An Alternative BEM for Fracture Mechanics

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Abstract. An alternative single domain boundary element formulation and its numerical implementation are presented for the analysis of two-dimensional cracked bodies. The problem is formulated employing the classical displacement boundary integral representation and a novel integral equation based on the stress or Airy's function. This integral equation written on the crack provides the relations needed to determine the problem solution in the framework of linear elastic fracture mechanics. Results are presented for typical problems in terms of stress intensity factors and they show the accuracy and efficiency of the approach.

The Boundary Element Method for Fracture Mechanics

The Somigliana identity is the fundamental relation giving the boundary integral representation of the elastic response in the elastic domain Ω having contour Γ. Indeed it links the displacements at the point P_0 to the displacements $\mathbf{u}$ and tractions $\mathbf{p}$ on the boundary through a fictitious elastic system due to a concentrated body force acting at the point P_0. Denoting by $\mathbf{u}_j$ and $\mathbf{p}_j$ the displacements and tractions of the fictitious system, respectively, The Somigliana identity is written [3, 4]

$$\mathbf{u}(P_0) = \int_\Gamma \left(\mathbf{u}_j^T \mathbf{p} - \mathbf{p}_j^T \mathbf{u}\right) d\Gamma + \int_\Omega \mathbf{u}_j^T \mathbf{f}\, d\Omega \tag{1}$$

where f are the body forces applied in the domain. The eq (1) is the boundary integral representation of the displacement field inside the continuum Ω. If the point P_0 belongs to the boundary Γ, by a suitable limit procedure [3], one obtains the boundary integral equation which, taking the prescribed boundary conditions into account, allows the solution of the elastic problem in terms of displacements and tractions on the boundary [3, 4]. One has

$$\mathbf{c}\,\mathbf{u}(P_0) = \int_\Gamma \left(\mathbf{u}_j^T \mathbf{p} - \mathbf{p}_j^T \mathbf{u}\right) d\Gamma + \int_\Omega \mathbf{u}_j^T \mathbf{f}\, d\Omega \tag{2}$$

where the coefficient matrix $\mathbf{c}$ is given by

$$\mathbf{c} = -\int_\Gamma \mathbf{p}_j^T\, d\Gamma \tag{3}$$

The eq (2) is the basis for the numerical solution of the problem by the Boundary Element Method; However, in the framework of Fracture Mechanics when cracks are located in the domain, the eq (2) needs to be revised by taking the unknowns relative displacements along the cracks into account; the eq (2) becomes [5, 6, 12]

$$\mathbf{c}\,\mathbf{u}(P_0) = \int_\Gamma \left(\mathbf{u}_j^T \mathbf{p} - \mathbf{p}_j^T \mathbf{u}\right) d\Gamma - \int_{\Gamma_f} \mathbf{p}_j^T \Delta\mathbf{u}\, d\Gamma + \int_\Omega \mathbf{u}_j^T \mathbf{f}\, d\Omega \tag{4}$$

where Γ_f is the boundary representative of the crack and $\Delta\mathbf{u}$ are the relative displacements along it. It straight away appears that eq (4) in its numerical application originates a system with more unknowns than equations. To overcome this drawback many approaches have been proposed among which there are the Green's function, the multidomain method and the Dual Boundary Element Method (DBEM) [6]. The Green's function method even is very accurate is limited to very simple problems [12], whereas the other two approaches are general and therefore they are the most employed. The multidomain method requires a partition of the investigated domain into suitable subregions so that each face of the cracks belongs to the boundary of distinct subregions. Restoring the continuity conditions between the considered subregions the number of integral equations written is equal to the number of unknowns and the problem can be modelled without limitations [8, 9]. Nevertheless the resolving system arising from the multidomain approach has higher order than that strictly needed to solve the problem with the consequent higher computational effort required. On the other hand the Dual Boundary Element Method (DBEM) does not require any partition of the investigated domain [10]; it recovers the further equations for the problem solution by expressing the tractions acting on the crack faces by means of the relative boundary integral representations. The main difficulty of this single domain approach is due to the hypersingular kernels occurring in the traction integral equation which need particular care in their numerical integration [11].

Stress function approach

For an homogeneous, isotropic two-dimensional body the stress field can be derived from a single function, the so-called stress function or Airy function $\Phi = \Phi(x, y)$, so that the equilibrium equations are trivially fulfilled; in the absence of body forces one has [13]

$$\boldsymbol{\sigma} = \begin{bmatrix} \sigma_{xx} & \sigma_{yy} & \sigma_{xy} \end{bmatrix}^T = \mathbf{C}\Phi \tag{5}$$

where

$$\mathbf{C}^T = \begin{bmatrix} \dfrac{\partial^2}{\partial y^2} & \dfrac{\partial^2}{\partial x^2} & -\dfrac{\partial^2}{\partial x \partial y} \end{bmatrix} \tag{6}$$

Besides, the compatibility conditions requires that the stress function Φ satisfies the following governing equation

$$\mathbf{C}^T \mathbf{E}^{-1} \mathbf{C}\, \Phi = 0 \tag{7}$$

where **E** denotes the elasticity matrix. From eq (7) one deduces that the stress function Φ is biharmonic. Once the stress function is introduced the displacement field can be expressed by

$$\mathbf{u} = \mathbf{v} - \frac{1}{2G}\mathbf{S}\,\Phi \tag{8}$$

where $\mathbf{S} = \begin{bmatrix} \partial/\partial x & \partial/\partial y \end{bmatrix}^T$ is the gradient operator; G is the shear modulus and **v** is a vector, whose components v_1 and v_2 are conjugate harmonic functions. The boundary tractions **p** are expressed by the following relationship

$$\mathbf{p} = \frac{\partial}{\partial s}\mathbf{H}\mathbf{S}\,\Phi \tag{9}$$

where $\partial/\partial s$ indicates the tangent derivative, whereas the matrix **H** is defined by

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \tag{10}$$

In terms of stress function the integral equation (4) becomes

$$\mathbf{c}\,\mathbf{v}(P_0) - \frac{1}{2G}\mathbf{c}\mathbf{S}\,\Phi(P_0) = \int_\Gamma \left(\mathbf{u}_j^T \mathbf{p} - \mathbf{p}_j^T \mathbf{u}\right) d\Gamma - \int_{\Gamma_f} \mathbf{p}_j^T\, \Delta\mathbf{u}\, d\Gamma \tag{11}$$

Given that the components of **v** are harmonic functions, applying the Green theorem one has

$$\mathbf{c}\,\mathbf{v}(P_0) = \int_\Gamma \left(\varphi \frac{\partial \mathbf{v}}{\partial n} - \frac{\partial \varphi}{\partial n}\mathbf{v} \right) d\Gamma \tag{12}$$

where

$$\varphi = \ln r(P, P_0) \tag{13}$$

and $r(P, P_0)$ is the distance between the domain point P and the point P_0. Remembering that v_1 and v_2 are conjugate and taking eq (8) into account, eq (12) becomes

$$\mathbf{c}\,\mathbf{v}(P_0) = \int_\Gamma \left(\mathbf{u}^*\,\mathbf{p} - \mathbf{p}^*\,\mathbf{u}\right) d\Gamma - \frac{1}{2G}\int_\Gamma \mathbf{S}\Phi \frac{\partial \varphi}{\partial n} d\Gamma \tag{14}$$

where

$$\mathbf{u}^* = \frac{1}{2G}\begin{bmatrix} \varphi & 0 \\ 0 & \varphi \end{bmatrix} \tag{15}$$

$$\mathbf{p}^* = \begin{bmatrix} \dfrac{\partial \varphi}{\partial n} & -\dfrac{\partial \varphi}{\partial s} \\ \dfrac{\partial \varphi}{\partial s} & \dfrac{\partial \varphi}{\partial n} \end{bmatrix} \tag{16}$$

Finally, by using eq (14), the integral equation (11) is written as

$$\begin{aligned} \frac{1}{2G}\mathbf{c}\mathbf{S}\Phi(P_0) = {} & \int_\Gamma \left(\mathbf{p}_j^T\,\mathbf{u} - \mathbf{u}_j^T\,\mathbf{p}\right) d\Gamma + \int_{\Gamma_f} \mathbf{p}_j^T\,\Delta\mathbf{u}\,d\Gamma + \\ & + \int_\Gamma \left(\mathbf{u}^*\,\mathbf{p} - \mathbf{p}^*\,\mathbf{u}\right) d\Gamma - \int_{\Gamma_f} \mathbf{p}^*\,\Delta\mathbf{u}\,d\Gamma - \frac{1}{2G}\int_\Gamma \mathbf{S}\Phi \frac{\partial \varphi}{\partial n} d\Gamma \end{aligned} \tag{17}$$

Recalling that the components of the resultant of the tractions applied between the point P_0 and a generic point P_A are defined as

$$\mathbf{R} = \int_{P_0}^{P_A} \mathbf{p}\,d\Gamma \tag{18}$$

by integration of the eq (9) one obtains

$$\mathbf{S}\Phi = \mathbf{H}^{-1}\,\mathbf{R} + \mathbf{k} \tag{19}$$

where **k** is a vector whose components are arbitrary constants. For a point P_0 belonging to the crack line the integral equation (17) becomes

$$\begin{aligned} \frac{1}{G}\mathbf{c}\,\mathbf{k} = {} & \int_\Gamma \left(\mathbf{p}_j^T\,\mathbf{u} - \mathbf{u}_j^T\,\mathbf{p}\right) d\Gamma + \int_{\Gamma_f} \mathbf{p}_j^T\,\Delta\mathbf{u}\,d\Gamma + \\ & + \int_\Gamma \left(\mathbf{u}^*\,\mathbf{p} - \mathbf{p}^*\,\mathbf{u}\right) d\Gamma - \int_{\Gamma_f} \mathbf{p}^*\,\Delta\mathbf{u}\,d\Gamma - \frac{\mathbf{H}^{-1}}{2G}\int_\Gamma \mathbf{R} \frac{\partial \varphi}{\partial n} d\Gamma \end{aligned} \tag{20}$$

This equation allows the problem solution through the boundary element method. After the discretization by boundary elements of the boundaries Γ and Γ_f [5], one obtains the resolving system by collocating the Eq. (4) at the nodes on the boundary Γ and the Eq. (20) at the points belonging to Γ_f. Once the displacements **u** and the tractions **p** on the boundary Γ and the relative displacements $\Delta\mathbf{u}$ along Γ_f are determined, the Fracture Mechanics parameters, specifically the stress intensity factor, can be calculated by standard procedures [5, 8, 14].

Applications

To validate the proposed approach and prove its efficacy and potentiality some numerical results are presented for classical Fracture Mechanics problems. The first application deals with the computation of the stress intensity factors for a crack of length $2a$ embedded in an infinite domain. The results for horizontal and 45° inclined crack are shown in Table 1 where the comparison with the analytical solution is also presented. This comparison clearly shows the accuracy of the proposed approach to compute the stress intensity factor. In the second example a rectangular panel having $h/w = 2$ with a central crack inclined of 45° is analysed. The results obtained in terms of stress intensity factor for different crack length are given in Tables 2 and 3. Again the comparison of the present results with those found in the literature shows the accuracy and efficiency of the proposed method. Finally a finite rectangular plate $h/w = 0.5$ with an edge crack of length a has been analyzed. The calculated stress intensity factors are given in Table 4. Once again the comparison between the present results and those found in the literature confirms the soundness of the method for both its accuracy and efficiency.

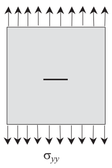
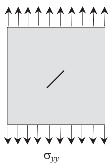
	Stress Intensity Factor (SIF)		
	SIF	Present	Analytic
	$K_I/\sigma_{yy}\sqrt{\pi a}$	1.00	1.00
	$K_I/\sigma_{yy}\sqrt{\pi a}$	0.50	0.50
	$K_{II}/\sigma_{yy}\sqrt{\pi a}$	0.50	0.50

Table 1. SIFs for horizontal and 45° inclined crack in an infinite domain

a/w	Present	Ref. [5]	Ref. [15]
0.2	0.51	0.53	0.52
0.3	0.52	0.55	0.54
0.4	0.56	0.59	0.57
0.5	0.60	0.63	0.61
0.6	0.65	0.69	0.66

Table 2. $K_I/\left(\sigma_{yy}/\sqrt{\pi a}\right)$ for a 45° central crack in a finite rectangular plate with $h/w=2$

a/w	Present	Ref. [5]	Ref. [15]
0.2	0.49	0.52	0.51
0.3	0.50	0.53	0.52
0.4	0.52	0.54	0.53
0.5	0.54	0.56	0.55
0.6	0.56	0.58	0.57

Table 3. $K_{II}/\left(\sigma_{yy}/\sqrt{\pi a}\right)$ for a 45° central crack in a finite rectangular plate with $h/w=2$

a/w	Present	Ref. [5]	Ref. [15]
0.2	1.48	1.57	1.49
0.3	1.86	1.96	1.85
0.4	2.34	2.23	2.32
0.5	3.04	3.27	3.01

Table 4. $K_I \Big/ \left(\sigma_{yy} \big/ \sqrt{\pi a}\right)$ for finite rectangular plate having $h/w = 0.5$ with edge crack.

Conclusions

A single domain boundary element method for two dimensional elastic solids has been presented with the aim of overcoming the computational drawbacks of classical BEM approaches for fracture mechanics. The method rests on the use of additional integral equations deduced in terms of stress function which collocated on the crack provide the relations needed to determine the solution. These integral equations do not involve hypersingular integrals with the resulting simplification in numerical implementation. The numerical results obtained show the accuracy, efficiency and usefulness of the proposed approach to determine the characteristic parameters of fracture mechanics

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