Numerical Optimization of Low Eigenvalues of the Dirichlet and Neumann Laplacians

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Abstract

We perform a numerical optimization of the first ten nontrivial eigenvalues of the Neumann Laplacian for planar Euclidean domains. The optimization procedure is done via a gradient method, while the computation of the eigenvalues themselves is done by means of an efficient meshless numerical method allowing for the computation of the eigenvalues for large numbers of domains within a reasonable time frame. The Dirichlet problem, previously studied by Oudet using a different numerical method, is also studied and we obtain similar (but improved) results for a larger number of eigenvalues. These results reveal an underlying structure to the optimizers regarding symmetry and connectedness, for instance, but also show that there are exceptions to these preventing general results from holding.

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1   Introduction

The complexity of some of the problems encountered within the scope of the spectral theory of the Laplace and related operators has recently spanned an interest in the development and usage of numerical methods allowing for

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the processing of a large number of domains. Examples of this are not only the ability to provide compelling numerical evidence in the case of long standing conjectures, but also to allow the formulation of new conjectures relating different spectral and geometric quantities in ways which we believe to be beyond the reach of current analytic methods. Two examples of this approach may be found in [1] and [2], in relation to bounds for the first Dirichlet eigenvalue and the spectral gap conjecture, respectively. In these examples the optimal domains, that is, the domains for which one has equality in the inequalities studied, are the ball and the infinite strip. However, in many shape optimization problems, it is not to be expected in general that the optimizer be a domain whose boundary can be described explicitly in terms of known functions. Perhaps the best example of this is the problem of minimizing the second Dirichlet eigenvalue of the Laplace operator subject to a convexity constraint. In this case, and for a long time, the convex hull of two identical tangent disks (the stadium) was thought to be the minimizer. Recently, Henrot and Oudet disproved it by showing that the optimizer cannot contain arcs of circles [3]. Thus, on the one hand, it is now known that the stadium is not the minimizer for this problem and, on the other hand, there is no *good* candidate to replace it, in the sense that the boundary of the optimizer is not expected to be known in analytic form. Thus, what one may expect to obtain are qualitative characterizations, and to prove basic properties of the minimizer, such as symmetries and whether or not the boundary contains line segments, for instance, the latter being a natural result of a convex restriction.

In this respect, robust precise numerical optimization plays an important role not only in providing an idea of the shape of optimizers (in the example above, that the minimizer is indeed close to the stadium), but also giving an indication as to whether the properties mentioned above are expected to hold or not [4].

Part of the purpose of the present paper falls thus into the spirit of the above paragraphs. By performing a numerical optimization of low eigenvalues of the Dirichlet and Neumann Laplacians, it is our objective, more than to just obtain candidates for the optimizers, to provide a panorama of the results for these two problems and of the properties satisfied by the corresponding optimizers. Thus, the aim of this work is to contribute to uncovering the underlying structure of this classical optimization problem which is far from being well-understood. This includes multiplicities, symmetries and connectivity properties of optimizers, for instance, suggesting conjectures and giving indications for possible future lines of research.

Another important point here is to show that the simplicity and very fast convergence of the numerical methods used make them quite appropriate for dealing with a large number of domains while keeping the accuracy within the required levels.

The plan of the paper is as follows. The eigenvalue problems together with some useful basic results are stated in the next section, while the optimization procedure is described in Section 3. This consists of a mixture of several methods which include a genetic algorithm to get the process started, a classical gradient method to approach the optimizers, and finally some specific *ad hoc* methods to deal with problems such as multiplicities. This section also includes another important ingredient which is the way in which the domains in question are described analytically. While in the case of the optimization carried out in [4] this was done via level sets, here we have opted for a different

representation which is based on a truncated Fourier series of the boundary, as described in Section 3.2.1. Although this restricts the numerical optimization to star–shaped domains, the optimization procedure never approached sets which were not strictly star–shaped with respect to some point. This gives a clear indication that, except when the optimal domain is not connected such as is the case for the second and fourth Neumann eigenvalues, for instance, the optimizers should be star–shaped. Sections 4 and 5 then present the results of the process in the Neumann and Dirichlet cases, respectively, while a discussion of the results obtained is given in the last section. The Appendix contains a list of the coefficients describing the boundary of each of the domains found numerically.

2 The eigenvalue problems

Let $\Omega \subset \mathbb{R}^2$ be a bounded domain not necessarily connected. We will consider the Dirichlet and Neumann eigenvalue problems,

$$\begin{cases} \Delta u_i + \lambda_i u_i = 0 & \text{in } \Omega, \\ u_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{1}$$

$$\begin{cases} \Delta u_i + \mu_i u_i = 0 & \text{in } \Omega, \\ \partial_n u_i = 0 & \text{on } \partial\Omega, \end{cases} \tag{2}$$

where Δ denotes the Laplace operator and ∂_n the derivative with respect to the outward normal derivative. It is well known [5, 6] that both problems have discrete spectrum diverging to infinity and satisfying

$$0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \dots$$

and

$$0 = \mu_0 \leq \mu_1 \leq \mu_2 \leq \mu_3 \leq \dots$$

where each eigenvalue is repeated according to its multiplicity. In this paper, we will be interested in the numerical solution of the optimization problems

$$\lambda_i^* = \min\left\{ \lambda_i(\Omega), \ \Omega \subset \mathbb{R}^2, \ |\Omega| = 1 \right\}, \ \text{for } i = 1, 2, \dots \tag{3}$$

and

$$\mu_i^\star = \max\left\{ \mu_i(\Omega), \ \Omega \subset \mathbb{R}^2, \ |\Omega| = 1 \right\}, \ \text{for } i = 1, 2, \dots \tag{4}$$

Some of these shape optimization problems for low values of i have already been solved. The first Dirichlet eigenvalue is minimized by the ball, as proven by Faber and Krahn [7, 8]. The second Dirichlet eigenvalue is minimized by two balls of the same area. This result follows directly from the minimization of the first eigenvalue and is usually

attributed to Szegö [9], but was already published by Krahn [10]. It has long been conjectured that the ball minimizes λ_3, but there has not been much progress in this direction. For higher eigenvalues, in fact, not even existence of minimizers has been proven [6]. For the Neumann problem, we know that the ball maximizes μ_1. The result had been conjectured by Kornhauser and Stakgold in [11] and was proved by Szegö in [12] for Lipschitz simply connected planar domains and generalized by Weinberger in [13] to arbitrary domains, and any dimension. More recently, Girouard, Nadirashvili and Polterovich proved that the maximum of μ_2 among simply connected bounded domains is attained by two disjoint balls of equal area [14].

Taking the above scenario into account, one might expect the optimization problems (3) and (4) to have the same solutions in the sense that the set which minimizes a specific Dirichlet eigenvalue would also maximize the corresponding Neumann eigenvalue. However, it is sufficient to look at the third eigenvalue to convince ourselves that the solutions of both problems are not always the same. As mentioned above, it is conjectured that the third Dirichlet eigenvalue is minimized at the ball [15, 4, 16]. However, the ball cannot be the solution for the Neumann problem with μ_3. Indeed, there exist rectangles for which the third Neumann eigenvalue is larger than the corresponding value of the ball. Denoting by $\mathcal{B}$ the ball with unit area and by $\mathcal{R}$ a rectangle with unit area and for which the ratio of the lengths of the sides is $\sqrt{3}$, we have

$$\mu_3(\mathcal{B}) \approx 29.308 < 29.610 \approx 3\pi^2 = \mu_3(\mathcal{R}).$$

Now we note that we can obtain problems which are equivalent to (3) and (4) while avoiding the area constraint. We know that, if $(t\Omega)$ denotes the scaling of Ω by a factor of t, then the eigenvalues satisfy

$$\lambda_k(t\Omega) = t^{-2}\lambda_k(\Omega).$$

Therefore, problems (3) and (4) are respectively equivalent to the optimization problems

$$\lambda_i^* = \min\left\{\lambda_i(\Omega)|\Omega|,\ \Omega \subset \mathbb{R}^2\right\},\ \text{for } i = 1, 2, \dots \tag{5}$$

and

$$\mu_i^\star = \max\left\{\mu_i(\Omega)|\Omega|,\ \Omega \subset \mathbb{R}^2\right\},\ \text{for } i = 1, 2, \dots \tag{6}$$

which are easier to handle numerically. The existence of a minimizer for the Dirichlet problem (5) in the class of quasi-open sets contained in a bounded box was obtained in [17]. Extremal problems in starlike sets were considered in [18], with extra restrictions on the perimeter and inradius. Some partial results for question of existence of a maximizer of the Neumann problem (6) have also been obtained, but the proof of the existence of an open set that maximizes the i-th Neumann eigenvalue remains an open problem for $i \geq 3$ [6].

A very useful mathematical tool when dealing with optimization problems of this type is the Wolf-Keller theo-

rem [16], which allows us to deal with disconnected sets in a simple way. The extension to the Neumann case is due to Poliquin and Roy-Fortin [19].

Theorem 2.1. *Let Ω_i^* and $\Omega_i^\star$ be disconnected sets for which $\lambda_i^* = \lambda_i(\Omega_i^*)|\Omega_i^*|$ and $\mu_i^\star = \mu_i(\Omega_i^\star)|\Omega_i^\star|$. Then*

$$\lambda_i(\Omega_i^*) = \min_{1 \leq j \leq (i-1)/2} \left(\lambda_j^* + \lambda_{i-j}^* \right),$$

$$\mu_i(\Omega_i^\star) = \max_{1 \leq j \leq (i-1)/2} \left(\mu_j^* + \mu_{i-j}^* \right).$$

The following result, which was recently proved by Colbois and El Soufi [20], will allow us to partially check the validity of our results.

Theorem 2.2. *The optimal eigenvalues λ_j^* and μ_j^* satisfy*

$$\lambda_{j+1}^* - \lambda_j^* \leq \pi j_{01}^2 \approx 18.168, \ j = 1, 2, ...$$

and

$$\mu_{j+1}^* - \mu_j^* \geq \pi j_{11}^2 \approx 10.65, \ j = 1, 2, ...$$

In particular, we will see that the main difference between the domains found by us and those in [4], namely the minimizer of λ_7, is that the corresponding optimal value is now in agreement with the above restriction, while that was not the case for the domain given in [4] – see Section 5 below.

3 General description of the optimization procedure

The computational procedure for solving the optimization problem is, as usual, divided in two steps. The first, the so–called direct problem, consists of calculating some of the eigenvalues of a given domain. The other step is the optimization procedure of determining a domain which optimizes some quantity involving the eigenvalues of the Laplacian. In this work, we will use the Method of Fundamental Solutions (MFS) [21, 22] for solving the direct problem, while the optimization procedure is performed with a genetic algorithm [23] and a gradient method [24, 25].

3.1 Solving the direct problem

The direct problem can be solved by any numerical method for partial differential equations, such as classical methods with finite differences, finite elements or the boundary element method, for example. We will use the MFS which is very attractive for solving shape optimization problems. The MFS is a meshless numerical method which thus avoids the construction of a mesh at each iteration. This is an expensive computational task used by several numerical methods, as the finite element method, for instance. On the other hand, the MFS solves the eigenvalue problems with

high accuracy [22]. In particular, it provides accurate approximations for the gradients of the eigenfunctions on the boundary, which is crucial for the robustness and fast convergence of the gradient method.

We describe the application of the MFS briefly and refer to [22] for details. Let $\Omega \subset \mathbb{R}^2$ be a domain with smooth boundary $\Gamma = \partial\Omega$. Note that the eigenvalue problem (1) is equivalent to the eigenfrequency problem with the Helmholtz equation,

$$
\begin{cases}
\Delta u_i + \kappa_i^2 u_i = 0 & \text{in } \Omega, \\
u_i = 0 & \text{on } \Gamma,
\end{cases}
\tag{7}
$$

with $\lambda_i = \kappa_i^2$, and in a similar fashion for the Neumann eigenvalue problem where $\mu_i = \kappa_i^2$. We take the fundamental solution of the Helmholtz equation

$$
\Phi_\kappa(x) = \frac{i}{4} H_0^{(1)}(\kappa \|x\|),
\tag{8}
$$

where $H_0^{(1)}$ is the first Hänkel function, κ is the frequency and $\|.\|$ denotes the euclidean norm in $\mathbb{R}^2$. The MFS approximation for an eigenfunction is a linear combination

$$
u_k(x) \approx \tilde{u}(x) = \sum_{i=1}^{N} \alpha_i \phi_j(x),
$$

where

$$
\phi_j = \Phi_\kappa(\cdot - y_j)
\tag{9}
$$

are N point sources centered at some points y_j which are placed on an admissible source set which does not intersect $\bar{\Omega}$. By construction, the MFS approximation satisfies the partial differential equation of the problem and the coefficients are determined fitting the boundary conditions. We take N collocation points on Γ, and imposing the boundary conditions of the problems, we obtain a homogenous system of equations

$$
A(\kappa).\overrightarrow{\alpha} = \overrightarrow{0},
$$

where $A(\kappa)$ is a $N \times N$ matrix that depends on κ. The numerical approximations for the eigenfrequencies are the frequencies κ for which the matrix $A(\kappa)$ is singular. To locate them, we consider the evolution of the logarithm of the absolute value of the determinant of the system matrix which is a function of κ. The values κ for which there exists a singularity are the numerical approximations for the eigenfrequencies. As in [22], these are calculated by the golden ratio search. The multiplicity of the eigenfrequency can then be calculated by studying the dimension of the kernel of the matrix.

Having determined an approximated eigenfrequency κ, a corresponding eigenfunction is calculated using a collo-

cation technique on $n+1$ points, with $x_1, \cdots, x_n$ on $\partial\Omega$ and a point $x_{n+1} \in \Omega$, solving the system

$$\tilde{u}(x_i) = \delta_{i,n+1}, \; i = 1, \ldots, n+1,$$

(10)

where $\delta_{i,j}$ is the Kronecker delta. This procedure excludes the zero function.

3.2 Solving the optimization problem

In this section, we describe the main tools we have used to build an efficient algorithm for solving the optimization problems.

3.2.1 Definition of the domains

We will consider the class of star shaped domains $\mathcal{D}$ whose boundary can be parameterized by

$$\{r(t)\left(\cos(t), \sin(t)\right), \; t \in [0, 2\pi[\}\,,$$

(11)

where r is assumed to be 2π-periodic continuous and strictly positive function. To approximate the function r, we consider $M \in \mathbb{N}$ and the expansion

$$r(t) \approx \tilde{r}(t) = \sum_{j=0}^{M} a_j \cos(j\ t) + \sum_{j=1}^{M} b_j \sin(j\ t),$$

(12)

where the expansion coefficients a_j, b_j are to be determined. Then, each point $\mathcal{C} := (a_0, a_1, ..., a_M, b_1, b_2, ..., b_M)$ defines the boundary of a domain using (11) and (12), and thus the optimization problems (5) and (6) are solved searching for optimal points $\mathcal{C}$.

3.2.2 Initialization of the optimization procedure

As was already mentioned in [4], in this type of optimization problems and due to the existence of local maxima and minima, it is important to start the gradient method with a domain which is not too far from the global optimizer. As in [4], we have used a genetic algorithm to choose good candidates to initialize the iterative process. Moreover, for each eigenvalue we apply the gradient method several times starting with different domains to minimize the chance of having just a local optimizer and not the global optimizer.

3.2.3 The gradient method

The key ingredient for the gradient method is the Hadamard formula of derivation with respect to the domain [6, 26]. Consider an application $\Psi(t)$ such that

$$\Psi : t \in [0, T[\to W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N) \text{ is differentiable at 0 with } \Psi(0) = I, \ \Psi'(0) = V,$$

where $W^{1,\infty}(\mathbb{R}^N, \mathbb{R}^N)$ is the set of bounded Lipschitz maps from $\mathbb{R}^N$ into itself, I is the identity and V is a deformation field. We denote by $\Omega_t = \Psi(t)(\Omega)$, $\lambda_k(t) = \lambda_k(\Omega_t)$, and by u an associated normalized eigenfunction in $H_0^1(\Omega)$. If we assume that Ω be of class C^2 and $\lambda_k(\Omega)$ be simple, then

$$\left(\lambda_k(\Omega)|\Omega| \right)'(0) = \int_{\partial\Omega} \left[\lambda_k - \left(\frac{\partial u}{\partial n} \right)^2 |\Omega| \right] V.n d\sigma. \tag{13}$$

For the Neumann case, assuming that Ω be of class C^3, μ_k be simple and u be the associated normalized eigenfunction, we have

$$\left(\mu_k(\Omega)|\Omega| \right)'(0) = \int_{\partial\Omega} \left[\mu_k + |\Omega| \, |\nabla u|^2 - \mu_k u^2 |\Omega| \right] V.n d\sigma. \tag{14}$$

By (13) and (14), it is evident that the robustness of the numerical method for solving the optimization problem is related to the accuracy in the calculation of the gradient of the eigenfunction. This fact is one of the main reasons to choose the MFS as a forward solver for this type of problems. Once we have computed the gradient d, we have a direction along which we will determine the next point $\mathcal{C}_{n+1}$ by

$$\mathcal{C}_{n+1} = \mathcal{C}_n \pm \beta d.$$

The sign $\pm$ is equal to $-$ and $+$ respectively in Dirichlet and Neumann cases. The step length β determines the optimal distance along some direction defined by d and is calculated using the golden ratio search [22].

3.2.4 The case of multiple eigenvalues

As was reported in [4], when applying the gradient method, we must deal with multiple eigenvalues. Moreover, a priori we do not know which is the multiplicity at the optimal domain. In the Dirichlet case, for every $i > 1$ we start minimizing the quantity $|\Omega|\lambda_i(\Omega)$. As soon as we obtain $|\Omega|\lambda_{i-1}$ too close to $|\Omega|\lambda_i$,

$$|\Omega| \left(\lambda_i - \lambda_{i-1} \right) < \epsilon$$

for some parameter ϵ, we modify the cost function and try to minimize

$$|\Omega| \left(\delta_i \lambda_i + \delta_{i-1} \lambda_{i-1} \right)$$

for a suitable choice of constants δ_i and δ_{i-1} which may be adjusted to ensure the convergence of the numerical method. Then, once we have

$$|\Omega|\,(\lambda_i - \lambda_{i-1}) < \epsilon \quad \text{and} \quad |\Omega|\,(\lambda_{i-1} - \lambda_{i-2}) < \epsilon,$$

we change the cost function to

$$|\Omega|\,(\delta_i \lambda_i + \delta_{i-1}\lambda_{i-1} + \delta_{i-2}\lambda_{i-2}) \tag{15}$$

and continue applying this process, adding more eigenvalues to the linear combination which defines the cost function, until we find the optimizer and the multiplicity of the corresponding eigenvalue. This kind of procedure can be related to penalty methods. For example, another good strategy would be the use of a logarithmic barrier method [24]. Instead of minimizing the cost function (15), we could solve a sequence of minimization problems of the objective function

$$|\Omega|\,(\lambda_i + \omega_{i-1}\log(\lambda_i - \lambda_{i-1}) + \omega_{i-2}\log(\lambda_{i-1} - \lambda_{i-2})), \tag{16}$$

for decreasing values of ω_{i-1} and ω_{i-2}. The process in the Neumann case is analogous.

4 The Neumann problem

In this section, we present the main results obtained with our numerical algorithm for solving the optimization problem (6). Figure 1 shows the connected optimizers for μ_i, i=3,4,...,10 obtained in this way. Note that, because of the way in which the boundary of the domain is defined (cf. Section 3.2.1), the coefficients a_j, b_j are not determined uniquely, in the sense that different coefficients may correspond to the same domain after a rigid transformation. Moreover, this definition of the boundary restricts the optimization to star–shaped domains with respect to the origin of the polar coordinates. This restriction could be a handicap for the algorithm if, at some step in the iterative process, the origin became too close to the boundary of the domain, or if a ray connecting the origin to another point on the domain became tangent to the boundary. In the first case, it might become necessary to perform a change of variables to move the origin, while in the second it might just mean that the optimizer is not star–shaped, at least with respect to the origin being considered. However, this was not necessary in the optimizations under consideration. The optimal coefficients that were calculated correspond to domains which have the origin sufficiently far from the boundary. Each picture of an optimizer that we show in Figure 1 was obtained after a suitable rotation. We also recall that the optimization is performed within the class of star shaped domains as described in Section 3.2.1. In particular, our algorithm does not include the case of disconnected sets. However, using Theorem 2.1, it is possible to include the disconnected case in our study, because if some eigenvalue μ_i is maximized for some disconnected set $\Omega_i^\star$, then $\Omega_i^\star$ is the union of domains belonging to the set of optimizers of lower eigenvalues. In particular, we note that μ_4, μ_5 and μ_7 are not maximized by the domains obtained numerically for these specific eigenvalues, but by combinations of previous maximizers. Figure 2 shows the corresponding maximizers which were built using Theorem 2.1. In the

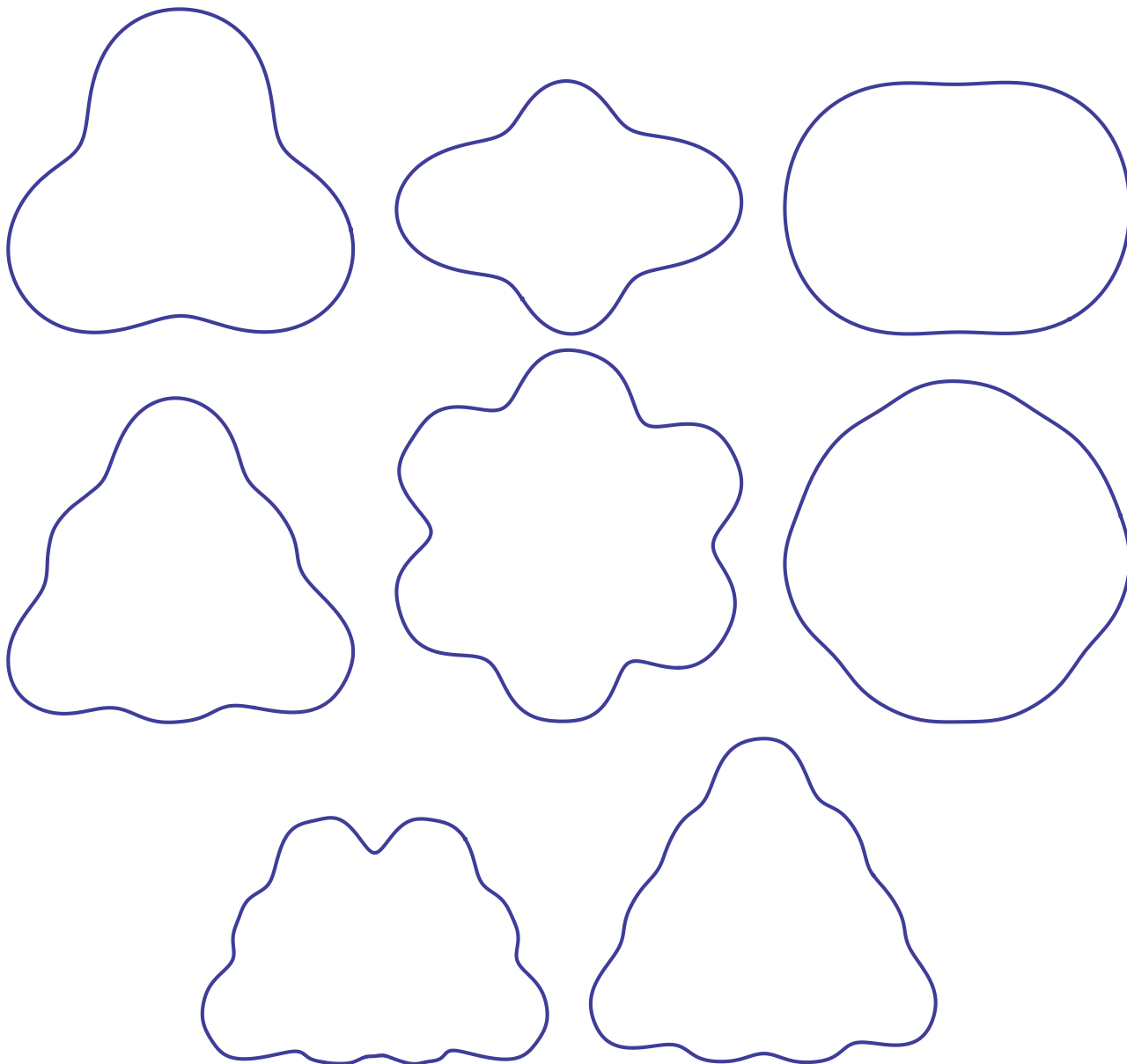remaining cases, our results point to optimizers being connected.

Figure 1: The numerical connected maximizers obtained with our numerical algorithm for the Neumann eigenvalue problem (6) with i=3,4,..,10.

In Table 1, we show the optimizers and the corresponding optimal value obtained via the numerical procedure described above. In all cases, the iterative process was stopped once the difference between the eigenvalue involved in the optimization and the corresponding value obtained at the previous step was small enough. This was then truncated to have two decimal digits and thus, the value which is presented is actually a lower bound for the optimal value. The optimal numerical values obtained in this way satisfy the bound of Theorem 2.2. In the cases of μ_i, with $i = 4, 5, 7$, for which the optimizer is disconnected, we also address the value obtained with our algorithm for the (connected) domain plotted in Figure 1. In the last column, we show the best result obtained with unions of disks calculated in [19]. Our gradient method revealed to be an effective tool for solving the optimization problems with accuracy.
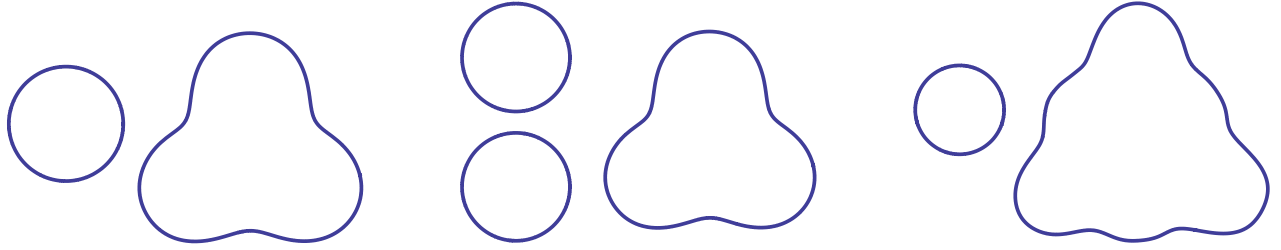
Figure 2: The numerical maximizers (built using Theorem 2.1) for the Neumann eigenvalue problem (6) with i=4,5,7.

However, when optimizing μ_8, it did not allow to obtain two digits of accuracy that we show in the Table. Indeed, it revealed to have slow convergence in the neighborhood of the optimizer. This effect may be related to the fact that the optimizer is similar to a ball, which may imply that the cost function has a complex region of multiplicities that does not allow the gradient method to converge faster. In that case, we simply have considered random perturbations of the domain obtained by the gradient method. To define a perturbed domain $\tilde{\Omega}$, we simply pick the vector $\mathcal{C}$ defining $\partial\Omega$ and perturb each component of this vector. Denoting by $\mathcal{C}_i$ and $\tilde{\mathcal{C}}_i$ (respectively) the ith components of $\mathcal{C}$ and $\tilde{\mathcal{C}}$, we have considered

$$\tilde{\mathcal{C}}_i = \mathcal{C}_i\left(1 + \eta_i\right),$$

where η_i is a random number generated in the interval [-0.05,0.05]. Then, if the eigenvalue of $\tilde{\Omega}$ is larger than the corresponding value of Ω, we define a new vector $\mathcal{C} = \tilde{\mathcal{C}}$ and repeat the process until an accuracy of two digits had been obtained.

i	Ω	multiplicity	$\mu_i^\star$	$\mu_i^\star(C)$	$\mu_i^\star(UD)$
3		3	**32.79**		31.95
4		5	**43.43**	43.20	42.60
5		7	**54.08**	52.97	53.25
6		4	**67.04**		63.90
7		6	**77.68**	77.24	74.55
8		4	**89.22**		88.85
9		4	**101.73**		99.50
10		5	**113.86**		110.15

Table 1: *The Neumann maximizers with the optimal values for $\mu_i^\star$ and the corresponding multiplicity; the last column shows the optimal value for unions of discs.*

5 The Dirichlet problem

Now we present our results for the Dirichlet case. As stated in the Introduction, a similar numerical study using a different method had already been performed by Oudet in [4] for the first ten eigenvalues. With our method we were able to obtain better results than those presented in that study. Moreover, we propose numerical optimizers for more five eigenvalues. In Figure 3 we plot these optimizers for λ_i, i=5,6,...,15. Except for one case, there is agreement between the optimal shapes that we obtained and those presented in [4]. The exception is related to the minimization of λ_7, for which the optimal shape proposed by Oudet is disconeccted and was built using the Wolf-Keller theorem. On the other hand, we have obtained the connected set which is plotted in Figure 3 and has a smaller eigenvalue. This difference between Oudet's results and ours may be related with the bound of Theorem 2.2. We note that while all our numerical optimal values satisfy that bound, Oudet's results for λ_6^* and λ_7^* do not. In Table 2, we show the numerical values obtained here and those obtained in [4]. In this study, we only aimed at an accuracy of two decimal digits. The MFS with an adequate choice for the point sources is a highly accurate numerical method, specially for smooth domains as those we deal with in this optimization procedure [22, 27]. We thus believe that the numerical approximations for the eigenvalues of our numerical optimizers have at least two decimal digits of accuracy, which could be confirmed using Moler-Payne Theorem [28, 22]. All the values indicated were obtained rounding up our numerical values and are thus upper bounds for the optimal value.

We remark also that with some extra computational time the method employed can easily provide more accuracy in the calculation of the optimal eigenvalue. To illustrate this, we have considered the domain optimizing λ_{10}. This has multiplicity 4 and we should thus have

$$\lambda_{10}(\Omega_{10}^*) = \lambda_9(\Omega_{10}^*) = \lambda_8(\Omega_{10}^*) = \lambda_7(\Omega_{10}^*).$$

With our algorithm, we obtained a domain for which $\lambda_7 \approx 142.7171281625934$ and $\lambda_{10} \approx 142.7171281626059$, the difference between the two values being 1.25×10^{-11}.

As in the Neumann case, the way in which the domains were defined, described in Section 3.2.1, restricts the numerical optimization to star–shaped domains with respect to the origin. Again, this could be a limitation if any of the situations mentioned above occurred. However, and as in the Neumann case, the coefficients calculated by our numerical algorithm correspond to domains having the origin sufficiently far from the boundary, and with no rays becoming tangent to it. To illustrate this fact, in Figure 4 we plot the numerical optimizers for λ_{13} and λ_{15}. In both cases we mark also the origin and the center of mass of the domain.

We also note that the optimizer for λ_{13} does not seem to have any kind of symmetry. It would be natural to assume this to be an artifact of the algorithm, due to the fact that the location of the origin of our coordinate system is *far* from the center of mass of the domain. However, this does not seem to be the case, as can be seen by performing a change of variables to move the origin to the center of mass and then re-start the optimization procedure. When we

i	Ω	multiplicity	λ_i^*	Oudet's result
5		2	**78.20**	78.47
6		3	**88.52**	88.96
7		3	**106.14**	107.47
8		3	**118.90**	119.9
9		3	**132.68**	133.52
10		4	**142.72**	143.45
11		4	**159.39**	-
12		4	**172.85**	-
13		4	**186.97**	-
14		4	**198.96**	-
15		5	**209.63**	-

Table 2: *Dirichlet minimizers with the optimal values for λ_i^* and the corresponding multiplicity.*

do this, we find that the results obtained do not differ in a significant way and, in particular, the numerical optimizer for λ_{13} remains without any symmetries.

6 Symmetries, multiplicities and TRIANGULAR domains

An analysis of the optimizers obtained suggests several remarks and directions for future study, both numerically and analytically. One first issue is related to symmetry. It is part of the *folklore* of this subject that optimizers should have some sort of symmetry. Although this seems to be the case in most situations, we found one example, λ_{13}, for which there seems to be no symmetry involved. Due to the high multiplicies involved and to the complexity of the optimization procedure we can't, of course, ensure that there does not exist another domain - which does not necessarily have to be close to this one - for which λ_{13} is lower than the one given here. We have considered the optimization of λ_{13} among domains which are symmetric by reflection with respect to some line. Instead of the expansion (12), we have considered

$$r(t) \approx \tilde{r}(t) = \sum_{j=0}^{M} a_j \cos(j\ t) \tag{17}$$

and then optimized the cooefficients $a_j,\ j = 0, ..., M$ to minimize $\lambda_{13}|\Omega|$. Our symmetric numerical optimizer is plotted in Figure 5 together with the optimizer obtained without symmetry constraint. For this symmetric domain, we obtained $\lambda_{13} = 187.92$ which, due to the high accuracy of the MFS, we believe to be significantly larger than 186.97

which was obtained without symmetry constraint.

This should be a matter for further study since, as mentioned in the Introduction, proving the existence of symmetries of optimizers is one of the important aspects from a theoretical point of view.

In all other cases, both for the Dirichlet and Neumann problems, the examples considered suggest the existence of either a reflection or $\mathbb{Z}_3$ symmetry (or both). We note that this can be checked in a more precise way than by just looking at the picture, as we will now illustrate. The picture for the optimizer of λ_{15} strongly suggests that Ω_{15}^* has $\mathbb{Z}_3$ symmetry. To confirm this, we performed a change of variables such that the domain has an expansion of type (11) with the origin of the polar coordinates placed at the center of mass of the domain. In Figure 6-left, we have then plotted the functions $r(t)$ (continuous line) and $r(t + 2\pi/3)$ (dashed line). In the right plot of the same figure we show the difference of the previous functions and we see that the order of magnitude of the agreement between the two graphs is smaller than the order of magnitude of the accuracy considered for the optimal value.

One other aspect that stands out by looking at the shape of the optimizers is the fact that the optimizer of μ_k seem to be close to a domain composed of k equal balls. This is emphasised by the appearance of what we might call *triangular domains* corresponding to the triangular numbers $k(k + 1)/2$. A similar effect also seems to be present in the Dirichlet case (see the optimizers for λ_6, λ_{10} and λ_{15}), although here the relation to the number of balls is not so straightforward.

In the Dirichlet case, these triangular domains also seem to be related to a change in multiplicity of the minimizer, which within the range considered takes place at $k = 6, 10$ and 15 – see Table 2. The exception here is the case of λ_4, which is not connected and has multiplicity 3 already.

All of this seems to point in the direction that at least for low eigenvalues, such as those under consideration here, although there seems to exist an underlying structure to the optimal solutions, there might exist a number of exceptions preventing results related to symmetry or connectedness to hold in full generality. This makes it all the more important for further numerical tests to be carried out to confirm (or not) the results found in this paper. At this level, and taking into account the exceptional behaviour mentioned above, it is worth noticing that all the optimizers found do satisfy Pólya's conjecture [29] by a clear margin. More precisely, we have

$$\lambda_k^* > \frac{4k\pi}{A}, \ k = 1, ..., 15 \quad \text{and} \quad \mu_k^* < \frac{4k\pi}{A}, \ k = 1, ..., 10,$$

implying that the conjecture holds in the range considered.

7 Conclusions

In this study, we have illustrated the possibility of taking advantage of the capacity of a meshless method to deal with problems demanding a lot of computational power while keeping accuracy within required levels, by applying it to the optimization of low Dirichlet and Neumann eigenvalues of the Laplace operator.

We have confirmed most of the results from a previous study of the Dirichlet case by Oudet using different methods [4], and provided a domain with a better value in the case of λ_7. Besides this, we have determined the candidates for optimizers for five more eigenvalues up to λ_{15}.

In the Neumann case, more challenging from a computational point of view, we determined numerical candidates for maximizers up to μ_{10}.

The numerical coefficients defining the numerical Dirichlet and Neumann optimizers are provided in the Appendix.

The results obtained reveal a rich structure behind the optimizers pertaining to symmetry, connectedness and multiplicities. However, we also found some exceptions which we believe to be of interest, such as the fact that it is likely that the optimizer of the 13$^{\text{th}}$ Dirichlet eigenvalue will not have any kind of symmetry. As far as we are aware, it is the first example of this type which appears in the literature. In view of these results, it would seem that although optimizers do possess an underlying structure, it might not be possible to establish general results due to the existence of exceptions.

It is possible, of course, to do some variations around the cases considered here. Although we have considered mainly the case of star-shaped domains, the case of a disconnected domain composed of several star-shaped components was also included by means of the Wolf-Keller Theorem. However, we did not consider domains with holes, as these are not expected to yield better values than simply–connected domains. If desired, the MFS can also be applied in that case [30] and thus, with the appropriate modifications, this study could be extended to include multiply connected domains.

A more interesting problem is the extension of the methods in this paper to the three-dimensional case. This is a much more challenging situation, which is currently under research. We believe that the Method of Fundamental Solutions with an appropriate choice for the source-points, as in [31], will also allow for a solution of this optimization problem in reasonable computational time.
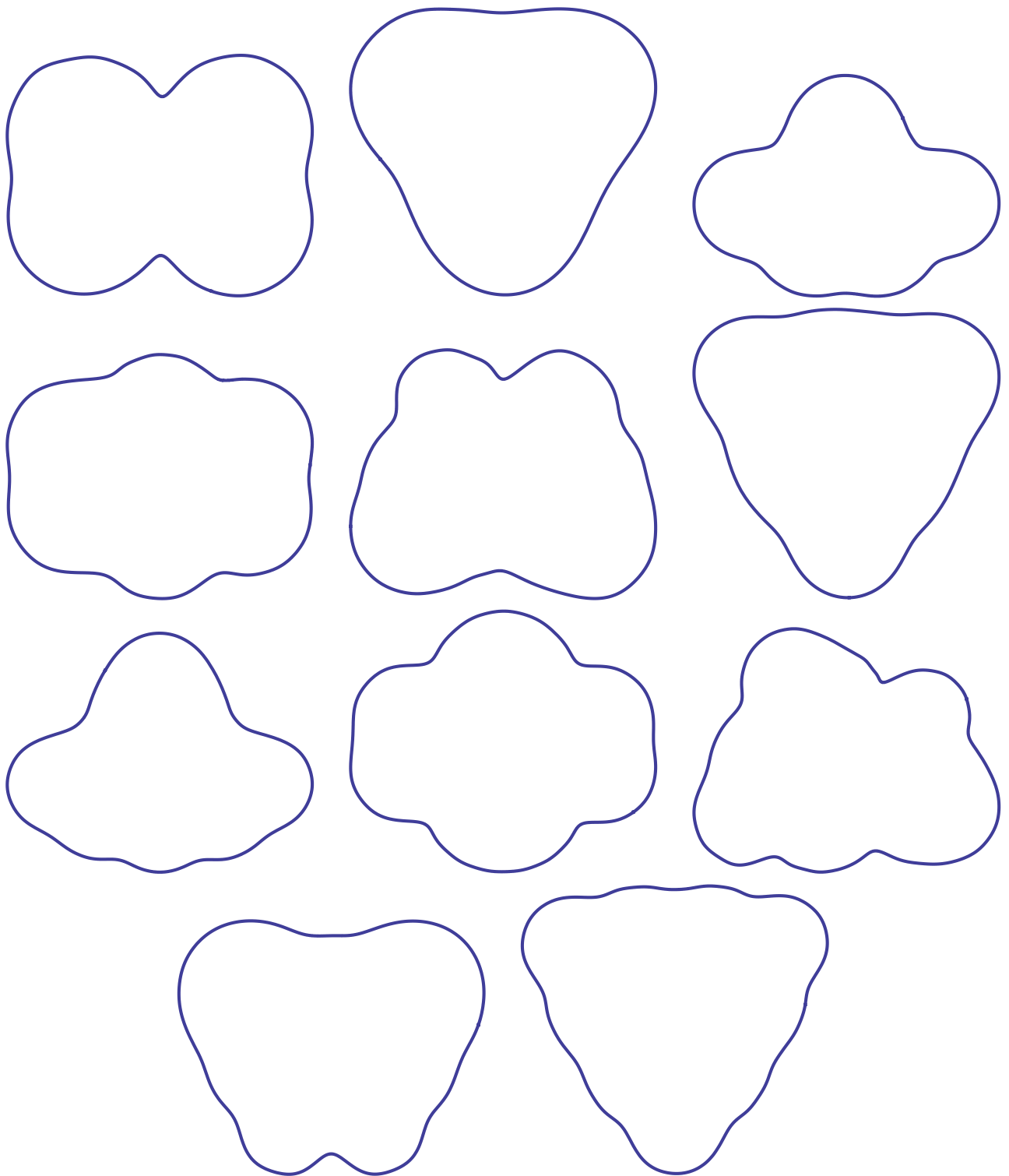
Figure 3: The optimizers for the Dirichlet eigenvalue problem (5) with i=5,6,...,15.
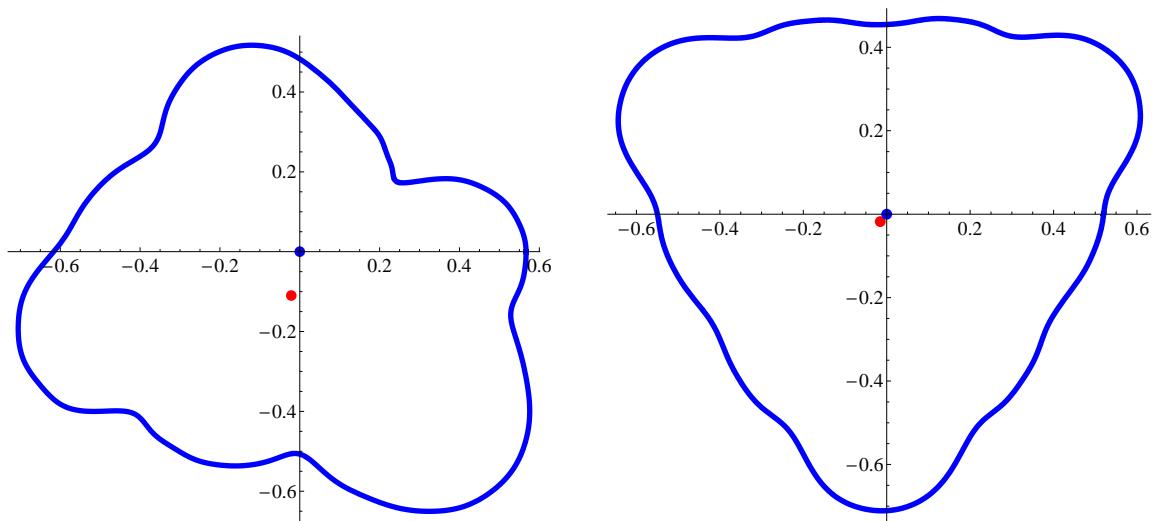
Figure 4: The numerical optimizers for λ_{13} and λ_{15} and the corresponding center of mass.
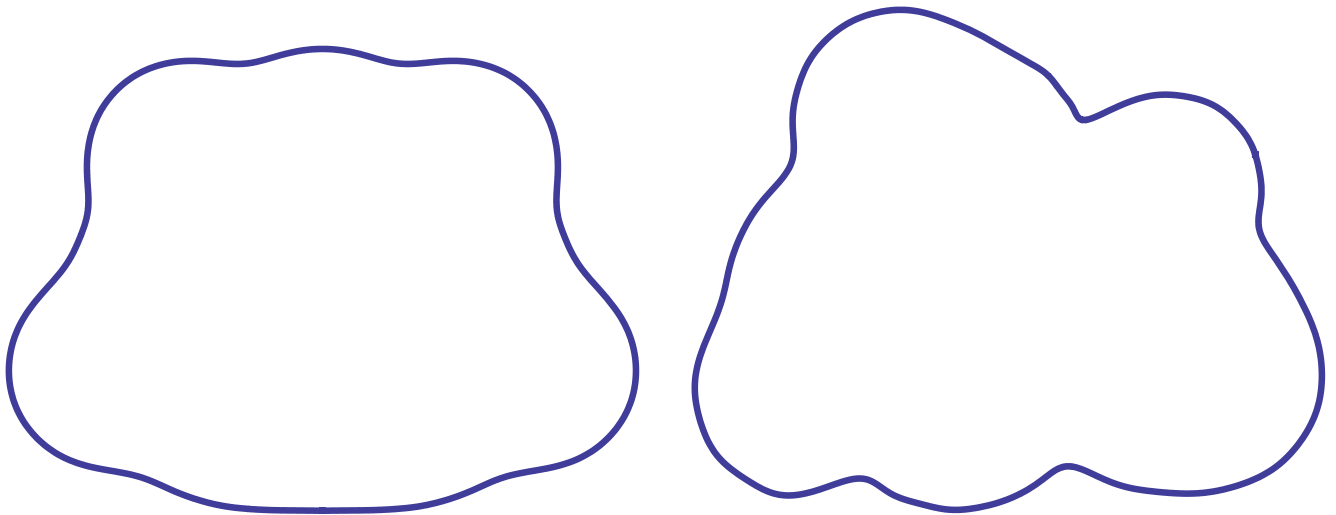
Figure 5: Symmetric numerical minimizer of λ_{13} and the optimizer obtained without symmetry constraint.
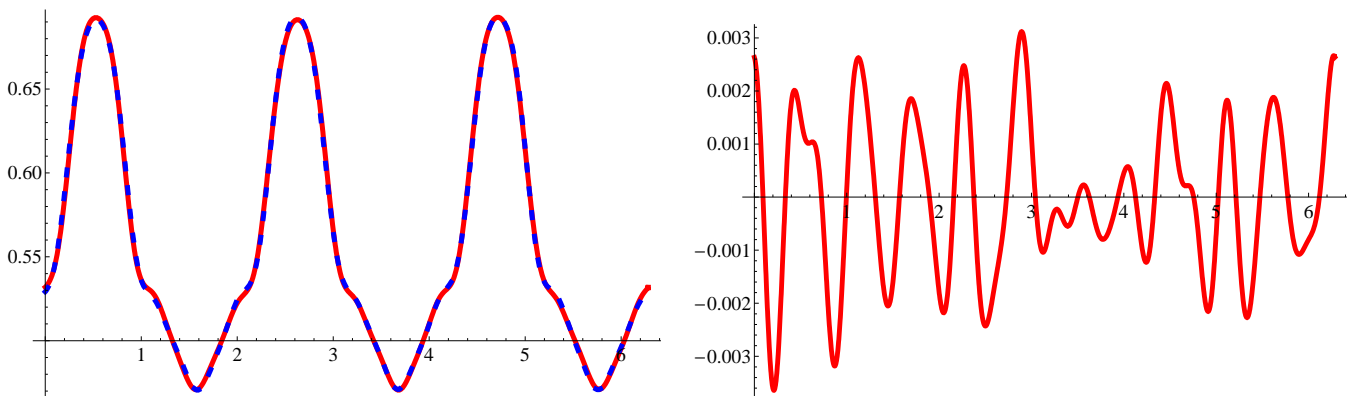
Figure 6: Plot of $r(t)$ (continuous line) and $r(t + 2\pi/3)$ (dashed line) for λ_{15}, and of the difference of these two functions.

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Appendix

	λ_5		λ_6		λ_7		λ_8	
i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i
0	0.55438079		0.540969934		0.555917408		0.52402981	
1	0.013238729	0.0057015487	-0.0254618108	0.20040578	-0.0396571523	0.0142910591	-0.133073983	-0.237327033
2	-0.077358408	0.092876235	-0.0441431399	-0.0456353393	-0.0378443687	-0.069786321	0.0524340446	0.0850167772
3	-0.0051545105	-0.0024247559	-0.0573759273	0.023490155	-0.0159573574	0.059561064	-0.0123987019	-0.0376173185
4	0.014513074	0.080309088	-0.0106613347	-0.0482010972	-0.0563539206	0.0490047495	-0.0260590608	0.00874912734
5	-0.0060402322	0.0077267846	0.0138861761	0.000120165574	-0.00279806649	-0.00185896562	-0.00308877567	-0.00973987438
6	0.0080050385	0.0023040320	0.000823401662	-0.00116176266	0.028061714	0.00907415168	-0.00634415879	0.014659245
7	-0.00041462756	0.0055027319	0.000493117601	0.000994308524	-0.00621703077	0.00398846543	-0.00742943607	0.00144421808
8	0.011735634	-0.0056190380	0.00160548125	-0.00108261442	0.00163688539	0.00365127087	0.0144983946	0.00753703381
9	0.0039973580	0.0034615840	-0.0000526885146	0.000351429203	0.00321047409	-0.0037258508	-0.00642345426	-0.013566282
10	-0.00022151852	-0.0026638205	0.000112772803	-0.0000539188865	0.0066201419	-0.00279887602	-0.00391602052	0.0101569665
11	0.0027374112	0.00075580032	-0.000110774901	0.000119156464	-0.00362005727	0.000298194234	0.00186757844	-0.00127916948
12	-0.0017708460	-0.0030278106	-0.000109752183	-0.000168572897	-0.00258236878	-0.00312578819	0.0024586174	0.00171021319
13	0.00096320890	-0.0020044397	$7.12030548 \times 10^{-6}$	-0.0000117806279	0.00193116396	0.0000971958322	-0.00172422653	-0.00153156433
14	-0.000035727810	0.00044155670	$7.24155168 \times 10^{-6}$	$-3.84022077 \times 10^{-6}$	-0.00131702223	-0.000360018056	0.00181204829	0.00094827332
15	-0.00053585540	-0.0017723963	$-4.96715037 \times 10^{-6}$	$-3.61610015 \times 10^{-6}$	-0.00172076406	0.000688255022	-0.00166054256	-0.00127832356
16	-0.00012345948	0.00096474668	$3.08139935 \times 10^{-7}$	$-2.03467268 \times 10^{-7}$	-0.00115709151	0.000109810932	0.000431561493	0.00130351922
17	-0.0013188383	-0.00070898727	$1.33654316 \times 10^{-7}$	$1.43346737 \times 10^{-7}$	0.00149459761	0.000610453469	0.0000346589834	0.000803396577
18	0.00022774519	0.000042899804	$-1.13982504 \times 10^{-7}$	$5.93369227 \times 10^{-8}$	-0.000305459945	0.00101611823	0.00188952975	-0.00135359719
19	-0.00050075176	0.000026338167			-0.000427708258	-0.000303869144	-0.00221559806	-0.000232152954
20	-0.00016570749	0.00018779596			0.000678645943	0.000276590019	0.00103793066	0.000830350291
21					0.000569909436	-0.000046937353	-0.000313475612	-0.000209535319
22					0.0000654203457	0.0000144614045	0.000450916946	-0.000139709455
23					-0.000351309674	-0.000371289997	-0.000211391115	0.0001928562
24					0.000351733859	-0.000136480239	0.000217311751	-0.000318264501

	λ_9		λ_{10}		λ_{11}		λ_{12}	
i	a_i	b_i	a_i	b_i	a_i	b_i	a_i	b_i
0	0.550655939		0.5595999481423		0.556134818		0.560041023	
1	-0.0998751448	0.0773421852	0.0138865410279	-0.008909759517606	-0.0368731391	0.0239315354	-0.0379604707	0.0121487257
2	0.0550157453	-0.0430957094	0.002833319704712	0.002227574982416	-0.0182053322	0.0378034906	0.0387729478	0.0507913179
3	0.0399594874	0.0593875966	-0.008349328761967	0.0957550910192	-0.00715126258	-0.101591534	-0.0015923326	0.00266165893
4	0.00142349384	0.0425900486	0.005723472594822	-0.002967862015974	-0.0297483526	-0.0418656294	-0.0146793258	0.0490107094
5	-0.0139749255	-0.0408617143	0.002652563387632	0.002447234406475	0.0241573412	0.011609453	0.00402780473	-0.0090959291
6	-0.00367223638	0.0134695479	-0.004599950115557	0.02627159459515	0.0164630521	-0.00243660354	0.0172020883	-0.00377988974
7	0.0082532258	0.00455014924	-0.002226983087317	0.000837603292585	-0.00634085209	0.00454033198	0.000684280074	0.00988820653
8	-0.00895830418	0.00176185763	0.00158099064617	0.00129569319721	-0.00412733089	0.0136032862	0.0176838961	0.0101877652
9	0.00783739536	0.00272856592	0.000470007408465	-0.00146767096302	-0.00119221343	-0.00583869131	-0.00650954246	-0.00231789514
10	0.00224548719	-0.00470960161	-0.00032089848269	-0.0000433152786976	-0.000348634021	0.0023618601	0.00215815089	0.000591548356
11	-0.00333457231	0.00120461234	-0.000417096748991	-0.000549411311543	-0.00458553797	-0.000155987857	0.00268858702	-0.000746677809
12	0.00200923924	0.00134340334	0.00118106640714	-0.00326484110276	0.00156859838	-0.00100931829	0.00277871992	-0.00316449624
13	-0.00128480877	-0.00273791879	0.0000474565496931	0.0000106513746558	0.00246569204	-0.000345244088	-0.00204072395	0.00161161335
14	0.000462592585	0.00386907619	-0.000325293746937	-0.000447970381423	-0.000674880003	-0.00290011162	-0.0000851755712	-0.00180079941
15	0.00220397703	-0.00328595287	-0.0000233520418177	$-2.21572997176 \times 10^{-6}$	0.000577306217	0.0013066555	0.000479952574	-0.00218761547
16	-0.00159326685	0.000960521718	0.000145484964248	0.0000270002808451	-0.000130890779	-0.000367319041	-0.00119840099	-0.00172733708
17	0.000330803919	0.00030383098... 0.000303830978	0.0000346822452809	0.0000929323504196	-0.00153045766	-0.0000950422371	0.00130704997	0.00124434852
18	-0.000225563872	-0.00129473302	-0.000323698743562	0.000580384511426	0.000111369245	0.0000164522889	-0.00102042771	-0.00100033718
19	-0.000189994632	0.000379332968	$5.83949533850 \times 10^{-6}$	$-3.49905606658 \times 10^{-6}$	0.000181076275	-0.000228614786	-0.000755371705	0.0000336785202
20	0.00114449161	0.00603512001	0.0000728114477851	0.000158698077368	0.0000332405057	-0.00075412169	-0.000423177546	0.000174586669
21	-0.00107566805	-0.000685698552	$-6.75766392730 \times 10^{-9}$	$8.82905652608 \times 10^{-9}$	0.0000453530838	0.000211630126	0.000529771728	-0.000195846058
22			$-4.71286357457 \times 10^{-9}$	$-1.34781200234 \times 10^{-9}$	0.000790026567	0.00042327352	-0.000711361356	0.000555750456
23			$4.91209220912 \times 10^{-9}$	$-1.56660058965 \times 10^{-9}$	-0.000395412526	-0.0000237025238	-0.000200059087	0.000361447102
24			$1.34638456165 \times 10^{-8}$	$-1.42615694109 \times 10^{-8}$	-0.000221380759	0.000184318601	$6.73027374 \times 10^{-6}$	0.000121028574
25			$-2.01122360346 \times 10^{-9}$	$-3.63972837821 \times 10^{-10}$				
26			$-5.44330996748 \times 10^{-9}$	$-6.68514460573 \times 10^{-9}$				
27			$1.73465493720 \times 10^{-9}$	$-1.98827703541 \times 10^{-9}$				
28			$1.26990184897 \times 10^{-9}$	$6.02497710742 \times 10^{-10}$				
29			$-6.50409422882 \times 10^{-10}$	$5.20844861007 \times 10^{-10}$				
30			$-4.33794173095 \times 10^{-9}$	$2.75350252810 \times 10^{-9}$				

i	λ_{13} a_i	b_i	λ_{14} a_i	b_i	λ_{15} a_i	b_i
0	0.551503971		0.557612527		0.559420408	
1	-0.0338396841	-0.114953381	0.0207678515	-0.0631536572	$6.04658369 \times 10^{-6}$	0.0000124733917
2	0.0303279427	-0.0574495473	0.058792891	-0.00649772139	-0.0000161762498	$5.76792466 \times 10^{-6}$
3	-0.0130257251	-0.0745641579	0.00531111557	0.0725253769	-0.00258516561	0.098348305
4	0.00554729211	0.0470925562	-0.0238412727	0.00370821936	0.0000123512599	-0.000190815291
5	-0.000623410989	0.0156226091	-0.00423481014	0.0177123045	-0.000180152753	-0.000020711265
6	0.017655584	0.0236085722	0.0103355869	0.00370104625	-0.0272911266	-0.00107299012
7	0.0157809243	0.0041362414	0.00304837882	-0.014888849	0.0000811807213	0.000119491243
8	-0.0102727122	0.00626879621	-0.00901708416	-0.00469316105	0.0000417425306	-0.000256107706
9	0.00478087159	0.0101645762	-0.00164226506	0.0112119165	0.00153912059	-0.0158989451
10	-0.00238858367	-0.00266695979	0.00355556561	0.00270396021	-0.0000614558942	-0.000447492308
11	-0.00105832356	-0.00408004353	0.000321931966	-0.00139053897	0.000866001007	0.00027348823
12	-0.000322435582	-0.00132277381	0.000904139462	-0.000968101304	-0.00316306275	-0.000317748954
13	-0.000213314175	-0.00418096453	-0.000540135261	0.00109132915	-0.000144984378	-0.0000655946379
14	0.00272342258	0.000515668982	0.00225122256	0.00138225898	0.0000138295682	0.0000408821376
15	0.00227560059	-0.00387180384	0.00168277384	-0.00245914572	0.000674764382	-0.00440846194
16	-0.00134117822	0.000421056294	-0.0020040944	-0.00149449634	0.000187045993	-0.0000389099159
17	0.00206300976	0.00168123748	-0.00133504032	0.00121246955	0.000295365946	0.000120717313
18	-0.000134937278	-0.000594329311	0.000932926429	0.000698600595	0.00201216141	0.000363000545
19	-0.000198444012	0.000832646684	0.000619420867	-0.000348513148	0.0000688156095	-0.0000196772322
20	-0.00102360784	0.0000520444273	-0.000149699453	-0.000278429144	-0.0000745226863	0.00023283672
21	-0.001672364	0.000149383067	-0.0000699271742	$-5.4335581 \times 10^{-6}$	0.000139260778	-0.000810604043
22	-0.0000358586874	0.000530041321	0.0000435046984	0.0000790199395	0.0000853310383	-0.0000589423176
23	-0.000396920563	-0.00174332362	0.000122963842	-0.0000894017094	$-9.68881282 \times 10^{-6}$	-0.0000137876809
24	-0.000877487074	0.000231158959	-0.000096357168	-0.000130817975	0.000859421574	0.000254606817
25	0.00081526229	-0.0000801225237			0.000192873826	0.00041027021
26	-0.0000126186682	-0.000184605499			-0.0000242683214	0.000159546939
27	0.000719525244	0.00045645044			-0.0000380641833	0.0000291315401
28	0.0000402334404	0.0000752523567				
29	-0.000231434094	0.000612933714				
30	0.000248027728	0.000375568407				

i	μ_3 a_i	b_i	μ_4 a_i	b_i	μ_5 a_i	b_i
0	0.549954737		0.554098864		0.560519732	
1	-0.000578925467	0.122335927	0.00784197933	0.00418017833	$2.70058702 \times 10^{-6}$	$-7.43462854 \times 10^{-7}$
2	0.0268566607	0.000218577103	-0.0567066976	-0.085617596	$-2.82208992 \times 10^{-6}$	0.0891304954
3	0.000496273706	-0.119430145	0.0000940623335	0.00223713046	$-2.99864606 \times 10^{-6}$	$3.78746736 \times 10^{-6}$
4	0.0393289299	0.000347273779	-0.0418915905	0.1008615	0.0176267143	$-1.07708404 \times 10^{-6}$
5	0.000137109558	0.0106100598	-0.00119189439	0.000890469706	$2.41171388 \times 10^{-6}$	$2.88093629 \times 10^{-6}$
6	0.00283111302	-0.000494884753	-0.000628688812	-0.000171550392	$-3.02951992 \times 10^{-7}$	0.000287510816
7	0.000178562726	0.00858917695	0.000993259124	0.000551021581	$-1.89820251 \times 10^{-6}$	$1.35592844 \times 10^{-6}$
8	-0.0044258337	-0.000117746227	0.00172088603	0.00189395041	-0.0000537048883	$-3.03014709 \times 10^{-7}$
9	-0.000153351823	-0.00245969402	0.0000211410532	0.000121738707	$1.11865906 \times 10^{-8}$	1.5845649×10^{-8}
10	0.000262693154	0.00011309051	0.0000861800292	-0.000464999058	$-7.95639895 \times 10^{-10}$	$1.16121376 \times 10^{-8}$
11	0.0000273742248	-0.000637837391	$-2.92218066 \times 10^{-6}$	$3.16893374 \times 10^{-6}$	$-9.58692935 \times 10^{-9}$	$7.27334368 \times 10^{-9}$
12	0.000277665815	0.000118370769	0.000038215104	$-1.87411565 \times 10^{-6}$	$-1.01220124 \times 10^{-8}$	$-2.3451735 \times 10^{-10}$
13	$4.89182689 \times 10^{-7}$	$-1.22498345 \times 10^{-6}$	$5.19443562 \times 10^{-7}$	$-6.26013908 \times 10^{-6}$	$-8.39746687 \times 10^{-10}$	$-1.23147374 \times 10^{-9}$
14	$-7.17958187 \times 10^{-7}$	$-5.83425372 \times 10^{-7}$	$-2.72365156 \times 10^{-8}$	$4.12250759 \times 10^{-8}$		
15	$9.84379966 \times 10^{-7}$	$-1.28371466 \times 10^{-6}$	$-3.8438191 \times 10^{-7}$	1.2958845×10^{-7}		
16	1.0499127×10^{-6}	$3.93269146 \times 10^{-7}$	$-1.20147834 \times 10^{-7}$	$-8.02744085 \times 10^{-8}$		
17	$-5.21573808 \times 10^{-8}$	1.3948447×10^{-7}				
18	$4.29340708 \times 10^{-7}$	$-7.32876443 \times 10^{-8}$				

i	μ_6		μ_7		μ_8	
	a_i	b_i	a_i	b_i	a_i	b_i
0	0.544423006		0.561303447		0.563472327	
1	0.154549722	0.074592044	-0.0296105015	0.00929288081	-0.0135281387	-0.0125695822
2	-0.0238297433	-0.0134221544	0.00292713876	0.000461527094	-0.00100221576	0.000360922605
3	0.0465889872	0.0735117289	-0.00581168985	0.00629595371	-0.000758596993	-0.00183183995
4	0.00798942924	0.0640407746	0.000521812939	0.000923350669	0.0108989574	-0.00829182165
5	-0.00631739792	-0.018003381	-0.00154983934	0.0094039286	-0.000820543007	-0.000308128932
6	-0.0137357316	0.0204107763	-0.00501494717	0.0700259021	0.000874631017	-0.00117284521
7	-0.0216354835	0.0151204116	-0.00243458512	-0.012367072	-0.00888157387	-0.00150292924
8	-0.00906075347	0.00115927695	0.00113180953	0.000671399157	0.000674798064	0.000729850385
9	-0.008733436	0.000345375567	0.00256031464	0.00298770473	-0.0000147809707	0.000267662456
10	0.00229280469	-0.00114925445	0.00279992286	-0.000175531225	-0.00054844287	-0.000335737527
11	0.00315240014	0.00284257911	0.00354013286	0.00287589877	0.00119052948	-0.000881687946
12	-0.00163249012	0.00100770067	0.0137409909	0.000577609805	0.000195303616	0.0000981460855
13	-0.0000192004444	0.00315093287	-0.0052461102	0.00145050804	0.000174689747	0.000173203814
14	-0.00187726514	0.0026655375	0.00146193752	-0.0000610893421	-0.00126202335	-0.000458787859
15	-0.00216069733	0.00147185804	0.00022791128	-0.0021485341	0.0000822630097	0.000478005166
16	-0.00226807794	0.00129214283	-0.0000970608746	0.000143359763	$1.09927346 \times 10^{-6}$	0.0000300340072
17	-0.00096819417	-0.000488799805	0.000144426536	-0.00032671668	$-9.92661542 \times 10^{-6}$	$-2.66527939 \times 10^{-6}$
18	-0.000118656413	-0.00018293087	0.0000407362083	-0.000176673501	0.0000515695341	$-9.26540504 \times 10^{-6}$
19	-0.000192492127	0.0000493303754	-0.0000273822824	0.000171666015	$-7.89252523 \times 10^{-6}$	$-6.26345568 \times 10^{-6}$
20	0.000263760246	0.000350170164	0.0000432423872	-0.00012765244	$2.41000428 \times 10^{-6}$	$-2.3749484 \times 10^{-6}$
21	$1.16558991 \times 10^{-6}$	$4.96712883 \times 10^{-6}$			$-5.85227252 \times 10^{-6}$	$-6.79087762 \times 10^{-6}$
22	$-1.63244689 \times 10^{-6}$	$6.47331955 \times 10^{-6}$			$-3.33045158 \times 10^{-6}$	$7.49271783 \times 10^{-6}$
23	$-3.4729084 \times 10^{-6}$	$5.95322586 \times 10^{-6}$			$1.88193258 \times 10^{-6}$	$-3.98305519 \times 10^{-7}$
24	$-4.96630548 \times 10^{-6}$	$1.72968518 \times 10^{-6}$			$-2.36175431 \times 10^{-18}$	$2.51061885 \times 10^{-7}$
25	$5.12755036 \times 10^{-8}$	$2.14945715 \times 10^{-7}$				
26	$-3.07397112 \times 10^{-8}$	$2.19743806 \times 10^{-7}$				
27	$-6.99669598 \times 10^{-8}$	$1.77349366 \times 10^{-7}$				

i	μ_9		μ_{10}	
	a_i	b_i	a_i	b_i
0	0.546107642		0.557982648	
1	0.132184562	0.0832860664	-0.0107226947	-0.00560550488
2	-0.0422564242	-0.081406976	-0.00243459937	0.00272185071
3	-0.00288494405	0.0494657212	-0.106247514	-0.0216645017
4	-0.00990364257	0.0141660798	0.00371146609	-0.00230659129
5	0.044032033	-0.02072678	0.0000381990189	0.00245899729
6	0.0162253823	0.00284538146	0.0299771487	0.017040206
7	-0.00684008357	-0.00453911102	-0.0033437228	0.00348204544
8	0.0113448501	0.0332865286	0.000299992113	-0.00319195161
9	-0.00329408749	0.0181081652	-0.0217413819	-0.0152507074
10	-0.001807493	0.000642929037	0.000428329649	0.00273289835
11	-0.00649512127	0.00171952483	0.000587136389	-0.0000355914864
12	0.00379245547	0.00021410323	0.0000998793688	-0.000710946153
13	-0.000881363531	-0.00148150927	0.00168461816	-0.00141128391
14	-0.00288415573	-0.0106440172	-0.000916519918	0.000996942438
15	0.000669513161	-0.0000464470469	0.00151666691	0.00459881141
16	0.00163822411	-0.000542216743	-0.00150433091	0.000444760945
17	0.0087891035	-0.00109922112	0.00109118785	-0.000873399337
18	0.000888529246	0.000976724264	-0.00103745245	-0.00391413716
19	0.000561401536	0.00191176013	0.000136745573	0.000451314701
20	0.000172718181	0.00178646218	-0.000328686271	0.0000923832995
21	-0.0000607718052	-0.00115913985	0.0000607676685	0.000247990695
22	-0.00224226772	0.000812563001	0.000109497006	-0.0000372289029
23	-0.0026768554	0.0000114857432	-0.0000756102401	0.0000118837984
24	0.00107005965	-0.000253133326	-0.000067632927	0.000143418041