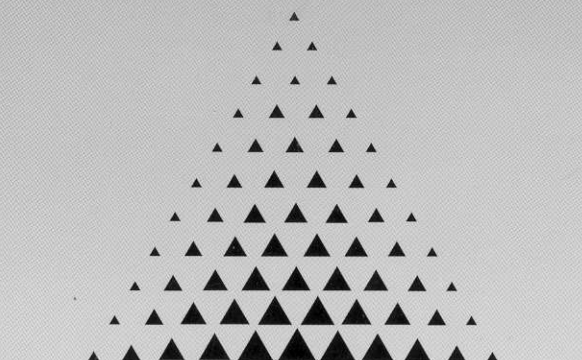
Building Research Journal

Institute of Construction and Architecture
Slovak Academy of Sciences



Volume 52, Number 1, 2004

ISSN 1335–8863 MIČ 49624

Building Research Journal

Volume 52, Number 1, 2004

CONTENTS

Building Research Journal

A scientific journal; aims at:

- providing a publication channel for new developments and achievements in structural engineering and building science,
- wide distribution at affordable subscription rates,
- fast reviewing and publication of manuscripts submitted,
- broad scope for wide participation and dissemination of new research results.

The main topics of the Journal are items of (1) mechanics, (2) structural science, (3) materials, (4) building physics and (5) experimental research in civil engineering.

Focused are theoretical, numerical and experimental problems of structural mechanics, design of metal, concrete and wood structures, the effects of wind, earthquake and wave loadings on structures, fluid- and soil-structure interactions, structural optimization, safety, reliability and control of structures, technology and physicochemical mechanics of new structural materials as well as the problems of daylighting, acoustics and thermal performance of structures.

Submissions of original papers from top scientists, researchers and practising engineers from all countries are particularly encouraged.

Building Research Journal

Editors in Chief

Ján Sládek **Alexander Tesár**

Institute of Constructions and Architecture
Slovak Academy of Sciences
Dúbravská cesta
842 20 Bratislava
SLOVAK REPUBLIC
Tel: 004212-54788662
004212-59309212
Fax: 004212-54773548

Registr. no. 7094

Subscription information

The Building Research Journal is published quarterly. The subscription price for 2004, including postage by surface mail, is US$98 for institutional subscription and US$ 64 for personal subscription. Subscription orders and/or enquiries should be sent to editorial office – Institute of Constructions and Architecture, Dúbravská cesta, 842 20 Bratislava, Slovak Republic.

BUILDING RESEARCH JOURNAL

VOLUME 52, 2004 NUMBER 1

ON INVERTIBILITY OF BOUNDARY INTEGRAL EQUATION SYSTEMS IN LINEAR ELASTICITY

ROMAN VODIČKA[1], VLADISLAV MANTIČ[2]

The existence of various types of non-uniqueness in the solution of elastostatic problems by means of systems formed by both displacement and traction boundary integral equations is presented. Such solutions may appear whenever a rigid-body motion is allowed in the solution of a boundary value problem, or the size of the domain is special giving arise to a phenomenon of critical scale, or the traction equation is used on a cavity boundary. In order to obtain uniquely solvable boundary integral equation systems a technique of the non-uniqueness removal, based on the Fredholm theory of integral operators, is proposed and analyzed. A numerical example presented shows various situations where these non-uniqueness appear and studies applicability of the approach suggested.

Keywords: Boundary Integral Equation, Rigid Body Motion, Solution uniqueness, Domain with cavities, Critical scale, Symmetric Galerkin Boundary Element Method, BIE of the second kind

1 Introduction

The present work deals with the solution of Boundary Value Problems (BVPs) of linear elasticity which may lead to non-uniquely solvable Boundary Integral Equations (BIEs). These non-unique solutions can be naturally related to rigid-body motions (RBM) of the body not restricted by the boundary conditions prescribed — the original BVP has the non-unique solution, too. However, there are some other special configurations when the BVP has a unique solution, while the associated BIE system does not.

One of these particular situations is related to the so-called *critical scale factors* of a region. This kind of problems appears solely in plane BVPs, see (Chen et al., 2002; Constanda, 1994; Heise, 1987; Vodička and Mantič, 2004). It is known, that the exterior

[1] Technical University of Košice, Faculty of Civil Engineering, Department of Mathematics, Vysokoškolská 4, 042 02 Košice, Slovak Republic. e-mail: Roman.Vodicka@tuke.sk

[2] University of Seville, School of Engineering, Group of Elasticity and Strength of Materials, Camino de los Descubrimientos s/n, 41092 Sevilla, Spain, e-mail: mantic@esi.us.es

BVP with Dirichlet boundary conditions is uniquely solvable when a suitable *radiation condition*, i.e. a condition on behaviour of displacements at infinity, is prescribed. Nevertheless, there are one or two scales for each domain in plane BVPs, where the corresponding exterior BVP with homogeneous boundary conditions and a radiation condition has a non-trivial solution. Then the associated BIE is not-invertible. However, not only the BIE associated to the exterior BVP is not-invertible but also that associated to the interior BVP defined on the domain, possibly with cavities, which outer contour is a boundary of the critical domain.

Another kind of non-unique solutions appears, when a system of BIEs, which uses the traction equation (Ugodchicoff and Khutoryanskiy, 1986), is applied. The kernels of the integral operators in the traction equation have some interesting properties, see (Chen and Zhou, 1992). In particular, if this equation is used to solve a BVP on a domain with cavities, then both, the displacements which are prescribed at one cavity boundary by a RBM and by zero on the rest of the boundary, and the tractions in the BVP with these displacements prescribed on the boundary, belong to the nullspaces of the pertinent integral operators generated by the traction equation, depending on the type of prescribed boundary data, see (Greenbaum et al., 1993; Vodička et al., subm.). The system of BIEs, which combines the traction equation with the classical displacement BIE may produce then the non-unique solutions, too. This is also the case of Symmetric Galerkin BEM (SGBEM), as was recently shown in (Pérez-Gavilán and Aliabadi, 2001; Vodička et al., subm.).

The non-invertibility may obviously cause problems in numerical solution. However, a suitable modification of the non-invertible BIE systems may remove this unpleasant property and generate a system with a unique solution. Fredholm theory, applied to boundary integral operators in (Chen and Zhou, 1992), seems to be an excellent framework for development of such techniques. The integral operators may be modified either by augmenting the system by some auxiliary equations or by adding an operator with a degenerated integral kernel. The application of such modification techniques were shown in (Blázquez et al., 1996) with an emphasis on the former approach and in (Vodička et al., subm.) which was focused on the latter one.

The above specified non-unique solutions are studied in the paper. It will be shown how the BIE systems can be modified to obtain uniquely solvable problems. Numerical tests will demonstrate the examples of all aforementioned cases of non-invertible BIE systems. The modifications will be based on adding an operator in such a way that the resulting system will be uniquely solvable.

2 BIE systems

Let us consider an elastic body occupying a region $\Omega \subset \mathbb{R}^D$ $(D = 2, 3)$ with a bounded Lipschitz boundary $\partial\Omega = \Gamma$. An example, rather general, of such a bounded body in 2D is shown in Fig. 1. Let us denote the cavities in the region Ω by symbols Ω_i, $i = 1, \ldots, H$ and their boundaries (i.e. connected subsets of Γ) Γ_i, $i = 1, \ldots, H$, where H denotes the total number of cavities. The sum of all regions Ω_i together with their boundaries Γ_i forms a 'cavitiless' region Ω_0 with a boundary Γ_0. Symbolically we

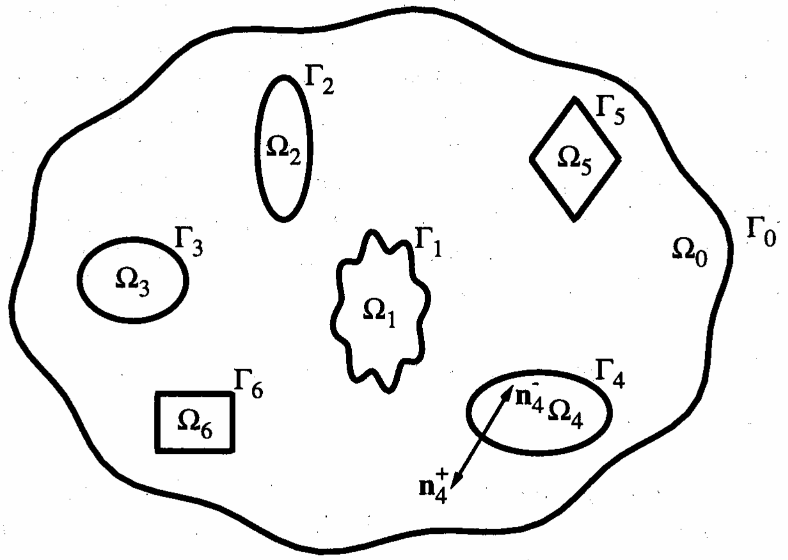


Fig. 1. A general bounded region in a plane.

may write

$$\Omega = \Omega_0 \setminus \left[\bigcup_{i=1}^{H} (\Omega_i \cup \Gamma_i) \right], \quad \Omega_i^+ = \Omega_i,\ \Omega_i^- = \mathbb{R}^D \setminus (\Omega_i \cup \Gamma_i),\ i = 0, \ldots, H, \\ \Gamma_i = \partial\Omega_i,\ i = 0, \ldots, H,\ \Gamma_i = \overline{\Gamma}_{iu} \cup \overline{\Gamma}_{it},\ \Gamma_{iu} \cap \Gamma_{it} = \emptyset. \tag{1}$$

The last relation introduces a boundary split via prescribed boundary conditions of the solved BVP, i.e. prescribed displacements $\boldsymbol{u}$ (Dirichlet boundary conditions)

$$u_k(x) = [\boldsymbol{u}_{iu}]_k(x), \quad x \in \Gamma_{iu}, \quad k = 1, \ldots, D \tag{2u}$$

and given load prescribed in tractions $\boldsymbol{t}$ (Neumann boundary conditions)

$$t_k(x) = [\boldsymbol{t}_{it}]_k(x), \quad x \in \Gamma_{it}. \tag{2t}$$

Two normal vectors are defined on each boundary Γ_i, $\boldsymbol{n}_i^{\pm}$, as the outward unit normal vectors of the regions $\Omega_i^{\pm}$, see also Fig. 1.

The BVP on the region Ω will be solved by two different BIE systems, obtained according to prescribed boundary conditions using the two following BIEs. The first BIE is the Somigliana displacement identity

$$C_{kl}(x)u_l(x) = \int_\Gamma U_{kl}(x,y)t_l(y)\mathrm{d}\Gamma(y) - \fint_\Gamma T_{kl}(x,y)u_l(y)\mathrm{d}\Gamma(y),\ x \in \Gamma, \\ k,l = 1, \ldots, D, \tag{3u}$$

where C_{kl} is a known free term matrix (Mantič, 1993), U_{kl} is a fundamental solution of the Navier equation and T_{kl} are pertinent fundamental tractions, i.e. $\boldsymbol{T}(x,y) = \mathcal{T}_{\boldsymbol{n}(y)}\boldsymbol{U}(x,y)$ and $\mathcal{T}_{\boldsymbol{n}(y)}$ is an operator relating tractions $\boldsymbol{t}$ with the displacements $\boldsymbol{u}$: $\boldsymbol{t}(x) = \mathcal{T}_{\boldsymbol{n}(x)}\boldsymbol{u}(x)$; $\boldsymbol{n}(x)$ being a normal vector of the traction plane. The last integral on the right-hand side may be evaluated, however, only as a Cauchy principal value.

The second BIE is the Somigliana traction identity (Ugodchicoff and Khutoryanskiy, 1986). It should be noted that in the present form it is valid only for the points of the boundary Γ_R (Mantič and París, 1995) — the parts with a smooth distribution of curvature and tangential vector. It reads

$$\frac{1}{2}t_k(x) = \int\!\!\!\!\!\!-_{\Gamma} T^*_{kl}(x,y)t_l(y)\mathrm{d}\Gamma(y) - \int\!\!\!\!\!\!=_{\Gamma} S_{kl}(x,y)u_l(y)\mathrm{d}\Gamma(y), \quad x \in \Gamma_R, \qquad k,l = 1,\dots,D, \tag{3t}$$

with $\boldsymbol{T}^*(x,y) = \mathcal{T}_{\boldsymbol{n}(x)}\boldsymbol{U}(x,y)$ and $\boldsymbol{S}(x,y) = \mathcal{T}_{\boldsymbol{n}(x)}\boldsymbol{T}(x,y)$. The first integral exists as a Cauchy principal value, the second as a Hadamard finite part of a hypersingular integral (Mantič and París, 1995).

These equations also hold for an unbounded domain (a case when $\Gamma_0 = \emptyset$) for problems with the solution satisfying the condition, see (Baláš et al., 1989),

$$u_k(x) = U_{kl}(x,0)b_l + \mathrm{O}(\|x\|^{1-D}), \ \|x\| \to \infty, \ b_k = \int_{\Gamma} t_k(y)\mathrm{d}\Gamma(y). \tag{4}$$

Let us introduce the following notation, in order to present the BIE systems in a transparent and compact way:

$$[(\boldsymbol{U}_{ia,jb}\boldsymbol{t}_{jb})(x)]_k = \int_{\Gamma_{jb}} U_{kl}(x,y)t_l(y)\mathrm{d}\Gamma(y), \quad x \in \Gamma_{ia}, \qquad i,j \in \{0,1,\dots,H\}, \ k,l \in 1,\dots,D, \ a,b = u,t, \tag{5U}$$

$$[(\boldsymbol{T}^{\pm}_{ia,jb}\boldsymbol{u}_{jb})(x)]_k = \int_{\Gamma_{jb}} T^{\pm}_{kl}(x,y)u_l(y)\mathrm{d}\Gamma(y), \quad ia \neq jb, \quad x \in \Gamma_{ia}, \qquad [(\boldsymbol{T}^{\pm}_{ia,ia}\boldsymbol{u}_{ia})(x)]_k = \int\!\!\!\!\!\!-_{\Gamma_{ia}} T^{\pm}_{kl}(x,y)u_l(y)\mathrm{d}\Gamma(y), \tag{5T}$$

$$[(\boldsymbol{T}^{*\pm}_{ia,jb}\boldsymbol{t}_{jb})(x)]_k = \int_{\Gamma_{jb}} T^{*\pm}_{kl}(x,y)t_l(y)\mathrm{d}\Gamma(y), \quad ia \neq jb, \quad x \in \Gamma_{ia}, \qquad [(\boldsymbol{T}^{*\pm}_{ia,ia}\boldsymbol{t}_{ia})(x)]_k = \int\!\!\!\!\!\!-_{\Gamma_{ia}} T^{*\pm}_{kl}(x,y)t_l(y)\mathrm{d}\Gamma(y), \tag{5D}$$

$$[(\boldsymbol{S}_{ia,jb}\boldsymbol{u}_{jb})(x)]_k = \int_{\Gamma_{jb}} S_{kl}(x,y)u_l(y)\mathrm{d}\Gamma(y), \quad ia \neq jb, \quad x \in \Gamma_{ia}, \qquad [(\boldsymbol{S}_{ia,ia}\boldsymbol{u}_{ia})(x)]_k = \int\!\!\!\!\!\!=_{\Gamma_{ia}} S_{kl}(x,y)u_l(y)\mathrm{d}\Gamma(y). \tag{5S}$$

In all of these expressions any of the indices i, j, a, b can be omitted, with a clear meaning of the remaining symbol. The integral kernels $T^{\pm}_{kl}$ and $T^{*\pm}_{kl}$ are equivalent to functions T_{kl} and T^{*}_{kl} in Eq. (3) expressed with respect to normal vectors $\boldsymbol{n}^{\pm}_{i}$, respectively. In the integral kernel S_{kl} the normals are always defined as outward vectors to pertinent regions Ω_i or Ω_j.

Two BIE systems are introduced and analyzed. They can be formally written as

$$(\mathcal{A}_z \boldsymbol{y})(x) = (\mathcal{B}_z \boldsymbol{b})(x), \tag{6}$$

where index z is either 'f' for the first-kind BIE system or 's' for the second-kind BIE system. The vector function $\boldsymbol{y}$ consists of the unknown boundary functions and the function $\boldsymbol{b}$ contains the prescribed boundary conditions

$$\begin{aligned} \boldsymbol{y}^T &= \{\boldsymbol{u}_{0t}\ \boldsymbol{u}_{1t}\ \cdots\ \boldsymbol{u}_{Ht}\ \boldsymbol{t}_{0u}\ \ \boldsymbol{t}_{1u}\ \ \cdots\ \boldsymbol{t}_{Hu}\}, \\ \boldsymbol{b}^T &= \{\boldsymbol{t}_{0t}\ \ \boldsymbol{t}_{1t}\ \ \cdots\ \boldsymbol{t}_{Ht}\ \ \boldsymbol{u}_{0u}\ \boldsymbol{u}_{1u}\ \cdots\ \boldsymbol{u}_{Hu}\}. \end{aligned} \tag{7}$$

A well-known SGBEM approach can be obtained applying both BIEs Eq. (3) so that a BIE system of the first kind is created. The operators for this system ($z = f$) have the form

$$\mathcal{A}_f = \begin{bmatrix} \boldsymbol{S}_{0t,0t} & \boldsymbol{S}_{0t,1t} & \cdots & \boldsymbol{S}_{0t,Ht} & -\boldsymbol{T}^{*+}_{0t,0u} & -\boldsymbol{T}^{*+}_{0t,1u} & \cdots & -\boldsymbol{T}^{*+}_{0t,Hu} \\ \boldsymbol{S}_{1t,0t} & \boldsymbol{S}_{1t,1t} & \cdots & \boldsymbol{S}_{1t,Ht} & -\boldsymbol{T}^{*-}_{1t,0u} & -\boldsymbol{T}^{*-}_{1t,1u} & \cdots & -\boldsymbol{T}^{*-}_{1t,Hu} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{S}_{Ht,0t} & \boldsymbol{S}_{Ht,1t} & \cdots & \boldsymbol{S}_{Ht,Ht} & -\boldsymbol{T}^{*-}_{Ht,0u} & -\boldsymbol{T}^{*-}_{Ht,1u} & \cdots & -\boldsymbol{T}^{*-}_{Ht,Hu} \\ -\boldsymbol{T}^{+}_{0u,0t} & -\boldsymbol{T}^{-}_{0u,1t} & \cdots & -\boldsymbol{T}^{-}_{0u,Ht} & \boldsymbol{U}_{0u,0u} & \boldsymbol{U}_{0u,1u} & \cdots & \boldsymbol{U}_{0u,Hu} \\ -\boldsymbol{T}^{+}_{1u,0t} & -\boldsymbol{T}^{-}_{1u,1t} & \cdots & -\boldsymbol{T}^{-}_{1u,Ht} & \boldsymbol{U}_{1u,0u} & \boldsymbol{U}_{1u,1u} & \cdots & \boldsymbol{U}_{1u,Hu} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\boldsymbol{T}^{+}_{Hu,0t} & -\boldsymbol{T}^{-}_{Hu,1t} & \cdots & -\boldsymbol{T}^{-}_{Hu,Ht} & \boldsymbol{U}_{Hu,0u} & \boldsymbol{U}_{Hu,1u} & \cdots & \boldsymbol{U}_{Hu,Hu} \end{bmatrix}, \tag{8f-A}$$

$$\mathcal{B}_f =$$

$$= \begin{bmatrix} -\frac{1}{2} + \boldsymbol{T}^{*+}_{0t,0t} & \boldsymbol{T}^{*+}_{0t,1t} & \cdots & \boldsymbol{T}^{*+}_{0t,Ht} & -\boldsymbol{S}_{0t,0u} & -\boldsymbol{S}_{0t,1u} & \cdots & -\boldsymbol{S}_{0t,Hu} \\ \boldsymbol{T}^{*-}_{1t,0t} & -\frac{1}{2} + \boldsymbol{T}^{*-}_{1t,1t} & \cdots & \boldsymbol{T}^{*-}_{1t,Ht} & -\boldsymbol{S}_{1t,0u} & -\boldsymbol{S}_{1t,1u} & \cdots & -\boldsymbol{S}_{1t,Hu} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{T}^{*-}_{Ht,0t} & \boldsymbol{T}^{*-}_{Ht,1t} & \cdots & -\frac{1}{2} + \boldsymbol{T}^{*-}_{Ht,Ht} & -\boldsymbol{S}_{Ht,0u} & -\boldsymbol{S}_{Ht,1u} & \cdots & -\boldsymbol{S}_{Ht,Hu} \\ -\boldsymbol{U}_{0u,0t} & -\boldsymbol{U}_{0u,1t} & \cdots & -\boldsymbol{U}_{0u,Ht} & \frac{1}{2} + \boldsymbol{T}^{+}_{0u,0u} & \boldsymbol{T}^{-}_{0u,1u} & \cdots & \boldsymbol{T}^{-}_{0u,Hu} \\ -\boldsymbol{U}_{1u,0t} & -\boldsymbol{U}_{1u,1t} & \cdots & -\boldsymbol{U}_{1u,Ht} & \boldsymbol{T}^{+}_{1u,0u} & \frac{1}{2} + \boldsymbol{T}^{-}_{1u,1u} & \cdots & \boldsymbol{T}^{-}_{1u,Hu} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -\boldsymbol{U}_{Hu,0t} & -\boldsymbol{U}_{Hu,1t} & \cdots & -\boldsymbol{U}_{Hu,Ht} & \boldsymbol{T}^{+}_{Hu,0u} & \boldsymbol{T}^{-}_{Hu,1u} & \cdots & \frac{1}{2} + \boldsymbol{T}^{-}_{Hu,Ht} \end{bmatrix}. \tag{8f-B}$$

The second-kind BIE system uses the operators introduced by the relations

$$\mathcal{A}_s = \\ = \begin{bmatrix} \frac{1}{2}+\boldsymbol{T}^+_{0t,0t} & \boldsymbol{T}^-_{0t,1t} & \cdots & \boldsymbol{T}^-_{0t,Ht} & -\boldsymbol{U}_{0t,0u} & -\boldsymbol{U}_{0t,1u} & \cdots & -\boldsymbol{U}_{0t,Hu} \\ \boldsymbol{T}^+_{1t,0t} & \frac{1}{2}+\boldsymbol{T}^-_{1t,1t} & \cdots & \boldsymbol{T}^-_{1t,Ht} & -\boldsymbol{U}_{1t,0u} & -\boldsymbol{U}_{1t,1u} & \cdots & -\boldsymbol{U}_{1t,Hu} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{T}^+_{Ht,0t} & \boldsymbol{T}^-_{Ht,1t} & \cdots & \frac{1}{2}+\boldsymbol{T}^-_{Ht,Ht} & -\boldsymbol{U}_{Ht,0u} & -\boldsymbol{U}_{Ht,1u} & \cdots & -\boldsymbol{U}_{Ht,Hu} \\ \boldsymbol{S}_{0u,0t} & \boldsymbol{S}_{0u,1t} & \cdots & \boldsymbol{S}_{0u,Ht} & \frac{1}{2}-\boldsymbol{T}^{*+}_{0u,0u} & -\boldsymbol{T}^{*+}_{0u,1u} & \cdots & -\boldsymbol{T}^{*+}_{0u,Hu} \\ \boldsymbol{S}_{1u,0t} & \boldsymbol{S}_{1u,1t} & \cdots & \boldsymbol{S}_{1u,Ht} & -\boldsymbol{T}^{*-}_{1u,0u} & \frac{1}{2}-\boldsymbol{T}^{*-}_{1u,1u} & \cdots & -\boldsymbol{T}^{*-}_{1u,Hu} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{S}_{Hu,0t} & \boldsymbol{S}_{Hu,1t} & \cdots & \boldsymbol{S}_{Hu,Ht} & -\boldsymbol{T}^{*-}_{Hu,0u} & -\boldsymbol{T}^{*-}_{Hu,1u} & \cdots & \frac{1}{2}-\boldsymbol{T}^{*-}_{Hu,Hu} \end{bmatrix}, \tag{8s-A}$$

$$\mathcal{B}_s = \begin{bmatrix} \boldsymbol{U}_{0t,0t} & \boldsymbol{U}_{0t,1t} & \cdots & \boldsymbol{U}_{0t,Ht} & -\boldsymbol{T}^+_{0t,0u} & -\boldsymbol{T}^-_{0t,1u} & \cdots & -\boldsymbol{T}^-_{0t,Hu} \\ \boldsymbol{U}_{1t,0t} & \boldsymbol{U}_{1t,1t} & \cdots & \boldsymbol{U}_{1t,Ht} & -\boldsymbol{T}^+_{1t,0u} & -\boldsymbol{T}^-_{1t,1u} & \cdots & -\boldsymbol{T}^-_{1t,Hu} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{U}_{Ht,0t} & \boldsymbol{U}_{Ht,1t} & \cdots & \boldsymbol{U}_{Ht,Ht} & -\boldsymbol{T}^+_{Ht,0u} & -\boldsymbol{T}^-_{Ht,1u} & \cdots & -\boldsymbol{T}^-_{Ht,Hu} \\ \boldsymbol{T}^{*+}_{0u,0t} & \boldsymbol{T}^{*+}_{0u,1t} & \cdots & \boldsymbol{T}^{*+}_{0u,Ht} & -\boldsymbol{S}_{0u,0u} & -\boldsymbol{S}_{0u,1u} & \cdots & -\boldsymbol{S}_{0u,Hu} \\ \boldsymbol{T}^{*-}_{1u,0t} & \boldsymbol{T}^{*-}_{1u,1t} & \cdots & \boldsymbol{T}^{*-}_{1u,Ht} & -\boldsymbol{S}_{1u,0u} & -\boldsymbol{S}_{1u,1u} & \cdots & -\boldsymbol{S}_{1u,Hu} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{T}^{*-}_{Hu,0t} & \boldsymbol{T}^{*-}_{Hu,1t} & \cdots & \boldsymbol{T}^{*-}_{Hu,Ht} & -\boldsymbol{S}_{Hu,0u} & -\boldsymbol{S}_{Hu,1u} & \cdots & -\boldsymbol{S}_{Hu,Hu} \end{bmatrix}. \tag{8s-B}$$

The pros and cons of both abovementioned BIE formulations are known, they are not analyzed here, see e. g. (Vodička and Mantič, 2001). The attention will be paid to those BVPs which lead to multiple solutions of the system in Eq. (6).

3 BVPs with non-invertible operators in BIE formulations

It was already mentioned that some BVPs cannot be solved uniquely. A natural case of such non-uniqueness appears in the solution of interior Neumann BVP. In the case of elasticity the boundary conditions are prescribed in tractions $\boldsymbol{t}$. The solution space of the homogeneous BVP is formed by RBM. Let us introduce the notation for them — $\mu^\alpha_k(x)$ for $x \in \Omega \cup \Gamma$ and $\alpha = 1, \ldots, n_D$, $n_D = \frac{D(D+1)}{2}$. Thus for plane we may write

$$\boldsymbol{\mu}^1(x) = \begin{Bmatrix} 1 \\ 0 \end{Bmatrix}, \quad \boldsymbol{\mu}^2(x) = \begin{Bmatrix} 0 \\ 1 \end{Bmatrix}, \quad \boldsymbol{\mu}^3(x) = \begin{Bmatrix} -x_2 \\ x_1 \end{Bmatrix}. \tag{9}$$

Moreover, let us denote Θ^α_i, $i = 1, \ldots, H$ a BVP with boundary conditions prescribed as follows

$$\Gamma_i = \Gamma_{iu}, \quad \boldsymbol{\mu}_{iu} = \mu^\alpha, \quad \boldsymbol{u}_{ju} = 0,\ j \neq i, \quad \boldsymbol{t}_{jt} = 0,\ j \neq i,\ j = 0, \ldots, H. \tag{10}$$

A classification of BVPs that lead to BIE system with a non-unique solution can be done, finding following classes of problems, (Blázquez et al., 1996; Chen and Zhou, 1992; Vodička and Mantič, 1995; Vodička and Mantič, 2004; Vodička et al., subm.):

(n) The solution of interior Neumann BVP is non-unique. RBMs in Eq. (9) belong to the kernel of the operators $\mathcal{A}_z$ $(z = f, s)$. [1]

(f) If there exists an index i such that the boundary conditions of Θ_i^α with reference to their types are opposite (i.e. $\boldsymbol{u}$ instead of $\boldsymbol{t}$ and vice versa) to the conditions of the solved BVP, the prescribed boundary conditions of Θ_i^α form vectors from the kernel of the operator $\mathcal{A}_f$, Eq. (8f-A). Let $\boldsymbol{H}_f$ be a set of indices i, which enable at least one such BVP Θ_i^α. Let n_{H_f} be the total number of all such problems. Then n_{H_f} gives the number of linearly independent functions from the kernel of the operator $\mathcal{A}_f$ diminished by enabled RBMs if any, see the item (n).

(s) If there exists an index i such that the boundary conditions of Θ_i^α with reference to their types are the same as the conditions of our BVP, the solution of Θ_i^α forms a vector from the kernel of the operator $\mathcal{A}_s$, Eq. (8s-A). Let $\boldsymbol{H}_s$ be a set of indices i, which enable at least one such BVP Θ_i^α. Let n_{H_s} be the total number of all such problems. Then it is the number of linearly independent functions from the kernel of the operator $\mathcal{A}_s$.

(c) The BIE solution of the 2D-Dirichlet exterior BVP or of any interior BVP with Dirichlet conditions prescribed on Γ_0 obtained by system Eq. (6) for $z = f$ is multiple provided that the size of the domain is special. According to (Vodička and Mantič, 2004) for each domain Ω with a boundary Γ there exist one or two *critical scales* ρ such that the solution of the BIE system Eq. (6) with $z = f$ on the set $\rho\Gamma = \{\rho x \in \mathbb{R}^2 : x \in \Gamma\}$ is not unique. The dimension n_c of the kernel of the operator $\mathcal{A}_f$ is one or two.

Thus the solution of the system in Eq. (6) is not unique whenever one of the above conditions is satisfied. It is possible, however, to modify the system using some techniques described in detail in (Blázquez et al., 1996; Chen and Zhou, 1992; Vodička et al., subm.). The modified systems are then uniquely solvable and the solution of the modified system is also the solution of the original system. Here we present an example of such a modification, without a detailed proof. The modification is based on the theory of Fredholm operators as all present operators are Fredholm of index zero, see (Chen and Zhou, 1992).

For each Θ_r^α, $r \in \mathbf{H}_z$ satisfying the conditions either of the item (f) or (s), according to z, let us define functions $\boldsymbol{v}_r^\alpha$ and $\boldsymbol{w}_r^\alpha$ for some fixed points $x_r^\alpha \in \Omega_r^+$ and for a critical BVP of the class (c) the same functions, but for the point $x_0^\alpha \in \Omega_0^-$ or $x_0^\alpha \in \Omega$ of the interior or exterior BVP, respectively, using the following relations

$$[\boldsymbol{v}_r^\alpha]_l(x) = \begin{cases} T_{kl}^-(x_r^\alpha, x), & \text{if } x \in \Gamma_{it}, i \neq 0 \\ T_{kl}^+(x_r^\alpha, x), & \text{if } x \in \Gamma_{0t} \\ -U_{kl}(x_r^\alpha, x), & \text{if } x \in \Gamma_{iu} \end{cases}, \tag{11v}$$

[1] Some kinds of boundary conditions, e.g. symmetry conditions, inhibit only some RBMs. In such a case not all RBMs appear in the kernel of the pertinent operator. This case is not treated hereinafter for the sake of simplicity.

$$[\boldsymbol{w}_r^\alpha]_l(x) = \begin{cases} -T_{kl}^-(x_r^\alpha, x), & \text{if } x \in \Gamma_{iu}, i \neq 0 \\ -T_{kl}^+(x_r^\alpha, x), & \text{if } x \in \Gamma_{0u} \\ U_{kl}(x_r^\alpha, x), & \text{if } x \in \Gamma_{it} \end{cases}, \tag{11w}$$

where $k = 1, \ldots, D$, is a suitably chosen index for each fixed point x_s^α. The functions $\boldsymbol{v}^\alpha$ and $\boldsymbol{w}^\alpha$, however, can be chosen arbitrarily, only some conditions introduced in (Blázquez et al., 1996; Chen and Zhou, 1992), have to be satisfied, so they are not restricted to the present particular choice.

Let us take $z = f$, and create, instead of the system Eq. (6), a modified system

$$(\mathcal{A}_f\boldsymbol{y})\,(x)+\sum_{\hat{\alpha}=1}^{\hat{n}}\boldsymbol{v}^{\hat{\alpha}}(x)\int_\Gamma \boldsymbol{v}^{\hat{\alpha}}(z)\boldsymbol{y}(z)\mathrm{d}\Gamma(z)=(\mathcal{B}_f\boldsymbol{b})\,(x)+\sum_{\hat{\alpha}=1}^{\hat{n}}\boldsymbol{v}^{\hat{\alpha}}(x)\int_\Gamma \boldsymbol{w}^{\hat{\alpha}}(z)\boldsymbol{b}(z)\mathrm{d}\Gamma(z), \tag{12f}$$

where $\boldsymbol{v}^{\hat{\alpha}}$ and $\boldsymbol{w}^{\hat{\alpha}}$ are just appropriately reordered functions $\boldsymbol{v}_r^\alpha$ and $\boldsymbol{w}_r^\alpha$ and $\hat{n}$ is equal to n_{H_f} if the BVP belongs to the class (f) or it is equal to n_c if the BVP is critical of the class (c).

Similarly, a modified system for $z = s$ can be introduced for a BVP belonging to the class (s)

$$(\mathcal{A}_s\boldsymbol{y})\,(x)+\sum_{\hat{\alpha}=1}^{n_{H_d}}\boldsymbol{w}^{\hat{\alpha}}(x)\int_\Gamma \boldsymbol{v}^{\hat{\alpha}}(z)\boldsymbol{y}(z)\mathrm{d}\Gamma(z)=(\mathcal{B}_s\boldsymbol{b})\,(x)+\sum_{\hat{\alpha}=1}^{n_{H_d}}\boldsymbol{w}^{\hat{\alpha}}(x)\int_\Gamma \boldsymbol{w}^{\hat{\alpha}}(z)\boldsymbol{b}(z)\mathrm{d}\Gamma(z). \tag{12s}$$

The system in Eq. (6) for the Neumann BVP (the class (n)) should be modified in a different way, as the above modification does not remove the RBMs from the kernel of the pertinent operator, but the RBM itself can be used for modification as follows

$$(\mathcal{A}_z\boldsymbol{y})\,(x) + \sum_{\alpha=1}^{n_D} \boldsymbol{\mu}^\alpha(x) \int_\Gamma \boldsymbol{\mu}^\alpha(z)\boldsymbol{y}(z)\mathrm{d}\Gamma(z) = (\mathcal{B}_z\boldsymbol{b})\,(x). \tag{12n}$$

It should be noted that a Neumann BVP on a region with cavities solved by SGBEM belongs to both classes (n) and (f). Thus the modification must be performed using both modifying parts of Eq. (12f) and Eq. (12n).

4 Some notes about the discretisation

The system of integral equations in Eq. (6), can be discretized by the BEM and written in a form of linear equation system as follows

$$\mathbf{A}_z\mathbf{y} = \mathbf{B}_z\mathbf{b}. \tag{13}$$

Let us denote N the number of unknowns in the discretized system, thus $\mathbf{y} \in \mathbb{R}^N$, and M the number of known nodal quantities in the same system according to the prescribed boundary data, hence $\mathbf{b} \in \mathbb{R}^M$. The left-hand side matrix of Eq. (13) is therefore a square matrix $\mathbf{A}_z \in \mathbb{R}^{N\times N}$, while the right-hand side matrix $\mathbf{B}_z \in \mathbb{R}^{N\times M}$ does not have to be.

Similarly, any modified systems Eq. (12) can be discretized to obtain

$$\left[\mathbf{A}_f + \mathbf{V}\mathbf{V}^T\right]\mathbf{y} = \left[\mathbf{B}_f + \mathbf{V}\mathbf{W}^T\right]\mathbf{b}, \tag{14f}$$

$$\left[\mathbf{A}_s + \tilde{\mathbf{W}}\mathbf{V}^T\right]\mathbf{y} = \left[\mathbf{B}_s + \tilde{\mathbf{W}}\mathbf{W}^T\right]\mathbf{b}, \tag{14s}$$

$$\left[\mathbf{A}_z + \mathbf{M}\mathbf{M}^T\right]\mathbf{y} = \mathbf{B}_z\mathbf{b}. \tag{14n}$$

The dimensions of matrices $\mathbf{V}$, $\mathbf{W}$ and $\mathbf{M}$ can be deduced from Eq. (12) — referring to the part added to the original system — and from the dimensions of matrices in Eq. (13): $\mathbf{V} \in \mathbb{R}^{N\times n_{H_z}}$, $\mathbf{W} \in \mathbb{R}^{M\times n_{H_z}}$, $\mathbf{M} \in \mathbb{R}^{N\times n_D}$. When the modification is used for a (c)-class BIE, the relevant matrices belong to $\mathbf{V} \in \mathbb{R}^{N\times n_c}$, $\mathbf{W} \in \mathbb{R}^{M\times n_c}$. The 'tilded' matrix $\tilde{\mathbf{W}} \in \mathbb{R}^{N\times n_{H_s}}$ is introduced just due to the discretisation, it corresponds to the same function in Eq. (12) as the matrix without tilde.

5 Notes about numerical implementation

A program BEMGAL, described in (Vodička, 1999) will be used to demonstrate the applicability of aforementioned modification techniques. Although, the developed theory can be applied to both 2D and 3D problems, the program is designed to solve only 2D problems. The code is then applied to numerical examples of the next section. Some important features of the code will be summarized below:

- Linear continuous elements are used to approximate both displacements and tractions. The tractions are, however, allowed to be discontinuous if necessary.
- Galerkin method is applied for the solution of BIE. The basis functions are the same as used for the approximation of the boundary data.
- Connected components of the boundary are approximated by polygons.
- Integrals in the influence matrices are calculated analytically.
- The discretized linear equation system is solved by Gauss elimination.
- All calculations are performed in double precision arithmetic.

6 An example

The example will present all classes of non-invertible boundary integral operators introduced in Section 3. Let us take a domain according to Fig. 2 of a material with $E = 10^4$MPa and $\nu = 0.25$. No 'reasonable' (meant in engineering sense) load produces a known analytical solution, hence a solution given by the following Airy stress function will be considered:

$$F(x_1, x_2) = F_0 \sin\left(\frac{x_1}{h}\right)\exp\left(-\frac{x_2}{h}\right), \quad F_0 = 10^4\text{N}, \; h = 100 \text{ mm}. \tag{14}$$

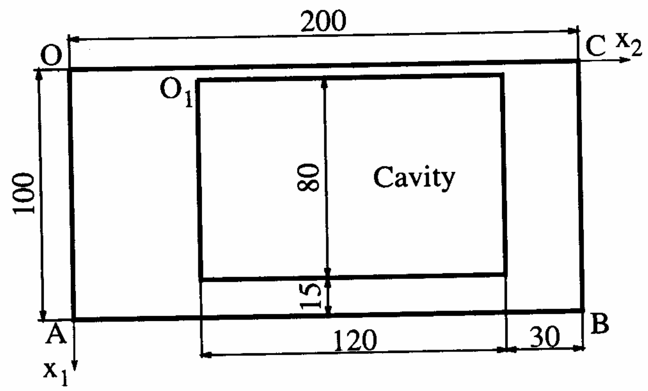


Fig. 2. Domain description.

Four instances of boundary conditions will be treated with, their values are chosen according to the stress function from Eq. (14) but their types are different. The first case is the first BVP of the elasticity theory, i. e. a Neumann type BVP. The case will be distinguished by 'NEU' mark. The second case is the second BVP of the elasticity theory, i. e. a Dirichlet type BVP, distinguished by 'DIR' mark. The last two cases are mixed BVPs: the mark 'NiDo' is used for a problem with tractions prescribed at inner boundary (Ni – Neumann in) and the displacements given on the outline contour (Do – Dirichlet out), 'NoDi' mark denotes a BVP with oposite definition of the boundary conditions. A uniform boundary element mesh will be used. The size of elements in the mesh is 10mm. Plane strain state will be considered for the calculation.

Besides that, the parameters for the modified systems Eq. (12) introduced by Eq. (11) have been chosen such that the classes (f) and (s) ($r = 1$ and $\alpha = 1, 2, 3$) use $x_1^1(35, 90)$ and $k = 1$, $x_1^2(35, 90)$ with $k = 2$ and $x_1^3(55, 130)$ with $k = 1$. Contrary, the class (c) uses $x_0^1(1000, 1000)$ and $k = 1$ for the critical scale, calculated according to (Vodička and Mantič, 2004), $\rho_1 = 0.0136279$.

Let us first check the BIE systems for possible non-invertibility of the pertinent integral operator. Table 1 gathers some singular values (obtained by SVD) of the matrices, belonging to the chosen discretisation. The rows 'System (Eq.)' inform about the used integral equation system, whether a modified system was used or not. At each instance three parameters are referred: the maximum of all singular values σ_{max}, the minimum of all positive singular values (zeroes skipped) σ_{min} and the number of zeroes appearing in SVD. The cases of our special interest are those, which have a non-trivial kernel, i. e. with some zero singular values.

The natural non-trivial kernel is present in a solution of Neumann BVP as the RBMs belong to such kernel (in plane there exist three linearly independent). The system Eq. (6) with $z = s$ confirms exactly this fact. However, the other type system with $z = f$ belongs, due to the present cavity with Neumann type boundary conditions, additionally to the class (f) of Section 3, therefore the kernel dimension of the matrix, and also of the operator, is six.

Table 1. Parameters obtained from SVD.

BVP case	NEU					
System (Eq.)	(6), $z = f$	(12f) + (12n)			(6), $z = s$	(12n), $z = s$
σ_{max}	1.36885	1.38455			0.139058	0.139058
Kernel dim.	6	0			3	0
$10^3\sigma_{min} \neq 0$	0.432967	0.429556			0.0529876	0.0529876
BVP case	DIR					
System (Eq.)	(6), $z = f$		(6), $z = f$, ρ_1	(12f), $\hat{n} = 1$, ρ_1	(6), $z = s$	(12s)
σ_{max}	0.0628024		0.115170	0.115093	0.132709	0.153533
Kernel dim.	0		1	0	3	0
$10^3\sigma_{min} \neq 0$	0.184506		0.342699	0.342661	2.34089	1.185462
BVP case	NiDo					
System (Eq.)	(6), $z = f$	(12f), $\hat{n} = 3$	(6), $z = f$, ρ_1	(12f), $\hat{n} = 4$, ρ_1	(6), $z = s$	
σ_{max}	0.889932	0.893306	0.890093	0.897909	0.639323	
Kernel dim.	3	0	4	0	0	
$10^3\sigma_{min} \neq 0$	0.190779	0.190779	0.354292	0.354287	0.889057	
BVP case	NoDi					
System (Eq.)	(6), $z = f$				(6), $z = s$	(12s)
σ_{max}	0.888114				0.640282	0.659178
Kernel dim.	0				3	0
$10^3\sigma_{min} \neq 0$	0.1910274				0.889063	0.889062

The second case of BVP describes a Dirichlet BVP. We can observe that this kind of boundary conditions causes the system Eq. (6) for the index $z = s$ to be non-invertible. There are three linearly independent functions in the kernel of the left-hand side operator. They are the solutions of the problem Θ_1^α as the BVP represents a problem of the class (s). The other system for $z = f$ is uniquely solvable. However, it does not have to be true generally. Each Dirichlet BVP can belong to the class (c) if the region is appropriately scaled. The present region has two critical scales $\rho_1 = 0.0136279$ and $\rho_2 = 0.01557632$. The former was chosen to demonstrate the existence a non-trivial kernel of the integral operator belonging to the class (c). It should be noted that the non-uniqueness can be removed from the solution in this case also in a simpler way, by modifying the integral kernel $U_{kl}(x, y)$ of the operator $U_{ia,jb}$ in Eq. (5U). The influences of this kind of operator modification can also be seen in (Vodička and Mantič, 2004).

The next case, a mixed BVP of the 'NiDo' case appears also rather interesting at least for the system Eq. (6) with the index $z = f$. While the other system has a unique solution, the first-kind BIE system does not. The Neumann type boundary condition at the cavity normally cause the kernel to have the dimension three. However, as the conditions of the outer countour are Dirichlet, the problem may also be of the class (c). Actually, if the size of the domain is ρ_1 scaled, the dimension of the kernel is increased by one. Nevertheless, in both cases a suitable modification of the system eliminates non-trivial functions from the kernel, so that the resulting system has a unique solution.

The last treated case of boundary conditions, the 'NoDi' case, confirms the results of the above paragraphs and also the classification of the BVPs in Section 3. The first-kind BIE system, Eq. (6) with $z = f$, is uniquely solvable, while the other BIE system presents non-invertibility of the class (s). It causes the second-kind BIE system to have a non-trivial kernel of the dimension three.

A general observation can be presented, too. The singular values does not change significantly, when, in a numerical solution, any of the systems Eq. (14) is used instead of Eq. (13). Of course, with an exception of the zero singular values which are pushed to a positive magnitude. It can be shown, for example, that the modification used for the solution of the Neumann type BVPs does not vary the singular values at all, except of the zeros, naturally, see (Heise, 1981).

Six pictures describe the behavior of the solution of BIE systems, especially, of the modified systems. The same key marks in the pictures have been used: The marks 'f' and 's' refer to the index z in Eq. (6), 'm' means that the system has been modified to obtain a uniquely solvable system. The used symbols representing values of the evaluated functions are the same for one BIE type, however the first component, u_1 or t_n, is displayed by unfilled symbol, the second component, u_2 or t_s, uses filled symbol. Displacements are depicted in global coordinate as they are continuous along the boundary, while the tractions do not have to be continuous and are expressed locally with respect to normal and tangential vectors. Moreover, l is a measure of length along the boundary. It is measured clockwise from point O_1 in the case of cavity boundary or from point A to B for the outmost boundary.

Fig. 3 presents the solution of the 'NEU' case at the cavity boundary. Only the solution of the modified systems is plotted. The numerical results agree with the analytical ones, with small distinctions caused probably by an eigen-solution close to a RBM. This influence is larger in the 'f' case.

The 'DIR' example, unlike the previous one, shows also a solution of the unmodified 's' system, Fig. 4. It does not have a unique solution and the kernel functions are more complicated than RBMs are. We can observe and it can be proved, too, that such functions are singular at reentrant corners. Contrary, the modified 's' system and the unmodified 'f' system offer solutions in accordance with the solution obtained analytically. On the other hand, the 'DIR' case is interesting also when it is scaled to the critical scale. We used ρ_1 to obtain the solutions of the unmodified and the modified 'f' systems. They are plotted in Fig. 5 ('c' in the key refers to critical scale) for the segment AB of the boundary. The presence of a kernel function in the solution is obvious for the unmodified system. The kernel function is singular at the corners, being a solution of the exterior BVP with the same boundary as the outline curve is, see (Vodička and Mantič, 2004).

The case 'NiDo' belongs to the class (f) if it is not scaled. The 'f' system has to be modified, as it is apparent from Fig. 6. The obtained solution of unmodified system contains a kernel function and lies mostly out of the range of the graph. Moreover, if the domain is scaled by the factor ρ_1, i. e. to its critical scale we obtain another kind

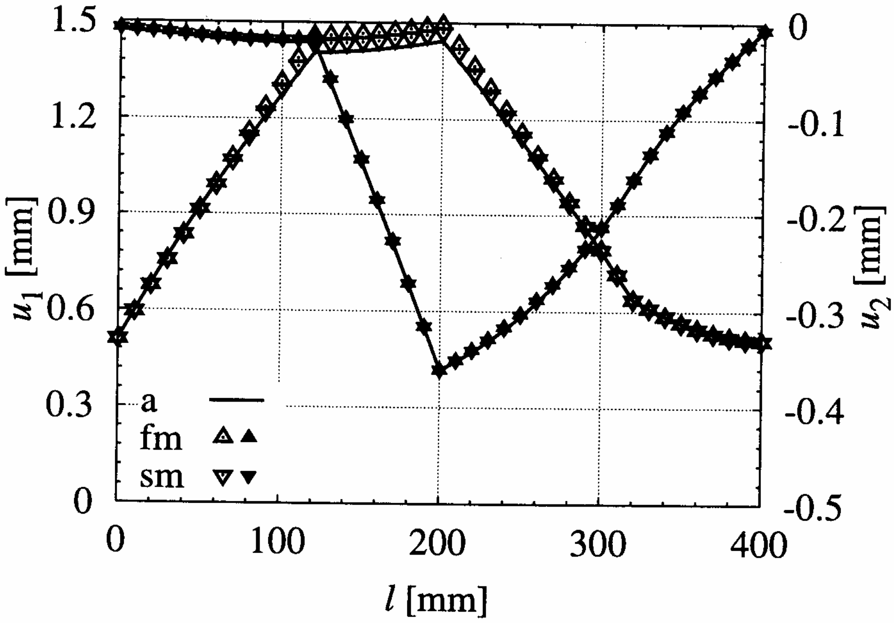


Fig. 3. A comparison of the solution of the modified system with the analytical solution, Neumann problem.

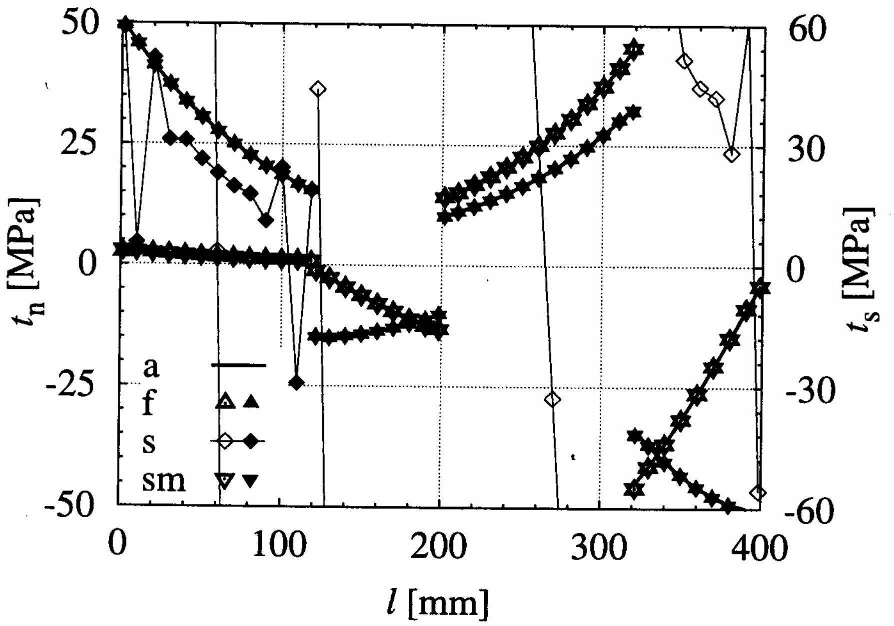


Fig. 4. A comparison of the solution of the modified system with the analytical solution, Dirichlet problem.

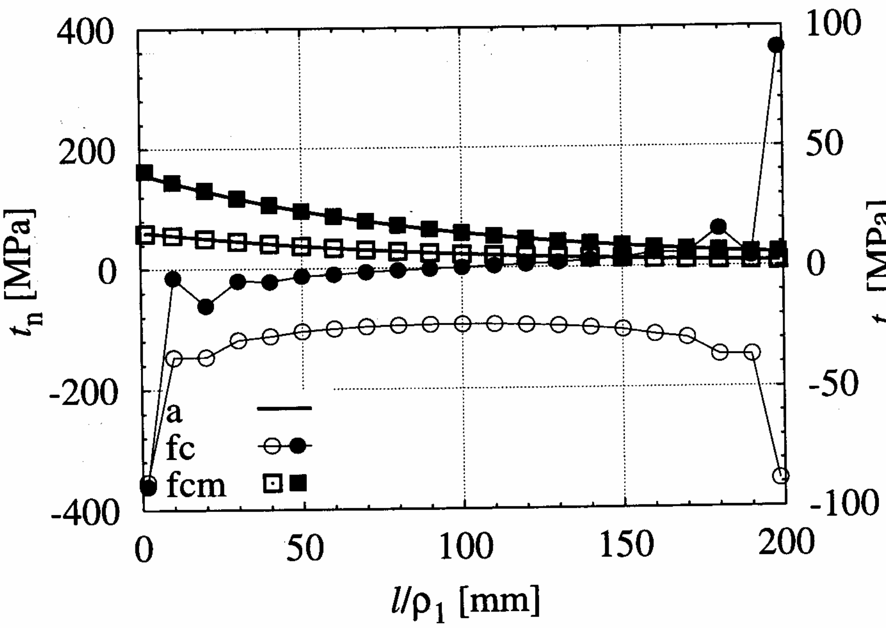


Fig. 5. A comparison of the solution of the unmodified and the modified systems with the analytical solution, when the domain is scaled by the factor ρ_1, Dirichlet problem.

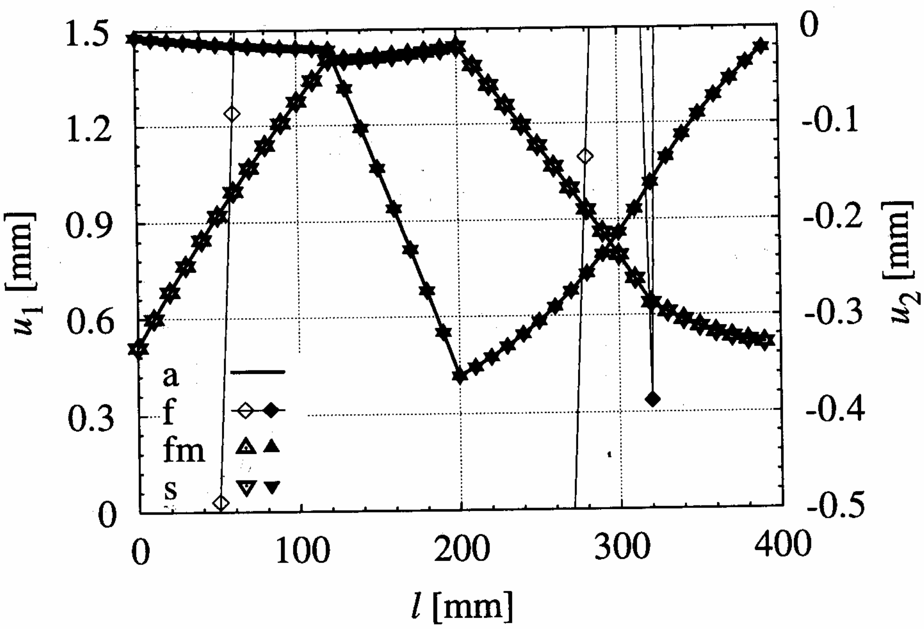


Fig. 6. A comparison of the solution of the modified system with the analytical solution, mixed BVP 'NiDo'.

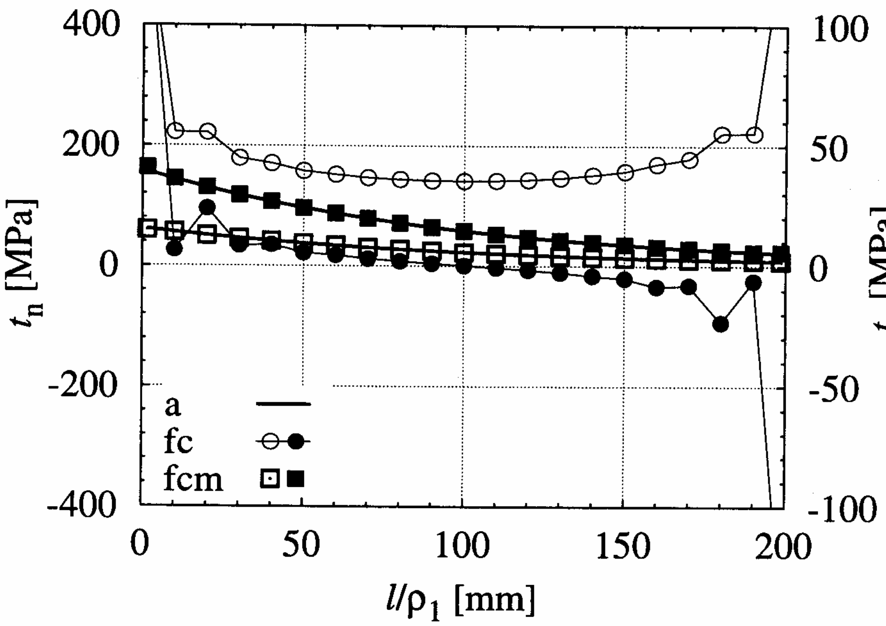


Fig. 7. A comparison of the solution of the unmodified and the modified systems with the analytical solution, when the domain is scaled by the factor ρ_1, 'NiDo' case.

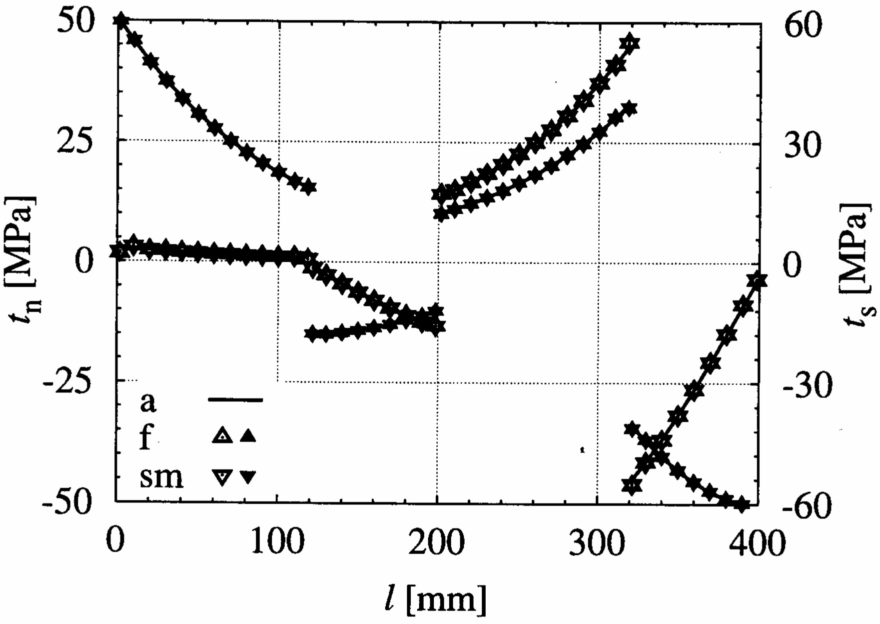


Fig. 8. A comparison of the solution of the modified system with the analytical solution, mixed BVP 'NoDi'.

of non-unique solution. It is plotted in Fig. 7 ('c' in the key refers to critical scale) for the segment AB of the boundary. The presence of a kernel function in the solution is clearly seen for the unmodified system, again it is singular at the corners.

The last selected type of mixed BVP, case 'NoDi', again confirms the above discussion. The problem is of the class (s), thus 's' system had to be modified. Nevertheless, the solutions of both possible uniquely solvable BIE systems are the same as the analytical solution is as it can be seen in Fig. 8.

7 Conclusions

The presented example shows various possibilities of non-invertibility of the operator appearing in BIE formulations. A natural situation appears in the solution of Neumann BVPs. Then not only the BIE system but also the original BVP have multiple solutions. An unexpected problem can occur for a domain with cavities. Then the dimension of the integral operator kernel of the system Eq. (6) with $z = f$ is higher than the number of allowed RBMs. This system is used for the SGBEM approach. The proposed technique which removes the non-trivial functions from the kernel seems to be quite successful and works in accordance with theoretical predictions.

Together with this, according to the present authors' opinion, most crucial result also other BIE formulation, leading to the system of BIEs of the second kind, Eq. (6) with $z = s$ can cause similar problems, but for different types of BVP. For engineers it is not so important, as this formulation is less applied. Surprising may only be the fact that the non-unique solution appears in Dirichlet BVP.

Another speciality of some BVPs is the phenomenon of critical scales. Though it is actually a speciality and can be avoided more comfortably by other means, it was treated in the same way as other cases of non-invertible integral operators in BIE systems. The proposed method works quite satisfactorily also in this instance.

Acknowledgements. The work was supported by the Scientific Grant Agency of Slovak Republic (Grants No. 1/8033/01 and 1/1089/04) and the Spanish Ministry of Education and Culture (Grant No. MAT2000-1115). The paper was presented at the 9th International Conference on Numerical methods in Continuum Mechanics, Žilina, Slovakia, 9–12 September 2003 and its shorter form was published in the Conference CD Proceedings.

REFERENCES

[1] BALÁŠ, J., SLÁDEK, J., SLÁDEK, V. (1989), *Stress analysis by Boundary Element Method*, Elsevier, Amsterdam — Oxford — New York — Tokyo

[2] BLÁZQUEZ, A., MANTIČ, V., PARÍS, F., CAÑAS, J. (1996), "On the removal of rigid body motions in the solution of elastostatic problems by direct BEM", *Int. J. Num. Meth. Engrg.*, **39**, 4021–4038

[3] CHEN, G., ZHOU, J. (1992), *Boundary Element Methods*, Academic Press, London

[4] CHEN, J. T., KUO, S. R., LIN, J. H. (2002), "Analytical study and numerical experiments for degenerate scale problems in the boundary element method for two-dimensional elasticity", *Int. J. Num. Meth. Engrg.*, **54**, 1669–1681,

[5] CONSTANDA, C. (1994), "On non-unique solutions of weakly singular integral equations in plane elasticity", *Quart. J. Mech. and Appl. Math.*, **47**, 261 – 268,

[6] GREENBAUM, A., GREENGARD, L., MCFADDEN, G. (1993), "Laplace's equation and the Dirichlet-Neumann map in multiply connected domains", *J. Comput. Phys.*, **105**, 267 – 278

[7] HEISE, U. (1981), "Removal of the zero eigenvalues of integral operators in elastostatic boundary value problems", *Acta Mechanica*, **41**, 41–61

[8] HEISE, U. (1987), "Dependence of the round-off error in the solution of boundary integral equations on geometrical scale factor", *Comput. Methods Appl. Mech. and Engrg.*, **62**, 115–126

[9] MANTIČ, V. (1993), "A new formula for the C–matrix in the Somigliana identity", *J. Elast.*, **33**, 191–201

[10] MANTIČ, V., PARÍS, F. (1995), "Existence and evaluation of the free terms in the hypersingular boundary integral equation of potential theory", *Engrg. Anal. Boundary Element*, **16**, 253–260

[11] PÉREZ-GAVILÁN, J., ALIABADI, M. (2001), "A symmetric Galerkin BEM for multi-connected bodies: a new approach", *Engrg. Anal. Boundary Element*, **25**, 633–638

[12] UGODCHICOFF, A. G., KHUTORYANSKIY, N. M. (1986), *Boundary element method in solid mechanics*, Kazan University Press, (in Russian)

[13] VODIČKA, R. (1999), *Coupled formulations of boundary integral equations for solving contact problems of elasticity*, PhD thesis, TU Košice, Faculty of Mechanical Engineering, (in Slovak)

[14] VODIČKA, R., MANTIČ, V. (1995), A comparison of three basic formulations of BIEs of potential theory, Report, Technical University of Košice, University of Sevilla

[15] VODIČKA, R., MANTIČ, V., (2004), "On invertibility of elastic single layer potential operator", *J. Elast.* in print

[16] VODIČKA, R., MANTIČ, V., PARÍS, F. (submitted), "On the removal of non-uniqueness in the solution of SGBEM for bodies with holes", *Int. J. Num. Meth. Engrg.*

[17] VODIČKA, R., MANTIČ, V. (2001), "A comparative study of three systems of boundary integral equations in the potential theory", In T. Burczynski, editor, *IUTAM/IACM/IABEM Symposium on Advanced Mathematical and Computational Mechanics Aspects of the Boundary Element Method*, pages 377 – 394, Kluwer Academic Publishers