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Analytical integrations in 3D BEM: preliminaries

A. Carini, A. Salvadori

Abstract This work provides a preliminary contribution in the context of analytical integrations of strongly and hyper singular kernels in boundary element methods (BEMs) in 3D. It concerns the integral of $1/r^3$ over a triangle in $\mathbf{R}^3$, that plays a fundamental role in BEMs in 3D, especially for the Galerkin implementation. Because the existence of the aforementioned integral depends on the position of the source point, all significant instances of the position of the source point will be considered and detailed. For its interest in the context of BEM, the integral is also considered in the more general sense of finite part of Hadamard.

Keywords Boundary element method, Analytical integration

1 Introduction

Boundary integral equations [1, 2] represent a classical formulation for many engineering problems. Their numerical solution, towards the boundary element method (BEM), reveals computationally effective when non-linear phenomena (if any) take place only along the boundaries. Despite this lack of generality, BEM is widely used in potential problems and linear elasticity [3], both in static and dynamic [4, 5], in fracture mechanics problems, even in presence of internal pressure [6] and frictional contact [7], in multidomain problems with non-linear interfaces [8]. Reviews on the various applications of BEM can be found, among others, in [9, 10]. The boundary integral formulation of linear elasticity is taken as a prototype in the frame of the present work.

Consider therefore a homogeneous solid with domain $\Omega \subset \mathbf{R}^3$ and with boundary $\Gamma = \Gamma_u \cup \Gamma_p$. Assuming small strains and displacements, its response to quasi-static external actions: tractions $\bar{\mathbf{p}}(\mathbf{x})$ on Γ_p, displacements $\bar{\mathbf{u}}(\mathbf{x})$ on Γ_u and domain forces $\bar{\mathbf{f}}(\mathbf{x})$ in Ω is studied. The well known Somigliana's identity, which stems from Green's second theorem, is the boundary integral representation of displacements, in the interior of the domain, $\mathbf{x} \in \Omega$, for the aforementioned linear elastic problem. The Somigliana's identity is based on Green's functions (also called kernels, see Appendix 2) which represent components u_i of the displacement vector $\mathbf{u}$ in a point $\mathbf{x}$ due to: (i) a unit force concentrated in space (point $\mathbf{y}$) and acting on the unbounded elastic space Ω_∞ in direction j (such functions are gathered in matrix $\mathbf{G}_{uu}(\mathbf{x}-\mathbf{y})$); (ii) a unit relative displacement concentrated in space (at a point $\mathbf{y}$), crossing a surface with normal $\mathbf{l}(\mathbf{y})$ and acting on the unbounded elastic space Ω_∞ (in direction j) (gathered in matrix $\mathbf{G}_{up}(\mathbf{x}-\mathbf{y})$).

Because all above introduced kernels are infinitely smooth in their domain, which is the whole space $\mathbf{R}^3$ with exception of the origin (that is when $\mathbf{x} \neq \mathbf{y}$), the traction operator can be applied to Somigliana's identity, thus obtaining the boundary integral representation of tractions on a surface of normal $\mathbf{n}(\mathbf{x})$ in the interior of the domain. Such a representation formula (by some authors named "hypersingular identity" [11]) involves Green's functions (collected in matrices $\mathbf{G}_{pu}$ and $\mathbf{G}_{pp}$) which describe components (p_i) of the traction vector $\mathbf{p}$ on a surface of normal $\mathbf{n}(\mathbf{x})$ due to: (i) a unit force concentrated in space (point $\mathbf{y}$) and acting on the unbounded elastic space Ω_∞ in direction j; (ii) a unit relative displacement concentrated in space (at a point $\mathbf{y}$) , crossing a surface with normal $\mathbf{l}(\mathbf{y})$ and acting on the unbounded elastic space Ω_∞ (in direction j).

Boundary integral equations (BIEs) for the linear elastic problem can be derived from the aforementioned two representation formulae performing the boundary limit $\Omega \ni \mathbf{x} \to \mathbf{x}^o \in \Gamma$. For the hypersingular identity, the boundary limit must be considered at a smooth point $\mathbf{x}^o$ with a well defined normal vector $\mathbf{n}(\mathbf{x}^o)$ [12]. The two integral equations, usually referred to as displacements and traction equations, are also called "dual" boundary integral equations [13].

In the limit process, singularities of Green's functions are triggered off. Kernel $\mathbf{G}_{uu}$ shows a singularity (named "weak") of $O(r^{-1})$; kernels $\mathbf{G}_{up}$ and $\mathbf{G}_{pu}$ present a strong singularity of $O(r^{-2})$; kernel $\mathbf{G}_{pp}$ is usually named hypersingular because it shows a singularity of $O(r^{-3})$ greater than the dimension of the integral [14]. Following the approach of [15], all singular terms cancel out in the limit process (and without the recourse to any a-priori interpretation in the finite part sense). Though, there exists an intimate relationship between hypersingular BIEs and finite part integrals (HFP) in the sense of Hadamard [16]: in [14] and [18] among others, it has been proved that a hypersingular integral can be interpreted as a HFP in the limit as an internal source point approaches the boundary. In [17], the same conclusion has been obtained by a different definition of HFP, without the need for any limit process.

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The numerical solution of the discrete set of integral equations is generally performed toward two different techniques: the "collocation" method [9] and the Galerkin approach [10], which is stated on the weak form of the integral equations [19, 20]. In recent years, the increasing use of the Symmetric Galerkin BEM stimulated a considerable amount of research in the area of efficient evaluation of double "integrals" containing singular and hypersingular kernel functions. In fact, the evaluation of (hyper) singular integrals still remains the highest difficulty within the implementation of Galerkin BEM.

Three main techniques (regularization methods, numerical approximations and analytical integrations) have been proposed for the evaluation of singular and hypersingular integrals. Analytical integrations have been basically performed in 2D (only a few works appeared in the 3D context, see e.g. [21–23]), towards different schemes. In the first scheme (see e.g. [11, 15, 24–26]), the source point is fixed, while the boundary around the source point is temporarily deformed to allow an analytical evaluation of contributions from singular and hypersingular kernels, and then the limit is taken as the deformed boundary shrinks back to the actual boundary. All singular and hypersingular integrations are performed analytically, while standard quadrature formulae are used for non-singular integrals. In a second approach, see among others [27–29], the source point $\mathbf{x}$ is first moved away from the boundary; integrals are evaluated analytically and a limit process is then performed to bring the source point back to the boundary. In a third fashion, the direct evaluation of the HFP and of the CPV has been performed in [30, 31].

The present note is preliminary to the complete analytical integration of kernels in 3D, that will be published in a forthcoming paper (see also [32]). This work only concerns the nature and the analytical integration of $1/r^3$ over a triangle, say T_j, that plays a major role in strong and hypersingular kernels. Because the aforementioned integral depends on the position of the source point $\mathbf{x}$ with respect to T_j, all significant instances of the position of the source point will be analyzed. In particular, when the source point $\mathbf{x}$ belongs to the triangle T_j, the integral does not exist in a classical sense. The HFP of such a divergent integral has a perfect meaning though and an interesting property of continuity (with respect to the source point) between the HFP and the Lebesgue integral is shown. To this aim, the HFP has been directly evaluated as first; further, the limit process to the boundary $\Omega \ni \mathbf{x} \to \mathbf{x}^o \in \Gamma$ has been performed.

In Sect. 2 notations and the local orthogonal reference adopted in the subsequent paragraphs are explained. The Lebesgue integral is thereafter performed in Sect. 3, discussing separately the two items of source point outside of the plane of the triangle T_j and of source point inside of such a plane. The HFP is analyzed in Sect. 4 considering a square neighborhood around the source point (the equivalence with a circular neighborhood can be found in [23]). Remarks of Sect. 5 conclude the work, whereas Appendix 1 includes a property of the $\arctan(x)$ function that is relevant to the proposed analytical integration.

2 Notation

Let Γ_h be a triangulation of the boundary Γ and let T_j be the generic triangle of Γ_h. Let $\mathscr{L} \equiv \{y_1, y_2, y_3\}$ define a local coordinate system such that: (i) a vertex of T_j is the origin; (ii) the plane $y_1 = 0$ contains T_j; (iii) the plane $y_3 = 0$ is orthogonal to the side of T_j opposite to the origin (see Fig. 1). In $\mathscr{L}$, T_j is defined by:

$$T_j := \{\mathbf{y} \in \mathbf{R}^3 \text{ s.t. } y_1 = 0; 0 \le y_2 \le \bar{y}_2; \quad ay_2 - y_3 \le 0; by_2 - y_3 \ge 0\}$$

where a and b denote the slopes of the two sides of T_j that cross the origin (see again Fig. 1). In $\mathscr{L}$ consider the field point $\mathbf{y} \in T_j$, the source point $\mathbf{x} \in \mathbf{R}^3$, the vector $\mathbf{d} = (\mathbf{y} - \mathbf{x})$ and denote with $r = \|\mathbf{x} - \mathbf{y}\| = \sqrt{d_1^2 + d_2^2 + d_3^2}$ the usual norm of $\mathbf{d}$ in $\mathbf{R}^3$.

This paper is mainly focused on the evaluation of the integral:

$$F(\mathbf{x}) = \int_{T_j} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^3} \, d\mathbf{y} = \int_0^{\bar{y}_2} \int_{ax_2}^{bx_2} \frac{1}{\|\mathbf{x} - \mathbf{y}\|^3} \, dy_3 \, dy_2 \quad (1)$$

in the coordinate system $\mathscr{L}$. Adopting the new variable $\mathbf{d} = \mathbf{y} - \mathbf{x}$, integral (1) becomes:

$$F(\mathbf{x}) = \int_{-x_2}^{\bar{y}_2 - x_2} \int_{ad_2 + k_a}^{bd_2 + k_b} \frac{1}{r^3} \, dd_3 \, dd_2 \quad (2)$$

in which $k_a := ax_2 - x_3$ and $k_b := bx_2 - x_3$. By the definition of a and b, $k_a = 0$ and $k_b = 0$ are the equations of the two sides of T_j that cross the origin. The integral (2) will be expressed as the sum of the two factors

$$F(\mathbf{x}) = f(\mathbf{x}, \bar{y}_2 - x_2) - f(\mathbf{x}, -x_2)$$

where $f : \{\mathbf{R}^3 \backslash T_j \times [-x_2, \bar{y}_2 - x_2]\} \to \mathbf{R}$ is defined by:

$$f(\mathbf{x}, d_2) = \int dd_2 \int_{ad_2 + k_a}^{bd_2 + k_b} \frac{1}{r^3} \, dd_3 = f^b(\mathbf{x}, d_2) - f^a(\mathbf{x}, d_2) \quad (3)$$

Integral (1) takes sense when $\mathbf{x} \notin T_j$ and is evaluated in Sect. 3. To make such section lighter, a flow chart of the

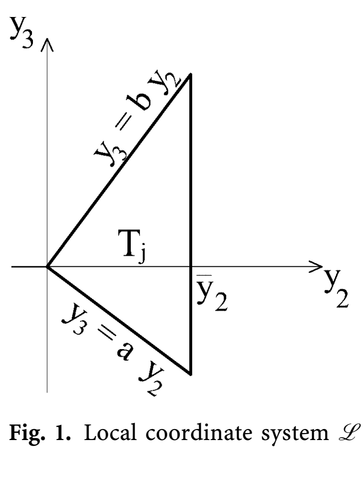

Fig. 1. Local coordinate system $\mathscr{L}$

if $x_1 \neq 0$ (*that is, the field point* $\mathbf{x}$ *does not lie in the plane of the triangle* T_j; z_1, z_2, z_{13}, z_{23}, η_{13}, η_{23}, f_1^b, f_2^b *as in paragraphs 3.1.1, 3.1.2*)

if $x_1^2(1+b^2) < k_b^2$ **then** $f^b = \begin{cases} -\infty < d_2 \leq z_{13} & f_2^b \\ z_{13} \leq d_2 \leq z_{23} & f_1^b + \frac{\pi}{4x_1}\mathrm{sgn}(\eta_{13}) \\ z_{23} \leq d_2 < +\infty & f_2^b + \frac{\pi}{4x_1}(\mathrm{sgn}(\eta_{13}) - \mathrm{sgn}(\eta_{23})) \end{cases}$

else if $x_1^2(1-b^2) \geq k_b^2$ **then** $f^b = f_1^b$

else if $z_3 < z_1 < z_2$ **then** $f^b = \begin{cases} -\infty < d_2 \leq z_{13} & f_1^b \\ z_{13} \leq d_2 < +\infty & f_2^b - \frac{\pi}{4x_1}\mathrm{sgn}(\eta_{13}) \end{cases}$

else $f^b = \begin{cases} -\infty < d_2 \leq z_{23} & f_2^b \\ z_{23} \leq d_2 < +\infty & f_1^b + \frac{\pi}{4x_1}\mathrm{sgn}(\eta_{23}) \end{cases}$

else (*that is, the field point* $\mathbf{x}$ *lies in the plane of the triangle* T_j, *see section 3.2*)

if $k_b \neq 0$ **then** $f^b = -\frac{\sqrt{d_2^2+(bd_2+k_b)^2}}{d_2 k_b}$

else $f^b = -\frac{b}{\sqrt{d_2^2+b^2\,d_2^2}}$

Fig. 2. A flow chart of $f^b(\mathbf{x}, d_2)$

expressions of $f^b(\mathbf{x}, d_2)$ is presented in Fig. 2. For its interest in the context of BEM, integral (1) is evaluated also in the sense of finite part of Hadamard for $\mathbf{x} \in T_j$ in Sect. 4. A relationship between the two instances is set performing the "limit to the boundary" $T_j \not\ni \mathbf{x} \to \mathbf{x}_0 \in T_j$ (Sect. 4).

3
Lebesgue integral

By the assumptions $\mathbf{x} \notin T_j$ and $\mathbf{y} \in T_j$, the function $1/r^3$ is infinitely smooth and from the calculus fundamental theorem the function $f(\mathbf{x}, d_2) \in C^\infty(\mathbf{R}^3 \backslash T_j, [-x_2, \bar{y}_2 - x_2])$. Keeping in mind that $r = \sqrt{d_1^2 + d_2^2 + d_3^2}$, it holds:

$$i_0(\mathbf{x}, d_2, d_3) := \int \frac{1}{r^3}\mathrm{d}d_3 = \frac{d_3}{d_1^2 + d_2^2}\frac{1}{r}$$

It is immediate to see that i_0 is not defined when:

(i) $r = 0$, never fulfilled.
(ii) $d_1^2 + d_2^2 = 0$, a new condition which has nothing to do with $\mathbf{x} \notin T_j$ (a suitable choice of x_3 is sufficient to prove it). In fact, if $d_1^2 + d_2^2 = 0$ one has:

$$i_0 = \int d_3^{-3/2}\mathrm{d}d_3 = -\frac{1}{2}\frac{\mathrm{sign}(d_3)}{d_3^2} \tag{4}$$

where $\mathrm{sign}(x) := x/|x|$.

Taking into account of Eq. (4), it's easy to show that:

$$I_0(\mathbf{x}, d_2) := \int_{ad_2+k_a}^{bd_2+k_b} \tfrac{1}{r^3}\mathrm{d}d_3 = \begin{cases} \frac{d_3}{d_1^2+d_2^2}\frac{1}{r}\Big|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} & \text{when } d_1^2 + d_2^2 \neq 0 \\ -\frac{1}{2}\frac{\mathrm{sgn}(d_3)}{d_3^2}\Big|_{d_3=ad_2+k_a}^{d_3=bd_2+k_b} & \text{when } d_1^2 + d_2^2 = 0 \end{cases} \tag{5}$$

with $I_0(\mathbf{x}, d_2) \in C^\infty(\mathbf{R}^3 \backslash T_j, [-x_2, \bar{y}_2 - x_2])$. Equation (5b) requires $k_a \neq 0$, $k_b \neq 0$, which are always fulfilled when $\mathbf{x} \notin T_j$.

In order to show how to perform the outer integral of Eq. (3), that is

$$f(\mathbf{x}, d_2) = \int I_0(\mathbf{x}, d_2)\mathrm{d}d_2$$

it is useful to separately consider the two items of field point $\mathbf{x}$ lying or not lying in the plane of T_j.

3.1
The field point x does not lie in the plane of the triangle

For being $d_1 \neq 0$, from Eq. (5) one writes:

$$I_0(\mathbf{x}, d_2) = I_0^b(\mathbf{x}, d_2) - I_0^a(\mathbf{x}, d_2);$$

$$I_0^b := \frac{d_3}{d_1^2 + d_2^2}\frac{1}{r}\Big|^{d_3=bd_2+k_b};$$

$$I_0^a := \frac{d_3}{d_1^2 + d_2^2}\frac{1}{r}\Big|_{d_3=ad_2+k_a}$$

Focusing on the upper limit $d_3 = bd_2 + k_b$ in Eq. (5), the two candidate functions to be a primitive for I_0^b are:

$$f_1^b(\mathbf{x}) = \int I_0^b(\mathbf{x}, d_2)\mathrm{d}d_2 = \frac{1}{2d_1} \times \arctan\frac{2d_1(bd_1^2 - k_b d_2)\sqrt{d_1^2 + d_2^2 + (bd_2 + k_b)^2}}{(b^2-1)d_1^4 + (k_b d_2)^2 - d_1^2((1+b^2)d_2^2 + 4bd_2k_b + k_b^2)}$$

$$f_2^b(\mathbf{x}) = \int I_0^b(\mathbf{x}, d_2)\mathrm{d}d_2 = -\frac{1}{2d_1} \times \arctan\frac{(b^2-1)d_1^4 + (k_b d_2)^2 - d_1^2((1+b^2)d_2^2 + 4bd_2k_b + k_b^2)}{2d_1(bd_1^2 - k_b d_2)\sqrt{d_1^2 + d_2^2 + (bd_2 + k_b)^2}}$$

A property of the arctan function, shortly discussed in Appendix 1, has been used to obtain this result. f_1^b and f_2^b are linked by the following identity:

$$f_1^b = f_2^b + \frac{\pi}{4d_1} \times \operatorname{sgn} \frac{(b^2-1)d_1^4 + (k_b d_2)^2 - d_1^2((1+b^2)d_2^2 + 4bd_2k_b + k_b^2)}{2d_1(bd_1^2 - k_b d_2)\sqrt{d_1^2 + d_2^2 + (bd_2 + k_b)^2}} \tag{6}$$

The (unique) primitive f^b of I_0^b can be caught after studying the domain in which f_1^b and f_2^b are defined. Within a domain where both f_1^b and f_2^b are defined, they have the same derivative, I_0^b, for they differ by a constant. Within a domain in which only f_1^b (f_2^b) is everywhere defined, f_1^b (f_2^b) is the unique primitive.

3.1.1
b = 0

Consider first the easier case of $b = 0$. If $k_b \neq 0$, the existence domain for f_2^b with respect to d_2 is the whole real axis except the point $z_3 = 0$.

The existence domain for f_1^b with respect to d_2 is the whole real axis except the two points:

$$z_{1,2} = \pm d_1 \sqrt{\frac{k_b^2 + d_1^2}{k_b^2 - d_1^2}}$$

When $d_1^2 \geq k_b^2$, f_1^b is defined with respect to d_2 along the whole real axis and again it represents the primitive for I_0^b. This item includes also $k_b = 0$.

When $d_1^2 < k_b^2$, the primitive $f^b(\mathbf{x}, y_1, y_2)$ (which must be smooth) is a suitable "glue" of f_1^b and f_2^b. Denote (and therefore complete their definition) z_1 and z_2 such that $z_1 \leq z_2$. Defining with:

$$z_{13} := \frac{z_1}{2}, \quad z_{23} := \frac{z_2}{2}$$

$$\eta_{13} := \frac{-d_1^4 + (k_b z_{13})^2 - d_1^2(z_{13}^2 + k_b^2)}{-2d_1 k_b z_{13}\sqrt{d_1^2 + z_{13}^2 + k_b^2}},$$

$$\eta_{23} := \frac{-d_1^4 + (k_b z_{23})^2 - d_1^2(z_{23}^2 + k_b^2)}{-2d_1 k_b z_{23}\sqrt{d_1^2 + z_{23}^2 + k_b^2}}$$

f^b reads as follows:

$$f^b = \begin{cases} -\infty \leq d_2 \leq z_{13} & f_2^b \\ z_{13} \leq d_2 \leq z_{23} & f_1^b - \frac{\pi}{4d_1}\operatorname{sgn}(\eta_{13}) \\ z_{23} \leq d_2 \leq +\infty & f_2^b - \frac{\pi}{4d_1}(\operatorname{sgn}(\eta_{13}) - \operatorname{sgn}(\eta_{23})) \end{cases} \tag{7}$$

In Fig. 3 the subregions of the $d_1 \times d_2$ plane in which f^b is defined by f_1^b or f_2^b are depicted.

3.1.2
b ≠ 0

In the more general case of $b \neq 0$, the existence domain for f_2^b with respect to d_2 is the whole real axis except the point:

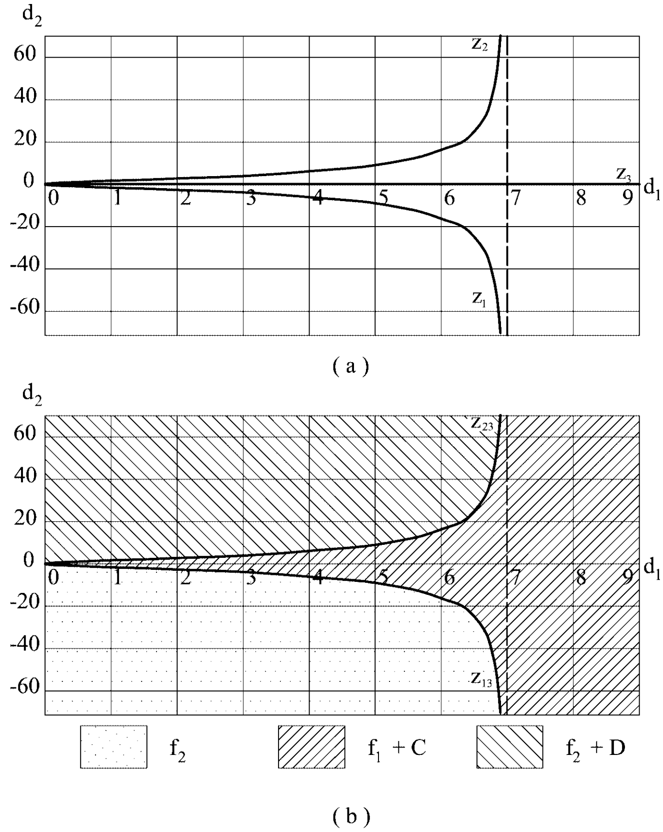

Fig. 3. Subregions of f. Results refer to the following data: $b = 0, x_2 = 5, x_3 = 7$

$$z_3 = \frac{bd_1^2}{k_b} \tag{8}$$

When $k_b = 0$, f_2^b is everywhere defined provided that $d_1 \neq 0$. Therefore f_2^b is the primitive of I_0^b.

The existence domain for f_1^b with respect to d_2 is the whole real axis except the zeroes of the quadratic polynomial

$$p(d_2) = (b^2 - 1)d_1^4 + (k_b d_2)^2 - d_1^2((1 + b^2)d_2^2 + 4bd_2k_b + k_b^2) \tag{9}$$

When $(1 - b^2)d_1^2 - k_b^2 > 0$, $p(d_2)$ has no roots. Accordingly, f_1^b is well defined $\forall\, d_2 \in \mathbf{R}$ and it is the primitive of I_0^b.

When $d_1^2(1 + b^2) - k_b^2 = 0$, $p(d_2)$ becomes linear. As a consequence, the existence domain for f_1^b with respect to d_2 is the whole real axis except the point

$$z_1 = -\frac{k_b}{2b(1 + b^2)} = -\frac{1}{2b^2} z_3 \tag{10}$$

The two roots z_1 and z_3 lie on opposite sides of the real axis.

If none of the previous items holds, the zeroes of $p(d_2)$ are the two points:

$$z_{1,2} = -\frac{2bd_1^2 k_b}{d_1^2(1 + b^2) - k_b^2} \pm \frac{d_1}{d_1^2(1 + b^2) - k_b^2}\sqrt{(k_b^2 + b^2 d_1^2)^2 - d_1^4} \tag{11}$$

Because of the properties $z_1(-d_1) = z_2(d_1)$ and $z_3(-d_1) = z_3(d_1)$, only the subdomain $d_1 > 0$ will be considered. Results on the complementary subdomain $d_1 < 0$ will be easily recovered. As before, denote z_1 and z_2 such that $z_1 \leq z_2$. The lowest root in (11) depends on the quantity

$$\kappa = \frac{d_1}{d_1^2(1+b^2) - k_b^2}$$

One has in fact:

$$\begin{cases} \kappa > 0 & \begin{cases} z_1 = -2b\kappa d_1 k_b - \kappa\sqrt{(k_b^2 + b^2 d_1^2)^2 - d_1^4} \\ z_2 = -2b\kappa d_1 k_b + \kappa\sqrt{(k_b^2 + b^2 d_1^2)^2 - d_1^4} \end{cases} \\ \kappa < 0 & \begin{cases} z_1 = -2b\kappa d_1 k_b + \kappa\sqrt{(k_b^2 + b^2 d_1^2)^2 - d_1^4} \\ z_2 = -2b\kappa d_1 k_b - \kappa\sqrt{(k_b^2 + b^2 d_1^2)^2 - d_1^4} \end{cases} \end{cases}$$

With regard to the mutual position of z_1, z_2 and z_3, from Eq. (9) one finds:

$$p(z_3) = -\frac{d_1^2(b^4 d_1^4 + k_b^2(d_1^2 + k_b^2) + b^2(d_1^4 + 2d_1^2 k_b^2))}{k_b^2} < 0$$

Therefore "around" z_3 there is always a neighborhood in which the primitive is f_1^b. Moreover, it can be easily checked that:

$$\lim_{d_2 \to \infty} p(d_2) = -\mathrm{sgn}(d_1^2(1+b^2) - k_b^2)(+\infty)$$

Because $p(d_2)$ is a quadratic polynomial, the mutual position of z_1, z_2 and z_3 can be summarized in this way, by means of the Weierstrass theorem.

$$\begin{aligned} d_1^2(1+b^2) - k_b^2 < 0 &\rightarrow z_1 < z_3 < z_2 \\ d_1^2(1+b^2) - k_b^2 > 0 &\rightarrow z_3 < z_1 < z_2 \text{ or } z_1 < z_2 < z_3 \end{aligned} \tag{12}$$

As mentioned, the function $f^b(\mathbf{x}, y_2)$ is smooth. Because around z_3 the primitive is f_1^b, and around z_1 and z_2 the primitive must be f_2^b, the whole primitive will be a "glue" of f_1^b and f_2^b smooth by construction. Defining with:

$$z_{13} := \frac{z_1 + z_3}{2}, \quad z_{23} := \frac{z_3 + z_2}{2}$$

$$\eta_{13} := \frac{(b^2-1)d_1^4 + (k_b z_{13})^2 - d_1^2((1+b^2)z_{13}^2 + 4bz_{13}k_b + k_b^2)}{2d_1(bd_1^2 - k_b z_{13})\sqrt{d_1^2 + z_{13}^2 + (bz_{13} + k_b)^2}}$$

$$\eta_{23} := \frac{(b^2-1)d_1^4 + (k_b z_{23})^2 - d_1^2((1+b^2)z_{23}^2 + 4bz_{23}k_b + k_b^2)}{2d_1(bd_1^2 - k_b z_{23})\sqrt{d_1^2 + z_{23}^2 + (bz_{23} + k_b)^2}}$$

and making use of Eq. (6), we have the following instances:

- When $d_1^2(1+b^2) - k_b^2 < 0 \rightarrow z_1 < z_3 < z_2$:

$$f^b = \begin{cases} -\infty < d_2 \leq z_{13} & f_2^b \\ z_{13} \leq d_2 \leq z_{23} & f_1^b - \frac{\pi}{4d_1}\mathrm{sgn}(\eta_{13}) \\ z_{23} \leq d_2 < +\infty & f_2^b - \frac{\pi}{4d_1}(\mathrm{sgn}(\eta_{13}) - \mathrm{sgn}(\eta_{23})) \end{cases} \tag{13}$$

- When $d_1^2(1+b^2) - k_b^2 \geq 0$ and $(1-b^2)d_1^2 - k_b^2 \leq 0$:

$$\begin{cases} z_3 < z_1 < z_2 \\ \quad \rightarrow f^b = \begin{cases} -\infty < d_2 \leq z_{13} & f_1^b \\ z_{13} \leq d_2 < +\infty & f_2^b + \frac{\pi}{4d_1}\mathrm{sgn}(\eta_{13}) \end{cases} \\ z_1 < z_2 < z_3 \\ \quad \rightarrow f^b = \begin{cases} -\infty < d_2 \leq z_{23} & f_2^b \\ z_{23} \leq d_2 < +\infty & f_1^b - \frac{\pi}{4d_1}\mathrm{sgn}(\eta_{23}) \end{cases} \end{cases} \tag{14}$$

- When $(1-b^2)d_1^2 - k_b^2 > 0$:
  $f^b := f_1^b$.

In Figs. 4 and 5 the subregions of the $d_1 \times d_2$ plane in which f^b is defined as above are depicted.

3.2
The field point x lies in the plane of the triangle

Within the hypothesis $\mathbf{x} \notin \bar{T}_j$, consider the point $\mathbf{x}$ lying on the same plane of the triangle T_j, i.e. $d_1 = 0$. Depending on the position of the point $\mathbf{x}$, different values of d_2, k_a, k_b may occur, as shown in Fig. 6.

$$I_0 = \frac{d_3}{d_2^2}\frac{1}{\sqrt{d_2^2 + d_3^2}}\Bigg|_{d_3 = ad_2 + k_a}^{d_3 = bd_2 + k_b}$$

is again an integrable function. In fact, because $\mathbf{x} \notin \bar{T}_j$ implies either $d_2 \neq 0$ or $\mathrm{sgn}(k_a) = \mathrm{sgn}(k_b)$, the asymptotic expansion of I_0 around $d_2 = 0$ holds:

$$I_0(d_2) = \frac{1}{d_2^2}\left(-\frac{k_a}{\sqrt{k_a^2}} + \frac{k_b}{\sqrt{k_b^2}}\right) + O(1) = O(1)$$

When $k_b \neq 0$ and $k_a \neq 0$, it turns out:

$$\int_{-x_2}^{\bar{y}_2 - x_2} I_0 \, \mathrm{d}d_2 = \left[\frac{\sqrt{d_2^2 + (ad_2 + k_a)^2}}{d_2 k_a} - \frac{\sqrt{d_2^2 + (bd_2 + k_b)^2}}{d_2 k_b}\right]\Bigg|_{d_2 = -x_2}^{d_2 = \bar{y}_2 - x_2} \tag{15}$$

When $k_b = 0$ the quantity $-\sqrt{d_2^2 + (bd_2 + k_b)^2}/d_2 k_b$ must be substituted with $-b/\sqrt{d_2^2 + b^2 d_2^2}$ in Eq. (15). Analogously when $k_a = 0$.

One could obtain such a result by a limit process:

$$\int_{-x_2}^{\bar{y}_2 - x_2} I_0 \, \mathrm{d}d_2 = \lim_{d_1 \to 0}\left(f^b(\mathbf{x}, d_2) - f^a(\mathbf{x}, d_2)\right)\Big|_{d_2 = -x_2}^{d_2 = \bar{y}_2 - x_2}$$

From Eq. (12), it turns out that for $d_1 \to 0$:

$$d_1^2(1+b^2) - k_b^2 < 0 \rightarrow z_1 < z_3 < z_2$$

The asymptotic expansion:

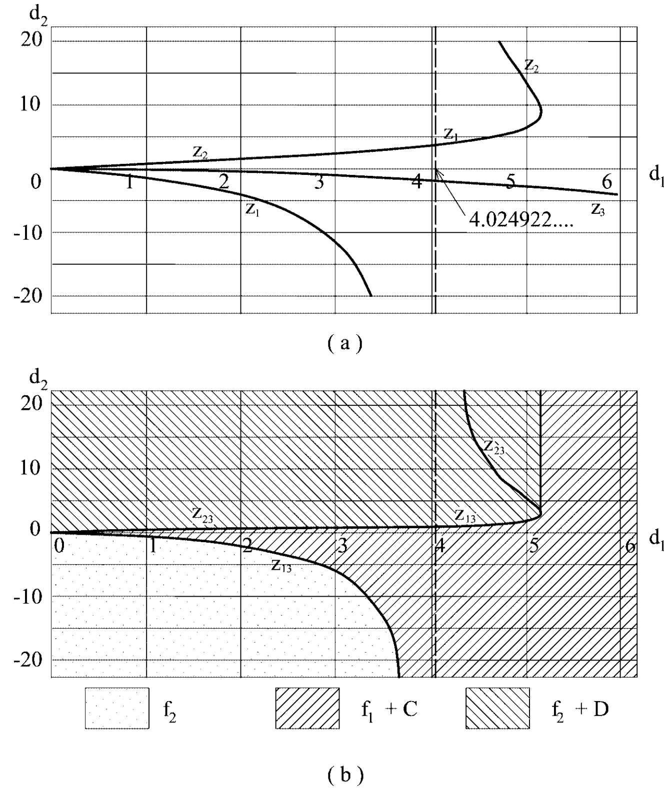

Fig. 4. Subregions of f. Results refer to the following data: $b = 0.5, x_2 = 5, x_3 = 7$

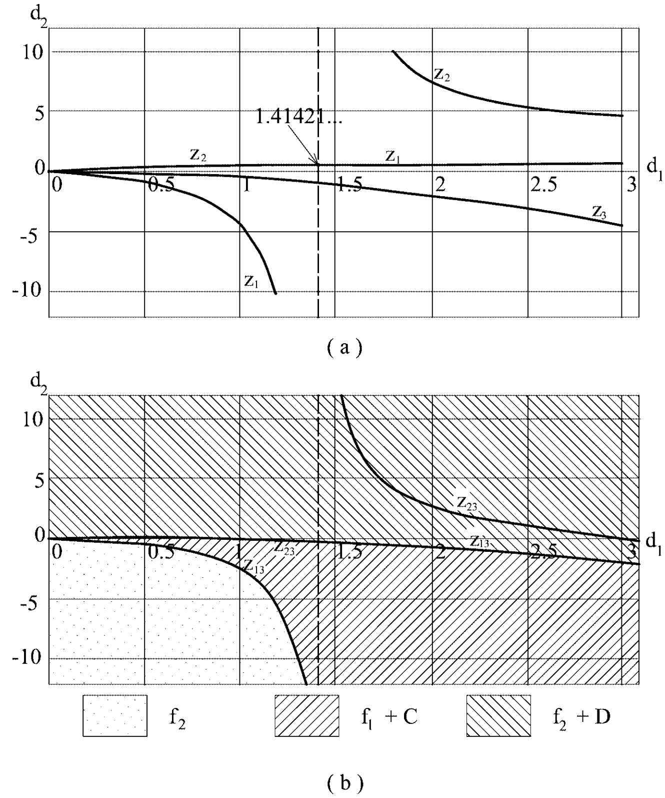

Fig. 5. Subregions of f. Results refer to the following data: $b = 1, x_2 = 5, x_3 = 7$

$$
\begin{gathered}
z_1 = -d_1 + O(d_1^2); \quad z_3 = \frac{b}{k_b} d_1^2; \quad z_2 = d_1 + O(d_1^2) \\
-\tfrac{1}{2} d_1 + O(d_1^2) = z_{13} < 0 < z_{23} = \tfrac{1}{2} d_1 + O(d_1^2) \\
\eta_{13} = -\tfrac{3}{4}\operatorname{sgn}(k_b) + O(d_1); \quad \eta_{23} = \tfrac{3}{4}\operatorname{sgn}(k_b) + O(d_1)
\end{gathered} \tag{16}
$$

can be easily derived from Eqs. (8)–(11). As a consequence:

$$
f^b(\mathbf{x}, d_2) \sim \begin{cases} -\infty < d_2 \le z_{13} & f_2^b \\ z_{13} \le d_2 \le z_{23} & f_1^b + \frac{\pi}{4d_1}\operatorname{sgn}(k_b) \\ z_{23} \le d_2 < +\infty & f_2^b + \frac{\pi}{2d_1}\operatorname{sgn}(k_b) \end{cases}
$$

Equation (16) implies that for $x_2 > \bar{y}_2$ it will exist a d_1^* s.t. $\forall\ d_1 < d_1^* \Rightarrow -x_2 < z_{13}$ and it will exist a $d_1^\dagger$ s.t. $\forall\ d_1 < d_1^\dagger \Rightarrow \bar{y}_2 - x_2 < z_{13}$. Accordingly,

$$
\begin{aligned}
&\lim_{d_1 \to 0} \left(f^b(\mathbf{x}, d_2) - f^a(\mathbf{x}, d_2) \right) \Big|_{d_2 = -x_2}^{d_2 = \bar{y}_2 - x_2} \\
&= \left[\frac{\sqrt{{d_2}^2 + (ad_2 + k_a)^2}}{d_2 k_a} - \frac{\sqrt{{d_2}^2 + (bd_2 + k_b)^2}}{d_2 k_b} \right]_{d_2 = -x_2}^{d_2 = \bar{y}_2 - x_2} \\
&\quad + \left[\frac{\pi}{4d_1} (\operatorname{sgn}(d_2 k_b) - \operatorname{sgn}(d_2 k_a)) \right]_{d_2 = -x_2}^{d_2 = \bar{y}_2 - x_2} + O(d_1)
\end{aligned}
$$

with

$$
\left[\frac{\pi}{4d_1} (\operatorname{sgn}(d_2 k_b) - \operatorname{sgn}(d_2 k_a)) \right]_{d_2 = -x_2}^{d_2 = \bar{y}_2 - x_2} = 0
$$

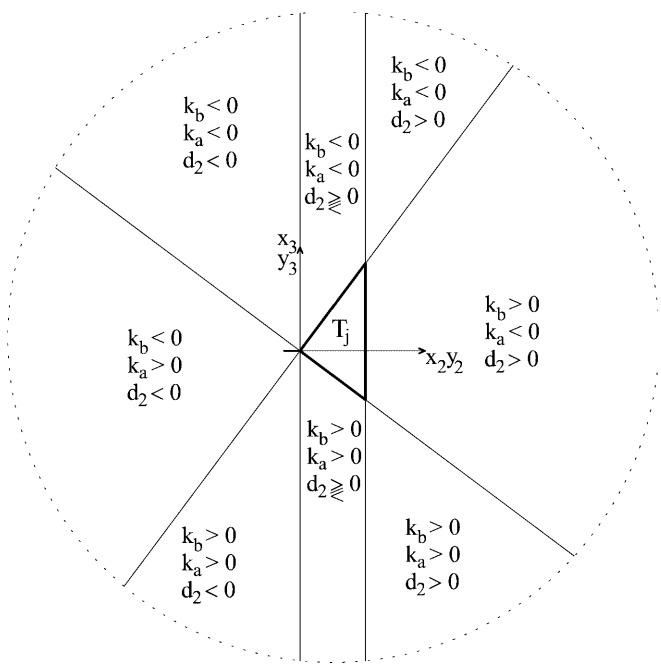

Fig. 6. Different values of d_2, k_a, k_b depending on the position of the point $\mathbf{x}$

Similar considerations hold when $x_2 < 0$. Again from Eq. (16), when $0 < x_2 < \bar{y}_2$ it will exist a d_1^* s.t. $\forall\ d_1 < d_1^* \Rightarrow -x_2 < z_{13}$ and it will exist a $d_1^\dagger$ s.t. $\forall\ d_1 < d_1^\dagger \Rightarrow \bar{y}_2 - x_2 > z_{23}$. Accordingly,

$$\lim_{d_1\to 0}\left(f^b(\mathbf{x},d_2)-f^a(\mathbf{x},d_2)\right)\Big|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}$$
$$=-\lim_{d_1\to 0}\left(f_2^b(\mathbf{x},-x_2)-f_2^a(\mathbf{x},-x_2)\right)$$
$$+\lim_{d_1\to 0}\Big[f_2^b(\mathbf{x},\bar{y}_2-x_2)-f_2^a(\mathbf{x},\bar{y}_2-x_2)$$
$$+\frac{\pi}{2d_1}(\mathrm{sgn}(k_b)-\mathrm{sgn}(k_a))\Big]$$
$$=\left[\frac{\pi}{4d_1}(\mathrm{sgn}(d_2k_b)-\mathrm{sgn}(d_2k_a))\right]_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}$$
$$+\left[\frac{\pi}{2d_1}(\mathrm{sgn}(k_b)-\mathrm{sgn}(k_a))\right]$$
$$+\left[\frac{\sqrt{d_2^2+(ad_2+k_a)^2}}{d_2k_a}-\frac{\sqrt{d_2^2+(bd_2+k_b)^2}}{d_2k_b}\right]_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}+O(d_1)$$

Because $\mathbf{x}\notin\bar{T}_j$, it holds (see also Fig. 6):

$$\left[\frac{\pi}{4d_1}(\mathrm{sgn}(d_2k_b)-\mathrm{sgn}(d_2k_a))\right]_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}=0;$$
$$\mathrm{sgn}(k_b)-\mathrm{sgn}(k_a)=0 \tag{17}$$

Analogous considerations hold in case of $k_b=0$ or $k_a=0$.

4 Hadamard's finite part

Let $F(\varepsilon)$ denote a complex-valued function which is continuous in $(0,\varepsilon_0)$ and assume that

$$F(\varepsilon)=F_0+F_1\log(\varepsilon)+\sum_{j=2}^{m}F_j\,\varepsilon^{1-j}+o(1);\quad \varepsilon\to 0$$

Then F_0 is called the finite part *p.f.* of $F(\varepsilon)$ and one writes $F_0=p.f.F$ [21].

Considering $\mathbf{x}\in T_j$, let us define as in Fig. 7:

$$T_j^\varepsilon := \{\mathbf{y}\in T_j : |y_2-x_2|<\varepsilon \text{ and } |y_3-x_3|<\varepsilon\}$$
$$I_{\boxed{\epsilon}}(\mathbf{x}) := \int_{T_j\setminus T_j^\varepsilon}\frac{1}{r^3}\,\mathrm{d}\mathbf{y}$$

By direct integration, it can be proved that for $\varepsilon\to 0$:

$$I_{\boxed{\epsilon}}(\mathbf{x})=\frac{4\sqrt{2}}{\varepsilon}+\left[\frac{\sqrt{d_2^2+(k_a+ad_2)^2}}{d_2k_a}-\frac{\sqrt{d_2^2+(k_b+bd_2)^2}}{d_2k_b}\right]\Bigg|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2} \tag{18}$$

Therefore, by definition of finite part we obtain:

$$p.f.\int_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}\int_{ad_2+k_a}^{bd_2+k_b}\frac{1}{r^3}\,\mathrm{d}d_3\,\mathrm{d}d_2=\left[\frac{\sqrt{d_2^2+(ad_2+k_a)^2}}{d_2k_a}-\frac{\sqrt{d_2^2+(bd_2+k_b)^2}}{d_2k_b}\right]\Bigg|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2} \tag{19}$$

the same expression as (15). This result can be viewed as a continuity property (with respect to $\mathbf{x}$) of the finite part of Hadamard, that coincides with the Lebesgue integral for every integrable function. 183

In the framework of the BEM, it is interesting to perform the limit process,

$$\lim_{d_1\to 0}\left(f^b(\mathbf{x},d_2)-f^a(\mathbf{x},d_2)\right)\Big|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2};\quad \mathbf{x}\in T_j$$

In this case, $0<x_2<\bar{y}_2$ and it exist a d_1^* s.t. $\forall\ d_1<d_1^*\Rightarrow -x_2<z_{13}$. Moreover it exist a $d_1^\dagger$ s.t. $\forall\ d_1<d_1^\dagger\Rightarrow \bar{y}_2-x_2>z_{23}$. Accordingly,

$$\lim_{d_1\to 0}\left(f^b(\mathbf{x},d_2)-f^a(\mathbf{x},d_2)\right)\Big|_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}$$
$$=\lim_{d_1\to 0}\Big(f_2^b(\mathbf{x},\bar{y}_2-x_2)+\frac{\pi}{2d_1}\mathrm{sgn}(k_b)$$
$$-f_2^a(\mathbf{x},\bar{y}_2-x_2)-\frac{\pi}{2d_1}\mathrm{sgn}(k_a)\Big)$$
$$-\lim_{d_1\to 0}\left(f_2^b(\mathbf{x},-x_2)-f_2^a(\mathbf{x},-x_2)\right)$$
$$=\left[\frac{\pi}{4d_1}(\mathrm{sgn}(d_2k_b)-\mathrm{sgn}(d_2k_a))\right]_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}$$
$$+\left[\frac{\pi}{2d_1}(\mathrm{sgn}(k_b)-\mathrm{sgn}(k_a))\right]$$
$$+\left[\frac{\sqrt{d_2^2+(ad_2+k_a)^2}}{d_2k_a}-\frac{\sqrt{d_2^2+(bd_2+k_b)^2}}{d_2k_b}\right]_{d_2=-x_2}^{d_2=\bar{y}_2-x_2}+O(d_1) \tag{20}$$

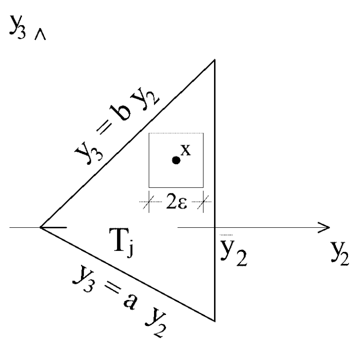

Fig. 7. Geometrical description of a square neighborhood

Differently from (17), for being $\mathbf{x} \in \bar{T}_j$ one has this time (see also Fig. 6):

$$\left[\frac{\pi}{4d_1}(\mathrm{sgn}(d_2k_b) - \mathrm{sgn}(d_2k_a))\right]_{d_2=-x_2}^{d_2=\bar{y}_2-x_2} + \left[\frac{\pi}{2d_1}(\mathrm{sgn}(k_b) - \mathrm{sgn}(k_a))\right] = \frac{2\pi}{d_1} \quad (21)$$

which is an expected divergent term, as the integral

$$\int_{-x_2}^{\bar{y}_2-x_2} I_0(\mathbf{x}, d_2)\mathrm{d}d_2; \quad \mathbf{x} \in T_j$$

does not exist.

5 Concluding remarks

As mentioned in the introduction, integral (1) plays a fundamental role in the analytical integration of the Green's functions pertaining to the BEM, especially for symmetric Galerkin BEM. As a matter of fact, in BEM one usually deals with integrals of the following form:

$$\int_{\Gamma_s} \mathbf{G}_{rs}(\mathbf{x}-\mathbf{y})\mathbf{\Phi}(\mathbf{y})\mathrm{d}\Gamma(\mathbf{y}) \quad r = u, \ \ s = u, p \quad (22)$$

where $\mathbf{\Phi}(\mathbf{y})$ are matrices of shape functions for the approximation of the displacement and traction fields, and $\mathbf{G}_{rs}(\mathbf{x}-\mathbf{y})$ are generic kernels (see Appendix 2). It can be proved that integral (22) admits of the following closed form expression in the local coordinate system $\mathscr{L}$:

$$\left\{\left[\log\left(\frac{d_2+bd_3}{\sqrt{1+b^2}}+r\right)\mathbf{L}^{rs} + \log(d_3+r)\mathbf{L}_2^{rs} + \mathrm{arctanh}\left(\frac{d_3}{r}\right)\mathbf{A}^{rs} + f(\mathbf{x}, d_2)\mathbf{F}^{rs} + \mathbf{R}^{rs}r + \mathbf{P}^{rs} + \mathbf{S}^{rs}\frac{1}{r} + \mathbf{H}^{rs}\frac{1}{r^3}\right]_{d_3=ad_2+k_a}^{d_3=bd_2+k_b}\right\}_{d_2=-x_2}^{d_2=\bar{y}_2-x_2} \quad (23)$$

where $f(\mathbf{x}, d_2)$ has been defined by integral (3) and $\mathbf{L}^{rs}$, $\mathbf{L}_2^{rs}$, $\mathbf{A}^{rs}$, $\mathbf{F}^{rs}$, $\mathbf{R}^{rs}$, $\mathbf{P}^{rs}$, $\mathbf{S}^{rs}$, $\mathbf{H}^{rs}$ are suitable matrices. In particular, integral (3) is responsible for the free terms that arise in the limit process to the boundary $\Omega \ni \mathbf{x} \to \mathbf{x}^o \in \Gamma$, which are due to the term (21). For details on these subjects, the reader is referred to a forthcoming paper.

It has been shown [23] that for potential problems, the integral (3) does not affect the result of the integration of the hypersingular kernel. This surprising fact is due to the coefficient $\mathbf{F}^{pp}$ that vanishes for the potential hypersingular kernel [23].

All aforementioned integrals have been implemented to verify the capability of the proposed formulation and their computational interest. Some benchmarks are included in [8] whereas engineering applications, pertaining to a jaw-teeth stress analysis are in progress [34].

Appendix 1 – A property of arctan *x*

It is a fact that:

$$\frac{\mathrm{d}}{\mathrm{d}x}\arctan x = -\frac{\mathrm{d}}{\mathrm{d}x}\arctan\frac{1}{x} = \frac{1}{1+x^2} \quad (24)$$

The function

$$\frac{1}{1+x^2} \quad (25)$$

is obviously infinitely smooth in $\mathbf{R}$ and the calculus fundamental theorem guarantees that there exists one (and only one) family of primitives, which differ by a constant. Equation (24) seems to say that

$$\arctan x + \arctan\frac{1}{x} = C$$

which is not true. It is well known in fact that

$$\arctan x + \arctan\frac{1}{x} = \frac{\pi}{2}\mathrm{sgn}(x) \quad (26)$$

and (of course) this fact does not contradict the fundamental theorem. As an hypothesis of the fundamental theorem, the primitive must be differentiable over the given domain. $\frac{\pi}{2}\mathrm{sgn}(x) - \arctan\frac{1}{x}$ is neither defined nor differentiable at $x = 0$. Therefore, in every domain which contains zero, the primitive of (25) is $\arctan x$ and not $\arctan\frac{1}{x}$. In every domain which does not contain zero, the two functions do differ by a constant and are elements of the same (unique) family of primitive for (25).

Appendix 2 – Green's functions for 3D linear elasticity

The expressions of Green's functions for 3D linear elasticity follows (see also [33]). $\mathbf{n}(\mathbf{x})$ and $\mathbf{l}(\mathbf{y})$ are the normals at the boundary at $\mathbf{x}$ and $\mathbf{y}$, respectively. Vector $\mathbf{d} = (\mathbf{y} - \mathbf{x})$ has been defined in Sect. 2

$$\mathbf{G}_{uu}(\mathbf{d}) = \frac{1}{16\pi}\frac{1}{G(1-\nu)}\frac{1}{r}\left(\frac{\mathbf{d}\otimes\mathbf{d}}{r^2} + (3-4\nu)\,\mathbf{I}\right)$$

$$\mathbf{G}_{pu}(\mathbf{d};\mathbf{n}(\mathbf{x})) = -\frac{1}{8\pi}\frac{1}{(1-\nu)}\frac{1}{r^3}\left[(1-2\nu)(2\,\mathbf{SKW}(\mathbf{d}\otimes\mathbf{n}) - (\mathbf{d}\cdot\mathbf{n})\,\mathbf{I}) - 3(\mathbf{d}\cdot\mathbf{n})\frac{\mathbf{d}\otimes\mathbf{d}}{r^2}\right]$$

$$\mathbf{G}_{up}(\mathbf{d};\mathbf{l}(\mathbf{y})) = -\frac{1}{8\pi}\frac{1}{(1-\nu)}\frac{1}{r^3}\left[(1-2\nu)(2\,\mathbf{SKW}(\mathbf{d}\otimes\mathbf{l}) + (\mathbf{d}\cdot\mathbf{l})\,\mathbf{I}) + 3\,(\mathbf{d}\cdot\mathbf{l})\frac{\mathbf{d}\otimes\mathbf{d}}{r^2}\right]$$

$$\begin{aligned}\mathbf{G}_{pp}&(\mathbf{d};\mathbf{n}(\mathbf{x});\mathbf{l}(\mathbf{y})) \\ &= \frac{G\nu}{4\pi(1-\nu)}\frac{1}{r^3}\left\{2\mathbf{SYM}(\mathbf{l}\otimes\mathbf{n}) + 2\mathbf{SKW}(\mathbf{l}\otimes\mathbf{n})\frac{3\nu-1}{\nu}\right. \\ &+ 3\frac{(3\nu-1)}{\nu}\left[\mathbf{SKW}(\mathbf{d}\otimes\mathbf{l})\frac{\mathbf{d}\cdot\mathbf{n}}{r^2} - \mathbf{SKW}(\mathbf{d}\otimes\mathbf{n})\frac{\mathbf{d}\cdot\mathbf{l}}{r^2}\right] \\ &+ 3\frac{(1-\nu)}{\nu}\left[\mathbf{SYM}(\mathbf{d}\otimes\mathbf{l})\frac{\mathbf{d}\cdot\mathbf{n}}{r^2} + \mathbf{SYM}(\mathbf{d}\otimes\mathbf{n})\frac{\mathbf{d}\cdot\mathbf{l}}{r^2}\right] \\ &+ 3\frac{\mathbf{d}\otimes\mathbf{d}}{r^2}\left[(\mathbf{l}\cdot\mathbf{n}) - \frac{5}{\nu}\frac{(\mathbf{d}\cdot\mathbf{n})(\mathbf{d}\cdot\mathbf{l})}{r^2}\right] \\ &\left.+ \left[3\frac{(\mathbf{d}\cdot\mathbf{n})(\mathbf{d}\cdot\mathbf{l})}{r^2} + (\mathbf{l}\cdot\mathbf{n})\frac{(1-2\nu)}{\nu}\right]\mathbf{I}\right\}\end{aligned}$$

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