt{1-k^2}}}\max\left\{2e^{-\sqrt{1-k^2}(1-y)}\left(\ln\frac{1}{\sqrt[3]{\alpha}}\right)^{-p}, \alpha^{\frac{2}{3}-\frac{y}{3}}(1-k^2)^{-\frac{p}{2}}\right\}.$$

Therefore, for $0 < k < 1$ and fixed $0 < y \leqslant 1$, by (3.12), we derive that the first conclusion of Eq. (3.13).

In the following, we consider the case $k \geqslant 1$. Note that (2.6) is equivalent to

$$\left\|\frac{\partial^p u(\cdot,1)}{\partial y^p}\right\|^2 = \begin{cases} \sum_{n=1}^{[k]} |(\varphi,X_n)|^2(k^2-n^2)^p\sin^2\left(\sqrt{k^2-n^2}\right) + \\ \sum_{[k]+1}^{\infty} |(\varphi,X_n)|^2(n^2-k^2)^p\sinh^2\left(\sqrt{n^2-k^2}\right) \leqslant E^2, \quad p \quad \text{is} \quad \text{odd}, \\ \sum_{n=1}^{[k]} |(\varphi,X_n)|^2(k^2-n^2)^p\cos^2\left(\sqrt{k^2-n^2}\right) \\ + \sum_{[k]+1}^{\infty} |(\varphi,X_n)|^2(n^2-k^2)^p\cosh^2\left(\sqrt{n^2-k^2}\right) \leqslant E^2, \quad p \quad \text{is} \quad \text{even}. \end{cases} \tag{3.23}$$

By the second equation in (2.7), (3.9) and (3.15), we have

$$\|u^{\delta}_{*\alpha}(\cdot,y)-u(\cdot,y)\| \leqslant \left(\sum_{n=1}^{[k]} |(\varphi_\delta-\varphi,X_n)|^2\cos^2\left(\sqrt{k^2-n^2}y\right)\right)^{\frac{1}{2}} + \left(\sum_{n=[k]+1}^{\infty}\frac{\cosh^2\left(\sqrt{n^2-k^2}y\right)}{\left(1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)\right)^2}|(\varphi_\delta-\varphi,X_n)|^2\right)^{\frac{1}{2}} + \left(\sum_{n=[k]+1}^{\infty}\left(\frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)}-\cosh\left(\sqrt{n^2-k^2}y\right)\right)^2|(\varphi,X_n)|^2\right)^{\frac{1}{2}}$$
$$\leqslant \delta+\delta\sup_{n\geqslant[k]+1}\frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)} + \left(\sum_{n=[k]+1}^{\infty}\frac{\alpha^2\cosh^2\left(\sqrt{n^2-k^2}y\right)\cosh^4\left(\sqrt{n^2-k^2}\right)}{\left(1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)\right)^2}|(\varphi,X_n)|^2\right)^{\frac{1}{2}}$$
$$:= \delta+\delta\sup_{n\geqslant[k]+1}A1(n)+B1(n), \tag{3.24}$$

Similar to the case $0<k<1$, we have

$$\delta\sup_{n\geqslant[k]+1}A1(n)\leqslant 2^y\delta\alpha^{-\frac{y}{2}}. \tag{3.25}$$

In the following, we estimate $B1(n)$. When p is even, by the second equation in (3.23), we have

$$B1(n)\leqslant E\sup_{n\geqslant[k]+1}\left|\frac{\alpha\cosh\left(\sqrt{n^2-k^2}y\right)\cosh\left(\sqrt{n^2-k^2}\right)}{\left(1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)\right)(n^2-k^2)^{\frac{p}{2}}}\right| = E\sup_{n\geqslant[k]+1}\left|\frac{\alpha\cosh(\xi y)\cosh\xi}{1+\alpha\cosh^2\xi}\xi^{-p}\right| := E\sup_{n\geqslant[k]+1}B2(n).$$

Similar to the case $0<k<1$, note that

$$\xi\geqslant\sqrt{([k]+1)^2-k^2}:=L, \tag{3.26}$$

we have

Case 1: For large values of n with $\xi=\sqrt{n^2-k^2}\geqslant\ln\frac{1}{\sqrt[3]{\alpha}}$, we have

$$B2(n)\leqslant\frac{\cosh(\xi y)}{\cosh\xi}\xi^{-p}\leqslant 2e^{-\xi(1-y)}\xi^{-p}\leqslant 2e^{-L(1-y)}\left(\ln\frac{1}{\sqrt[3]{\alpha}}\right)^{-p}.$$

Case 2: For $\xi=\sqrt{n^2-k^2}<\ln\frac{1}{\sqrt[3]{\alpha}}$, we have

$$B2(n)=\left|\frac{\alpha\cosh(\xi y)\cosh\xi}{1+\alpha\cosh^2\xi}\xi^{-p}\right|\leqslant\alpha e^{\xi(1+y)}\xi^{-p}\leqslant\alpha^{\frac{2}{3}-\frac{y}{3}}L^{-p}.$$

Combining Case 1 and Case 2, we obtain that

$$B1(n)\leqslant E\max\left\{\alpha^{\frac{2}{3}-\frac{y}{3}}L^{-p},2e^{-L(1-y)}\left(\ln\frac{1}{\sqrt[3]{\alpha}}\right)^{-p}\right\}. \tag{3.27}$$

In the following, we discuss the case when p is odd. By the first equation in (3.23), we have

$$B1(n)\leqslant E\sup_{n\geqslant[k]+1}\left|\frac{\alpha\cosh\left(\sqrt{n^2-k^2}y\right)\cosh^2\left(\sqrt{n^2-k^2}\right)}{\left(1+\alpha\cosh^2\left(\sqrt{n^2-k^2}\right)\right)\sinh(\sqrt{n^2-k^2})(n^2-k^2)^{\frac{p}{2}}}\right|$$
$$=E\sup_{n\geqslant[k]+1}\left|\frac{\alpha\cosh(\xi y)\cosh^2\xi}{\left(1+\alpha\cosh^2\xi\right)\sinh\xi}\xi^{-p}\right| := E\sup_{n\geqslant[k]+1}B3(n),$$

Now we estimate $B3(n)$. Similar to the case $0<k<1$, by (3.26), we have

$$B3(n)\leqslant\frac{2}{1-e^{-2L}}\left|\frac{\alpha\cosh(\xi y)\cosh\xi}{1+\alpha\cosh^2\xi}\xi^{-p}\right| = \frac{2}{1-e^{-2L}}B2(n).$$

According to Case 1 and Case 2, we obtain

$$B3(n) \leqslant \frac{2}{1-e^{-2L}} \max\left\{\alpha^{\frac{2}{3}-\frac{y}{3}}L^{-p}, 2e^{-L(1-y)}\left(\ln\frac{1}{\sqrt[3]{\alpha}}\right)^{-p}\right\}.$$

Then when p is odd, we obtain

$$B1(n) \leqslant \frac{2E}{1-e^{-2L}} \max\left\{\alpha^{\frac{2}{3}-\frac{y}{3}}L^{-p}, 2e^{-L(1-y)}\left(\ln\frac{1}{\sqrt[3]{\alpha}}\right)^{-p}\right\}. \tag{3.28}$$

Therefore, by (3.24), (3.25), (3.27) and (3.28), for $k \geqslant 1$ and fixed $0 < y \leqslant 1$, we have

$$\|u^{\delta}_{*\alpha}(\cdot,y) - u(\cdot,y)\| \leqslant \delta + 2^{y}\delta\alpha^{-\frac{y}{2}} + \frac{2E}{1-e^{-2L}} \max\left\{\alpha^{\frac{2}{3}-\frac{y}{3}}L^{-p}, 2e^{-L(1-y)}\left(\ln\frac{1}{\sqrt[3]{\alpha}}\right)^{-p}\right\}. \tag{3.29}$$

Consequently, by (3.12), the second estimate in (3.13) can be obtained. $\square$

Remark 3.2. In practice, the constant E is unknown. In such a case, we choose the regularization parameter

$$\alpha = \delta. \tag{3.30}$$

Then, for $p \geqslant 1$ and fixed $0 < y \leqslant 1$, the following convergence estimates hold,

$$\|u^{\delta}_{*\alpha}(\cdot,y) - u(\cdot,y)\| \leqslant \begin{cases} 2^{y}\delta^{1-\frac{y}{2}} + \frac{2E}{1-e^{-2\sqrt{1-k^2}}} \\ \max\left\{2e^{-\sqrt{1-k^2}(1-y)}\left(\ln\sqrt[3]{\frac{1}{\delta}}\right)^{-p}, \delta^{\frac{2}{3}-\frac{y}{3}}(1-k^2)^{-\frac{p}{2}}\right\}, & 0 < k < 1, \\ \delta + 2^{y}\delta^{1-\frac{y}{2}} + \\ \frac{2E}{1-e^{-2L}} \max\left\{2e^{-L(1-y)}\left(\ln\sqrt[3]{\frac{1}{\delta}}\right)^{-p}, \delta^{\frac{2}{3}-\frac{y}{3}}L^{-p}\right\}, & k \geqslant 1. \end{cases}$$

where L is given by (3.14).

Now we give the convergence estimate for the Cauchy problem of the Helmholtz equation with Neumann conditions at $x = 0$ and $x = \pi$. The proof can be obtained by similar procedure. We omit it in this paper.

Theorem 3.3. *Let u be the exact solution of problem (2.1)–(2.3) and 2.8 given by (2.9), and let $u^{*\delta}_{\alpha}$ be the regularized approximation solution given by (3.10). Suppose that the measured data φ_δ fulfills (2.5) and the exact solution u at $y = 1$ satisfies condition (2.6).*

If we choose the regularization parameter α by (3.12), then for fixed $0 < y \leqslant 1$, the following convergence estimate can be obtained:

$$\|u^{*\delta}_{\alpha}(\cdot,y) - u(\cdot,y)\| \leqslant \delta + 2^{y}E^{\frac{y}{2}}\delta^{1-\frac{y}{2}} + \frac{2}{1-e^{-2L}} \max\left\{2Ee^{-L(1-y)}\left(\ln\sqrt[3]{\frac{E}{\delta}}\right)^{-p}, \delta^{\frac{2}{3}-\frac{y}{3}}E^{\frac{1}{3}+\frac{y}{3}}L^{-p}\right\}, \quad k > 0,$$

where $L = \sqrt{([k]+1)^2 - k^2}$.

In the following, we give the convergence estimates for the Cauchy problem of the Helmholtz equation with Robin conditions at $x = 0$ and $x = \pi$.

Theorem 3.4. *Let u be the exact solution of problem (2.1)–(2.3), (2.11) and (2.12) given by (2.13), and let $u^{\delta}_{\alpha*}$ be the regularized approximation solution given by (3.11). Assume that the measured data φ_δ fulfills (2.5) and the exact solution u at $y = 1$ satisfies condition (2.6). We choose the regularization parameter α by (3.12).*

Then, for fixed $0 < y \leqslant 1$, we have

$$\|u^{\delta}_{\alpha*}(\cdot,y) - u(\cdot,y)\| \leqslant \begin{cases} 2^{y}E^{\frac{y}{2}}\delta^{1-\frac{y}{2}} + \frac{2}{1-e^{-2\sqrt{\beta_1^2-k^2}}} \\ \max\left\{2Ee^{-\sqrt{\beta_1^2-k^2}(1-y)}\left(\ln\sqrt[3]{\frac{E}{\delta}}\right)^{-p}, \frac{\delta^{\frac{2}{3}-\frac{y}{3}}E^{\frac{1}{3}+\frac{y}{3}}}{(\beta_1^2-k^2)^{\frac{p}{2}}}\right\}, & 0 < k < \beta_1, \\ \delta + 2^{y}E^{\frac{y}{2}}\delta^{1-\frac{y}{2}} + \frac{2}{1-e^{-2S}} \\ \max\left\{2Ee^{-S(1-y)}\left(\ln\sqrt[3]{\frac{E}{\delta}}\right)^{-p}, \frac{\delta^{\frac{2}{3}-\frac{y}{3}}E^{\frac{1}{3}+\frac{y}{3}}}{S^{p}}\right\}, & \beta_1 \leqslant k. \end{cases} \tag{3.31}$$

where $S = \sqrt{\beta_{s+1}^2 - k^2}$ and s is given by (2.16).

4. The truncation method

In this section, we give a truncation method and obtain the convergence estimates for $0 < y \leqslant 1$ under an *a-priori* assumption for the exact solution.

Firstly, we discuss the Cauchy problem for the Helmholtz equation with Dirichlet conditions at $x = 0$ and $x = \pi$.

From (2.7), we note that $\cosh(\sqrt{n^2 - k^2}y)$ tends to infinity as n tends to infinity, then in order to guarantee the convergence of solution u given by (2.7), the coefficient (φ, X_n) must decay rapidly. Usually such a decay is not likely to occur for the measured data φ_δ. Therefore, a natural way to obtain a stable approximation solution u is to eliminate the high frequencies and consider the solution u for $n \leqslant N$, where N is a positive integer. In the following, we assume $N \geqslant k + 1$ and define a regularization solution u_N^δ as a stable approximation of solution u, i.e.

$$u_N^\delta(x,y) = \begin{cases} \sum_{n=1}^{N} \cosh\left(\sqrt{n^2 - k^2}y\right)(\varphi_\delta, X_n)X_n, \quad 0 < k < 1, \\ \sum_{n=1}^{[k]} \cos\left(\sqrt{k^2 - n^2}y\right)(\varphi_\delta, X_n)X_n \\ + \sum_{n=[k]+1}^{N} \cosh\left(\sqrt{n^2 - k^2}y\right)(\varphi_\delta, X_n)X_n, \quad k \geqslant 1. \end{cases} \tag{4.1}$$

Secondly, for the Cauchy problem of the Helmholtz equation with Neumann conditions at $x = 0$ and $x = \pi$, we define a stable approximation $u_N^{*\delta}$ as follows,

$$u_N^{*\delta}(x,y) = \sum_{n=0}^{[k]} \cos\left(\sqrt{k^2 - n^2}y\right)(\varphi_\delta, Z_n)Z_n + \sum_{n=[k]+1}^{N} \cosh\left(\sqrt{n^2 - k^2}y\right)(\varphi_\delta, Z_n)Z_n, \quad k > 0, \tag{4.2}$$

Thirdly, for the Cauchy problem of the Helmholtz equation with Robin conditions at $x = 0$ and $x = \pi$, we define a stable approximation u_{N*}^δ as follows,

$$u_{N*}^\delta(x,y) = \begin{cases} \sum_{n=1}^{N} \cosh\left(\sqrt{\beta_n^2 - k^2}y\right)(\varphi_\delta, T_n)T_n, \quad 0 < k < \beta_1, \\ \sum_{n=1}^{s} \cos\left(\sqrt{k^2 - \beta_n^2}y\right)(\varphi_\delta, T_n)T_n \\ + \sum_{n=s+1}^{N} \cosh\left(\sqrt{\beta_n^2 - k^2}y\right)(\varphi_\delta, T_n)T_n, \quad k \geqslant \beta_1. \end{cases} \tag{4.3}$$

In the following theorem, we will give the convergence estimate between u_N^δ given by (4.1) and u given by (2.7).

Theorem 4.1. *Let the exact solution u and the regularization solution u_N^δ be given by (2.7) and (4.1), respectively. Assume that the measured data φ_δ satisfies (2.5) and the exact solution u fulfills the a-priori bound assumption (2.6). We choose the regularization parameter $N = [\beta]$, here*

$$\beta = \sqrt{\ln^2\left(\frac{2E}{\delta}\left(\ln\frac{2E}{\delta}\right)^{-p}\right) + k^2}. \tag{4.4}$$

Then, for the integer $p \geqslant 1$ and fixed $0 < y \leqslant 1$, we have the following convergence estimates,

$$\|u_N^\delta(\cdot,y) - u(\cdot,y)\| \leqslant \begin{cases} \left(1 + \frac{1}{1-e^{-2}}\left(\frac{\ln\frac{2E}{\delta}}{\ln\frac{2E}{\delta} + \ln(\ln\frac{2E}{\delta})^{-p}}\right)^p\right)(2E)^y\delta^{1-y}\left(\ln\frac{2E}{\delta}\right)^{-py}, \quad 0 < k < 1, \\ \delta + \left(1 + \frac{1}{1-e^{-2}}\left(\frac{\ln\frac{2E}{\delta}}{\ln\frac{2E}{\delta} + \ln(\ln\frac{2E}{\delta})^{-p}}\right)^p\right)(2E)^y\delta^{1-y}\left(\ln\frac{2E}{\delta}\right)^{-py}, \quad k \geqslant 1. \end{cases} \tag{4.5}$$

Proof. Note that condition (2.5) gives

$$\sum_{n=1}^{\infty} |(\varphi_\delta - \varphi, X_n)|^2 \leqslant \delta^2. \tag{4.6}$$

Firstly, we consider the case $0 < k < 1$. By the first equations in (2.7) and (4.1), and (4.6), we can obtain that

$$\begin{aligned}\|u_N^\delta(\cdot,y)-u(\cdot,y)\| &\leqslant \left(\sum_{n=1}^{N}\cosh^2\left(\sqrt{n^2-k^2}y\right)|(\varphi_\delta-\varphi,X_n)|^2\right)^{\frac{1}{2}}+\left(\sum_{n=N+1}^{\infty}\cosh^2\left(\sqrt{n^2-k^2}y\right)|(\varphi,X_n)|^2\right)^{\frac{1}{2}}\\ &\leqslant \delta\sup_{1\leqslant n\leqslant N}\cosh\left(\sqrt{n^2-k^2}y\right)+\left(\sum_{n=N+1}^{\infty}\cosh^2\left(\sqrt{n^2-k^2}y\right)|(\varphi,X_n)|^2\right)^{\frac{1}{2}}\\ &\leqslant \delta e^{\sqrt{N^2-k^2}y}+\left(\sum_{n=N+1}^{\infty}\cosh^2\left(\sqrt{n^2-k^2}y\right)|(\varphi,X_n)|^2\right)^{\frac{1}{2}},\end{aligned} \tag{4.7}$$

Denote $A(n)=\left(\sum_{n=N+1}^{\infty}\cosh^2(\sqrt{n^2-k^2}y)|(\varphi,X_n)|^2\right)^{\frac{1}{2}}$. In the following, we estimate $A(n)$.

When p is even, by the second equation in (3.16), we derive that

$$A(n)\leqslant E\sup_{n\geqslant N+1}\frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{\cosh\left(\sqrt{n^2-k^2}\right)}(n^2-k^2)^{-\frac{p}{2}}\leqslant 2E\sup_{n\geqslant N+1}\frac{e^{\sqrt{n^2-k^2}y}}{e^{\sqrt{n^2-k^2}}}(n^2-k^2)^{-\frac{p}{2}}\leqslant 2Ee^{-\sqrt{(N+1)^2-k^2}(1-y)}\left((N+1)^2-k^2\right)^{-\frac{p}{2}}. \tag{4.8}$$

When p is odd, by the first equation in (3.16), we have

$$\begin{aligned}A(n)&\leqslant E\sup_{n\geqslant N+1}\frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{\sinh\left(\sqrt{n^2-k^2}\right)}(n^2-k^2)^{-\frac{p}{2}}\leqslant 2E\sup_{n\geqslant N+1}\frac{e^{\sqrt{n^2-k^2}y}}{e^{\sqrt{n^2-k^2}}\left(1-e^{-2\sqrt{n^2-k^2}}\right)}(n^2-k^2)^{-\frac{p}{2}}\\ &\leqslant \frac{2E}{1-e^{-2}}e^{-\sqrt{(N+1)^2-k^2}(1-y)}\left((N+1)^2-k^2\right)^{-\frac{p}{2}}.\end{aligned} \tag{4.9}$$

Note that $N\leqslant\beta<N+1$, by (4.7), (4.8) and (4.9), we obtain

$$\begin{aligned}\|u_N^\delta(\cdot,y)-u(\cdot,y)\|&\leqslant \delta e^{\sqrt{N^2-k^2}y}+\frac{2E}{1-e^{-2}}e^{-\sqrt{(N+1)^2-k^2}(1-y)}\left((N+1)^2-k^2\right)^{-\frac{p}{2}}\\ &\leqslant \delta e^{\sqrt{\beta^2-k^2}y}+\frac{2E}{1-e^{-2}}e^{-\sqrt{\beta^2-k^2}(1-y)}(\beta^2-k^2)^{-\frac{p}{2}}.\end{aligned}$$

By (4.4), we obtain the first estimate in (4.5).

When $k\geqslant 1$, by the second equation in (2.7), (4.1), and (4.6), we can obtain that

$$\begin{aligned}\|u_N^\delta(\cdot,y)-u(\cdot,y)\|&\leqslant \left(\sum_{n=1}^{[k]}\cos^2\left(\sqrt{k^2-n^2}y\right)|(\varphi_\delta-\varphi,X_n)|^2\right)^{\frac{1}{2}}\\ &+\left(\sum_{n=[k]+1}^{N}\cosh^2\left(\sqrt{n^2-k^2}y\right)|(\varphi_\delta-\varphi,X_n)|^2\right)^{\frac{1}{2}}\\ &+\left(\sum_{n=N+1}^{\infty}\cosh^2\left(\sqrt{n^2-k^2}y\right)|(\varphi,X_n)|^2\right)^{\frac{1}{2}}\\ &\leqslant \delta+\delta\sup_{[k]+1\leqslant n\leqslant N}\cosh\left(\sqrt{n^2-k^2}y\right)+\left(\sum_{n=N+1}^{\infty}\cosh^2\left(\sqrt{n^2-k^2}y\right)|(\varphi,X_n)|^2\right)^{\frac{1}{2}}\\ &\leqslant \delta+\delta e^{\sqrt{N^2-k^2}y}+\left(\sum_{n=N+1}^{\infty}\cosh^2\left(\sqrt{n^2-k^2}y\right)|(\varphi,X_n)|^2\right)^{\frac{1}{2}},\end{aligned}$$

Similar to the case $0<k<1$, we have

$$\|u_N^\delta(\cdot,y)-u(\cdot,y)\|\leqslant \delta+\delta e^{\sqrt{\beta^2-k^2}y}+\frac{2E}{1-e^{-2}}e^{-\sqrt{\beta^2-k^2}(1-y)}(\beta^2-k^2)^{-\frac{p}{2}}.$$

Then by (4.4), the second estimate in (4.5) is obtained. □

Remark 4.2. The constant E is unknown in practice. In such a case, we choose the regularization parameter $N=[\beta]$, where

$$\beta=\sqrt{\ln^2\left(\frac{2}{\delta}\left(\ln\frac{2}{\delta}\right)^{-p}\right)+k^2}. \tag{4.10}$$

Then, for $p\geqslant 1$ and fixed $0<y<1$, the following convergence estimates hold,

$$\|u_N^\delta(\cdot,y)-u(\cdot,y)\| \leqslant \begin{cases} \left(1+\frac{E}{1-e^{-2}}\left(\frac{\ln\frac{2}{\delta}}{\ln\frac{2}{\delta}+\ln(\ln\frac{2}{\delta})^{-p}}\right)^p\right)2^y\delta^{1-y}\left(\ln\frac{2}{\delta}\right)^{-py}, & 0<k<1, \\ \delta+\left(1+\frac{E}{1-e^{-2}}\left(\frac{\ln\frac{2}{\delta}}{\ln\frac{2}{\delta}+\ln(\ln\frac{2}{\delta})^{-p}}\right)^p\right)2^y\delta^{1-y}\left(\ln\frac{2}{\delta}\right)^{-py}, & k\geqslant 1. \end{cases}$$

Similarly, we can also obtain the following convergence estimate between $u_N^{*\delta}$ given by (4.2) and u given by (2.9) for the Cauchy problem of the Helmholtz equation with Neumann conditions at $x=0$ and $x=\pi$.

Theorem 4.3. *Let the exact solution u and the regularization solution $u_N^{*\delta}$ be given by (2.9) and (4.2), respectively. Assume that the measured data φ_δ satisfies (2.5) and the exact solution u fulfills the a-priori assumption (2.6). We choose the regularization parameter $N=[\beta]$, here β is given by (4.4).*

Then, for the wave number $k>0$ and fixed $0<y\leqslant 1$, the following convergence estimate holds,

$$\|u_N^{*\delta}(\cdot,y)-u(\cdot,y)\| \leqslant \delta+\left(1+\frac{1}{1-e^{-2}}\left(\frac{\ln\frac{2E}{\delta}}{\ln\frac{2E}{\delta}+\ln(\ln\frac{2E}{\delta})^{-p}}\right)^p\right)(2E)^y\delta^{1-y}\left(\ln\frac{2E}{\delta}\right)^{-py}.$$

Now we give the convergence estimate for the Cauchy problem of the Helmholtz equation with Robin conditions at $x=0$ and $x=\pi$. The proof is similar to Theorem 4.1, we omit it in the following theorem.

Theorem 4.4. *Suppose that exact solution u and the regularization solution u_{N*}^δ are given by (2.13) and (4.3), respectively. Let the measured data φ_δ satisfy (2.5), and let the exact solution u fulfill the a-priori bound assumption (2.6). If the regularization parameter is chosen as $N=\min\{n\mid\beta_{n+1}>\beta\}$, here β is given by (4.4).*

Then, for fixed $0<y\leqslant 1$, we have

$$\|u_{N*}^\delta(\cdot,y)-u(\cdot,y)\| \leqslant \begin{cases} \left(1+\frac{1}{1-e^{-2}}\left(\frac{\ln\frac{2E}{\delta}}{\ln\frac{2E}{\delta}+\ln(\ln\frac{2E}{\delta})^{-p}}\right)^p\right)(2E)^y\delta^{1-y}\left(\ln\frac{2E}{\delta}\right)^{-py}, & 0<k<\beta_1, \\ \delta+\left(1+\frac{1}{1-e^{-2}}\left(\frac{\ln\frac{2E}{\delta}}{\ln\frac{2E}{\delta}+\ln(\ln\frac{2E}{\delta})^{-p}}\right)^p\right)(2E)^y\delta^{1-y}\left(\ln\frac{2E}{\delta}\right)^{-py}, & k\geqslant\beta_1. \end{cases} \tag{4.11}$$

5. Numerical examples

In this section, we present three numerical examples to test the effectiveness of the two proposed regularization methods.

In order to compare the proposed two regularized methods, we introduce the following relative root mean square error between the regularized solution u^δ and the exact solution u,

$$RMS := \frac{\sqrt{\frac{1}{(M_1+1)\times(M_2+1)}\sum_{i=1}^{M_1+1}\sum_{j=1}^{M_2+1}\left(u^\delta(x_i,y_j)-u(x_i,y_j)\right)^2}}{\sqrt{\frac{1}{(M_1+1)\times(M_2+1)}\sum_{i=1}^{M_1+1}\sum_{j=1}^{M_2+1}(u(x_i,y_j))^2}},$$

where $x_i=\frac{\pi}{M_1}(i-1),\ y_j=\frac{j-1}{M_2},\ i=1,2,\cdots,M_1+1,\ j=1,2,\cdots,M_2+1$. In our numerical computations, we always take $M_1=M_2=30$.

In the following, we will give the numerical results obtained by using the Tikhonov method (Tikh) and the truncation method (Trun).

Example 1. Consider the following direct problem for the Helmholtz equation at first:

$$\Delta u(x,y)+k^2u(x,y)=0,\quad 0<x<\pi,\quad 0<y<1, \tag{5.1}$$

$$u(x,1)=x^2(\pi-x)+\sin(2kx)\cosh(\sqrt{3}k),\quad 0\leqslant x\leqslant\pi, \tag{5.2}$$

$$u_y(x,0)=0,\quad 0\leqslant x\leqslant\pi, \tag{5.3}$$

$$u(0,y)=u(\pi,y)=0,\quad 0\leqslant y\leqslant 1. \tag{5.4}$$

Separation of variables leads to the solution of problem (5.1)–(5.4) as follows,

$$u(x,y)=\begin{cases} \sum\limits_{n=1}^{\infty}\frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{\cosh\left(\sqrt{n^2-k^2}\right)}(u(x,1),X_n)X_n, \quad 0<k<1, \\ \sum\limits_{n=1}^{[k]}\frac{\cos\left(\sqrt{k^2-n^2}y\right)}{\cos\left(\sqrt{k^2-n^2}\right)}(u(x,1),X_n)X_n \\ +\sum\limits_{n=[k]+1}^{\infty}\frac{\cosh\left(\sqrt{n^2-k^2}y\right)}{\cosh\left(\sqrt{n^2-k^2}\right)}(u(x,1),X_n)X_n, \quad k\geqslant 1, \quad \text{and} \\ k\neq\sqrt{\left(\frac{m}{2}\pi\right)^2+n^2}, \quad n=1,2,\cdots,[k] \quad \text{and} \quad m=1,2,\cdots, \end{cases} \tag{5.5}$$

where $X_n=\sqrt{\frac{2}{\pi}}\sin(nx)$.

Then, we choose

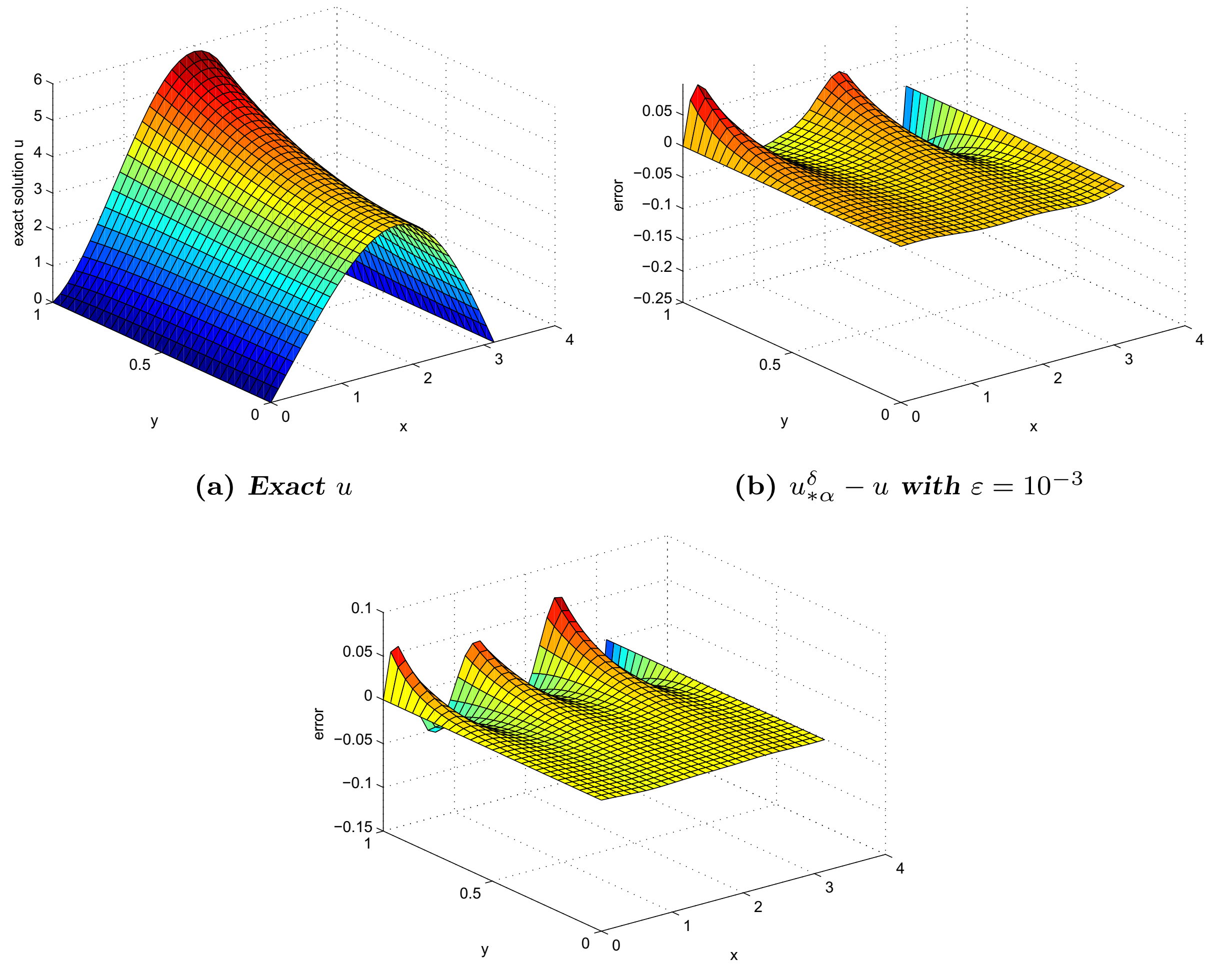
$$\varphi(x) = u(x,0) \approx \begin{cases} \sum_{n=1}^{m} \frac{(u(x,1),X_n)}{\cosh\left(\sqrt{n^2-k^2}\right)} X_n, & 0 < k < 1, \\ \sum_{n=1}^{[k]} \frac{(u(x,1),X_n)}{\cos\left(\sqrt{k^2-n^2}\right)} X_n + \sum_{n=[k]+1}^{m} \frac{(u(x,1),X_n)}{\cosh\left(\sqrt{n^2-k^2}\right)} X_n, & k \geqslant 1. \end{cases}$$

as the initial data for problem (2.1)–(2.4) with $m = 36$. The measured data $\varphi_\delta(x)$ is given by $\varphi_\delta(x) = \varphi(x) + \varepsilon(x-2)(4-x)\sin x$, where ε denotes the error level. Then we obtain the regularized solutions $u^\delta_{*\alpha}$ computed by (3.9) and u^δ_N computed by (4.1) for problem (2.1)–(2.4).

The relative root mean square error between the regularized solution and the exact solution is shown in Table 1 in which we take $k = 0.5$ with $\varepsilon = 10^{-4}, \quad 10^{-3}, \quad 10^{-2}$, and 5×10^{-2}.

Table 1
RMS with $k = 0.5$.

ε	10^{-4}	10^{-3}	10^{-2}	5×10^{-2}
Tikh	0.0039	0.0085	0.0312	0.1164
Trun	0.0017	0.0044	0.0185	0.0446



(a) ***Exact*** u **(b)** $u^\delta_{*\alpha} - u$ ***with*** $\varepsilon = 10^{-3}$

(c) $u^\delta_N - u$ ***with*** $\varepsilon = 10^{-3}$

Fig. 1. $k = 0.5$.

In Figs. 1–5 and Table 1, the regularized solution $u^{\delta}_{*\alpha}$ given by (3.9) and u^{δ}_{N} given by (4.1), in which we choose regularization parameter α by (3.30) and $N = [\beta]$, here β is given by (4.10) with $p = 1$.

In Fig. 1, we show the numerical results for exact solution u given by (5.5) and the error between the regularized solution $u^{\delta}_{*\alpha}$ (u^{δ}_{N}) and the exact solution u for $0 < y \leqslant 1$ with $k = 0.5$ and $\varepsilon = 10^{-3}$.

In Figs. 2 and 4, we show the numerical results at $y = 1$ for $\varepsilon = 10^{-3}$, 10^{-2} and 5×10^{-2} with $k = 0.5$ and $k = 1$, respectively.

In Fig. 3, we give the numerical results for $u(\cdot, y)$, $u^{\delta}_{*\alpha}(\cdot, y)$ and $u^{\delta}_{N}(\cdot, y)$ at $y = 0.2, 0.6, 0.9$ with $k = 1$ and $\varepsilon = 3 \times 10^{-2}$. From Fig. 3, we find that the numerical results become worse when y approaches to 1.

The numerical results at $y = 1$ for $k = 1.5$, 2 with $\varepsilon = 10^{-2}$ are shown in Fig. 5.

From Table 1 and Figs. 1–5, we find that the two proposed regularization methods work effectively and that the truncation method performs better than the Tikhonov method, which is in agreement with the convergence results. From Figs. 2 and 4, we also note that the numerical results become discouraging when the error level become a little high, which indicates that the proposed method is sensitive to the noise. From Figs. 4b and 5, we find that the proposed method become less accurate with the increase of the wave number k, which indicates that the proposed method is also sensitive to the wave number k.

Now we give another example to show that the proposed two regularized methods are also valid for the Cauchy problem of the Helmholtz equation with Neumann conditions at $x = 0$ and $x = \pi$.

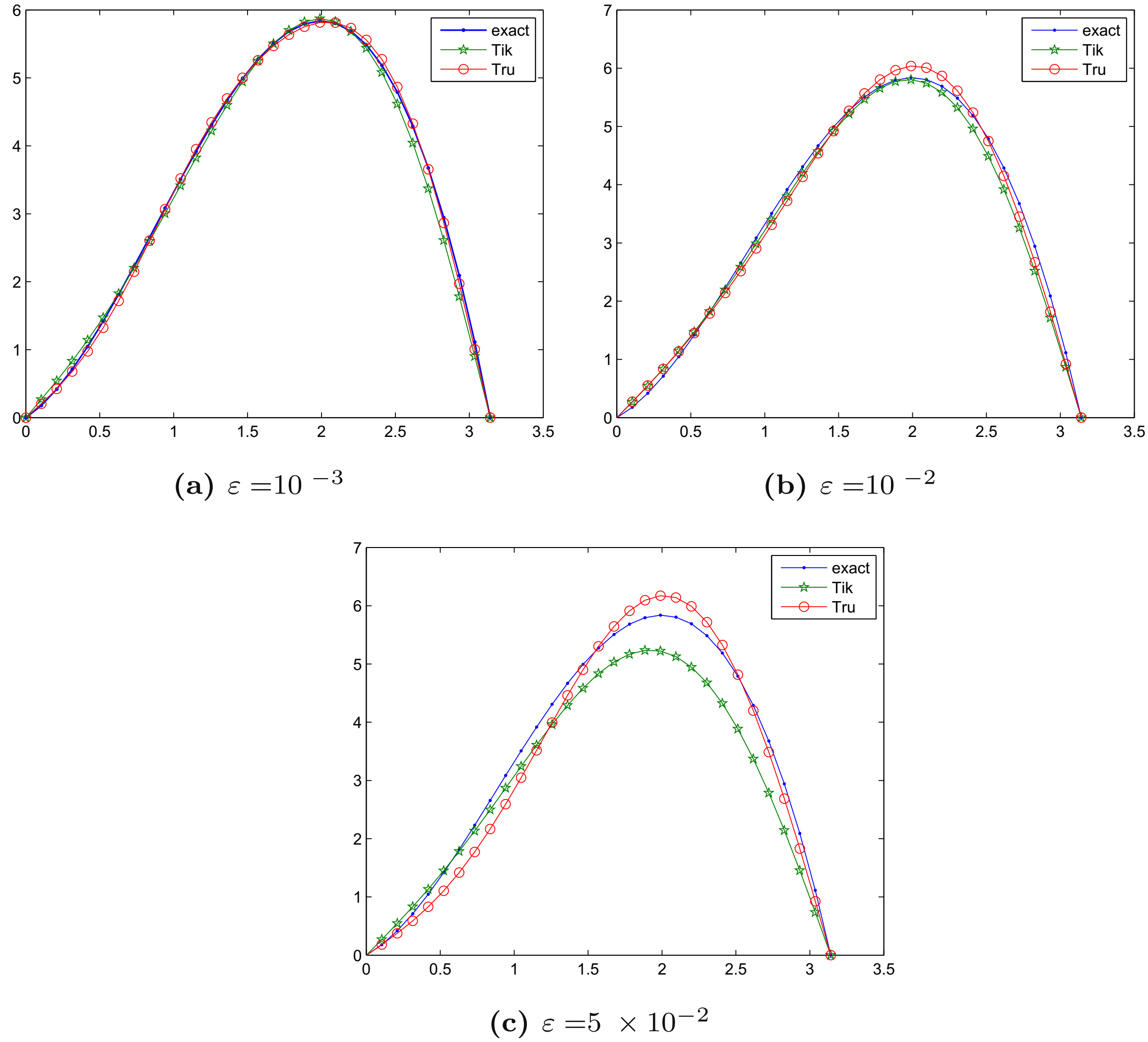


Fig. 2. $k = 0.5$.

Example 2. Firstly, solve the following direct problem for the Helmholtz equation:

$$\Delta u(x,y) + k^2 u(x,y) = 0, \quad 0 < x < \pi, \quad 0 < y < 1, \tag{5.6}$$

$$u(x,1) = x(\pi - x)\sin x + \cos(2kx)\cosh(\sqrt{3}k), \quad 0 \leqslant x \leqslant \pi, \tag{5.7}$$

$$u_y(x,0) = 0, \quad 0 \leqslant x \leqslant \pi, \tag{5.8}$$

$$u_x(0,y) = u_x(\pi,y) = 0, \quad 0 \leqslant y \leqslant 1. \tag{5.9}$$

By the method of separation of variables, the solution of problem (5.6), (5.7), (5.8), (5.9) can be obtained as follows,

$$u(x,y) = \sum_{n=0}^{[k]} \frac{\cos\left(\sqrt{k^2 - n^2}y\right)}{\cos\left(\sqrt{k^2 - n^2}\right)}(u(x,1), Z_n)Z_n \tag{5.10}$$

$$+ \sum_{n=[k]+1}^{\infty} \frac{\cosh\left(\sqrt{n^2 - k^2}y\right)}{\cosh\left(\sqrt{n^2 - k^2}\right)}(u(x,1), Z_n)Z_n, \quad k > 0, \quad \text{and}$$

$$k \neq \sqrt{\left(\frac{m}{2}\pi\right)^2 + n^2}, \quad n = 0, 1, \cdots, [k] \quad \text{and} \quad m = 1, 2, \cdots,$$

where $\{Z_n\}_{n=0}^{\infty}$ are given by (2.10).

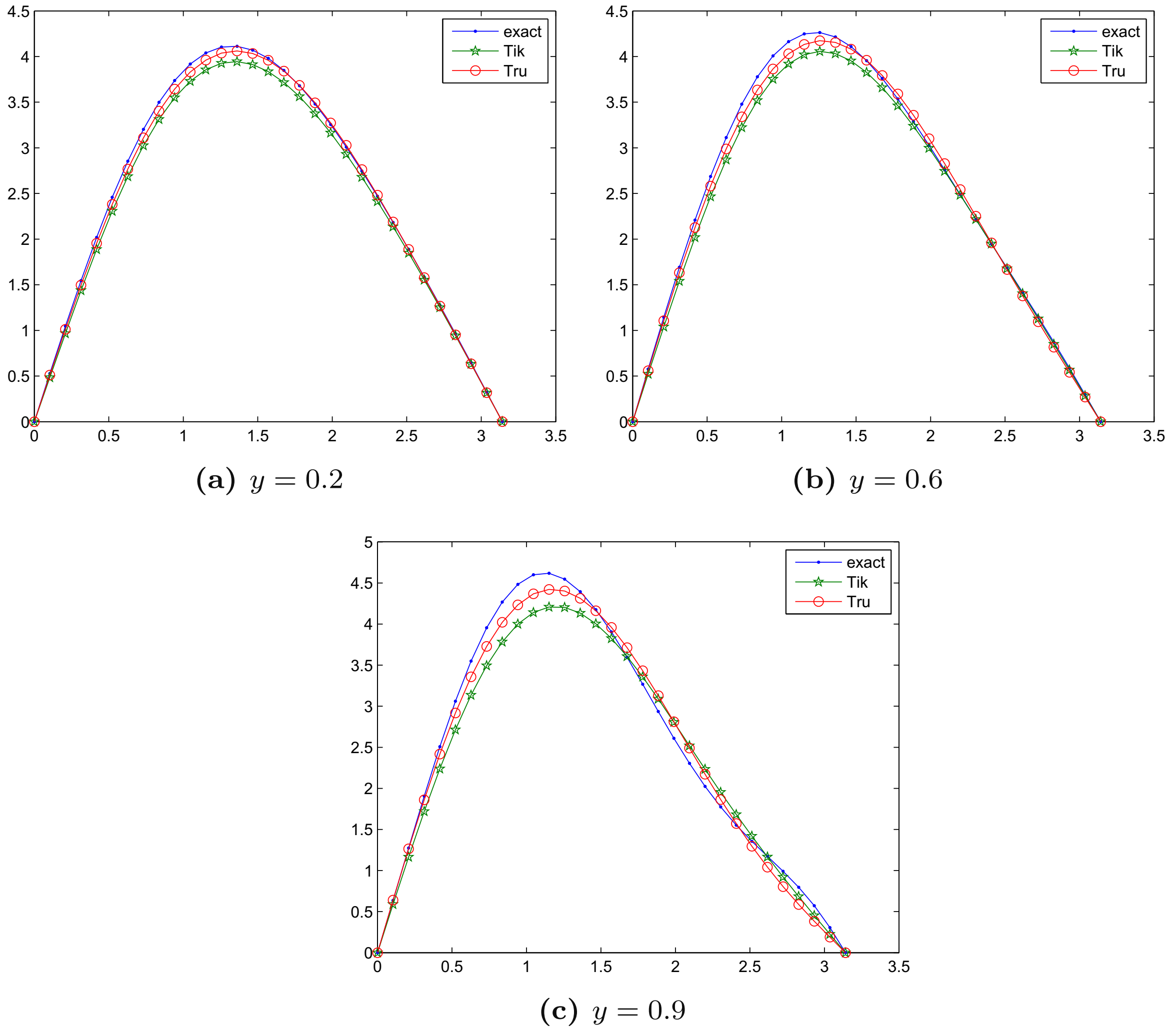


Fig. 3. $k = 1$.

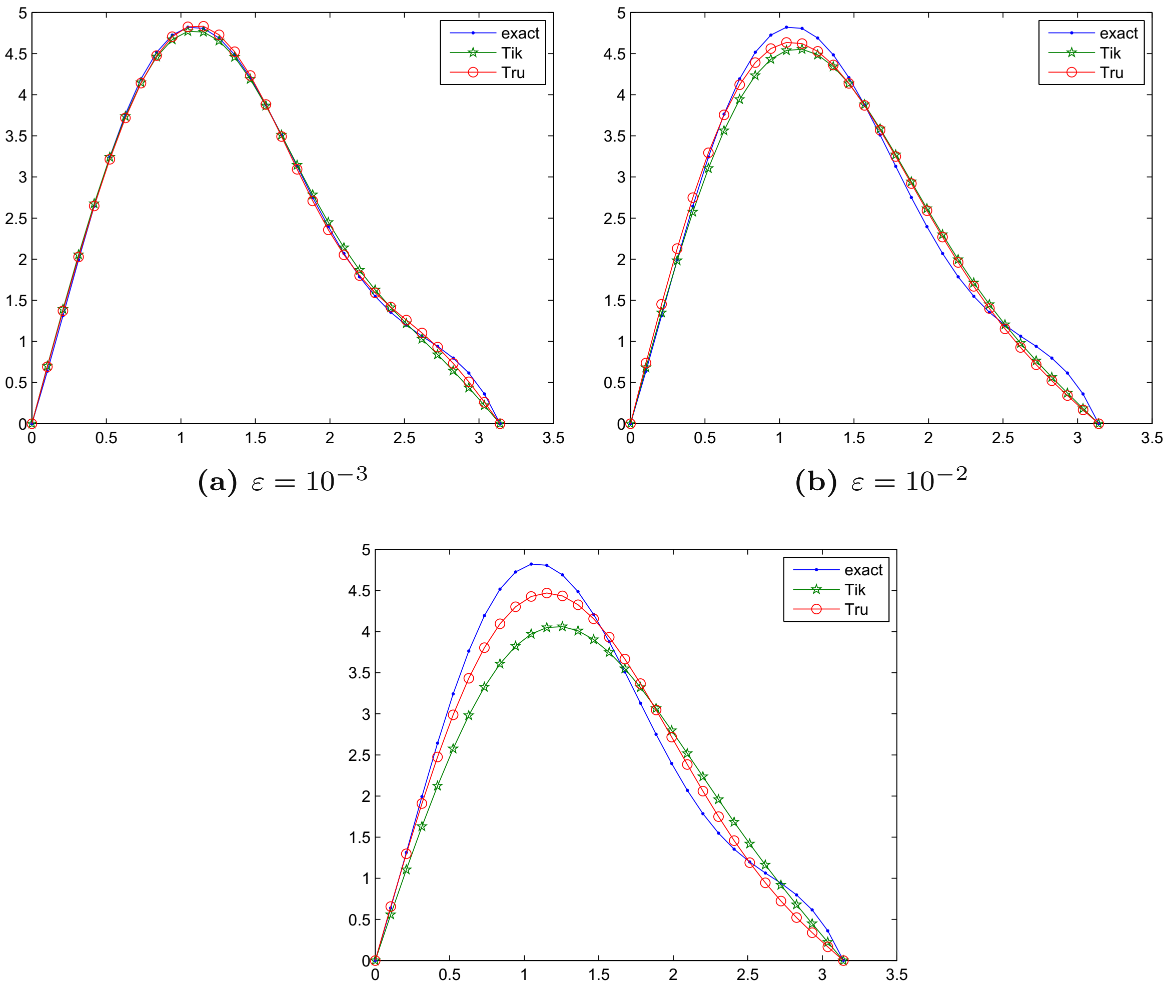
Then, we choose

$$\varphi(x) = u(x,0) \approx \sum_{n=0}^{[k]} \frac{(u(x,1), Z_n)}{\cos\left(\sqrt{k^2 - n^2}\right)} Z_n + \sum_{n=[k]+1}^{m} \frac{(u(x,1), Z_n)}{\cosh\left(\sqrt{n^2 - k^2}\right)} Z_n, \quad k > 0.$$

as the initial data for problem 2.1, 2.2, 2.3 and (2.8) with $m = 41$. The measured data $\varphi_\delta(x)$ is given by $\varphi_\delta(x) = \varphi(x) + \varepsilon x(1-x)$, where ε denotes the error level. Then we obtain the regularized solutions $u_\alpha^{*\delta}$ computed by (3.10) and $u_N^{*\delta}$ computed by (4.2) for problem (2.1)–(2.3) and (2.8).

The numerical results for $k = 0.5, \quad 1, \quad 1.5$ with $\varepsilon = 2 \times 10^{-2}$ and 5×10^{-2} are shown in Figs. 6 and 7. The regularized parameter α and N are chosen by the same ways as in Example 1.

From the numerical results in Figs. 6 and 7, we also note that the proposed regularized methods are accurate and effective for the Cauchy problem of the Helmholtz equation with Neumann conditions at $x = 0$ and $x = \pi$ and the truncation method performs more accurately than the Tikhonov method. Meanwhile, from subfigures (a)–(c) in Figs. 6 and 7, it can be seen that when y approaches to 1, the numerical result becomes less accurate. From subfigures (c) and (d) in Figs. 6 and 7, it can be noted that when the error level increases, the numerical result becomes less accurate. From Figs. 7c and Fig. 7e, we also note that the numerical result becomes discouraging with the increase of the wave number k, which indicates that the proposed regularized methods are only effective for the relative small wave number k.



(a) $\varepsilon = 10^{-3}$

(b) $\varepsilon = 10^{-2}$

(c) $\varepsilon = 5 \times 10^{-2}$

Fig. 4. $k = 1$.

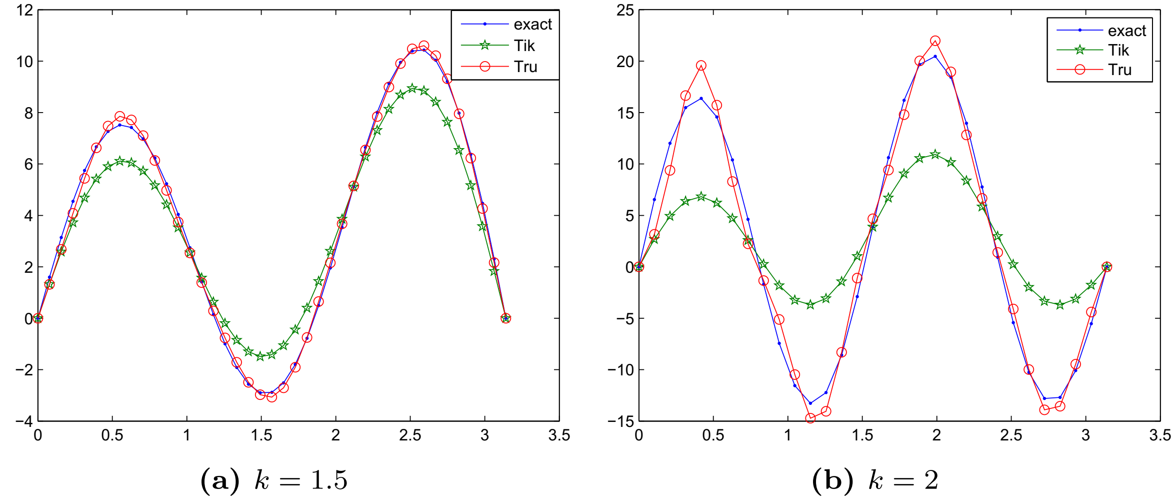


Fig. 5. $\varepsilon = 10^{-2}$.

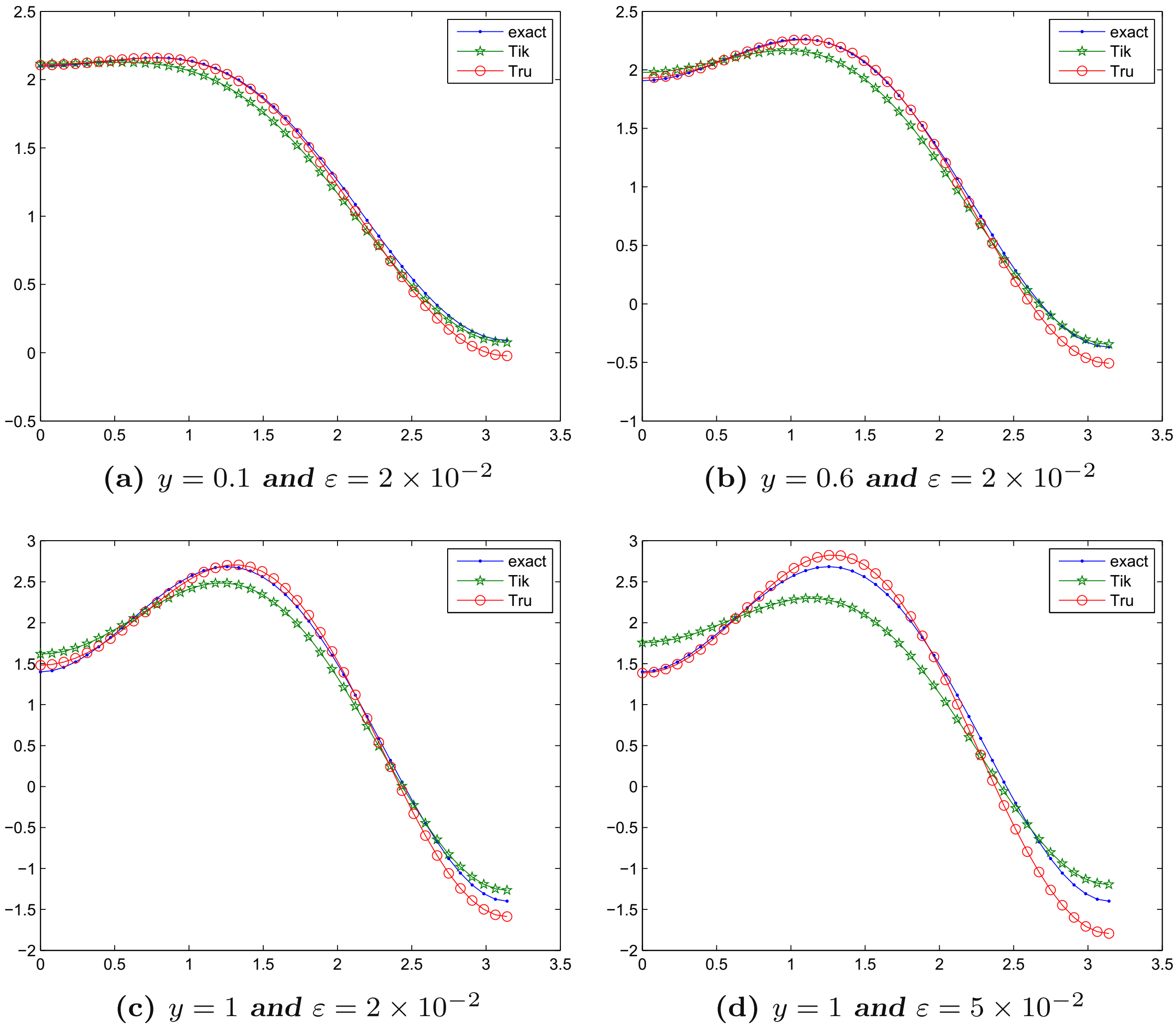


Fig. 6. $k = 0.5, u(\cdot, y), u_{\alpha}^{*\delta}(\cdot, y)$ and $u_N^{*\delta}(\cdot, y)$.

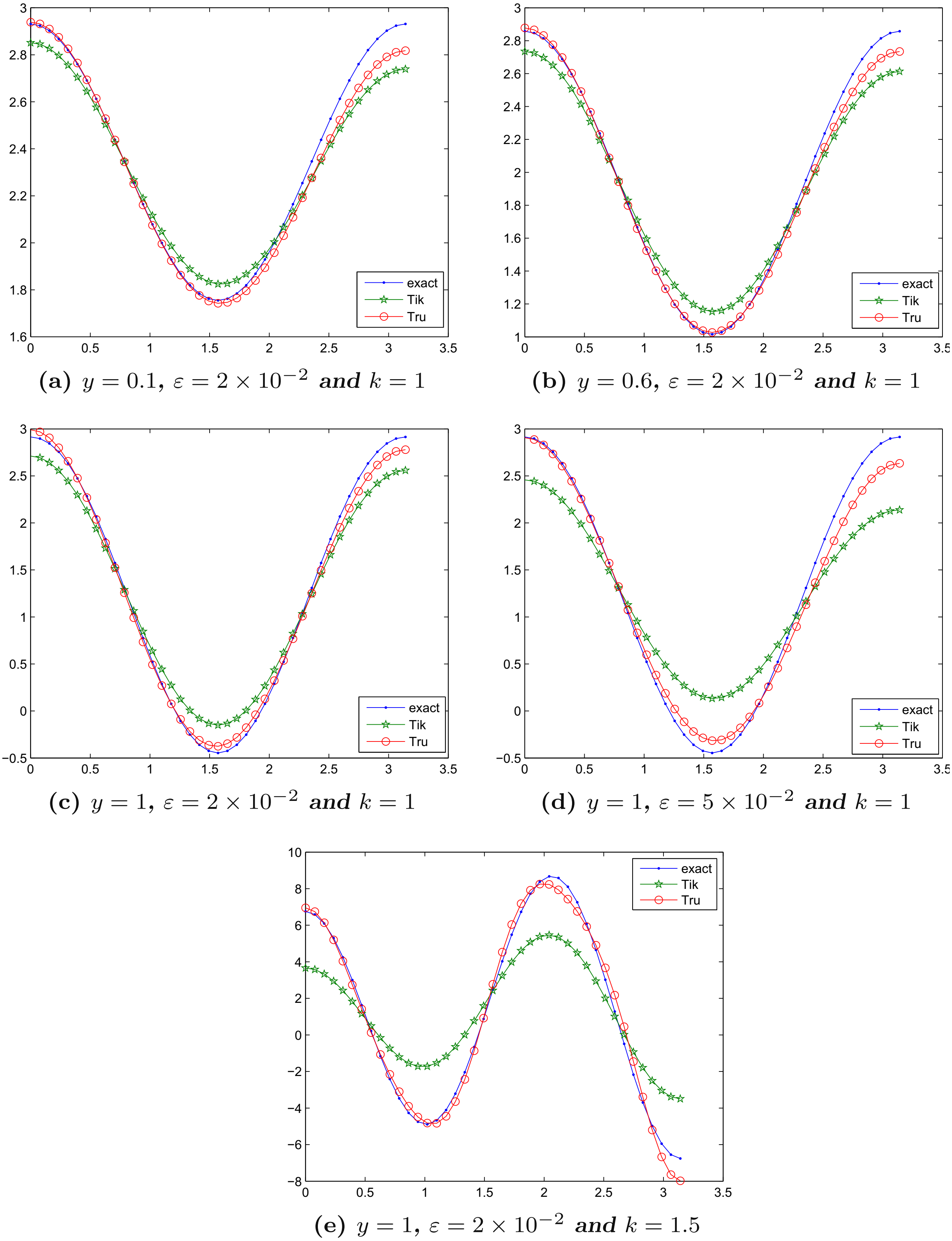
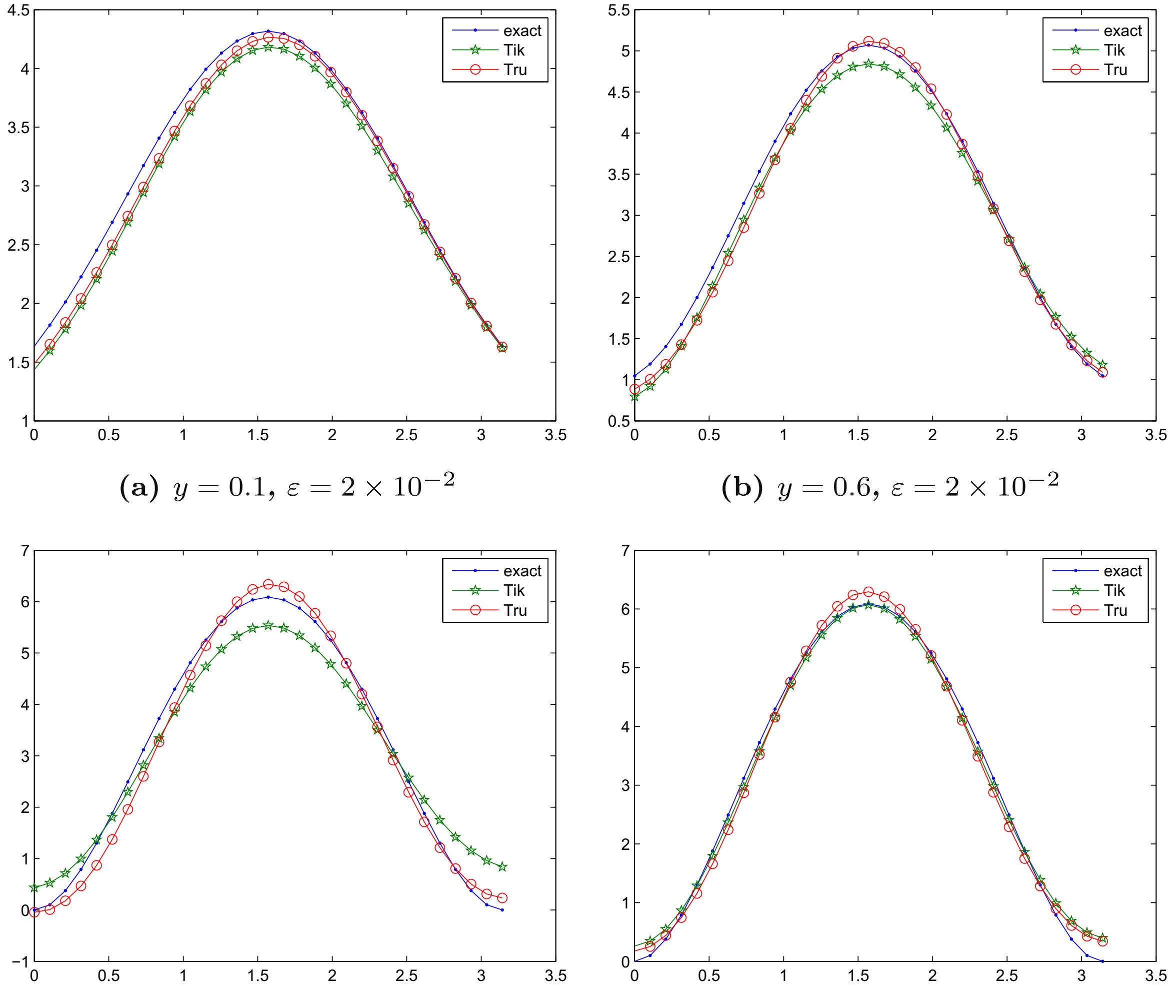


Fig. 7. $u(\cdot, y)$, $u_{\alpha}^{*\delta}(\cdot, y)$ and $u_{N}^{*\delta}(\cdot, y)$.



(a) $y = 0.1$, $\varepsilon = 2 \times 10^{-2}$ (b) $y = 0.6$, $\varepsilon = 2 \times 10^{-2}$

(c) $y = 1$, $\varepsilon = 2 \times 10^{-2}$ (d) $y = 1$, $\varepsilon = 5 \times 10^{-3}$

Fig. 8. $u(\cdot, y)$, $u^{\delta}_{\alpha*}(\cdot, y)$ and $u^{\delta}_{N*}(\cdot, y)$ with $k = 0.5$.

Example 3. At first, consider the following direct problem for the Helmholtz equation:

$$\Delta u(x,y) + k^2 u(x,y) = 0, \quad 0 < x < \pi, \quad 0 < y < 1, \tag{5.11}$$

$$u(x,1) = x^2(\pi - x)^2, \quad 0 \leqslant x \leqslant \pi, \tag{5.12}$$

$$u_y(x,0) = 0, \quad 0 \leqslant x \leqslant \pi, \tag{5.13}$$

$$-u_x(0,y) + au(0,y) = 0, \quad 0 \leqslant y \leqslant 1, \tag{5.14}$$

$$u_x(\pi,y) + bu(\pi,y) = 0, \quad 0 \leqslant y \leqslant 1. \tag{5.15}$$

By the method of separation of variables, the solution of problem (5.11)–(5.15) can be given by,

$$u(x,y) = \begin{cases} \sum_{n=1}^{\infty} \frac{\cosh\left(\sqrt{\beta_n^2 - k^2}\, y\right)}{\cosh\left(\sqrt{\beta_n^2 - k^2}\right)} (u(x,1), T_n) T_n, \quad 0 < k < \beta_1, \\ \sum_{n=1}^{s} \frac{\cos\left(\sqrt{k^2 - \beta_n^2}\, y\right)}{\cos\left(\sqrt{k^2 - \beta_n^2}\right)} (u(x,1), T_n) T_n \\ + \sum_{n=s+1}^{\infty} \frac{\cosh\left(\sqrt{\beta_n^2 - k^2}\, y\right)}{\cosh\left(\sqrt{\beta_n^2 - k^2}\right)} (u(x,1), T_n) T_n, \quad k \geqslant \beta_1, \quad \text{and} \\ \quad k \neq \sqrt{\left(\frac{m}{2}\pi\right)^2 + \beta_n^2}, \quad n = 1, 2, \cdots, s \quad \text{and} \quad m = 1, 2, \cdots, \end{cases} \tag{5.16}$$

where T_n is given by (2.14).

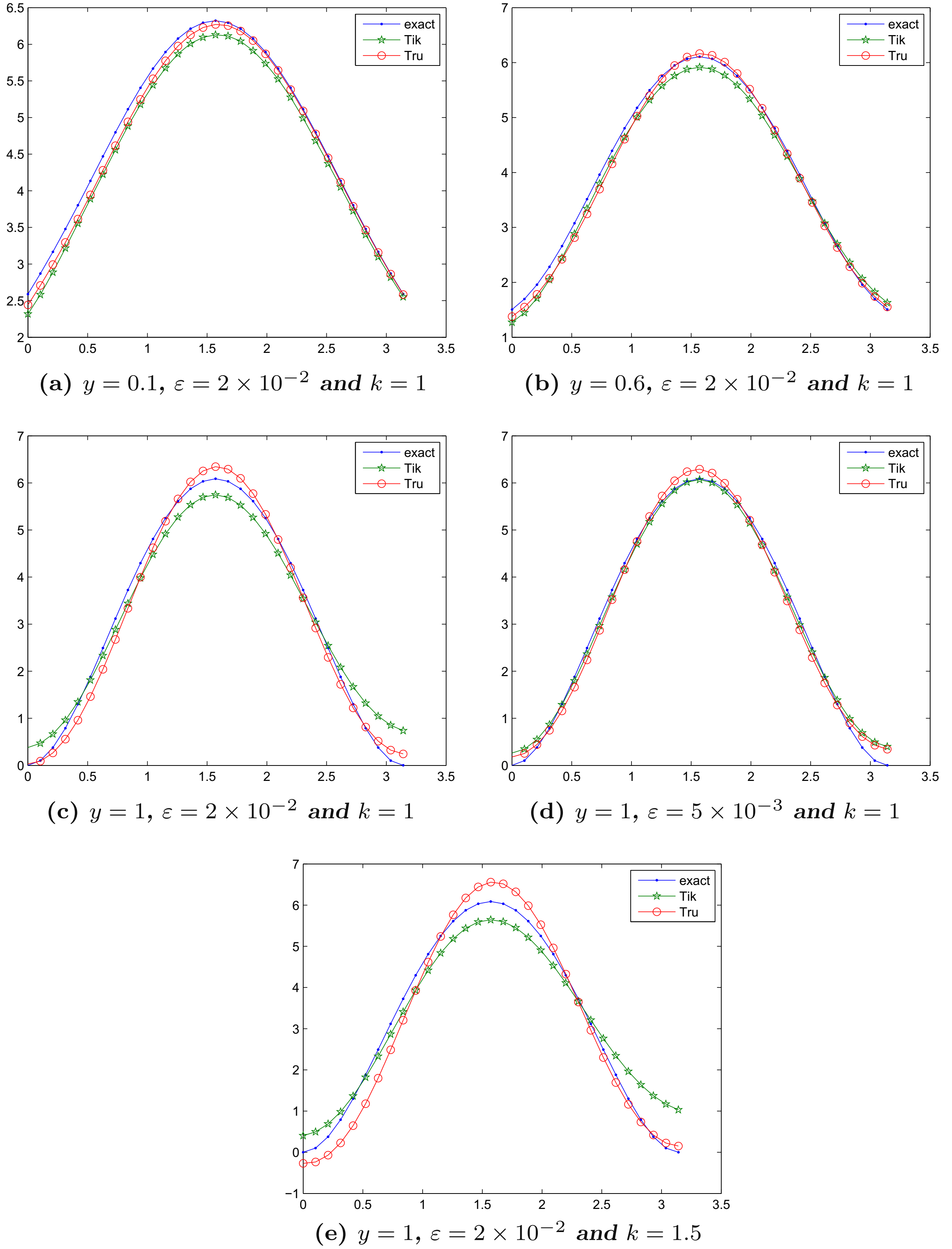


Fig. 9. $u(\cdot, y)$, $u^{\delta}_{\alpha*}(\cdot, y)$ and $u^{\delta}_{N*}(\cdot, y)$.

Then, we choose

$$\varphi(x) = u(x,0) \approx \begin{cases} \sum_{n=1}^{m} \frac{(u(x,1),T_n)}{\cosh\left(\sqrt{\beta_n^2-k^2}\right)} T_n, & 0 < k < \beta_1, \\ \sum_{n=1}^{s} \frac{(u(x,1),T_n)}{\cos\left(\sqrt{k^2-\beta_n^2}\right)} T_n + \sum\limits_{n=s+1}^{m} \frac{(u(x,1),T_n)}{\cosh\left(\sqrt{\beta_n^2-k^2}\right)} T_n, & k \geqslant \beta_1. \end{cases}$$

as the initial data for problem (2.1), (2.2), (2.3) and (2.11), (2.12) with $m = 31$. The measured data $\varphi_\delta(x)$ is given by $\varphi_\delta(x) = \varphi(x) + \varepsilon(x-3)(4-x)$, where ε denotes the error level. Then we obtain the regularized solutions $u^\delta_{\alpha*}$ computed by (3.11) and u^δ_{N*} computed by (4.3) for problem (2.1), (2.2), (2.3) and (2.11), (2.12).

In the following numerical computation, we always take $a = b = 1$.

The numerical results for $k = 0.5, \quad 1, \quad 1.5$ with $\varepsilon = 5 \times 10^{-3}$ and 2×10^{-2} are shown in Figs. 8 and 9. In the computation, s is taken by (2.16) and the regularized parameter α and N are chosen by the same ways as in Example 1, as explained above in Example 2. From Figs. 8 and 9, we also note that the two proposed methods are effective and stable, and the truncation method performs more accurately than the Tikhonov method.

6. Conclusion

In this paper, we use two regularization methods to solve the Cauchy problems for the Helmholtz equation in a rectangular domain with Dirichlet, Neumann and Robin boundary conditions at $x = 0$ and $x = \pi$, respectively. The convergence error estimates have been presented for $0 < y \leqslant 1$ under an *a-priori* bound assumption for the exact solution. The numerical results show that the proposed methods are effective and stable. It is obvious that the two proposed methods are easy to adapt to three-dimensional Cauchy problem for the Helmholtz equation in a cubic domain. However, to deal with the more complex and irregular domains, our proposed methods do not work again. The detailed comparison with the other techniques available in the literature will be investigated in our future work.

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