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Singularity in Darcy Flow Around a Cutoff Wall

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Abstract

The singularity problem in Darcy flow around cutoff walls is examined herein and a complete integral formulation is derived, which may be regarded as the fundamental equations of a Darcy or a potential flow field in a domain bounded by regular boundaries and/or boundaries of geometry degeneracy which enclose no area or volume. The paper further explores the adoption of the resulting formulation to the boundary element method and thereby solves a long-standing abstrusity of the boundary element method as is applied to the cutoff wall problem.

INTRODUCTION

In engineering practice , sheet piles or cutoff walls are often constructed to detour flows. For example, seepage in porous media such as soil underneath a dam can be blocked and dramatically cut down if a sheet pile wall is built from the dam to some depth. The flow velocity at the tip of the wall is known to go to infinity. This introduces singularity in the flow field, which may pose difficulties in the solution schemes, especially for problems of practical importance where the geometrical shapes of the ranges of the flow fields under consideration are more than often arbitrary and which inevitably indicate numerical treatment.

This paper first gives a complete formulation for Darcy flow in a finite or infinite domain where cutoff walls may be present. The formulation consists of a pair of dual boundary integral equations, which may be solved analytically or numerically. Introducing the concept of finite elements into the boundary integral equations yields the famous boundary element method (BEM). It is noted that although the BEM is well developed in the study of potential flows for simply-connected domains [1], it is no longer the case once a cutoff wall is installed in the interior of the domain. A cutoff wall is represented by a boundary of geometry degeneracy which encloses no area or volume and possesses sharp edges or borders, where singularity is expected. For problems of this nature there are no complete formulations existing and no BEM's which discretize the boundary only (without adding artificial boundaries to divide the domain into zones) are available. The paper by Lafe et al.[2] fully recognized the difficulty, reading that arrangement which places the boundary nodes on each side of the wall in the same coordinate results in a coefficient matrix that is singular. Even if a trick is played that the nodes of one side of the wall are intentionally placed separately from the nodes of the other side of the wall, it was found still of little help. In fact, the difficulty is indeed rooted in a fundamental point that the formulation devised for a problem without degenerate boundaries does not suffice to solve a problem with degenerate boundaries. In other words, the formulation must be always complete in order to secure a unique solution. The point is sharpened furthermore if we recognize that on the cutoff wall the number of the unknown is doubled and the number of the prescribed boundary conditions is also doubled, therefore it is obvious that the number of independent equations should be doubled in order to accomodate the increasing known boundary data and to secure a unique solution. The present paper not only derives a complete formulation of the so called dual boundary integral equations [3,4,5] but also explores the problem of singularity and its BEM scheme. It is seen that physical problem investigated leads naturally to principal values of singular integrals. The alleged super-singularity can thus be explained by the concept of Hadamard principal value [6]; although Hadamard [7] didn't treat exactly the same problem; similarities do exist, however.

DARCY FLOW AND CUTOFF WALL

From Darcy's law :

$$v_i(x) = -k_{ij} \frac{\partial \phi(x)}{\partial x_j}$$

and the continuity equation of incompressible fluids :

$$\frac{\partial v_i}{\partial x_i} = 0$$

the potential $\emptyset$ is governed by

$$k_{ij} \frac{\partial^2 \phi}{\partial x_i \partial x_j} = 0$$

where the Einstein's indicial summation convention is employed and kij denotes the permeability tensor of the anisotropic porous medium considered. If the coordinate axes are adjusted to be directed along the principal axes of kij and an appropriate scaling is employed for the coordinate or if the medium is isotropic in permeability, the above governing equation becomes

$$\nabla^2 \phi = 0$$

where ∇^2 denotes the Laplacian operator. Consider the problem of Darcy flow permeating through a simply-connected region D (Fig.1), which is bounded by regular boundaries S (S may be the infinite S_∞), and in which there may exist degenerate boundaries $C = C^+ + C^-$ of geometry degeneracy such as sheetpiles, cutoff walls, or thin foils or slits. The regular boundaries S and the degenerate boundaries C comprise the total boundaries B = S + C of the domain D. An appropriate boundary condition must be prescribed on the boundaries $S + C^+ + C^- = B$, over one portion S1 of which the boundary potential is prescribed (Dirichlet type), over another portion S2 the velocity normal to the boundary is known (Neumann type), and over the remainder S3 where the boundary is absorbent with/without distributed (line or surface)sources and sinks a boundary condition of Robin type is specified. It is noted that on the degenerate boundaries $C = C^+ + C^-$, boundary conditions should be imposed on both C^+ and C^-, and the conditions imposed can be of different types.

INTEGRAL EQUATIONS

For the aforementioned problem a boundary intergral equation can be obtained from Green's second identity as

$$2\pi\emptyset(x) = \int_B T(s,x)\emptyset(s)dB(s) - \int_B U(s,x) \frac{\partial\phi(s)}{\partial n} dB(s), \qquad (1)$$

where $\emptyset(x)$ is the potential of a point x in the domain D bounded by the boundaries B and, for the two-dimensional case,

$$U(s,x) = \ln(r),$$

$$T(s,x) = \frac{\partial}{\partial n_s}(\ln(r)),$$

in which r is the distance between the points s and x, and ns is the outer unit normal to the boundary at the point s. After applying a normal differentiation to the above equation, the other equation emerges :

$$2\pi \frac{\partial\phi(x)}{\partial n} = \int_B M(s,x)\phi(s)dB(s) - \int_B L(s,x) \frac{\partial\phi(s)}{\partial n} dB(s), \qquad (2)$$

where, for the two-dimensional case,

$$L(s,x) = \frac{\partial}{\partial n_x}(U(s,x)) = \frac{\partial}{\partial n_x}(\ln(r)),$$

$$M(s,x) = \frac{\partial}{\partial n_x}(T(s,x)) = \frac{\partial^2}{\partial n_x \partial n_s}(\ln(r)),$$

in which nx is the outer unit normal to the boundary at the point x. Eqs. 1 and 2 are called herein the dual integral equations for the domain point x.

In order to get a compatible relationship for the boundary unknowns, the point x of Eqs. 1 and 2 have to be on the boundary. This might induce the problem of singularity. Analogous to the treatment of complex contour integral involving poles, the boundary is detoured circularly or spherically around the point x of singularity and then shrunk all the way back to the point. In this way the strong singularity which arises in the first part of the right-hand side of Eq. 1, as well as in the second part of the right-hand side of Eq. 2, as x approaches the boundary, leads to a sum of an integral interpreted as its Cauchy principal value, which is finite, and a jump term. Whereas the super-singularity which arises in the first part of the right-hand side of Eq.2 as x tends to the boundary leads to an integral interpreted as its Hadamard principal value, which is finite, too. When one pushes the point x to the boundary, one indeed performs an operation of extention which extends or maps the function defined on the interior of the domain D onto a function defined on the boundary. The operator is sometimes called the trace operator (of order 0). Accordingly Eqs. 1 and 2 become

$$\alpha\,\phi(x) = ⨍_B T(s,x)\phi(s)dB(s) - \int_B U(s,x) \frac{\partial\phi(s)}{\partial n} dB(s), \qquad (3)$$

$$\alpha \frac{\partial\phi(x)}{\partial n} = ⨍_B M(s,x)\phi(s)dB(s) - \int_B L(s,x) \frac{\partial\phi(s)}{\partial n} dB(s), \qquad (4)$$

upon noting the property that the singular integral

$$\int_B \frac{\partial}{\partial n_s}(U(s,x)) + \frac{\partial}{\partial n_x}(U(s,x))\, dB(s) \qquad (5)$$

is continuous when x is across the boundary and the continuity property of the normal derivative

$$-\frac{\partial}{\partial n_x}\int_B T(s,x)\ \phi(s)\ dB(s) \qquad (6)$$

of the double layer potential [8]. In Eqs. 3 and 4 $\alpha(x)$ measures the solid angle at x, and $\fint$ and $\fint$ denote that the integrals are to be interpreted as their Cauchy and Hadamard principal values, respectively. It is worthwhile noting that Eq. 4 can alternatively be derived from Eq. 3 by a normal derivative operation. This suggests commutativity of the two operators, normal derivative and trace.

Equations 3 and 4 are called the dual boundary intergral equations, which give a complete formulation of relations of compatibility of the boundary potential and the normal derivative of the potential at the boundaries. For a simply-connected domain which has no degenerate boundaries, either Eq. 3 or Eq. 4 individually suffices to solve the problem posed. For illustration purposes we shall give a numerical example later on to make the point clear. On the other hand, if there exist degenerate boundaries $C = C^+ + C^-$, both equations 3 and 4 must be used to secure a unique solution.

FORMULATION FOR CUTOFF WALLS

In this section we are going to derive a formulation explicitly considering the presence of cutoff walls. Separating each of the boundary integrals of Eqs. 1 and 2 into three portions according to $B = S + C^+ + C^-$, and noting that the normal of C^+ is opposite to that of C^- and letting C^+ and C^- be thought to be one with the same normal of C^+, we have

$$2\pi\phi(x) = \int_S T(s,x)\phi(s)\ dB(s) - \int_S U(s,x)\frac{\partial\phi(s)}{\partial n}\ dB(s)$$
$$+\int_C T(s,x)\,\Delta\phi(s)\ dB(s) - \int_C U(s,x)\,\Sigma\frac{\partial\phi(s)}{\partial n}\ dB(s) \qquad (7)$$

$$2\pi\frac{\partial\phi}{\partial n} = \int_S M(s,x)\phi(s)\ dB(s) - \int_S L(s,x)\frac{\partial\phi(s)}{\partial n}\ dB(s)$$
$$+\int_C M(s,x)\,\Delta\phi(s)\ dB(s) - \int_C L(s,x)\,\Sigma\frac{\partial\phi(s)}{\partial n}\ dB(s) \qquad (8)$$

where

$$\Sigma\phi = \phi(x_{c^+}) + \phi(x_{c^-})$$
$$\Delta\phi = \phi(x_{c^+}) - \phi(x_{c^-})$$
$$\Sigma\frac{\partial\phi}{\partial n} = \left.\frac{\partial\phi}{\partial n}\right|_{x_{c^+}} + \left.\frac{\partial\phi}{\partial n}\right|_{x_{c^-}}$$
$$\Delta\frac{\partial\phi}{\partial n} = \left.\frac{\partial\phi}{\partial n}\right|_{x_c^+} - \left.\frac{\partial\phi}{\partial n}\right|_{x_c^-}$$

Pushing x to the regular boundary S and recalling the properties of the expressions 5 and 6, Eqs. 7 and 8 turn out to be

$$\alpha\, \emptyset(x) = \fint_S T(s,x)\emptyset(s)\ dB(s) - \int_S U(s,x)\frac{\partial \phi(s)}{\partial n}\ dB(s)$$
$$+\int_C T(s,x)\,\Delta\emptyset(s)\ dB(s) - \int_C U(s,x)\,\Sigma\frac{\partial \phi(s)}{\partial n} dB(s) \quad (9)$$
$$\alpha\,\frac{\partial \phi}{\partial n} = \mathop{=\!\!\!\!\!\!\int}_S M(s,x)\emptyset(s)\ dB(s) - \fint L(s,x)\frac{\partial \phi(s)}{\partial n}\ dB(s)$$
$$+\int_C M(s,x)\ \Delta\emptyset(s)\ dB(s) - \int_C L(s,x)\,\Sigma\frac{\partial \phi(s)}{\partial n} dB(s) \quad (10)$$

By the same token, but the point x approaches the degenerate boundary C, Eqs. 7 and 8 instead become

$$\pi\,\Sigma\emptyset(x) = \int_S T(s,x)\emptyset(s)\ dB(s) - \int_S U(s,x)\frac{\partial \phi(s)}{\partial n} dB(s)$$
$$+\fint_C T(s,x)\,\Delta\emptyset(s)\ dB(s) - \int_C U(s,x)\,\Sigma\frac{\partial \phi(s)}{\partial n} dB(s) \quad (11)$$
$$\pi\,\Delta\frac{\partial \phi}{\partial n} = \int_S M(s,x)\emptyset(s)\ dB(s) - \int_S L(s,x)\frac{\partial \phi(s)}{\partial n}\ dB(s)$$
$$+\mathop{=\!\!\!\!\!\!\int}_C M(s,x)\,\Delta\emptyset(s)\ dB(s) - \fint_C L(s,x)\,\Sigma\frac{\partial \phi(s)}{\partial n}\ dB(s) \quad (12)$$

if C is smooth at x. Using Eqs. 9, 11 and 12, or Eqs. 10, 11 and 12, we have enough independent equations to solve the unknowns of $\emptyset$ (resp. $\frac{\partial\phi}{\partial n}$) on S, and $\Sigma\emptyset$ and $\Delta\phi$ (resp. $\Sigma\frac{\partial\phi}{\partial n}$ and $\Delta\frac{\partial\phi}{\partial n}$) on C.

HADAMARD PRINCIPAL VALUE

To evaluate the Hadamard principal value

$$\mathop{=\!\!\!\!\!\!\int} M(s,x)\ \emptyset(s)\ dB(s) = \frac{\partial}{\partial n_x}\fint T(s,x)\ \emptyset(s)\ dB(s) \quad (13)$$

of the integrals in Eqs. 4, 10 and 12, we note that the explicit forms of M(s,x) and L(s,x) are, if $r_i = x_i - s_i$ and $r = (r_i r_i)^{1/2}$,

$$M(s,x) = \frac{2\, r_i r_j\, n_i(s) n_j(x)}{r^4} - \frac{n_i(s) n_i(x)}{r^2}, \quad (14)$$

$$T(s,x) = \frac{-r_i n_i(s)}{r^2}, \quad (15)$$

which simplify to

$$M(s,x) = \frac{-1}{(s-x)^2},$$

$$T(s,x) = 0,$$

if the boundary is taken to be straight, running from -1 to 1. Since the integration path is detoured circularly at the neighborhood of x and that of the integral of the right-hand side of Eq. 13 excludes the circular arc part, the nx of the right-hand side of Eq. 13 is therefore the normal to the very beginning of the arc, which is just the direction x. (Fig. 2). Having this and Eqs. 14 and 15 into Eq. 13 we have

$$\int_{-\ell}^{\ell} \frac{\phi(s)}{(x-s)^2}\, ds = \frac{d}{dx} \int_{-\ell}^{\ell} \frac{\phi(s)}{s-x}\, ds \qquad (16)$$

which using Leibnitz' rule becomes

$$\lim_{\varepsilon \to 0} \left\{ \frac{-2\phi(x)}{\varepsilon} + \left[\int_{\ell}^{x-\varepsilon} + \int_{x+\varepsilon}^{\ell} \right] \frac{\phi(s)}{(x-s)^2}\, ds \right\} \qquad (17)$$

It is understood that in the above the large constant $\frac{-2\phi(x)}{\varepsilon}$ cancels the large contributions from the integral near s = x [9]. A further manipulation of Eq. 17 is possible by integration by parts, giving formally

$$\int_{-\ell}^{\ell} \frac{\phi'(s)}{s-x}\, ds + \frac{\phi(\ell)}{x-\ell} - \frac{\phi(-\ell)}{x+\ell}$$

which is to exist if $-1 < x < 1$ and $\phi'(s)$ is Holder-Lipschitzian.

UNIFORM FLOW

For illustration purposes, a Darcy flow in a rectangular region D, as shown in Fig. 3, of two units long and one unit wide is considered. The upper and the lower boundaries are solid walls such that the normal flux is zero; that is, $\frac{\partial\phi}{\partial n} = 0$ on S1 ($x2 = -0.5$, $-1 < x1 < 1$ and $x2 = 0.5$, $-1 < x1 < 1$). Along the vertical boundaries S2 the potentials are prescribed as $\phi = -1$ on $x1 = -1$, $-0.5 < x2 < 0.5$ and as $\phi = 1$ on $x1 = 1$, $-0.5 < x2 < 0.5$. Note that B = S = S1 + S2 and that there is no absorbent boundary S3 and no degenerate boundary in the present example. The exact solution of this simple problem is $\phi(x1, x2) = x1$. In the present work the boundary integrals are discretized into superparametric constant elements and the collocation points are at the centers of the elements. As mentioned earlier this problem can be solved by either Eq. 3 or Eq. 4 alone. Table 1 shows a comparison between the numerical results of Eq. 3 and those of Eq. 4. Twenty four elements of 1/4 unit long were used . It is observed that the symmetry about the x1 axis and the anti-symmetry about the x2 axis were conserved to a great extent. The errors of the numerical solutions at the nodes relative to the exact values were acceptable in view of the crude discretization used.

CUTOFF WALL PROBLEM

Consider a problem as in Fig. 4 which is almost identical to the previous problem of the last section except that a cutoff wall of half a unit long is built from the mid-point of the upper solid wall vertically down to the center of the rectangular region. The cutoff wall is assumed to be impermeable. Using 30 elements of 1/5 unit long on the regular boundaries and 5 elements of 1/10 unit long on the degenerate boundary

(cutoff wall), we obtained the numerical results of boundary unknowns by utilizing Eqs. 9, 10 and 12, whereas we also calculated from 10, 11 and 12. Then the domain potential and the domain flux can be determined from Eqs. 7 and 8, respectively. Fig. 5 shows the flux $-\frac{\partial\phi}{\partial n}$ along the x1 direction at x1 = 0 from the tip of the cutoff vertically down to the mid-point of the lower solid wall. For comparison a result obtained from a numerical scheme of the Schwartz-Christoffel transformation [2] is also shown in the figure. Note that in the vicinity of the tip where singularity is expected and most schemes suffer much the agreement between the present solutions and the Schwartz-Christoffel transformation solution is remarkable.

CONCLUDING REMARK

The dual boundary integral equations have been derived to give a complete description of the relation of the boundary potential and the normal flux at the boundaries.The singularity problem has been solved by a careful derivation which leads naturally to a convergent formulation. The formulation can be written in more compact form if the notions and notations of Cauchy and Hadamard principal values are taken advantage of, as have been done in the present work. The adoption of the derived dual equations to the boundary element method has resulted in a powerful numerical scheme suitable for solution of a wide class of problems (see Fig. 1). For illustration we have presented two numerical examples, the results of which were found encouraging.

REFERENCE

1. Liggett, J. A. and Liu, P. L-F. The Boundary Integral Equation Method for Porous Media, Allen & Unwin, 1983.
2. Lafe, Q. E., etc Singularity in Darcy Flow Through Porous Media, J. of ASCE, Hydraulic Division, Vol.106, No.HY6, 1980.
3. Hong, H. K. and Chen, J. T. Derivation of Integral Equations in Elasticity, J. of ASCE, Eng. Mech. Div. Vol.114, No.6, pp. 1028-1044,1988.
4. Chen, J. T. and Hong, H. K. Application of Integral Equations with Superstrong Singularity to Steady State Heat Conduction, pp. 133-138, Proceeding of 9th ICTA Congress, Jerusalem,Israel, 1988, Elsevier.
5. Hong, H. K. and Chen J. T. Generality and Special Cases of Dual Integral Equations of Elasticity, J. of the Chinese Society of Mechanical Engineers, Vol.9, No.1, pp. 1-9, 1988.
6. Chen, J. T. On Hadamard Principal Value and Boundary Integral Formulation of Fracture Mechanics, Masters Thesis of Institute of Applied Mechanics, National

Taiwan University, Taiwan, 1986.
7. Hadamard, J. Lectures on Cauchy Problems in Linear Partial Differential Equations, Dover, 1952.
8. Gunter, N. M. Potential Theory and Its Applications to Basic Problems of Mathematical Physics, Frederick Ungar, 1967.
9. Tuck, E. O. Application and Solution of Cauchy Integral Equations, in " The Application and Numerical Solution of Integral Equations, (Ed. Anderson, R. S. et al), Sijthoff & Noordhoff, 1980.

TABLE 1. SOLUTION TO UNIFORM FLOW

Position of the node	Exact solution	numerical solution (error %) Solved by Eq. 3	Solved by Eq. 4
		$\partial\phi/\partial n$	
(1,0.375)	1	1.047671 (4.77)	1.081877 (8.19)
(1,0.125)	1	0.979321 (2.07)	1.013894 (1.39)
(1,-0.125)	1	0.979323 (2.07)	1.013893 (1.39)
(1,-0.375)	1	1.047670 (4.77)	1.081877 (8.19)
(-1,0.375)	-1	-1.047672 (4.77)	-1.081877 (8.19)
(-1,0.125)	-1	-0.979321 (2.07)	-1.013893 (1.39)
(-1,-0.125)	-1	-0.979323 (2.07)	-1.013893 (1.39)
(-1,-0.375)	-1	-1.047671 (4.77)	-1.081876 (8.19)
		ϕ	
(0.125,1)	0.125	0.125768 (0.62)	0.124220 (0.62)
(0.375,1)	0.375	0.377344 (0.63)	0.372394 (0.69)
(0.625,1)	0.625	0.629172 (0.67)	0.619416 (0.89)
(0.875,1)	0.875	0.882124 (0.81)	0.861658 (1.52)
(-0.125,1)	-0.125	-0.125768 (0.62)	-0.124220 (0.62)
(-0.375,1)	-0.375	-0.377344 (0.63)	-0.372394 (0.69)
(-0.625,1)	-0.625	-0.629172 (0.67)	-0.619416 (0.89)
(-0.875,1)	-0.875	-0.882124 (0.81)	-0.861658 (1.52)
(-0.125,-1)	-0.125	-0.125768 (0.62)	-0.124220 (0.62)
(-0.375,-1)	-0.375	-0.377344 (0.63)	-0.372394 (0.69)
(-0.625,-1)	-0.625	-0.629172 (0.67)	-0.619416 (0.89)
(-0.875,-1)	-0.875	-0.882124 (0.81)	-0.861658 (1.52)
(0.125,-1)	0.125	0.125768 (0.62)	0.124220 (0.62)
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(0.625,-1)	0.625	0.629172 (0.67)	0.619416 (0.89)
(0.875,-1)	0.875	0.882124 (0.81)	0.861658 (1.52)

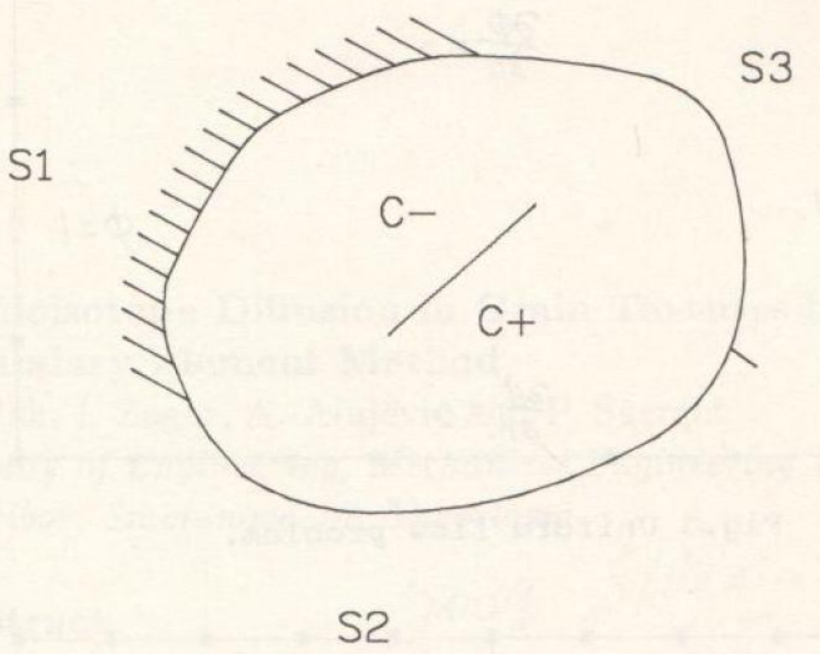

Fig.1 General Darcy Flow Problem.

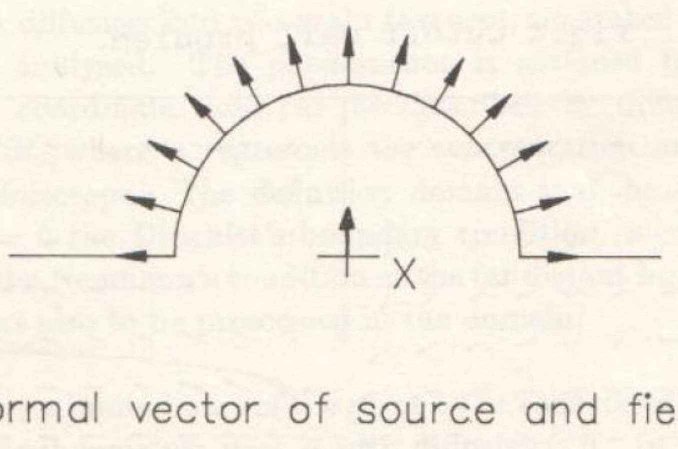

Fig.2 Normal vector of source and field point.

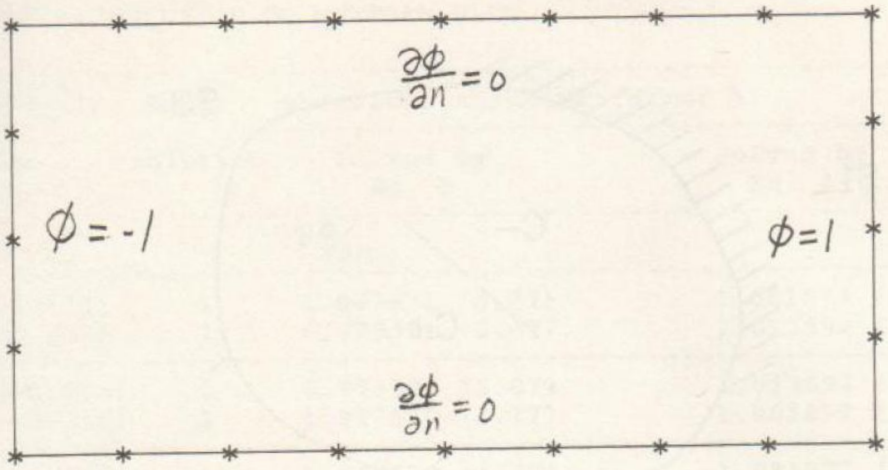

Fig.3 Uniform flow problem.

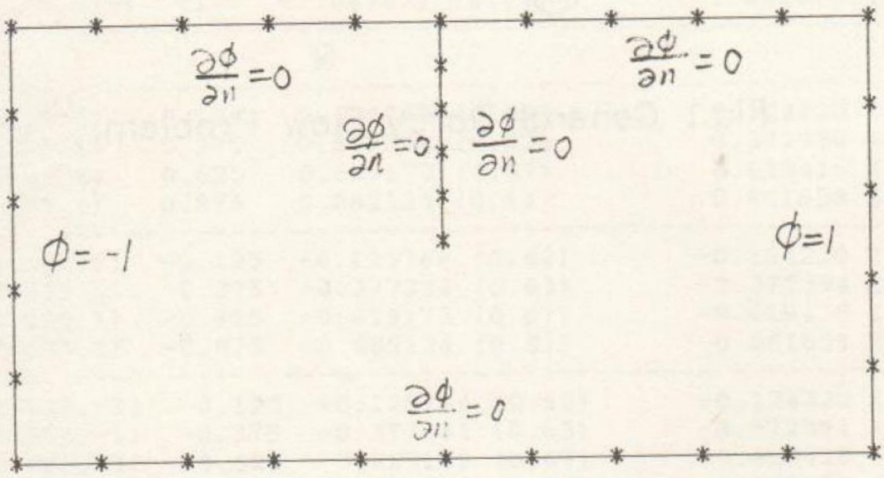

Fig.4 Cutoff wall problem.

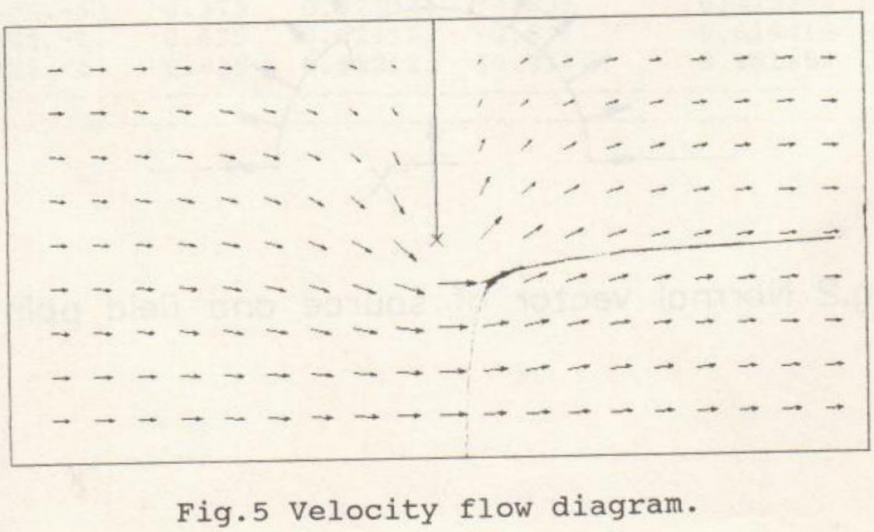

Fig.5 Velocity flow diagram.