Local errors in the constant and linear Boundary Element Method for potential problems

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Abstract

In this paper we investigate local errors in the Boundary Element Method (BEM) for potential problems. We start from the basics of BEM formulation for both constant and continuous linear elements to derive exact expressions for the local errors. Our analysis shows that the local error with constant or linear elements decreases quadratically with the boundary element mesh size. Examples are provided and the results of our numerical experiments confirm the theory.

1 Introduction

The subject of errors in the boundary element method (BEM) is still a very interesting one and some aspects have not yet been as explored as they are in other numerical methods like FEM and FDM. Errors in BEM solutions may be due to discretisation or to inaccuracies in the solver that involves the use of BEM matrices with high condition numbers. For a given discretisation there are several ways to implement BEM because of the choice in collocation and nodal points and the shape functions. These will all influence the resulting error in the solution. In most cases where error measurement has been performed, like in adaptive refinement, the main focus has been a guiding measure of the error. This paper presents recent results on an analysis of actual local errors in BEM solutions. The results presented will not only be helpful in choosing an implementation strategy but also in guiding adaptive refinement techniques.

Several techniques have been used to measure BEM errors in the area of adaptive refinement. For instance the discretisation error is estimated by the difference between two solutions obtained using different collocation points but the same discretisation [3]. Another technique uses the first (singular integral equation) and the second (hypersingular integral equation) kind formulations to provide an error estimate [6]. Data from the first kind of equation is substituted into the second kind to obtain a residual which is then used to estimate the error. Some authors have used a posteriori error estimation in FEM as a guiding tool to develop error estimates for BEM [1]. Unfortunately usually such techniques are restricted to Galerkin BEM. In our case we would like to start from the basics of the boundary integral equation (BIE) discretisation to develop error analyses for collocation BEM for potential problems. Though the ideas could easily be adapted to 3D, we will discuss 2D problems and the Laplace equation in particular.

This paper is outlined as follows. Section 2 gives a recapitulation of the BEM formulation. An outline of the integral equation formulation for a Laplace problem and its discretisation is presented. In Section 3 we present a discussion on local errors in the BEM and give the theory for the expected convergence rates of the local error for both constant and linear elements. It is shown here that the local errors for both formulations are of second order. Numerical examples and results for a Dirichlet and mixed problem are presented in Section 4 to illustrate the theory presented on local errors. The paper ends with a conclusion on our findings in Section 5.

2 BEM formulation for potential problems

In this section we briefly present a BEM formulation to pave way for our error analysis. The BEM is used to approximate solutions of boundary value problems that can be formulated as integral equations. For the sake of clarity, we consider potential problems governed by the Laplace equation on a simply connected domain Ω in 2D with boundary Γ. That is

$$\nabla^2 u(\mathbf{r}) = 0, \quad \mathbf{r} \in \Omega. \tag{1}$$

We will consider problems for which either the function u (Dirichlet) or its outward normal derivative $q := \partial u/\partial n$ (Neumann) is given on Γ, or the mixed problem where on one part of the boundary u is given and on the other q is given. The integral equation formulation of eq (1) is, see [7, 5, 8],

$$c(\mathbf{r})u(\mathbf{r}) + \int_\Gamma u(\mathbf{r}')\frac{\partial v}{\partial n'}(\mathbf{r};\mathbf{r}')d\Gamma_{r'} = \int_\Gamma q(\mathbf{r}')v(\mathbf{r};\mathbf{r}')d\Gamma_{r'}, \ \mathbf{r} \in \Gamma, \tag{2}$$

where

$$q(\mathbf{r}') := \frac{\partial u}{\partial n'}(\mathbf{r}') \tag{3}$$

and $\mathbf{r}$ and $\mathbf{r}'$ are points in the boundary and n' is the unit outward normal at $\mathbf{r}'$ in Γ. The function v,

$$v(\mathbf{r};\mathbf{r}') := \frac{1}{2\pi}\log\frac{1}{||\mathbf{r}-\mathbf{r}'||},$$

is the fundamental solution for the Laplace equation in 2D. The constant $c(\mathbf{r})$ is $1/2$ if Γ is a smooth boundary at $\mathbf{r}$. Thus eq (2) expresses the potential at any point $\mathbf{r}$ in terms of its values $u(\mathbf{r}')$ and the values of its outward normal derivatives $q(\mathbf{r}')$ on the boundary. Some of this information will be given as boundary conditions and the rest has to be solved for. We note that the most important step of BEM is to approximate the missing boundary data as accurately as possible since the final step of computing the unknown function in Ω is merely a case of postprocessing.

The boundary Γ is divided into N partitions Γ_j such that $\bigcup_{j=1}^{N} \Gamma_j = \Gamma$. Each Γ_j is then represented by a numerical boundary S_j, see Fig. 1 for a two dimensional case of rectilinear elements on a disc domain. Then on each S_j

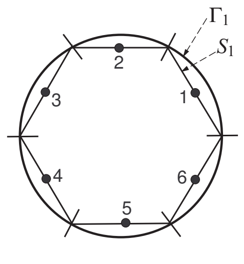

Fig. 1: A BEM discretisation in which a polygon of rectilinear elements is used to represent a circular boundary.

the functions $u(\mathbf{r}')$ and $q(\mathbf{r}')$ are assumed to vary as the so called shape functions. Let us denote these shape functions by $f_u(\xi)$ and $f_q(\xi)$ respectively, where ξ is a local coordinate on S_j. For instance $f_u(\xi)$ and $f_q(\xi)$ are constant functions in the case of constant elements and linear functions in the case of linear elements. Thus the discretised integral equation is

$$c(\mathbf{r}_i)u(\mathbf{r}_i) + \sum_{j=1}^{N} \int_{S_j} f_u(\mathbf{r}'(\xi)) \frac{\partial v}{\partial n'}(\mathbf{r}_i; \mathbf{r}'(\xi))\, dS = \sum_{j=1}^{N} \int_{S_j} v(\mathbf{r}_i; \mathbf{r}'(\xi)) f_q(\mathbf{r}'(\xi))\, dS \tag{4}$$

where $\mathbf{r}_i$ is the i-th collocation point.

At this stage we would like to note two important sources of numerical error. First, replacing the physical boundary by a numerical boundary and second, representing the unknown functions by shape functions. In some geometries or discretisations $S_j \equiv \Gamma_j$ thus eliminating one inherent source of error. The size of the local error will be small depending on how well $f_u(\xi)$ and $f_q(\xi)$ represent the original functions. Errors may also be due to integration and discontinuities at element boundaries but these can be reduced to negligible amounts by using suitable techniques during implementation. The integral equation eq (4) is written for N collocation points $\mathbf{r_i}$ and boundary conditions are appropriately applied to obtain a linear system of equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{5}$$

where $\mathbf{x}$ is a vector of the unknown values of either u or q at the boundary.

3 Local errors in potential problems

Consider a Dirichlet problem given by,

$$\begin{cases} \nabla^2 u(\mathbf{r}) = 0, & \mathbf{r} \in \Omega, \\ u(\mathbf{r}) = g(\mathbf{r}), & \mathbf{r} \in \Gamma. \end{cases} \tag{6}$$

The unknown in this case is the outward normal derivative $q(\mathbf{r})$. Thus, for a discretisation of N elements, what we want to solve for using BEM are the values of the normal flux at the boundary Γ. That is, $\mathbf{x} = \mathbf{q} = (q_1\ q_2\ \cdots\ q_N)^T$, a vector of the unknown values of $q(\mathbf{r})$ at the boundary. Here we have introduced the notation

$$u_i := u(\mathbf{r}_i) \quad \text{and} \quad q_i := q(\mathbf{r}_i).$$

Let $\tilde{\mathbf{q}} := (\tilde{q}_1\ \tilde{q}_2\ \cdots\ \tilde{q}_N)^T$ be the corresponding BEM solution and let $\mathbf{q}^* := (q_1^*\ q_2^*\ \cdots\ q_N^*)^T$ denote the exact solution at the corresponding nodes. The global error $\mathbf{e}$ is defined as

$$\mathbf{e} := \mathbf{q}^* - \tilde{\mathbf{q}}. \tag{7}$$

In order to advance our error investigations, let us define the exact values of the integrals on the j-th element from source node i as

$$I_{ij}^u := \int_{\Gamma_j} \frac{\partial v}{\partial n'}(\mathbf{r}_i; \mathbf{r}') u(\mathbf{r}')\, d\Gamma, \tag{8a}$$

$$I_{ij}^q := \int_{\Gamma_j} v(\mathbf{r}_i; \mathbf{r}') q(\mathbf{r}')\, d\Gamma. \tag{8b}$$

Let us also define the corresponding numerical estimates of the integrals in eq (8) in the BEM as

$$\tilde{I}_{ij}^u := \int_{S_j} \frac{\partial v}{\partial n'}(\mathbf{r}_i; \mathbf{r}') f_u(\mathbf{r}'(\xi))\, dS, \tag{9a}$$

$$\tilde{I}_{ij}^q := \int_{S_j} v(\mathbf{r}_i; \mathbf{r}') f_q(\mathbf{r}'(\xi))\, dS. \tag{9b}$$

We then define the two contributions to the local error on element j when the source node is i as

$$d^q_{ij} := I^q_{ij} - \tilde{I}^q_{ij}, \quad d^u_{ij} := I^u_{ij} - \tilde{I}^u_{ij}. \tag{10a}$$

The total local error on element j due to source node i is

$$d_{ij} := d^q_{ij} - d^u_{ij}. \tag{11}$$

Therefore the local error for the i-th equation eq (4) due to contributions from all the elements is given by

$$d_i := -\sum_{j=1}^{N} d^u_{ij} + \sum_{j=1}^{N} d^q_{ij}. \tag{12}$$

In what follows we will asses further these local errors.

Theorem 1 *The local error eq* (12) *in constant elements BEM with collocation at the midpoints is second order with respect to grid size.*

Proof. We need to show that the local error in eq (11) is third order in grid size. Take the case of an element with a Dirichlet boundary condition where the unknown is $q(\mathbf{r}')$. Consider the integral in eq (8b), that is

$$I^q_{ij} = \int_{\Gamma_j} v(\mathbf{r}_i;\mathbf{r}')q(\mathbf{r}')\,d\Gamma. \tag{13}$$

Let l_j be the length of S_j and ξ be a local coordinate on S_j such that

$$-l_j/2 \le \xi \le l_j/2.$$

Then eq (13) is transformed and evaluated in terms of ξ. Let ξ_j be the midpoint of S_j. Suppose we have a Taylor series expansion of $q(\xi)$ about ξ_j, that is,

$$q(\xi) = q(\xi_j) + q'(\xi_j)(\xi-\xi_j) + \frac{q''(\xi_j)}{2}(\xi-\xi_j)^2 + \frac{q^{(3)}(\xi_j)}{6}(\xi-\xi_j)^3 + \cdots. \tag{14}$$

Then, if we use eq (14) in eq (13) and also use the expression for $v(\mathbf{r};\mathbf{r}')$, we have

$$2\pi I^q_{ij} = \int_{-l_j/2}^{l_j/2} \ln[r(\xi)]q(\xi_j)\,d\xi + \int_{-l_j/2}^{l_j/2} \ln[r(\xi)]q'(\xi_j)(\xi-\xi_j)\,d\xi + \int_{-l_j/2}^{l_j/2} \ln[r(\xi)]\frac{q''(\xi_j)}{2}(\xi-\xi_j)^2\,d\xi + \cdots, \tag{15}$$

where

$$r(\xi) := ||\mathbf{r_i} - \mathbf{r}'(\xi)||.$$

In constant elements BEM we only use the first term, that is,

$$I^q_{ij} \approx \frac{1}{2\pi}\int_{-l_j/2}^{l_j/2} \ln[r(\xi)]q(\xi_j)\,d\xi. \tag{16}$$

So we have a truncation error whose principle term is the second term on the right of eq (15), here denoted E_0. That is,

$$E^{ij}_0 := \frac{1}{2\pi}\int_{-l_j/2}^{l_j/2} q'(\xi_j)\ln[r(\xi)](\xi-\xi_j)\,d\xi. \tag{17}$$

For points r_i outside Γ_j, the distance $r(\xi)$ can be expanded about the point ξ_j as

$$r(\xi) = r(\xi_j) + r'(\xi_j)(\xi-\xi_j) + \mathrm{O}((\xi-\xi_j)^2) \tag{18}$$

Let

$$\rho_0 := r(\xi_j), \quad \rho_1 := r'(\xi_j),$$

so that we can write eq (18) as

$$r(\xi) = \rho_0(1 + \frac{\rho_1}{\rho_0}(\xi - \xi_j) + O((\xi - \xi_j)^2).$$

Then

$$\ln[r(\xi)] = \ln(\rho_0) + \frac{\rho_1}{\rho_0}(\xi - \xi_j) + O((\xi - \xi_j)^2).$$

The integral in eq (17) can now be evaluated as

$$\begin{aligned} E_0^{ij} &= \frac{1}{2\pi}\int_{-l_j/2}^{l_j/2} q'(\xi_j)\ln[r(\xi)](\xi - \xi_j)\,d\xi \\ &= \frac{1}{2\pi}\int_{-l_j/2}^{l_j/2} \ln(\rho_0)q'(\xi_j)(\xi - \xi_j)\,d\xi + \frac{1}{2\pi}\int_{-l_j/2}^{l_j/2} q'(\xi_j)[\frac{\rho_1}{\rho_0}(\xi - \xi_j) + O((\xi - \xi_j)^2)](\xi - \xi_j)\,d\xi. \quad (19) \end{aligned}$$

Since we use the midpoint as ξ_j, the first integral on the right of eq (19) is zero so that we remain with

$$E_0^{ij} = \frac{\rho_1}{\rho_0}\frac{l_j^3}{24\pi}q'(\xi_j) + O(l_j^5). \quad (20)$$

For $i = j$, the distance $r(\xi) = |\xi|$ and the integral in eq (17) evaluates to zero.

This shows that midpoint constant elements BEM error is expected to be of second order.

■

Theorem 2 *The local error eq* (12) *in linear elements BEM is second order with respect to grid size.*

Proof. Likewise we need to show that the error in (11) is third order in grid size. In linear elements the unknown function is assumed to vary linearly on the element, that is,

$$q(\xi) \approx p_1(\xi) := \alpha_0 + \alpha_1\xi,$$

where α_0 and α_1 are constants and ξ a local coordinate on S_j. The error in this case is due to the error when we interpolate by an order one polynomial which is given by, see [4],

$$q(\xi) - p_1(\xi) = \frac{q''(\eta)}{2}(\xi - \xi_0)(\xi - \xi_1), \eta \in (\xi_0, \xi_1).$$

Using this result, the local error in linear elements BEM will be

$$E_1^{ij} = \frac{1}{4\pi}\int_{-l_j/2}^{l_j/2} \ln[r(\xi)]q''(\eta)(\xi - \xi_0)(\xi - \xi_1)\,d\xi, \quad (21)$$

where ξ_0 and ξ_1 are the interpolation coordinates. Again using the second mean value theorem, we have

$$\begin{aligned} E_1^{ij} &\approx \frac{\ln[r(\beta)]q''(\eta)}{4\pi}\int_{-l_j/2}^{l_j/2} (\xi - \xi_0)(\xi - \xi_1)\,d\xi \\ &= -\frac{l_j^3}{24\pi}\ln[r(\beta)]q''(\eta), \quad (22) \end{aligned}$$

where $\xi_0 = -l_j/2$, $\xi_1 = l_j/2$ and β an intermediate point. This result shows that linear elements BEM error is expected to be of second order.

■

4 Numerical examples

In this section we will perform numerical experiments to illustrate the above claims about the error. Results show that indeed we obtain second order convergence of the error. Our experiments are performed on a circle which rules out the errors caused by discontinuities at corners in other geometries like a square. We give an example of Dirichlet boundary conditions and Neumann boundary conditions. For the Neumann problem, we prescribe Dirichlet boundary conditions at one of the nodes so we can as well say it is a mixed boundary conditions problem.

Example 1 *Consider the Dirichlet boundary value problem*

$$\begin{cases} \nabla^2 u(\mathbf{r}) = 0, & \mathbf{r} \in \Omega := \{\mathbf{r} \in \mathbb{R}^2 : ||\mathbf{r}|| \leq 1.2\}, \\ u(\mathbf{r}') = g(\mathbf{r}'), & \mathbf{r}' \in \Gamma. \end{cases} \tag{23}$$

The boundary condition function $g(\mathbf{r}')$ *is chosen such that the exact solution is*

$$q(\mathbf{r}') = \frac{(\mathbf{r}' - \mathbf{r}_s) \cdot \mathbf{n}(\mathbf{r}')}{||\mathbf{r}' - \mathbf{r}_s||^2} \tag{24}$$

where the source point $\mathbf{r}_s = (0.36, 1.8)$ *is a fixed point outside* Ω *and* $\mathbf{n}(\mathbf{r}')$ *is the outward normal at* $\mathbf{r}'$.

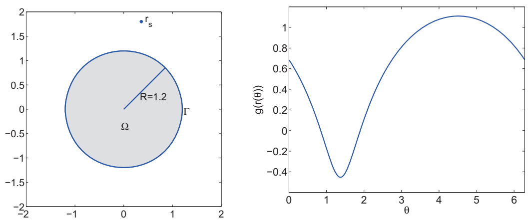

Fig. 2: Definition of the domain (left) and the boundary condition function (right) for Example 1.

Since we have the analytic expression for the unknown $q(\mathbf{r}')$, for each node i we can compute the exact value x_i^* of the unknown. Thus we can compute both the local residual error $\mathbf{d} := \mathbf{A}\mathbf{x}^* - \mathbf{b}$ and the global error $\mathbf{e} = \mathbf{x}^* - \mathbf{x}$ in eq (7).

As explored in [2], for these problems the operator is well conditioned and we see that the global error behaves the same as the local error.

In Tables 1 and 2, we show both the global and local errors for a Dirchlet problem using constant or linear elements. As we can see, by taking ratios of consecutive errors, both errors are of second order convergence.

Example 2 *Consider the Neumann boundary value problem*

$$\begin{cases} \nabla^2 u(\mathbf{r}) = 0, & \mathbf{r} \in \Omega := \{\mathbf{r} \in \mathbb{R}^2 : ||\mathbf{r}|| \leq 1.2\}, \\ q(\mathbf{r}') = h(\mathbf{r}'), & \mathbf{r}' \in \Gamma. \end{cases} \tag{25}$$

The boundary condition function $h(\mathbf{r}')$ *is chosen such that the exact solution is*

$$u(\mathbf{r}') = \log(||\mathbf{r}' - \mathbf{r}_s||) \tag{26}$$

where the source point $\mathbf{r}_s = (0.36, 1.8)$ *is a fixed point outside* Ω *and* $\mathbf{n}(\mathbf{r}')$ *is the outward normal at* $\mathbf{r}'$.

In the implementation, a Dirichlet boundary condition is prescribed at the last node $j = N$ as a remedy to avoid a singular system characteristic of a total Neumann problem. So in actual sense we solve a mixed boundary problem. The results in Tables 3 and 4 indeed show that we have second order error convergence.

N	$\|\|\mathbf{d}\|\|_2/\sqrt{N}$	$\|\|\mathbf{e}\|\|_2/\sqrt{N}$	error ratios	
5	1.62E-01	1.98E-01	10.44	15.19
15	1.55E-02	1.31E-02	5.68	4.49
45	2.73E-03	2.91E-03	7.92	7.39
135	3.45E-04	3.93E-04	8.66	8.49
405	3.99E-05	4.63E-05	8.89	8.80
1215	4.49E-06	5.27E-06	-	-

Table 1: Constant elements errors for Example 1 with Dirichlet boundary conditions. On the right we have the respective ratios of the error with N to that with $3N$.

N	$\|\|\mathbf{d}\|\|_2/\sqrt{N}$	$\|\|\mathbf{e}\|\|_2/\sqrt{N}$	error ratios	
5	4.24E-01	2.03E-01	16.26	7.59
15	2.61E-02	2.68E-02	6.30	5.24
45	4.14E-03	5.11E-03	8.22	8.29
135	5.04E-04	6.17E-04	8.78	8.77
405	5.74E-05	7.04E-05	8.93	8.89
1215	6.43E-06	7.91E-06	-	-

Table 2: Linear elements errors for Example 1 with Dirichlet boundary conditions. On the right we have the respective ratios of the error with N to that with $3N$.

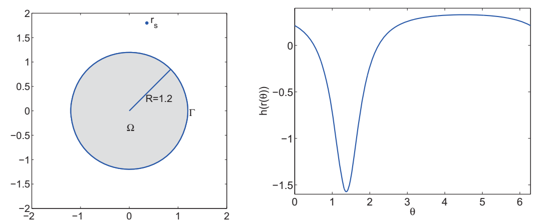

Fig. 3: Definition of the domain (left) and the boundary condition function (right) for Example 2.

N	$\|\|\mathbf{d}\|\|_2/\sqrt{N}$	$\|\|\mathbf{e}\|\|_2/\sqrt{N}$	error ratios	
5	3.35E-01	2.02E-01	8.86	15.98
15	3.78E-02	1.27E-02	7.15	6.73
45	5.29E-03	1.88E-03	8.52	8.54
135	6.21E-04	2.21E-04	8.86	8.87
405	7.01E-05	2.49E-05	8.96	8.96
1215	7.82E-06	2.78E-06	-	-

Table 3: Constant elements errors for Example 2 with Neumann boundary conditions except for one Dirichlet node. On the right we have the respective ratios of the error with N to that with $3N$.

N	$\|\|\mathbf{d}\|\|_2/\sqrt{N}$	$\|\|\mathbf{e}\|\|_2/\sqrt{N}$	error ratios	
5	3.88E-01	5.85E-01	16.18	63.42
15	2.40E-02	9.22E-03	5.98	6.93
45	4.01E-03	1.33E-03	8.05	8.06
135	4.99E-04	1.65E-04	8.71	8.70
405	5.71E-05	1.90E-05	8.91	8.90
1215	6.41E-06	2.13E-06	-	-

Table 4: Linear elements errors for Example 2 with Neumann boundary conditions except for one Dirichlet node. On the right we have the respective ratios of the error with N to that with $3N$.

5 Conclusions

We have presented a theory to show that the error when we use constant elements with collocation at the midpoints is second order in grid size. Likewise for linear elements we have shown that the local error is of second order with grid size. The results of our experiments agree with the theory. These results are not only important in guiding our choice of grid but also the choice between constant and linear elements.

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