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An efficient multi-point support-motion random vibration analysis technique

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Abstract

Random vibration analysis technique based on the mode superposition method requires computation of multiple static solutions for multi-point support-motion problems. These static solutions, known as quasi-static solutions, are utilized in the calculation of participation factors. For large practical problems, the quasi-static solutions may become expensive and time-consuming. The present paper shows that the needed participation factors can be computed from modal reactions, mode shapes and natural frequencies eliminating the need to solve for the quasi-static problem. It is also shown that a simple expression can be developed for the quasi-static solution in terms of modal reactions, mode shapes and natural frequencies, which can then be used in the expressions for the quasi-static and covariance components of the response power spectral densities. Thus, the need for quasi-static solution is completely eliminated without introducing any further assumption into the formulation.

Keywords: Random vibration; Modal analysis; Participation factor; Quasi-static solution; Power spectral density; Multi-point excitation

1. Introduction

In many engineering applications in aerospace, automotive and numerous other industries, the dynamic loading on the structures cannot be characterized in deterministic terms because of the uncertainties involved in the loading process. In these circumstances the loading on the structure is described by statistical means, viz., mean, mean square values or power spectral density (PSD) of the loading and random vibration analysis is then performed to compute statistical response of the structure.

The mode-based random vibration analysis for support excitation via finite element method (FEM) requires the computation of modal participation factors. Modal participation factors are indicators of the possible contributions of particular modes towards the stochastic response of the structure. The computation of modal participation factor requires the pseudo-static displacement solution of the structure due to a unit displacement of the support degrees of freedom (dof's) excited by the support motion. For multi-point support excitations, multiple pseudo-static displacement solutions must be performed before modal participation factors can be computed [3–5].

The pseudo-static displacements are also utilized in the computation of the quasi-static and covariance components of the mean square response when the total random response rather than the random response relative to the base is requested. However, once the participation factors and the total mean square responses are computed, the quasi-static displacements no longer serve any practical purpose and are not sought at the post-processing phase of the analysis. It would therefore

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be desirable to all-together avoid the computation of the quasi-static displacements if the participation factors as well as the quasi-static and covariance parts of the total mean square response could be determined by an alternative method. One would seek to avoid computation of the quasi-static displacements because these computations may become expensive for a medium to large model having for example over 100,000 dof's.

It has recently been shown [1] that the participation factors for support motion transient problems can be computed using modal solution quantities such as modal reaction forces and natural vibration frequencies. Chen et al. [1] have applied the method to deterministic problems in time domain. The present paper proposes to apply the alternative participation factor computation method to base-excitation multi-point random vibration analysis. A simple formula, based entirely on eigensolutions, is also developed for the quasi-static displacements. This formula is then used in the expression for the quasi-static and covariance components of the mean square response, thereby eliminating the need for quasi-static displacement solution.

The capabilities of the multi-point random vibration analysis module in the general-purpose finite element program such as ANSYS [5] may include:

1. multi-point nodal and base/support excitations,
2. fully correlated, uncorrelated and partially correlated random inputs,
3. spatially correlated random excitations,
4. random vibration under propagating excitation,
5. solutions for displacement (for example, structural dof's, stress components, reactions, etc.), velocities (for example, velocities for structural dof's, stress velocities, reaction velocities, etc.) and accelerations (for example, accelerations for structural dof's, stress accelerations, reaction accelerations, etc.),
6. contour display of response quantities over part or entire structure,
7. closed-form fail-proof integration of the response power spectral densities [2,4].

The random vibration capabilities of general-purpose finite element programs are routinely used by many corporations, agencies, and national laboratories in The United States and all over the world. For multi-point base-excitation case, the general-purpose programs presently carry out multiple quasi-static displacement solutions, which are then used in the computation of participation factors and quasi-static and covariance components of the response power spectral densities.

The formula for the alternative participation factor is derived in Section 2. It is then shown how this simple formula can be used to compute the "total" response power response spectral densities. A number of illustration problems are solved to demonstrate the validity of the formula.

2. Derivation of the participation factors

The mode-based random vibration analysis via FEM involves the following steps:

1. building the finite element model for the structure which includes geometric modeling, meshing, specifying material properties and boundary conditions,
2. performing modal analysis on the finite element model for the specified boundary conditions,
3. computing participation factors,
4. integrating modal covariance matrices with the participation factors,
5. combining modal covariance matrices with the mode shapes to compute the mean square responses,
6. computing response PSD for individual dof and covariance value for two different dof's,
7. displaying the mean square and response power spectral densities in post-processors.

The first and second steps can be completed using various features, processes and solution techniques available in the finite element program, yielding mode shape $\lfloor \boldsymbol{\phi}_j \ \mathbf{0} \rfloor^{\mathrm{T}}$ and natural frequencies ω_j for jth mode of vibration. The zero's in the mode shape vector represents homogeneous boundary conditions at the support. For nodal excitation problems, the participation factors can be simply computed from the mode shapes and the nodal excitation vector. However, when multiple supports of the structure are excited by partially correlated random inputs, the computation of the participation factors involves the quasi-static displacement solution. The discussion of the present paper will be limited to support excitation problems only. First, the dynamic equations of motion for discretized finite element system are written in partitioned form:

$$\begin{bmatrix} \mathbf{M} & \mathbf{M}_{\mathrm{C}} \\ \mathbf{M}_{\mathrm{C}}^{\mathrm{T}} & \mathbf{M}_{\mathrm{R}} \end{bmatrix} \begin{Bmatrix} \ddot{\mathbf{u}} \\ \ddot{\mathbf{u}}_{\mathrm{R}} \end{Bmatrix} + \begin{bmatrix} \mathbf{C} & \mathbf{C}_{\mathrm{C}} \\ \mathbf{C}_{\mathrm{C}}^{\mathrm{T}} & \mathbf{C}_{\mathrm{R}} \end{bmatrix} \begin{Bmatrix} \dot{\mathbf{u}} \\ \dot{\mathbf{u}}_{\mathrm{R}} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{K}_{\mathrm{C}} \\ \mathbf{K}_{\mathrm{C}}^{\mathrm{T}} & \mathbf{K}_{\mathrm{R}} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \mathbf{u}_{\mathrm{R}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{R} \end{Bmatrix} \quad (1)$$

where $\mathbf{M}$, $\mathbf{C}$ and $\mathbf{K}$: mass, damping and stiffness matrices corresponding to the free dof's $\mathbf{u}$; $\mathbf{M}_{\mathrm{R}}$, $\mathbf{C}_{\mathrm{R}}$ and $\mathbf{K}_{\mathrm{R}}$: mass, damping and stiffness matrices corresponding to the support dof's $\mathbf{u}_{\mathrm{R}}$ that are excited by random loading; $\mathbf{M}_{\mathrm{C}}$, $\mathbf{C}_{\mathrm{C}}$ and $\mathbf{K}_{\mathrm{C}}$: mass, damping and stiffness matrices that couple the free and restrained dof's; $\mathbf{R}$: the reaction vector.

It should be noted here that the support dof's that are not excited by any random excitation need not be included in Eq. (1).

For support excitation dynamic problems, the free displacements $\mathbf{u}$ are decomposed into pseudo-static and dynamic components:

$$\mathbf{u} = \mathbf{u}_s + \mathbf{u}_d \tag{2}$$

The pseudo-static displacements can be obtained from Eq. (1) after excluding the inertia and damping terms and by replacing $\mathbf{u}$ by $\mathbf{u}_s$:

$$\mathbf{u}_s = -\mathbf{K}^{-1}\mathbf{K}_C\mathbf{u}_R = \mathbf{A}\mathbf{u}_R \tag{3}$$

where

$$\mathbf{A} = -\mathbf{K}^{-1}\mathbf{K}_C \tag{4}$$

The elements of the ith column of the $\mathbf{A}$ matrix are the pseudo-static displacements due to a unit displacement of the support dof's excited by the ith base PSD input. It will be shown next that the $\mathbf{A}$ matrix is required to compute the participation factors. To this end, we substitute Eqs. (2) and (3) into (1) to arrive at:

$$\mathbf{M}\ddot{\mathbf{u}}_d + \mathbf{C}\dot{\mathbf{u}}_d + \mathbf{K}\mathbf{u}_d = -(\mathbf{M}\mathbf{A} + \mathbf{M}_C)\ddot{\mathbf{u}}_R - (\mathbf{C}\mathbf{A} + \mathbf{C}_C)\dot{\mathbf{u}}_R \tag{5}$$

The damping forces on the right hand side are small compared to the inertia forces and can be neglected:

$$\mathbf{M}\ddot{\mathbf{u}}_d + \mathbf{C}\dot{\mathbf{u}}_d + \mathbf{K}\mathbf{u}_d = -(\mathbf{M}\mathbf{A} + \mathbf{M}_C)\ddot{\mathbf{u}}_R \tag{6}$$

The damping forces on the right hand side of Eq. (5) are identically zero for stiffness proportional damping.

The undamped free vibration mode shapes $\lfloor \boldsymbol{\phi}_j \ \mathbf{0} \rfloor^T$ of the structure restrained at the support points can be used to decouple the equations of motion (6) to yield:

$$\ddot{y}_j + 2\xi_j\omega_j\dot{y}_j + \omega_j^2 y_j = \boldsymbol{\Gamma}_j^T\ddot{\mathbf{u}}_R \tag{7}$$

where the participation factors $\boldsymbol{\Gamma}_j$ are give by:

$$\boldsymbol{\Gamma}_j = -\frac{(\mathbf{M}\mathbf{A} + \mathbf{M}_C)^T\boldsymbol{\phi}_j}{\boldsymbol{\phi}_j^T\mathbf{M}\boldsymbol{\phi}_j} \tag{8}$$

and ξ_j are the modal damping ratio. The generalized coordinated y_j are related to the mode shapes $\boldsymbol{\phi}_j$ in the following manner:

$$\mathbf{u}_d = \boldsymbol{\phi}_j y_j \quad i = 1, n \tag{9}$$

n is the number of modes to be included in the random vibration analysis. To facilitate spectrum analysis downstream, general-purpose finite element programs [5] typically normalize the mode shapes to the mass matrix such that $\boldsymbol{\phi}_j^T\mathbf{M}\boldsymbol{\phi}_j = 1$. The participation factors are then given by:

$$\boldsymbol{\Gamma}_j = -(\mathbf{M}\mathbf{A} + \mathbf{M}_C)^T\boldsymbol{\phi}_j \tag{10}$$

It can be seen that the computation of participation factors will require:

(a) multiple static solutions, represented by the column vectors of the $\mathbf{A}$ matrix, for multi-point random vibration problem,
(b) the mass matrix $\mathbf{M}$ corresponding to free dof's,
(c) the so-called excited mass matrix $\mathbf{M}_C$ and
(d) the mode shapes $\boldsymbol{\phi}_j$.

The expression in Eq. (10) is currently used to compute participation factors for multi-point excitation random vibration analysis [2,3,5].

In order to express the participation factors entirely in terms of eigensolutions, the eigenproblem is written from equation of motion (1) in the following manner:

$$-\omega_j^2\begin{bmatrix} \mathbf{M} & \mathbf{M}_C \\ \mathbf{M}_C^T & \mathbf{M}_R \end{bmatrix}\begin{Bmatrix} \boldsymbol{\phi}_j \\ \mathbf{0} \end{Bmatrix} + \begin{bmatrix} \mathbf{K} & \mathbf{K}_C \\ \mathbf{K}_C^T & \mathbf{K}_R \end{bmatrix}\begin{Bmatrix} \boldsymbol{\phi}_j \\ \mathbf{0} \end{Bmatrix} = \begin{Bmatrix} \mathbf{0} \\ \mathbf{R}_j \end{Bmatrix} \tag{11}$$

Once again, $\mathbf{R}_j$ is the reaction vector for the jth mode of vibration at the support points that are excited by random loading. The eigenproblem can be divided into two sets of equations, one corresponding to the free dof's and the other corresponding to the support dof's excited by random loads:

$$-\omega_j^2\mathbf{M}\boldsymbol{\phi}_j + \mathbf{K}\boldsymbol{\phi}_j = \mathbf{0} \tag{12}$$

$$-\omega_j^2\mathbf{M}_C^T\boldsymbol{\phi}_j + \mathbf{K}_C^T\boldsymbol{\phi}_j = \mathbf{R}_j \tag{13}$$

We will require the transpose of these equations and so taking transpose we rewrite these two equations as:

$$\omega_j^2\boldsymbol{\phi}_j^T\mathbf{M} = \boldsymbol{\phi}_j^T\mathbf{K} \tag{14}$$

$$-\omega_j^2\boldsymbol{\phi}j^T\mathbf{M}_C + \boldsymbol{\phi}_j^T\mathbf{K}_C = \mathbf{R}_j^T \tag{15}$$

We will first express the quasi-static displacement solution $\mathbf{A}$ in terms of modal solutions. To this end, we post-multiply both sides of Eq. (14) by $\mathbf{u}_s$ and use the relation in Eq. (3) to obtain:

$$\omega_j^2\boldsymbol{\phi}_j^T\mathbf{M}\mathbf{u}_s = -\boldsymbol{\phi}_j^T\mathbf{K}_C\mathbf{u}_R$$

Substituting from Eq. (15) into the right hand side of this we arrive at:

$$-\omega_j^2\boldsymbol{\phi}_j^T[\mathbf{M} \quad \mathbf{M}_C]\begin{Bmatrix} \mathbf{u}_s \\ \mathbf{u}_R \end{Bmatrix} = \mathbf{R}_j^T\mathbf{u}_R \tag{16}$$

Performing matrix augmentation we rewrite the above as follows:

$$-\omega_j^2 \begin{Bmatrix} \boldsymbol{\phi}_j \\ 0 \end{Bmatrix}^{\mathrm{T}} \begin{bmatrix} \mathbf{M} & \mathbf{M}_{\mathrm{C}} \\ \mathbf{M}_{\mathrm{C}}^{\mathrm{T}} & \mathbf{M}_{\mathrm{R}} \end{bmatrix} \begin{Bmatrix} \mathbf{u}_{\mathrm{s}} \\ \mathbf{u}_{\mathrm{R}} \end{Bmatrix} = \mathbf{R}_j^{\mathrm{T}} \mathbf{u}_{\mathrm{R}} \tag{17}$$

Similar to the dynamic components displacement dof's (Eq. (9)) we can express the quasi-static part in terms of mode shapes and generalized coordinates:

$$\mathbf{u}_{\mathrm{s}} = \boldsymbol{\phi}_j z_j \tag{18}$$

If we now define these quasi-static generalized coordinates z_j as:

$$z_j = \boldsymbol{\phi}_j^{\mathrm{T}} [\mathbf{M} \quad \mathbf{M}_{\mathrm{C}}] \begin{Bmatrix} \mathbf{u}_{\mathrm{s}} \\ \mathbf{u}_{\mathrm{R}} \end{Bmatrix} \tag{19}$$

then, by virtue of Eq. (16), we can write this as:

$$z_j = -\frac{1}{\omega_j^2} \mathbf{R}_j^{\mathrm{T}} \mathbf{u}_{\mathrm{R}} \tag{20}$$

Substituting these generalized coordinates into Eq. (18), we obtain:

$$\mathbf{u}_{\mathrm{s}} = -\frac{1}{\omega_j^2} \boldsymbol{\phi}_j \mathbf{R}_j^{\mathrm{T}} \mathbf{u}_{\mathrm{R}} \tag{21}$$

Comparing this with the expression for the quasi-static solution (Eq. (3)), derived earlier, we are able to express the quasi-static matrix **A** in terms of mode shapes, natural frequencies and modal reactions:

$$\mathbf{A} = -\frac{1}{\omega_j^2} \boldsymbol{\phi}_j \mathbf{R}_j^{\mathrm{T}} \tag{22}$$

Let us now rewrite the expression for participation factor from Eq. (10) using the above redefinition of **A** matrix:

$$\begin{aligned} \boldsymbol{\Gamma}_j^{\mathrm{T}} &= -\boldsymbol{\phi}_j^{\mathrm{T}} \left(-\mathbf{M} \frac{1}{\omega_j^2} \boldsymbol{\phi}_j \mathbf{R}_j^{\mathrm{T}} + \mathbf{M}_{\mathrm{C}} \right) \\ &= \frac{1}{\omega_j^2} \boldsymbol{\phi}_j^{\mathrm{T}} \mathbf{M} \boldsymbol{\phi}_j \mathbf{R}_j^{\mathrm{T}} - \boldsymbol{\phi}_j^{\mathrm{T}} \mathbf{M}_{\mathrm{C}} \end{aligned} \tag{23}$$

Because the mass matrix is normalized such that $\boldsymbol{\phi}_j^{\mathrm{T}} \mathbf{M} \boldsymbol{\phi}_j = 1$, we get the following simplified expression for the computation of participation factor:

$$\boldsymbol{\Gamma}_j = \frac{\mathbf{R}_j}{\omega_j^2} - \mathbf{M}_{\mathrm{C}}^{\mathrm{T}} \boldsymbol{\phi}_j \tag{24}$$

This method of computation of participation factor eliminates the need for additional quasi-static or any other solution besides modal solutions, thereby saving significant computation cost.

3. Computation of "total" response power spectral density

In random vibration analysis, the random loading on the structure is represented in the form of PSD function. For a multi-point support or nodal excitation problem, the input PSD can be expressed in a general form as follows:

$$S(\omega) = \begin{bmatrix} S_{11}(\omega) & C_{12}(\omega) + \mathrm{i}Q_{12}(\omega) & \cdots \\ C_{12}(\omega) - \mathrm{i}Q_{12}(\omega) & S_{22}(\omega) & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix} \tag{25}$$

with $S_{ii}(\omega)$: auto-spectrum, $C_{12}(\omega)$: co-spectrum, and $Q_{12}(\omega)$ quad-spectrum [6].

The normalized cross PSD function, called the coherence function or correlation coefficient, is defined as:

$$\rho_{mn}(\omega) = \frac{|\mathrm{C}_{mn}(\omega) - \mathrm{i}Q_{mn}(\omega)|}{S_{mn}(\omega) S_{nn}(\omega)}, \quad -1 \leqslant \rho_{mn}(\omega) \leqslant 1 \tag{26}$$

ω is forcing frequency. For the special case in which all cross-terms are zero, the input spectra are said to be uncorrelated.

For some applications, the excitations constitute a homogeneous random field with a single propagating plane wave. In that case, the input PSD can be expressed as [3]:

$$S_{lm}(\omega) = S_0(\omega) |\gamma(D_{lm}, \omega)| (\mathrm{e}^{-\mathrm{i}\omega d_{lm}}) \tag{27}$$

where $\{D_{lm}\} = \{x_m - x_l\}$: separation vector between excitation points l and m, $d_{lm} = (\lfloor D_{lm} \rfloor \{V\})/V^2$: time delay for wave to propagate from l to m, $\{V\}$: velocity of propagation of the traveling wave, $S_0(\omega)$: input PSD of the excitation at any point, $|\gamma(D_{lm}, \omega)|$: absolute value of the coherency between the two excitations l and m, $\{x_l\}$, $\{x_m\}$: nodal coordinates of excitation points l and m.

This form of the input PSD's is widely used to characterize spatially varying support motion, and may also be useful for applications involving propagating pressure waves.

Particular forms of Eq. (27) may also be considered. For example, the degree of correlation between excited nodes may solely be based upon the distance between two excited nodes (Fig. 1):

$$[S(\omega)] = S_0(\omega) \begin{bmatrix} 1 & \alpha_{12} \\ \alpha_{12} & 1 \end{bmatrix} \tag{28}$$

where:

$$\alpha_{12} = \begin{cases} 0, & \text{if } D_{12} \geqslant R_{\max} \\ 1, & \text{if } D_{12} \leqslant R_{\min} \\ \dfrac{R_{\max} - D_{12}}{R_{\max} - R_{\min}}, & \text{if } R_{\min} < D_{12} < R_{\max} \end{cases}$$

D_{12}: distance between the two excitation points 1 and 2.

The homogeneous random field travelling with a propagating velocity can also be considered with the coherency function taken as unity, i.e., $|\gamma(D_{lm}, \omega)| = 1$ in Eq. (27).

The theory of random vibration, along with the method of mode superposition, can be employed to re-

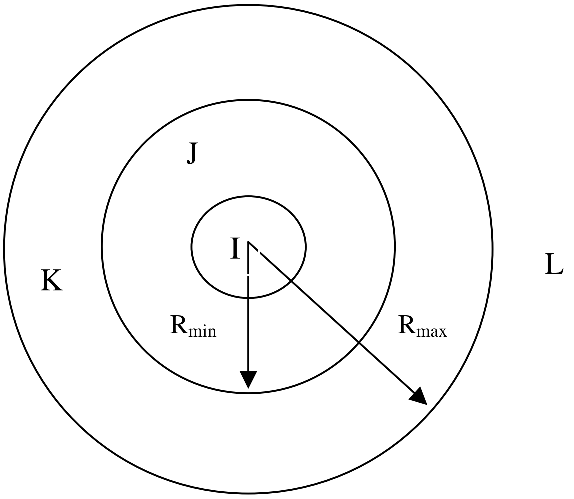

Fig. 1. Spatial correlation based on distance between excited nodes.

late the input PSD to output or response PSD at a particular free dof "i" on the structure:

Dynamic part:

$$S_{\mathrm{d}_i}(\omega) = \sum_{j=1}^{n}\sum_{k=1}^{n}\phi_{ij}\phi_{ik}\sum_{l=1}^{r}\sum_{m=1}^{r}\Gamma_{lj}\Gamma_{mk}H_j^*(\omega)H_k(\omega)S_{lm}(\omega) \tag{29}$$

Pseudo-static part:

$$S_{\mathrm{s}_i}(\omega) = \sum_{l=1}^{r}\sum_{m=1}^{r}A_{il}A_{im}\left(\frac{1}{\omega^4}S_{lm}(\omega)\right) \tag{30}$$

Covariance part:

$$S_{\mathrm{sd}_i}(\omega) = \sum_{j=1}^{n}\sum_{l=1}^{r}\sum_{m=1}^{r}\phi_{ij}A_{il}\left(-\frac{1}{\omega^2}\Gamma_{mj}H_j(\omega)S_{lm}(\omega)\right) \tag{31}$$

where n: number of mode shapes chosen for random vibration analysis; r: number of support excitation input PSD tables; $H_j(\omega)$: the jth modal transfer function with force or acceleration taken as the input and displacement taken as the output; $H_j^*(\omega)$: the complex conjugate of $H_j(\omega)$:

$$H_j(\omega) = \frac{1}{\omega_j^2 - \omega^2 + \mathrm{i}(2\xi_j\omega_j\omega)} \tag{32}$$

The "total" value, as opposed to the dynamic or relative value, of the mean square response of the ith free displacement component is then computed as:

$$\begin{aligned}\sigma_{\mathrm{u}_i}^2 &= \int_0^\infty S_{\mathrm{d}_i}(\omega)\,\mathrm{d}\omega + \int_0^\infty S_{\mathrm{s}_i}(\omega)\,\mathrm{d}\omega \\ &\quad + 2\left|\int_0^\infty S_{\mathrm{sd}_i}(\omega)\,\mathrm{d}\omega\right|_{\mathrm{Re}} \\ &= \sigma_{\mathrm{d}_i}^2 + \sigma_{\mathrm{s}_i}^2 + 2C_{\mathrm{v}}(u_{\mathrm{s}_i}, u_{\mathrm{d}_i})\end{aligned} \tag{33}$$

where $|\ |_{\mathrm{Re}}$: the real part of the argument; $\sigma_{\mathrm{d}_i}^2$: variance of the ith relative or dynamic free displacements; $\sigma_{\mathrm{s}_i}^2$: variance of the ith pseudo-static displacements; $C_{\mathrm{v}}(u_{\mathrm{s}_i}, u_{\mathrm{d}_i})$: covariance between the static and dynamic displacements.

It is seen that the computation of the quasi-static and the covariance parts requires the quasi-static matrix **A**. We can however use the simpler alternate definition of the quasi-static matrix **A**, given by Eq. (22) and rewrite the quasi-static and covariance components of the response power spectral densities given in Eqs. (30) and (31) as follows:

$$S_{\mathrm{s}_i}(\omega) = \sum_{j=1}^{n}\sum_{k=1}^{n}\phi_{ij}\phi_{ik}\frac{1}{\omega^4\omega_j^2\omega_k^2}\sum_{l=1}^{r}\sum_{m=1}^{r}R_{jl}R_{km}S_{lm}(\omega) \tag{34}$$

$$S_{\mathrm{sd}_i}(\omega) = \sum_{j=1}^{n}\sum_{k=1}^{n}\frac{1}{\omega^2\omega_k^2}\phi_{ij}\phi_{ik}\sum_{l=1}^{r}\sum_{m=1}^{r}R_{kl}\Gamma_{mj}H_j(\omega)S_{lm}(\omega) \tag{35}$$

With these, the need to compute the quasi-static matrix **A** is now completely eliminated.

4. Illustrative examples

Three example problems are solved here to demonstrate that the formula developed in the present paper (Eq. (24)) produces the same participation factors as computed by Eq. (10). The computation of participation factors using Eq. (10) involves the use of the quasi-static solution, the **A** matrix, as part of the computation. The new method involves gathering the modal reaction forces corresponding to the excited support dof's and dividing these reactions by the square of the circular natural frequency of vibration. The modal contribution from the so-called excited mass $\mathbf{M}_{\mathrm{C}}$ is then subtracted from this outcome to obtain the modal participation factors.

The first illustration problem is selected because the components of Eq. (10) can be computed by hand. For the second and third problems, the participation factors calculated using Eq. (24) is compared against those produced by Eq. (10).

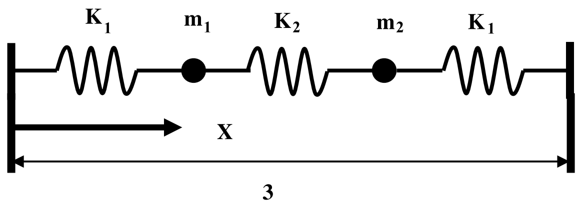

Fig. 2. Spring-mass supported at both ends.

4.1. Example 1

A simple one-dimensional spring-mass problem is first solved. This problem will allow us to perform all necessary computations by hand. The masses for this model (Fig. 2) are taken as $m_1 = 0.5$ and $m_2 = 1.0$, whereas the spring constants are $k_1 = 42{,}832$ and $k_2 = 32{,}416$. The model is supported at both ends. The two supports are excited by two uncorrelated random signals. Although the model has four nodes, it is actually a two dof's system represented by the two lumped masses m_1 and m_2. The assembled mass matrix is then given by:

$$[\mathbf{M}] = \begin{bmatrix} 0.0 & 0.0 & 0.0 & 0.0 \\ 0.0 & 0.5 & 0.0 & 0.0 \\ 0.0 & 0.0 & 1.0 & 0.0 \\ 0.0 & 0.0 & 0.0 & 0.0 \end{bmatrix}$$

In this case, the elements of the excited mass matrix $\mathbf{M}_C$ are identically zero. Thus, the participation factors can simply be computed by: $\Gamma_{lj} = -(M_{mn}A_{ml})^{\mathrm{T}}\phi_{nj}$ or, $\Gamma_{lj} = R_{lj}/\omega_j^2$. The frequencies of vibration are computed as: $f_1 = 36.835$ and $f_2 = 66.040$ Hz and the mode shapes are:

$$\boldsymbol{\phi}_1 = \lfloor 0.0 \quad 0.60464 \quad 0.90399 \quad 0.0 \rfloor$$
$$\boldsymbol{\phi}_2 = \lfloor 0.0 \quad 1.27840 \quad -0.42755 \quad 0.0 \rfloor$$

The modal reactions and the unit solution vectors are found to be:

$$R_{lj} = \begin{bmatrix} -25,898 & -54,758 \\ -38,720 & +18,318 \end{bmatrix}$$

$$A_{ij} = \begin{bmatrix} 1.00000 & 0.00000 \\ 0.69892 & 0.30108 \\ 0.30108 & 0.69892 \\ 0.00000 & 1.00000 \end{bmatrix}$$

Using Eq. (10), we can now compute the participation factors as:

$$\Gamma_{lj} = -(M_{mn}A_{ml})^{\mathrm{T}}\phi_{nj} = \begin{bmatrix} -0.48347 & -0.31802 \\ -0.72284 & +0.10637 \end{bmatrix}$$

We can now employ the new formula developed in this paper to compute the participation factors as follows:

$$\Gamma_{lj} = \frac{R_{lj}}{\omega_j^2} = \begin{bmatrix} \dfrac{-25,898}{(2\pi \times 36.835)^2} & \dfrac{-54,758}{(2\pi \times 66.040)^2} \\ \dfrac{-38,720}{(2\pi \times 36.835)^2} & \dfrac{+18,318}{(2\pi \times 66.040)^2} \end{bmatrix} = \begin{bmatrix} -0.48349 & -0.31803 \\ -0.72286 & +0.10639 \end{bmatrix}$$

which are seen to be the same, within the round-off tolerance, as computed above. This proves validity of the simpler formula proposed in the paper. Note that it is sufficient to prove the validity of the formula for the participation factors given by Eq. (24), which in turn validates the formula for the **A** matrix. However, it would be worthwhile to carry out the calculations required by Eq. (22) to arrive at **A** matrix. Using the natural frequency vector ω_j, mode shape matrix ϕ_{nj} and reaction matrix R_{lj}, we get the $A_n l$ matrix as:

$$A_{nl} = \begin{bmatrix} 0.00000 & 0.00000 \\ 0.69891 & 0.30106 \\ 0.30109 & 0.69895 \\ 0.00000 & 0.00000 \end{bmatrix}$$

which is seen to be the same as shown above within a level of tolerance. Note that the computed **A** matrix does not contain the prescribed unit displacements at the support dof's.

4.2. Example 2

The second example problem is a two-span simply-supported beam as shown in Fig. 3. The beam has unit cross-sectional area and thickness. The moment of inertia about the Z-axis is taken as 1000. The density, modulus of elasticity and Poisson's ratio are 0.7, 30E6 and 0.33 respectively. The beam is excited at the three supports in the Y-direction by three identical but uncorrelated random signals whose PSD is shown in Fig. 4.

The beam is discretized using 60 two-node beam elements having UX, UY and ROTZ dof's per node [5]. A modal analysis is performed with simply supported boundary conditions as shown in Fig. 3. A total of 20

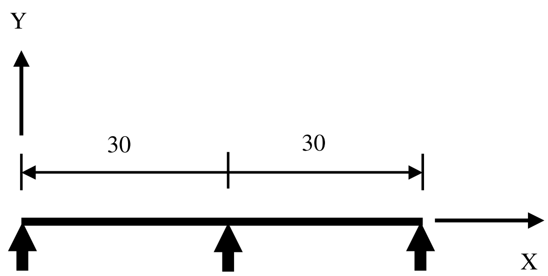

Fig. 3. Two-span beam excited by three uncorrelated random inputs.

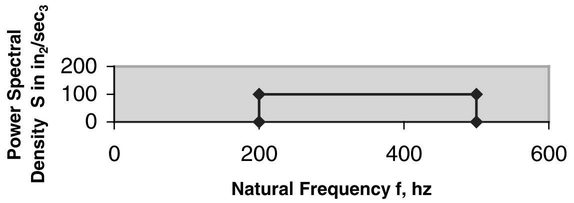

Fig. 4. Input PSD.

modes are extracted. The natural frequencies and the modal reactions are presented in Table 1. The modes were checked and were found to be of adequate quality given the level of discretization.

The participation factors for the random excitations at the three supports are computed using the natural frequencies, modal reactions, mode shape matrix and the so-called excited mass matrix (Eq. (24)) and are shown in Tables 2–4. The participation factors computed using Eq. (10) are also presented in these tables. The participation factor solutions, computed by Eq. (24), are seen to be practically identical to those given by Eq. (10).

Note that the participation factors for only a first few important modes are presented. The remaining modes are relatively unimportant and will not contribute appreciably to the final solution. The participation factors for the unimportant modes are relatively small numbers and may show large percent differences if they are compared against each other.

Table 2
Participation factors for two-span beam for excitation at the left support

Mode	Participation factors Γ_{1j}		Difference (%)
	Eq. (24)	Eq. (10)	
1	4.22E−01	4.22E−01	−0.01
4	7.37E−01	7.37E−01	−0.01
5	1.09E−01	1.09E−01	−0.01
8	6.20E−01	6.20E−01	−0.07
12	3.48E−01	3.48E−01	0.00
16	3.21E−01	3.20E−01	0.19
20	2.28E−01	2.28E−01	0.03

Table 3
Participation factors for two-span beam for excitation at the middle support

Mode	Participation factors Γ_{2j}		Difference (%)
	Eq. (24)	Eq. (10)	
4	−2.93E+00	−2.93E+00	0.01
8	−1.64E+00	−1.64E+00	0.07
12	−1.15E+00	−1.15E+00	0.00
16	−8.87E−01	−8.87E−01	0.02
20	−7.23E−01	−7.23E−01	−0.01

Table 1
Natural frequencies and modal reactions for two-span beam problem

Mode	Frequency (Hz)	Modal reactions		
		R_{1j}[a]	R_{2j}[a]	R_{3j}[a]
1	104.450468	0.21697E+07	−38.085	−0.21697E+07
2	109.158807	105.56	87.443	−6.0466
3	109.158807	223.63	166.64	191.27
4	150.552453	−0.45036E+07	−0.32596E+07	−0.45036E+07
5	215.772529	0.89202E+07	−78.027	−0.89189E+07
6	218.616944	1631.2	−935.77	2283.9
7	218.616945	−291.86	180.93	−1479.9
8	265.037145	0.13528E+08	−0.79371E+07	0.13527E+08
9	325.683271	−0.20033E+08	−12781	0.20024E+08
10	328.674522	−1179.7	−24.000	1254.1
11	328.674522	7.4203	−2.2150	−2.2091
12	376.385013	−0.26713E+08	−0.16034E+08	−0.26713E+08
13	435.206603	0.35280E+08	−6320.2	−0.35285E+08
14	439.633090	1749.4	−111.33	−868.61
15	439.633094	4604.9	−3485.5	−10796.
16	486.735841	0.43971E+08	−0.28905E+08	0.43948E+08
17	544.580574	−0.54327E+08	−5770.9	0.54325E+08
18	551.796032	−7895.8	1013.6	4624.8
19	551.796034	859.88	131.18	−1244.0
20	596.652484	−0.64816E+08	−0.47794E+08	−0.64818E+08

[a] l (letter l) = 1 for excitation at left support, l (letter l) = 2 for excitation at middle support, l (letter l) = 3 for excitation at right support.

Table 4
Participation factors for two-span beam for excitation at the right support

Mode	Participation factors Γ_{3j}		Difference (%)
	Eq. (24)	Eq. (10)	
1	−4.22E−01	−4.22E−01	−0.01
4	7.37E−01	7.37E−01	−0.01
5	−1.08E−01	−1.09E−01	−0.61
8	6.20E−01	6.20E−01	−0.12
12	3.48E−01	3.48E−01	0.00
16	3.19E−01	3.20E−01	−0.31
20	2.28E−01	2.28E−01	0.02

4.3. Example 3

Finally, a simply supported 16×8 rectangular plate is subjected to four uncorrelated random excitations in the vertical Z-direction (Fig. 5). The thickness of the plate surface is taken as unity. The Young's modulus, Poisson's ratio and density for the material of the plate are assumed respectively as 30E6, 0.33 and 0.7. The plate is discretized using 16×8 plate elements having six dof's per node. The plate is supported in the X, Y and Z directions at the four corners.

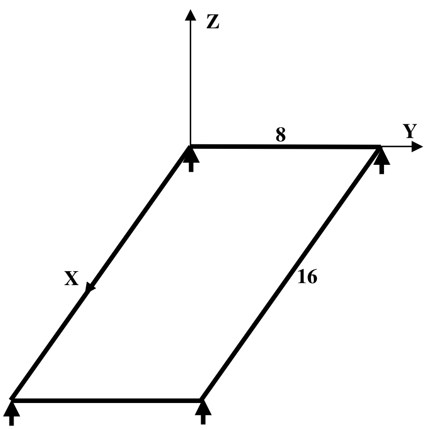

Fig. 5. Simply supported plate problem.

A modal analysis is performed on the plate finite element model and 20 modes are extracted. The natural frequencies for the first 13 important modes are shown in Table 5. The remaining modes will not participate appreciably in the response given the type of excitation considered here. The mode shapes for a few important modes (mode number 1, 3, 11, and 15) are plotted in Fig. 6(a–d). The modal vertical reaction sets and the appropriate rows for the excited mass are gathered for the four corner supports. The participation factors are then computed using Eq. (24) and are shown in Table 5. The participation factors computed by Eq. (10) are almost identical to the ones computed using Eq. (24) and so are not shown here.

A study is performed to determine the contribution of the excited mass term in Eqs. (10) and (24) to the final participation factors. Table 6 presents the results of such a study. It is seen that the contribution of the excited mass term is less than 5% for the 13 modes considered.

5. Conclusions

The present work demonstrates that the computation of participation factors for multi-point support-motion random vibration analysis via FEM, based on the mode superposition technique, does not require the so-called quasi-static solution. Once the eigenvalue solution has been performed, a simple expression can be developed for the quasi-static solution in terms of the modal reaction vector, mode shapes and natural frequencies.

Table 5
Natural frequencies and participation factors for the plate problem

Mode	Frequency (Hz)	Participation factors (Eq. (24))			
		Γ_{1j}	Γ_{2j}	Γ_{3j}	Γ_{4j}
1	11.518	−2.15930	−2.15930	−2.15930	−2.15930
2	33.647	1.23250	1.23250	−1.23250	−1.23250
3	41.060	1.23720	−1.23710	1.23710	−1.23710
4	64.504	−0.84976	−0.84976	−0.84976	−0.84976
5	78.922	−0.63090	0.63085	0.63090	−0.63085
7	87.693	0.36902	−0.36915	0.36919	−0.36905
9	122.864	−0.22921	−0.22916	−0.22914	−0.22909
11	139.754	−0.45245	−0.45264	0.45257	0.45249
12	154.757	−0.10770	−0.10783	−0.10761	−0.10782
13	184.669	0.29490	−0.29490	0.29490	−0.29490
15	206.713	−0.37142	0.37138	0.37143	−0.37136
18	254.419	0.23586	0.23599	−0.23590	−0.23586
19	264.624	−0.25070	−0.25091	−0.25088	−0.25097

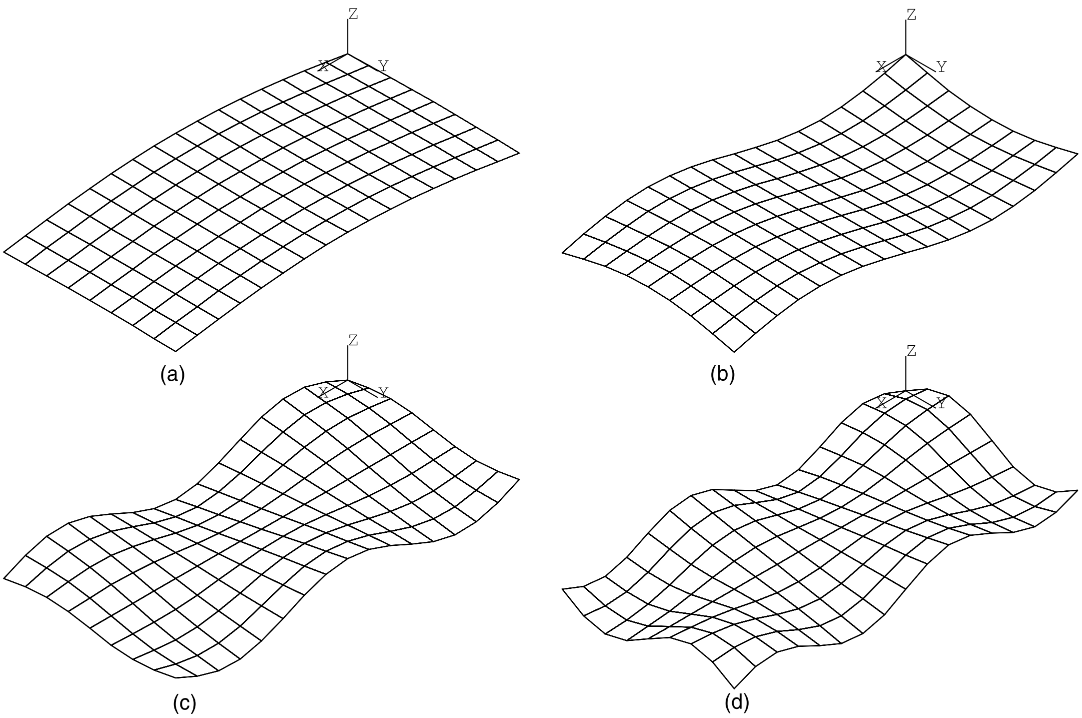

Fig. 6. PSD analysis for four uncorrelated loads on the plate (a) mode 1, (b) mode 3, (c) mode 11 and (d) mode 15.

Table 6
Comparison of participation factors for excitation at support 1 produced using Eq. (10) for the plate problem

Mode	Difference (%) in participation factors with excited mass term	Difference (%) in participation factors without excited mass term
1	0.00	0.09
2	0.00	0.2
3	0.00	0.38
4	0.00	0.68
5	−0.01	0.85
7	0.03	1.02
9	−0.02	1.65
11	0.03	2.04
12	0.02	1.91
13	0.00	2.84
15	−0.01	3.48
18	0.03	4.68
19	0.03	4.66

This simple expression can then be utilized (a) to formulate the expression for the participation factors and (b) to compute the quasi-static and covariance components of the response power spectral densities. This all together eliminates the need for multiple static solutions, which can become expensive for medium to large problems.

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