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An Efficient Galerkin BEM to Compute High Acoustic Eigenfrequencies

An efficient numerical method, using integral equations, is developed to calculate precisely the acoustic eigenfrequencies and their associated eigenvectors, located in a given high frequency interval. It is currently known that the real symmetric matrices are well adapted to numerical treatment. However, we show that this is not the case when using integral representations to determine with high accuracy the spectrum of elliptic, and other related operators. Functions are evaluated only in the boundary of the domain, so very fine discretizations may be chosen to obtain high eigenfrequencies. We discuss the stability and convergence of the proposed method. Finally we show some examples.
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1 Introduction

It is well known that the computation of eigenfrequencies is one of the most approached topics of the past decades, due to the need to determine the response of physical systems submitted to stresses. The existing literature is numerous and delivers valuable models to solve a wide number of spectral problems. From these works, for the finite elements methods, we may mention works by Babuška and Osborn [1], Zienkiewicz [2,3], Strang and Fix [4], Oden and Reddy [5], Wait and Mitchell [6], Mercier et al. [7], Raviart and Thomas [8], Kolata [9], Brezzi and Fortin [10], Rannacher [11], and Grégoire et al. [12,13]. Other useful bibliography includes works by Bernardi and Maday [14], Gottlieb and Orszag [15], Vandeven [16], Forsythe and Wasow [17], and related works of Banerjee et al. [18], Coyette and Fyfe [19], Ali et al. [20], and Kirkup and Amini [21]. Finally we may mention more recent related works by Chen and co-workers [22,23] and Alves et al. [24].

The motivation for this study is the computation of high frequencies in acoustics in two- or three-dimensional spaces. That is, we deal with very small wavelengths compared with the diameters of the propagation medium, which in practical applications means wavelengths that are 500 or more times smaller than the diameter of the domain. Such a problem is most difficult to solve by using techniques developed in the references above. We may nevertheless overcome this difficulty by using potential theory to reformulate the differential problem and then making use of the standard Galerkin method to obtain the discrete problem. This procedure gives us an effective numerical method, which is capable of finding the eigenfrequencies with great precision, without computing the nonuseful ones. The eigenfunctions can then be easily determined.

In architectural acoustics there are no step by step rules in designing rooms, since the physical phenomena involved are complex. Therefore, we have to settle for some approximate model, which best fits our particular case. The engineer's experience and intuition are often the best tools at hand.

There are many ways of evaluating room acoustic characteristics. We concentrate on the determination of normal frequencies or eigenfrequencies of the rooms whose wavelengths are such that stationary waves may appear. Once we determine the normal frequencies, along with the reverberation time, we obtain an almost full description of the room acoustic properties. A room will respond strongly to those sounds whose frequencies neighbor the normal ones. In fact, the room behaves as a resonator that has many vibration modes allowed. If there is a monofrequencial sound source present, we observe a forced oscillation. Thus the sound pressure amplitude at a given point will depend on the relation between the source frequency and the room normal frequencies. Other factors that affect this amplitude are the source output power, the damping factor (given by the absorption of the walls), the location of the source in the room, and, of course, the position where we wish to know the sound pressure amplitude.

We immediately note the importance of knowing these vibration modes, since we can deduce many acoustical properties from them, such as the normal frequency density and spacing, the Schroeder cut-off frequency, Borello's criterion, and the position of nodes and antinodes.

Unfortunately, we may determine the normal frequencies exactly or analytically only when we deal with certain regularly shaped rooms, such as rectangular or cylindrical ones. That is why most theoretical work has been done with rectangular rooms (see Refs. [25–27]). When we deal with irregularly shaped enclosures, we have to make use of other theories to describe the room, such as ray acoustics or statistical room acoustics. The drawback of these theories is that they only apply to limited frequency ranges or make use of experimental formulas. To find the normal frequencies of irregularly shaped rooms, we must use numerical methods. The problem is that most numerical methods cannot calculate high normal frequencies, since they are limited by the frequency wavelength. The shorter the wavelength, the finer the grid must be. In our test example (see Sec. 3.1) we use a two-dimensional domain of 2×1 (m^2), and we calculate eigenfrequencies with wavelengths close to 0.018 (m). Other methods fail to calculate such eigenfrequencies.

In Sec. 2 we present the model problem and its equivalent integral formulation. Section 3 is devoted to obtaining the discrete variational formulation, also called energy formulation, and to finding the matrix system associated with it. The numerical analysis, the discussion about the stability using integral representations (in acoustic and electromagnetic problems), and the results are

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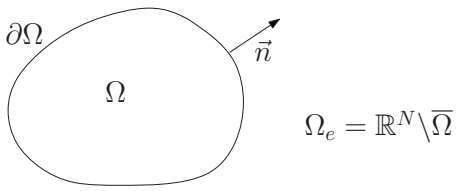

Fig. 1 Domain of wave propagation

given in Sec. 4. In Sec. 6 we give the conclusions and the discussion of the application of this technique to other problems of high complexity.

2 The Model Spectral Problem and the Integral Representation

Let Ω denote a bounded domain in $\mathbb{R}^N$ ($N=2$ or 3), and let $\partial\Omega$ denote its boundary, which we suppose to be smooth (in practice, the solid body represents the wave propagation medium (see Fig. 1)). We are interested in solving the following differential problem: Find $\lambda \in \mathbb{R}$ and a non-null complex valued function $u:\Omega \to \mathbb{C}$, which is a solution of

$$-\Delta u = \lambda u \quad \text{in } \Omega \tag{1a}$$

$$u = 0 \quad \text{on } \partial\Omega \tag{1b}$$

It is well known that Eq. (1*a*) is nothing but the acoustic wave propagation equation in periodic time dependence, as well as in absence of external sources. Relation (1*b*) is a Dirichlet homogeneous boundary condition. We are also interested in considering a Neumann homogeneous boundary condition, that is,

$$\frac{\partial u}{\partial n} = 0 \quad \text{on } \partial\Omega \tag{1c}$$

where the normal derivative is defined by $\partial u/\partial n = \nabla u \cdot \boldsymbol{n}$.

The parameter λ is the wave number, which is associated with the frequency f by the relation

$$\sqrt{\lambda} = \frac{2\pi f}{c} \tag{2}$$

where the real number c denotes the wave propagation speed. We call λ the eigenfrequency and $u(\cdot)$ the eigenfunction.

The only non-null solutions of Eqs. (1*a*) and (1*b*) are a countable sequence of pairs (λ_n, u_n), $n \in \mathbb{N}$ that satisfy the following properties:

$$0 < \lambda_1 \le \lambda_2 \le \lambda_3 \ldots \le \lambda_l \to +\infty \quad \text{as } l \to +\infty$$

$$\int_\Omega u_n u_m dx = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$$\int_\Omega \nabla u_n \cdot \nabla u_m dx = 0 \quad \forall\, n \neq m$$

Therefore, if $\lambda \neq \lambda_n$, $\forall n \ge 1$, then $u(\cdot)=0$ is the unique solution of Eqs. (1*a*) and (1*b*). The set of solutions associated with the Neumann homogeneous problem Eqs. (1*a*) and (1*c*) has the same properties.

The spectral problem (Eqs. (1*a*)–(1*c*)) is a classical elliptic one and the calculation of the first eigenfrequencies presents no problems. For this, efficient numerical methods already exist; among the most popular we have finite elements, finite differences, point collocation methods, and, recently, multiparameter extrapolation algorithms. Moreover, most of them have very good rate of convergence estimates for the spectral approximation. A particular case is the superconvergence phenomenon associated with several finite [7]. All of these numerical methods fail when we wish to obtain eigenfrequencies and eigenvectors linked to vibratory states produced by small wavelengths. In this case, discretized problems lead to huge matrices that are not adapted to numerical simulation producing instabilities and outlayer values.

To avoid this difficulty we propose the use of the potential theory. It is well known that in this theory, a function that satisfies Eq. (1*a*) may be written using integral representation [28–30]. In fact, $u(\cdot)$ satisfies Eq. (1*a*) if and only if

$$u(x) = \begin{cases} \displaystyle\int_{\partial\Omega} \frac{\partial u}{\partial n}(y) G_\lambda(x,y) ds(y) - \int_{\partial\Omega} u(y) \frac{\partial G_\lambda}{\partial n_y}(x,y) ds(y) & \forall x \in \Omega \\ \displaystyle 2\int_{\partial\Omega} \frac{\partial u}{\partial n}(y) G_\lambda(x,y) ds(y) - 2\int_{\partial\Omega} u(y) \frac{\partial G_\lambda}{\partial n_y}(x,y) ds(y) & \forall x \in \partial\Omega \end{cases} \tag{3}$$

where the integrals are defined on the boundary of Ω. Their associated integration variable is y, the symbol n_y gives the dependence of the unit normal vector on the variable y, and the function $G_\lambda(\cdot,\cdot)$ is the well known Green's function associated with the Helmholtz operator (see the references of above) which, for $N=2$ or 3, is given by

$$G_\lambda(x,y) = \begin{cases} \dfrac{i}{4} H_0^{(1)}(\sqrt{\lambda}|x-y|) & \text{if } N=2 \\ \dfrac{e^{i\sqrt{\lambda}|x-y|}}{4\pi|x-y|} & \text{if } N=3 \end{cases} \tag{4}$$

Here, the function $H_0^{(1)}$ is the classical Hankel function of order zero and of the first kind [31]. It is worth remarking that Eq. (3) yields the non-null solution $u(\cdot)$ defined in all Ω. In order to use these formulas, it suffices to know the values of $u(\cdot)$ and $\frac{\partial u}{\partial n}(\cdot)$ on $\partial\Omega$.

From a mathematical point of view, notice that we have considered a non-null function $u(\cdot)$, which vanishes outside Ω, that is

$$u(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ 0 & \text{if } x \in \Omega_e \end{cases} \tag{5}$$

where $\Omega_e = \mathbb{R}^N \backslash \Omega$ is the exterior domain.

Introducing the boundary condition (1*b*) and substituting in Eq. (3), it is easy to see that $u(\cdot)$ actually has an integral representation, namely, simple-layer potential, given by

$$u(x) = \int_{\partial\Omega} \frac{\partial u}{\partial n}(y) G_\lambda(x,y) ds(y) \quad \forall\, x \in \Omega \tag{6}$$

Since $G_\lambda(\cdot,\cdot)$ is given (see Eq. (4)), solving Eq. (6) is equivalent to finding $u(\cdot)$ in Ω or, alternatively finding $\frac{\partial u}{\partial n}(\cdot)$ on $\partial\Omega$. However, if we want to solve Eqs. (1*a*) and (1*b*), then $\frac{\partial u}{\partial n}(\cdot)$ should satisfy the integral equation

$$\int_{\partial\Omega} \frac{\partial u}{\partial n}(y) G_\lambda(x,y) ds(y) = 0 \quad \forall\, x \in \partial\Omega, \quad \forall\, \lambda \in \mathbb{R} \tag{7}$$

Thus, our method consists of computing a high accuracy approximation of the function $\partial u/\partial n(\cdot)$ and, next, searching for values of λ for which formula (7) is singular, that is, for those values such that Eq. (7) has at least one non-null solution.

Similarly, problem (1*a*) together with Neumann's homogeneous boundary condition (1*c*) may be formulated equivalently by integral representations. For this, the main idea is to consider an extension of $u(\cdot)$ other than Eq. (5). More precisely, if we denote the solution of the problem by $u_e:\Omega_e \to \mathbb{C}$,

$$-\Delta u_e = \lambda u_e \quad \text{in } \Omega$$

$$u_e = u \quad \text{on} \quad \partial\Omega_e$$

$$|u(x)| \to 0 \quad \text{as} \quad |x| \text{ goes to infinity}$$

$$|\nabla u(x)| \to 0 \quad \text{as} \quad |x| \text{ goes to infinity} \tag{8}$$

then the function $u_T(\cdot)$, defined by

$$u_T(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ u_e(x) & \text{if } x \in \Omega_e \end{cases} \tag{9}$$

satisfies the transmission problem

$$-\Delta u_T = \lambda u_T \quad \text{in} \quad \Omega \cup \Omega_e \tag{10a}$$

$$[u_T] = 0 \quad \text{on} \quad \partial\Omega \tag{10b}$$

$$\left[\frac{\partial u_T}{\partial n}\right] = q \quad \text{on} \quad \partial\Omega \tag{10c}$$

Here we have considered the jumps of the function as well as its normal derivative along the boundary $\partial\Omega$. We denote the jumps by

$$[u_T](x) = u(x) - u_e(x) \tag{11a}$$

$$\left[\frac{\partial u_T}{\partial n}\right](x) = \frac{\partial u}{\partial n}(x) - \frac{\partial u_e}{\partial n}(x) \tag{11b}$$

It is straightforward to see that $u_T(\cdot)$ restricted to Ω is none other than $u(\cdot)$. Also, it is easy to see that $u_T(\cdot)$ has an integral representation given by

$$u_T(x) = \int_{\partial\Omega} \left[\frac{\partial u_T}{\partial n}\right](y) G_\lambda(x,y) ds(y) - \int_{\partial\Omega} [u_T] \times (y) \frac{\partial G_\lambda}{\partial n_y}(x,y) ds(y) \quad \forall\, x \in \Omega_e \tag{12a}$$

$$\frac{1}{2}(u(x) + u_e(x)) = \int_{\partial\Omega} \left[\frac{\partial u_T}{\partial n}\right](y) G_\lambda(x,y) ds(y) - \int_{\partial\Omega} [u_T] \times (y) \frac{\partial G_\lambda}{\partial n_y}(x,y) ds(y) \quad \forall\, x \in \partial\Omega \tag{12b}$$

By introducing the transmission conditions (10*b*) and (10*c*), this representation actually reads

$$u_T(x) = \int_{\partial\Omega} q(y) G_\lambda(x,y) ds(y) \quad \forall\, x \in \mathbb{R}^N \tag{12c}$$

Taking into account the Neumann boundary condition (1*c*), we compute the interior normal derivative of expression (12*c*) on $\partial\Omega$. It yields

$$\left(\frac{\partial u_T}{\partial n}\right)_{\text{int}}(x) = \frac{1}{2} q(x) + \int_{\partial\Omega} q(y) \frac{\partial G_\lambda}{\partial n_x}(x,y) ds(y) \quad \forall\, x \in \partial\Omega \tag{13}$$

But

$$\left(\frac{\partial u_T}{\partial n}\right)_{\text{int}}(x) = \frac{\partial u}{\partial n}(x) = 0 \quad \text{on} \quad \partial\Omega$$

from which we deduce the integral equation

$$\frac{1}{2} q(x) + \int_{\partial\Omega} q(y) \frac{\partial G_\lambda}{\partial n_x}(x,y) ds(y) = 0 \quad \forall\, x \in \partial\Omega \tag{14}$$

Again, high accuracy in computing $q(\cdot)$ leads to those values of $\lambda > 0$ for which Eq. (14) is singular, and, as we said before, it has non-null solutions.

It is thinking about extension on Ω_e of the function $u(\cdot)$ in a different way of (1.5) and (1.9), respectively. In fact, we can consider $u_e(\cdot)$ such that

$$[u_T](x) = \varphi(x) \quad \text{on} \quad \partial\Omega \tag{15a}$$

$$\left[\frac{\partial u_T}{\partial n}\right](x) = 0 \quad \text{on} \quad \partial\Omega \tag{15b}$$

In that case, and for the Dirichlet boundary condition, the equation given by the values of the double-layer potential on the surface $\partial\Omega$ is

$$-\frac{1}{2}\varphi(x) + \int_{\partial\Omega} \varphi(y) \frac{\partial G_\lambda}{\partial n_y}(x,y) ds(y) = 0 \quad \forall\, x \in \partial\Omega \tag{16a}$$

which is very analogous to Eq. (14). A more difficult expression is obtained in the Neumann case, because a not usual Lebesgue integral appears. More precisely, the hypersingular integral equation is given by

$$\frac{\partial}{\partial n_x} \int_{\partial\Omega} \varphi(y) \frac{\partial G_\lambda}{\partial n_y}(x,y) ds(y) = 0 \quad \forall\, x \in \partial\Omega \tag{16b}$$

It makes sense as finite part of a distribution (see Refs. [32,33,28]), which is much more complex to implement and compute.

3 The Variational Formulation and its Discretization

We are interested in dealing with the following type of integral equation: Find $q: \partial\Omega \to \mathbb{C}$ such that

$$\int_{\partial\Omega} q(y) G_\lambda(x,y) ds(y) = u_0(x) \quad \forall\, x \in \partial\Omega \tag{17}$$

where $u_0: \partial\Omega \to \mathbb{C}$ is a given function. It is clear that in our case (7), the corresponding data is $u_0 = 0$. To tackle Eq. (17), we use the variational formulation, also called energy formulation. It is obtained by multiplying Eq. (17) by $\phi(\cdot)$ and integrating on $\partial\Omega$. This leads to the variational problem: Find

$$q: \partial\Omega \to \mathbb{C}$$

such that

$$\int_{\partial\Omega} \int_{\partial\Omega} q(y) \bar{\phi}(x) G_\lambda(x,y) ds(y) ds(x) = \int_{\partial\Omega} u_0(x) \bar{\phi}(x) ds(x) \quad \forall\, x \in \partial\Omega \tag{18}$$

for all regular function $\phi(\cdot)$. Linear system (18) has only one solution, except for the cases $\lambda = \lambda_n$, $n \geq 1$. It means that (18) becomes singular for these special values, and the associated homogeneous problem (7) has at least one non-null solution. Thus we approach by a finite discretization $q(\cdot)$ and look for those positive real values of λ for which the operator, the matrix in the discrete case, is singular.

As we have a variational formulation (Eq. (18)), it is quite natural to approximate the problem using the Galerkin method. For the sake of simplicity, in what follows, we expound on the discretization method by using $\mathbb{P}_0$-Lagrange finite element on $\partial\Omega$. However, this procedure could be adopted with any other degree of polynomial approximation or, another discrete scheme.

Let $\mathcal{T}_h$ be a regular surface mesh discretizing $\partial\Omega$ in classical sense [34]. Then we have

$$\partial\Omega_h = \bigcup_{j=1}^{M_h} \overline{T_j} \quad T_j \in \mathcal{T}_h$$

$$T_i \cap T_j = \emptyset \quad \forall\, i \neq j$$

The real positive parameter h measures the finesse of the discretization $\mathcal{T}_h$. Defining functions $\mathbb{I}_j$ as follows

$$\mathbb{I}_j(x) = \begin{cases} 1 & \text{if } x \in T_j \\ 0 & \text{if not} \end{cases}$$

we can approach $q(\cdot)$ by

$$q_h(y) = \sum_{j=1}^{M_h} \alpha_j \mathbb{I}_j(y) \quad \forall\, y \in \partial\Omega$$

which is equivalent to consider constant approximations on each element T_j. The discrete equation associated to Eq. (18) becomes

$$\sum_{j=1}^{M_h} \alpha_j \int_{\partial\Omega_h} \int_{\partial\Omega_h} \mathbb{I}_j(y)\, \overline{\phi}_h(x) G_\lambda(x,y) ds(y) ds(x) = \int_{\partial\Omega_h} u_0(x)\, \overline{\phi}_h(x) ds(x) \tag{19a}$$

for all regular discrete function $\phi_h(\cdot)$. In particular, taking $\phi_h(x) = \mathbb{I}_i(x)$, for all $i = 1, \ldots, M_h$, we obtain

$$\sum_{j=1}^{M_h} \alpha_j \int_{T_i} \int_{T_j} G_\lambda(x,y) ds(y) ds(x) = \int_{T_i} u_0(x) ds(x) \tag{19b}$$

We define the complex matrix and the complex vectors

$$\mathbf{A}(\lambda) = [a_{ij}(\lambda)] \quad 1 \leq i,j \leq M_h \tag{20a}$$

$$\boldsymbol{\alpha} = (\alpha_i) \quad 1 \leq i \leq M_h \tag{20b}$$

$$\boldsymbol{b} = (b_i) \quad 1 \leq i \leq M_h \tag{20c}$$

where the terms a_{ij} and b_i are computed as follows:

$$a_{ij}(\lambda) = \int_{T_i} \int_{T_j} G_\lambda(x,y) ds(y) ds(x) \tag{21a}$$

$$b_i = \int_{T_i} u_0(x) ds(x) \tag{21b}$$

If we adopt these notations, then Eq. (19*b*) reads

$$\mathbf{A}(\lambda)\boldsymbol{\alpha} = \boldsymbol{b} \tag{22}$$

It is important to remark that matrix $\mathbf{A}(\lambda)$ is symmetric and it has terms that must be computed carefully, because they include singularities. Making the calculation of $q_h(\cdot)$ with high accuracy implies: (i) taking M_h big enough ($h > 0$ small), and, (ii) determining $a_{ij}(\lambda)$ with adequate numerical integration formulas. The first question (i) is solved by comparing the diameter of Ω and the wavelengths corresponding to the values of the desired frequencies. The other question (ii) is much more complicated, despite it has a trivial part. If the distances between the centers of mass of T_i and T_j are greater or equal than five times the wavelength, then we use classical numerical integration techniques (for example, different Gauss' points on T_i and T_j). Otherwise, the elements T_i and T_j are too close and we use analytical development coupled to a fine numerical integration formula.

Under the above considerations, we can prove [35,28] that

$$\int_{\partial\Omega} |q(y) - q_h(y)|^2 ds(y) \leq C h^s \tag{23a}$$

for some $s > 0$, which implies that

$$\lim_{h \to 0} |\lambda_{lh} - \lambda_l| = 0 \quad \forall\, l \geq 1 \tag{23b}$$

Convergence rate estimates for spectral problems are much more difficult to derive. In this case, we refer to Ref. [35]. The general framework may be found in Refs. [1,7].

We finish this section by remarking that the discrete version of Eq. (14) is as follows: Find

$$q_h : \partial\Omega \to \mathbb{C}$$

such that

$$\frac{1}{2} \int_{\partial\Omega_h} q_h(x)\, \overline{\phi}_h(x) ds(x) + \int_{\partial\Omega_h} \int_{\partial\Omega_h} q_h(y)\, \overline{\phi}_h(x) \frac{\partial G_\lambda}{\partial n_x}(x,y) ds(y) ds(x) = 0 \quad \forall\, x \in \partial\Omega \tag{24}$$

or in matrix notation,

$$\left(\frac{1}{2}\mathbf{D} + \mathbf{B}(\lambda)\right)\boldsymbol{\alpha} = \mathbf{0} \tag{25}$$

where $\mathbf{D}$ is a diagonal matrix, having the i-diagonal term equal to the measure of $T_i \in \mathcal{T}_h$, whereas $\mathbf{B}(\lambda)$ is computed similarly to $\mathbf{A}(\lambda)$ as follows:

$$b_{ij}(\lambda) = \int_{T_i} \int_{T_j} \frac{\partial G_\lambda}{\partial n_x}(x,y) ds(y) ds(x) \tag{26}$$

4 Numerical Results

To compute the terms of the matrix defined in Eq. (20*a*) (or the singular part $\mathbf{B}(\lambda)$ of the matrix in (25)), we use the following rule:

$$a_{ij}(\lambda) = \begin{cases} \sum_{k,l} G_\lambda(x_k, y_l)\, \omega_k \omega_l & \text{if } \operatorname{dist}(T_i, T_j) \geq 5\mu \\ \sum_k F(k,\lambda)\, \omega_k & \text{if not} \end{cases} \tag{27}$$

where x_k, y_l are the points of some numerical integration rule, ω_k, ω_l are their corresponding weights, and μ is the wavelength considered. The complex function $F(k,\lambda)$ is given by

$$F(k,\lambda) = \int_{T_j} G_\lambda(x_k, y) ds(y) \tag{28}$$

and it must be determined very carefully. It seems that the best way to compute it is the analytical development. For the two-dimensional case, it is well known that $F(k,\lambda)$ can be computed exactly [32]. The three-dimensional case may be consulted in the works of Hamdi [33,36] and the references therein.

Factorizing matrix $\mathbf{A}(\lambda)$ in the *LU*-form, (it is also possible to factorize the matrix $\mathbf{A}(\lambda)$ in the *QR*-form) we may define the real-valued function as follows:

$$g(\lambda) = \frac{\max_{1 \leq i \leq M_h} |u_{ii}(\lambda)|}{\min_{1 \leq i \leq M_h} |u_{ii}(\lambda)|} \tag{29}$$

where u_{ii} is the ith diagonal element of matrix $\mathbf{U}$, for $i = 1, \ldots, M_h$. This function has a countable number of singularities, which correspond to the eigenfrequencies we are looking for.

4.1 The Test Example. We consider $\Omega =]0,2[\times]0,1[\subset \mathbb{R}^2$. It is well known that the solutions of Eqs. (1*a*) and (1*b*), $\forall\, n,m \geq 1$, are given by

$$u_{n,m}(x_1,x_2) = \sin\left(\frac{n\pi x_1}{2}\right) \sin(m\pi x_2) \tag{30a}$$

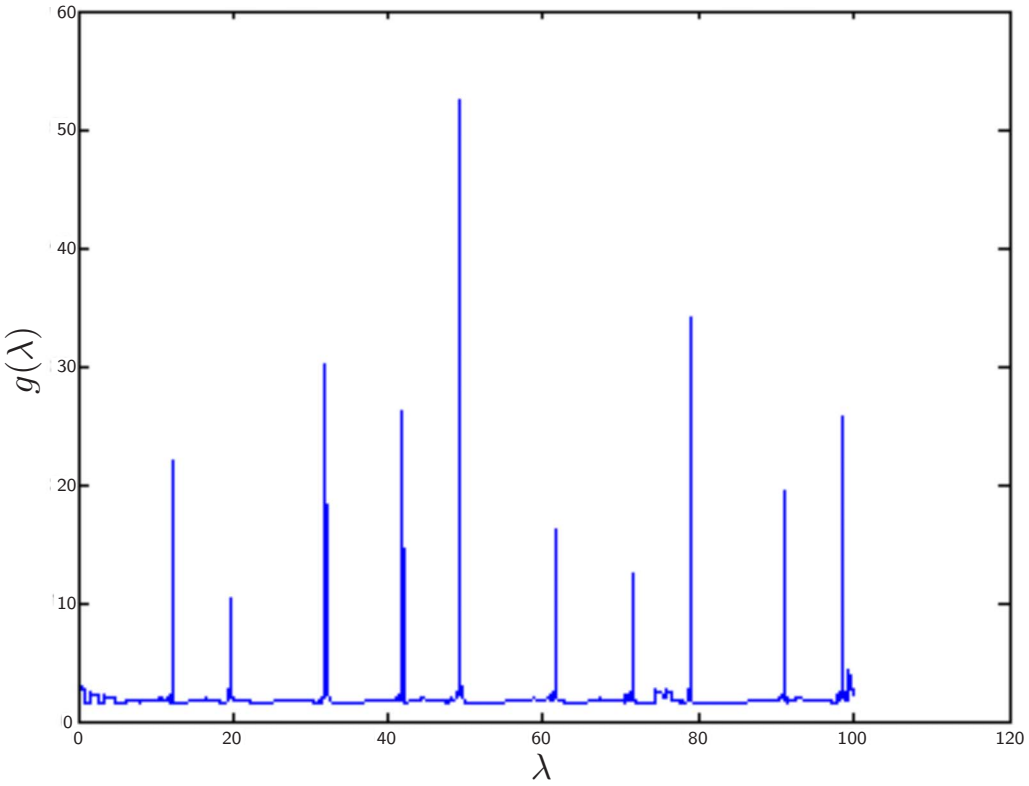

Fig. 2 The lowest eigenfrequencies of the test example (3.4)

$$\lambda_{n,m} = \left(\frac{n\pi}{2}\right)^2 + (m\pi)^2 \tag{30b}$$

Figure 2 (which resolution of increment of λ is 0.001) shows the lowest calculated eigenvalues of Eqs. (1*a*) and (1*b*), obtained for 96 degrees of freedom. All the measures are in meters and seconds. We verified the concordance and preciseness with the theoretical ones and are given in Table 1.

It is worth remarking that we have obtained relative errors of order 10^{-4} by computing with a low number of freedom degrees.

For the case of small wavelengths, we show Fig. 3 (whose resolution of increment of λ is 0.00001). As mentioned in Sec. 1, this corresponds to wavelengths closes to 0.018 (m) and it was obtained for 3336 freedom degrees. Table 2 shows the relative errors obtained.

4.2 About the Stability Using Integral Representations. Since the spectrum is real, in order to calculate the function $g(\lambda)$ defined in Eq. (29), it is natural to think of working only with the real parts of matrices defined in Eq. (22) or (25). In fact, on one hand we must compute a real spectrum, and on the other hand no radiation conditions are essentially needed. As it is tested numerically in Ref. [37], this choice is very inadequate because the discrete operator becomes very singular. We can compare Fig. 2 with Fig. 4 for the same number of degrees of freedom.

It is well known, for the Helmholtz problem, that the boundary integral equations for solving exterior (and interior) problems result in spurious eigenvalues (or eigenfrequencies). This phenomenon of instability is well explained by Chen and co-workers [22,23] and it is mainly due to the singularity of the kernel (associated to the integral equations) combined with the specified integral formulation used. Let us notice that this phenomenon of instability also appears while computing the Maxwell's eigenfrequencies by using integral equations. We will see this in Sec. 5.

Table 1 Theoretical and approximated eigenfrequencies

n	m	Theoretical eigenfrequencies	Calculated eigenfrequencies	Relative error
1	1	12.3370055	12.33	0.0005678
1	2	19.7392088	19.73	0.0004665
1	3	32.0762143	32.06	0.0005055
2	1	41.9458187	41.94	0.0001387
2	2	49.3480220	49.34	0.0001626
2	3	61.6850275	61.65	0.0005678
1	5	71.5546319	71.54	0.0002045
2	4	78.9568352	78.92	0.0004665
3	1	91.2938407	91.29	0.00004207
3	2	98.6960440	98.69	0.00006124

5 Instabilities in the Calculation of Maxwell's Eigenfrequencies

We compute the solution of 3D Maxwell equations for the case of a perfect conducting sphere illuminated by an incident plane wave. Thus, we search the solution of the following exterior problem:

$$\begin{aligned} &\text{rot } E - i\omega\mu H = 0 \quad \text{in } \Omega_e \\ &\text{rot } H + i\omega\varepsilon E = 0 \quad \text{in } \Omega_e \\ &E \wedge n = -E^{\text{inc}} \wedge n \quad \text{on } \partial\Omega \\ &\left|\sqrt{\varepsilon}E - \sqrt{\mu}H \wedge n\right| \leq \frac{c}{r^2} \quad \text{as } r \text{ goes to infinity} \end{aligned} \tag{31}$$

where $\Omega_e \equiv \mathbb{R}^3 \backslash \overline{B(0,1)}$, E^{inc} is a given incident wave with angular frequency ω, E is the electric field, H is the magnetic field, ε is the dielectric permittivity, and μ is the magnetic permeability.

Using an integral representation it is quite easy to obtain the following variational formulation [28]: Find

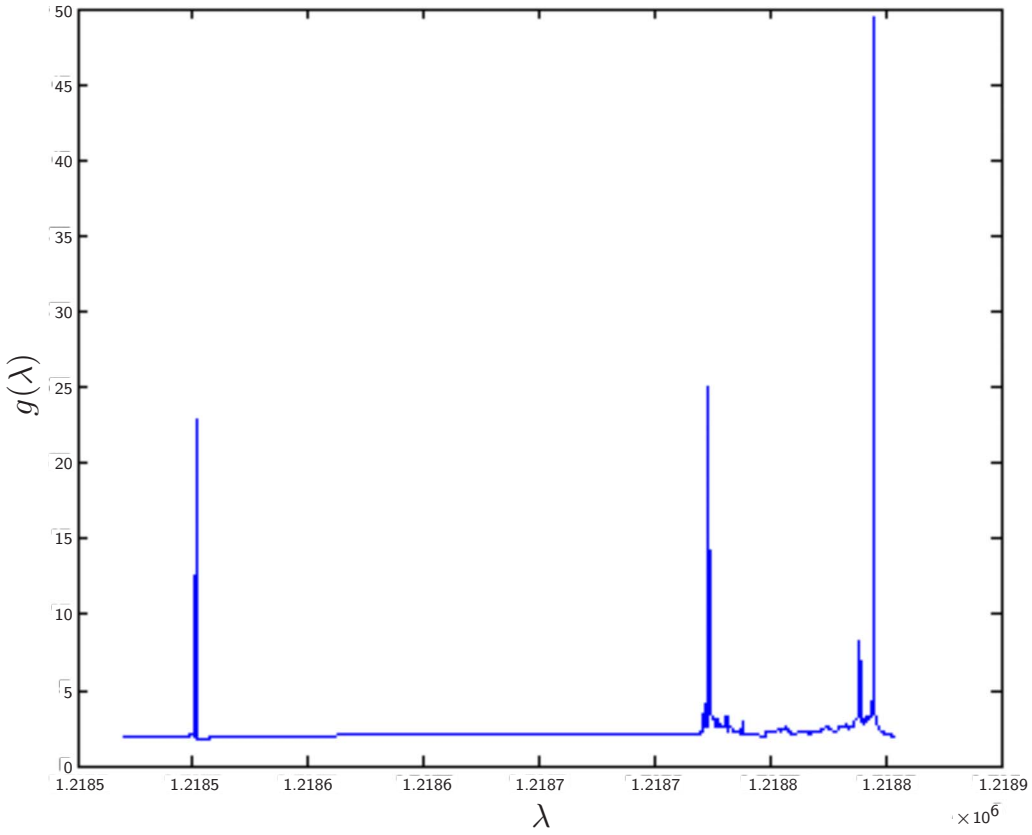

Fig. 3 The eigenfrequencies of the test example (3.4) for the case of small wavelengths

$$q \in H_{\text{div}}^{-1/2}(\partial\Omega)(q{:}\partial\Omega \to \mathbb{C})$$

such that

$$i\omega\mu \int_{\partial\Omega}\int_{\partial\Omega} G_\lambda(x,y) q(x)\cdot q^t(y) ds(x) ds(y) - \frac{i}{\omega\varepsilon}\int_{\partial\Omega}\int_{\partial\Omega} G_\lambda(x,y)\,\text{div}\, q(x)\,\text{div}\, q^t(y) ds(x) ds(y) = -\int_{\partial\Omega} E^{\text{inc}}(y)\cdot q^t(y) ds(y), \quad \forall\, q^t \in H_{\text{div}}^{-1/2}(\partial\Omega) \tag{32}$$

where the function $G_\lambda(\cdot,\cdot)$ is Green's function defined in Eq. (4) for $N=3$ and $\lambda=\omega^2\varepsilon\mu$. Actually, we deal with the wave number $k=\lambda^2$ instead of λ. Furthermore, the electric current $q(x)$ on the surface $\partial\Omega$ is defined as the trace of the tangential component of the magnetic field intensity inside the cavity, that is,

$$q(x) = (H \wedge n)(x), \quad x \in \partial\Omega \tag{33}$$

For the computation of these terms, we have used the discretization techniques given in Refs. [38,28].

Table 2 Theoretical and approximated eigenfrequencies for the case of small wavelengths

n	m	Theoretical eigenfrequencies	Calculated eigenfrequencies	Relative error
10	111	121850.1359	121850.1179	1.4772×10^{-7}
217	24	121872.3425	121872.3429	3.2821×10^{-9}
60	107	121879.7447	121879.4929	2.0660×10^{-6}

It is well known that the eigenfrequencies of the Maxwell system of equations on the unit ball are none other but the zeros of the following functions:

$$j_l(k)$$

$$k\frac{d}{dk}j_l(k) + j_l(k) \tag{34}$$

where j_l denotes the spherical Bessel function of order $l \geq 1$. The zeros of such functions are denoted by $k_{l,s}^*$ for the first one, and by $k_{l,s}'$ for the second one [1]. These zeros are specified in Table 3.

These values agree well with the approached eigenfrequencies or singularities obtained by computing the function $g(\lambda)$ using the whole Galerkin complex matrix $A(\lambda)$ (see Fig. 5). We considered 1440DOF. Considering the wave propagation speed of an electromagnetic field in vacuum and the number of degrees of freedom used, we can obtain good precision only for frequencies lower than 400 MHz, or equivalently, $k<8.377$ (at least five discrete triangles per wavelength). A very different situation occurs when computing the eigenfrequencies using only the real symmetric Galerkin matrix $A_R(\omega)$ (see Fig. 6).

5.1 Preconditioners for the Numerical Solution of Boundary Integral Equations. If the Garlerkin matrix $A(\lambda)$ of Eq. (21*a*) or $B(\lambda)$ of Eq. (26) is associated with an adequate preconditioning matrix $Z_h(\lambda)$, such that $Z_h(\lambda)A(\lambda)$ (or $Z_h(\lambda)B(\lambda)$) models a compact perturbation of the identity operator and such that the speed of convergence of the preconditioned system is larger than the speed of convergence of the nonpreconditioned system, then we can expect to have a very efficient algorithm for the calculated eigenvalues of Eqs. (1*a*) and (1*b*) or Eqs. (1*a*) and (1*c*). The application and the numerical evidence of the efficiency of the pre-

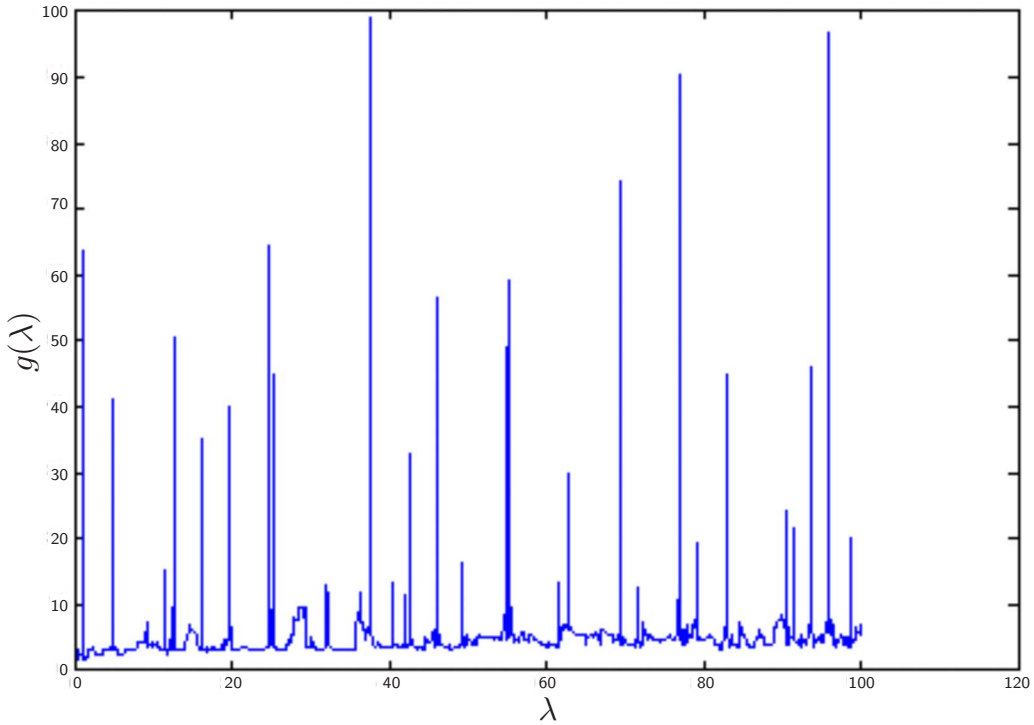

Fig. 4 The lowest eigenfrequencies computed by using real Galerkin matrices

Table 3 First eigenvalues for the spherical cavity of unitary radius

$k^*_{1,1}$	$k^*_{2,1}$	$k'_{1,1}$	$k^*_{3,1}$	$k'_{2,1}$	$k^*_{4,1}$	$k^*_{1,2}$	$k'_{3,1}$	$k^*_{5,1}$	$k^*_{2,2}$	$k'_{1,2}$	$k'_{4,1}$	$k^*_{6,1}$	$k^*_{3,2}$	$k'_{2,2}$	$k^*_{7,1}$	$k^*_{1,3}$	$k'_{5,1}$
2.74	3.87	4.49	4.97	5.76	6.06	6.11	6.98	7.14	7.44	7.72	8.18	8.21	8.72	9.09	9.27	9.31	9.35

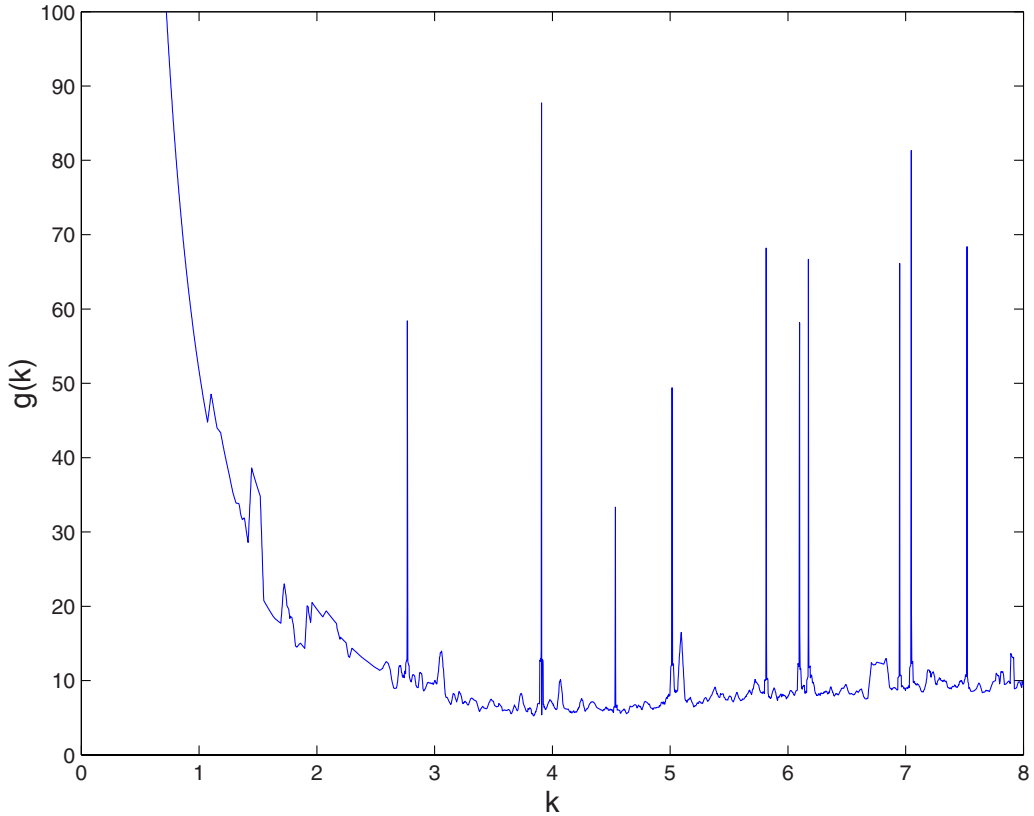

Fig. 5 Computed eigenfrequencies for the sphere using matrix $\boldsymbol{A}(\omega)$

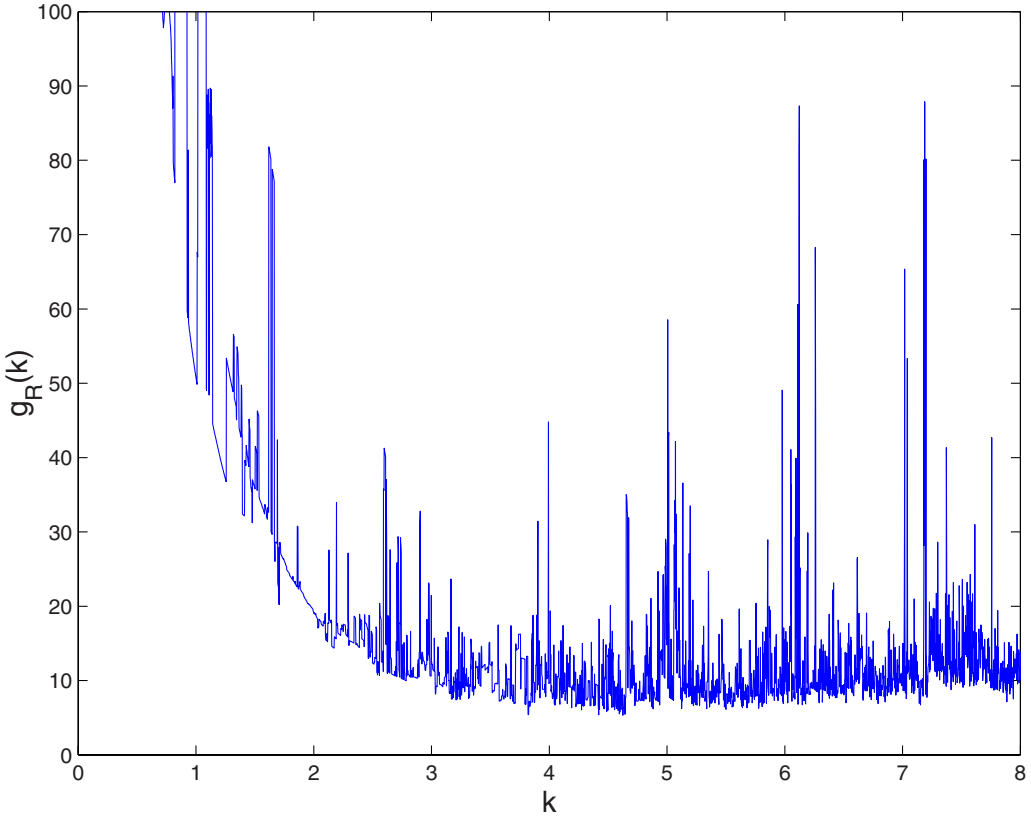

Fig. 6 Computed eigenfrequencies for the sphere using matrix $A_R(\omega)$

conditioning technique for the numerical solution of boundary integral equations from acoustics may be consulted in the work of Christiansen and Nédélec [39].

6 Conclusions

In this work, we have presented a method that has several advantages on the previous ones. It is capable of finding much higher eigenfrequencies (associated with very small wavelengths), it may be applied to unbounded domains and it gives an important tool to treat the three-dimensional domain case. We have shown its precision, which may be improved. It suffices to choose a finer step once we know approximately where the eigenfrequency lies, and search only in the surroundings of the peak.

One of the major improvements of this method is the use of complex matrices instead of real ones, which give us much more reliable results. Although simple, this fact is not quite known. The only drawback of this method may be that the computation time is longer, since matrices are dense than in other ones.

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