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ORIGINAL PAPER

Dual boundary integral equation formulation in antiplane elasticity using complex variable

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Abstract This paper investigates the dual boundary integral equation formulation in antiplane elasticity using complex variable. Four kinds of boundary integral equation (BIE) are studied, and they are the first complex variable BIE for the interior region, the second complex variable BIE for the interior region, the first complex variable BIE for the exterior region, and the second complex variable BIE for the exterior region. The first BIE for the interior region is derived from the Somigliana identity, or the Betti's reciprocal theorem in elasticity. A displacement versus traction operator is suggested. After using this operator, the second BIE for the interior region is derived. Similar derivations are performed for the first and second BIEs for the exterior region. In the case of the exterior boundary, two degenerate boundary cases are studied. One is the curved crack case, and other is the case of a deformable line. All kernels in the suggested BIEs are expressed in terms of complex variable.

Keywords Elasticity · Dual boundary integral equation · First complex variable BIE · Second complex variable BIE · Interior boundary value problem · Exterior boundary value problem · Curved crack problem · Curved deformable line problem

1 Introduction

The boundary integral equation (BIE) in elasticity was studied by some pioneer researchers in earlier years [1–3]. After discretization of the BIE, the boundary element method (BEM) will be formulated. A particular advantage of the method is that the method can considerably reduce the dimensionality of unknowns in the solution, if one compares it with the finite element method (FE). Therefore, BIE or BEM has attracted much attention by many researchers. However, the formulation of BIE is one side in the study, and the computation in BIE is other important task. The computation based on BIE considerably depends on modern computers. It was pointed out in [4] that "after the 'invention of the technology' in the late 1960s and early 1970s, the number of published literature was very small; but it was on an exponential growth rate, until it reached an inflection point around 1991". Therefore, it is not strange that the BIE study was promoted significantly in recent years. Depending on the adopted methods, two kinds of formulation were suggested. One is the direct method and other is indirect method [4].

In the direct method, for example, in plane elasticity, one can express the displacements at the domain points from the Somigliana identity, which are generally expressed in the form of integrals with boundary displacement and traction densities. Letting the domain point approach the boundary point, the first BIE is formulated [4,5].

It was pointed out that the integral equation based on the Somigliana identity is not sufficient to solve the general elastic crack problems [6,7]. Therefore, many authors suggested an additional integral equation, or so-called dual integral equation [6–10]. In the formulation of the second BIE, it is a necessary step to define a displacement versus traction operator at the domain point. Letting the domain point approach a boundary point, the second BIE can be derived from the traction expression. The second BIE also links the boundary traction to the boundary displacement. However, the forms of linkage between the boundary traction to the boundary displacement in the first and second BIEs are quite different. Particularly, in the case of a degenerate boundary in exterior

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problem, the second BIE can provide necessary equation to solve the crack problem.

The harmonic function is considered as a real part of an analytic function. The value of an analytic function at domain point is expressed in the form of a Cauchy type integral with density function provided by the function itself. A dual boundary element analysis using complex variable for Laplace equation was suggested [11]. A basic Dirichlet problem for the Laplace equation was considered. Let the harmonic function $u(x, y)$ be a real part of an analytic function $h(z)$, the complex variable boundary element method (CVBEM) was suggested [12]. In the method, the CVBEM function was assumed in a series form. A way for solving the boundary value problem was suggested, which was derived from the least-square procedure.

This paper investigates the dual boundary integral equation formulation in antiplane elasticity using complex variable. Four kinds of BIE are studied, and they are the first complex variable BIE for the interior region, the second complex variable BIE for the interior region, the first complex variable BIE for the exterior region, and the second complex variable BIE for the exterior region.

The first BIE for the interior region is derived from the Somigliana identity, or the Betti's reciprocal theorem in elasticity. A displacement versus traction operator is suggested. After using this operator, the second BIE for the interior region is derived.

Similar derivations are performed for the first and second BIEs for the exterior region. In the case of the exterior boundary, two degenerate boundary cases are studied. One is the curved crack case, and other is the case of a deformable line. For two degenerate boundary cases, the relevant BIEs are expressed in an explicit form. The advantage in the formulation is that all kernels in BIEs are expressed in an explicit form. One can easily distinguish the hypersingular, Cauchy singular and regular part in the kernels, if the complex variable BIE is used. Particularly, in the second BIE and degenerate boundary case, a hypersingular integral equation for crack problem is obtained.

2 The first complex variable BIE for the interior region

2.1 Some preliminary knowledge in complex variable method of antiplane elasticity

After using complex potential $\phi(z)$ in antiplane elasticity, all the physical quantities can be expressed through a complex potential $\phi(z)$ [13]

$$\phi(z) = Gw(x, y) + if(x, y),$$
$$\text{or} \quad Gw(x, y) = \frac{1}{2}(\phi(z) + \overline{\phi(z)}),$$

$$f(x, y) = \frac{1}{2i}(\phi(z) - \overline{\phi(z)}) \tag{1}$$

$$f(x, y) = \int_{z_o}^{z} \left(\sigma_{x\tilde{z}}\mathrm{d}y - \sigma_{y\tilde{z}}\mathrm{d}x\right), \tag{2}$$

$$G\frac{\partial w}{\partial x} + i\frac{\partial f}{\partial x} = \sigma_{x\tilde{z}} - i\sigma_{y\tilde{z}} = \Phi(z) = \phi'(z), \tag{3}$$

where G is the shear modulus of elasticity, "w" is the out of plane displacement, "f" is the longitudinal resultant force, and $\sigma_{x\tilde{z}}$ and $\sigma_{y\tilde{z}}$ are the stress components, $\tilde{z}$ is a coordinate, and $z = x + iy$. In Eq. (2), the integration is performed from a fixed point z_o to a moving point "z".

Clearly, the displacement component $w(x, y)$ satisfies the following Laplace equation

$$\nabla^2 w(x, y) = 0, \quad \text{where } \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \tag{4}$$

From Eq. (1), one more stress component $\sigma_{n\tilde{z}}$ is defined as follows (Fig. 1)

$$\sigma_{n\tilde{z}} = G\frac{\partial w}{\partial n} = \frac{1}{2}\frac{\partial(\phi(z) + \overline{\phi(z)})}{\partial n}$$
$$= \frac{1}{2}(\Phi(z)e^{i\delta} + \overline{\Phi(z)}\,e^{-i\delta}) \tag{5}$$

where $\sigma_{n\tilde{z}}$ represents the shear stress applied on an interval in the direction $z \to z + \mathrm{d}z$, and δ denotes the inclined angle of the direction $z \to z + \mathrm{d}z$ (Fig. 1).

In the following analysis, the following Cauchy type integral is useful [14]

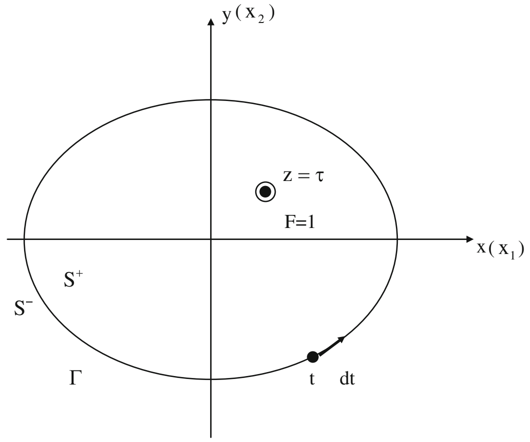
$$F(z) = \frac{1}{2\pi i}\int_L \frac{f(t)\mathrm{d}t}{t - z} \tag{6}$$

where L is a smooth curve or a closed contour Γ in Fig. 1. Also, we assume that the function $f(t)$ satisfy the Hölder condition [14]. Sometimes, the functions $f(t)$ is called the density function hereafter.

Generally speaking, the integral takes different values when $z \to t_0^+$ and $z \to t_0^-$, $(t_0 \in L)$, respectively. The limit values of the integral from the upper and lower sides of the curve L are found to be [14]

$$F^{\pm}(t_o) = \pm\frac{f(t_o)}{2} + \frac{1}{2\pi i}\int_L \frac{f(t)\mathrm{d}t}{t - t_o} \tag{7}$$

In Eq. (7), the integral should be understood in the sense of principal value. Note that the notations of $F(z)$ and $f(t)$ used in Eqs. (6) and (7) have no relation with those mentioned in other places.



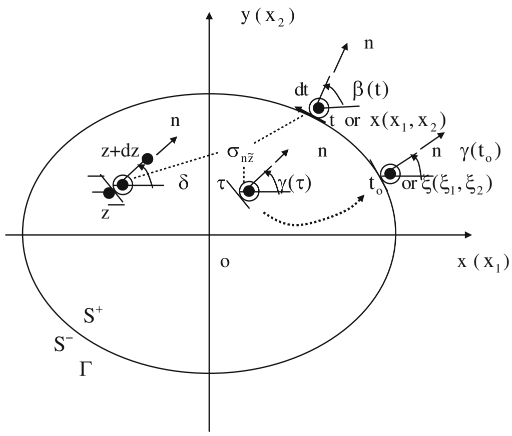


Fig. 1 **a** the α-field with a concentrated force applied at $z = \tau$, **b** the $\beta-$ field, or the physical field defined on a finite region

2.2 Formulation of the first complex variable BIE for the interior region

In the following analysis, the $\alpha-$ field shown by Fig. 1a is relating to the fundamental field caused by unit concentrated force at the point $z = \tau$. The relevant complex potential is as follows [13]

$$\phi(z) = -\frac{1}{2\pi}\ln(z-\tau), \quad \Phi(z) = \phi'(z) = -\frac{1}{2\pi}\frac{1}{z-\tau} \tag{8}$$

The complex potential shown by Eq. (8) is defined in a full infinite plane. From Eqs. (1) and (8), we can evaluate the relevant displacement at the boundary point "t" as follows (Fig. 1)

$$Gw_* = -\frac{1}{4\pi}(\ln(t-\tau) + \ln(\bar{t}-\bar{\tau})), \quad (t \in \Gamma, \tau \in S^+) \tag{9}$$

Similarly, from Eqs. (5) and (8) we can evaluate the relevant boundary traction at the boundary point "t" as follows (Fig. 1)

$$\sigma_{n\tilde{z}*} = G\frac{\partial w_*}{\partial n_t} = -\frac{1}{4\pi}\left(\frac{e^{i\beta(t)}}{t-\tau} + \frac{e^{-i\beta(t)}}{\bar{t}-\bar{\tau}}\right), \quad (t \in \Gamma, \tau \in S^+) \tag{10}$$

In Eqs. (9) and (10), the subscript "*" denotes that the arguments are derived from the fundamental solution, and $\beta(t)$ denotes the direction angle of normal at the boundary point "t" (Fig. 1).

After using the Betti's reciprocal theorem, or the Somigliana identity, between the fundamental field (or the $\alpha-$ field in Fig. 1a) and the physical field (or the $\beta-$ field in Fig. 1b), we have

$$w(\tau) + \int_\Gamma \sigma_{n\tilde{z}*}w(s)\mathrm{d}s = \int_\Gamma w_* p(s)\mathrm{d}s, \quad (\tau \in S^+) \tag{11}$$

where the left hand term represents the work done by traction from the fundamental field (the $\alpha-$ field) to the displacement of the physical field (the $\beta-$ field). In addition, the right hand term represents the work done by traction from the physical field to the displacement of the fundamental field. In Eq. (11), Γ denotes the outer boundary in the interior BVP (Fig. 1). This notation, or Γ, will be used from here to the end of third section.

Substituting Eqs. (9) and (10) into Eq. (11) yields

$$w(\tau) = \frac{1}{4\pi}\int_\Gamma S_1(t,\tau)w(s)\mathrm{d}s - \frac{1}{4\pi G}\int_\Gamma (\ln(t-\tau) + \ln(\bar{t}-\bar{\tau}))p(s)\mathrm{d}s, \quad (\tau \in S^+) \tag{12}$$

where

$$S_1(t,\tau) = \frac{e^{i\beta(t)}}{t-\tau} + \frac{e^{-i\beta(t)}}{\bar{t}-\bar{\tau}} \tag{13}$$

In Eqs. (12) and (13), "s" denotes a curve length coordinate, the $t(s)$ is a complex value, $\beta(t)$ denotes the inclined angle of normal to the boundary at the point "t" (Fig. 1b).

Before studying the formulation of BIE, the limit process $\tau \to t_o(\tau \in S^+, t_o \in \Gamma)$ for the following integral $I(\tau)$ is introduced. The integral $I(\tau)$ is defined by

$$I(\tau) = \frac{1}{4\pi}\int_\Gamma S_1(t,\tau)w(s)\mathrm{d}s = I_1(\tau) + I_2(\tau), \quad (\tau \in S^+) \tag{14}$$

where

$$I_1(\tau) = \frac{1}{4\pi}\int_\Gamma \frac{e^{i\beta(t)}}{t-\tau} w(s)ds,$$

$$I_2(\tau) = \overline{I_1(\tau)} = \frac{1}{4\pi}\int_\Gamma \frac{e^{-i\beta(t)}}{\bar{t}-\bar{\tau}} w(s)ds, \quad (\tau \in S^+) \tag{15}$$

Since $dt = ie^{i\beta(t)}ds$ (see Fig. 1), the integral $I_1(\tau)$ can be rewritten as

$$I_1(\tau) = \frac{1}{4\pi i}\int_\Gamma \frac{1}{t-\tau} w(s)dt, \quad (\tau \in S^+) \tag{16}$$

Letting $\tau \to t_o(\tau \in S^+, t_o \in \Gamma)$ and using the Sokhotski-Plemelj's formulae (7), we have

$$I_1^+(t_o) = \frac{w(t_o)}{4} + \frac{1}{4\pi i}\int_\Gamma \frac{1}{t-t_o} w(s)dt, \quad (t_o \in \Gamma) \tag{17}$$

Similarly, we have

$$I_2^+(t_o) = \overline{I_1^+(t_o)} = \frac{w(t_o)}{4} - \frac{1}{4\pi i}\int_\Gamma \frac{1}{\bar{t}-\bar{t}_o} w(s)d\bar{t}, \quad (t_o \in \Gamma) \tag{18}$$

and

$$I^+(t_o) = I_1^+(t_o) + I_2^+(t_o) = \frac{w(t_o)}{2} + \frac{1}{4\pi}\int_\Gamma S_1(t,t_o)w(s)ds, \quad (t_o \in \Gamma) \tag{19}$$

where

$$S_1(t,t_o) = \frac{e^{i\beta(t)}}{t-t_o} + \frac{e^{-i\beta(t)}}{\bar{t}-\bar{t}_o} \tag{20}$$

Letting $\tau \to t_o(\tau \in S^+, t_o \in \Gamma)$ and using Eqs. (12), (14) and (19) yields

$$\frac{w(t_o)}{2} = \frac{1}{4\pi}\int_\Gamma S_1(t,t_o)w(s)ds - \frac{1}{4\pi G}\int_\Gamma (\ln(t-t_o) + \ln(\bar{t}-\bar{t}_o))p(s)ds, \quad (t_o \in \Gamma) \tag{21}$$

It is noted here that, when taking the limit process $\tau \to t_o$ for the first integral at right hand side of Eq. (12), an additional term $w(t_o)/2$ was found. Thus, the left hand term in Eq. (21) becomes $w(t_o)/2$ (note that $w(t_o)/2 = w(t_o) - (w(t_o)/2)$). In the real variable BIE, this property has been obtained previously [15]. However, the property is obtained in a more explicit way in this paper, by using the Sokhotski-Plemelj's formulae (7).

Equation (21) represents the boundary integral equation using complex variable for the interior problem in antiplane elasticity based on the Somigliana identity. For compactness, we still name it the complex variable boundary integral equation (CVBIE). Alternatively, Eq. (21) is called the first complex variable BIE of antiplane elasticity.

After some manipulations, the following BIE for the interior boundary value problem in real variable was obtained [15] (here, to write the BIE in a slightly different form)

$$\frac{1}{2}w(\xi) + \int_\Gamma P^*(\xi,x)w(x)ds(x) = \int_\Gamma U^*(\xi,x)p(x)ds(x), \quad (\text{for } \xi \in \Gamma) \tag{22}$$

where the kernels $U^*(\xi,x)$ and $P^*(\xi,x)$ are defined by

$$U^*(\xi,x) = -\frac{1}{2\pi G}\ln r \quad (\text{with } r = |\xi - x|) \tag{23}$$

$$P^*(\xi,x) = G\frac{\partial U^*(\xi,x)}{\partial n_x} = -\frac{1}{2\pi}\frac{1}{r}(r_{,1}n_1 + r_{,2}n_2), \tag{24}$$

$$r = \sqrt{(x_1-\xi_1)^2 + (x_2-\xi_2)^2},$$

$$r_{,1} = \frac{x_1-\xi_1}{r} = \cos\alpha, \quad r_{,2} = \frac{x_2-\xi_2}{r} = \sin\alpha. \tag{25}$$

In Eq. (24) , the normal $n(n_1, n_2)$ at the boundary point "x" always directs at outward side. In Eq. (22), the point $x(x_1, x_2)$ (or t in Fig. 1b) is the field point, and $\xi(\xi_1, \xi_2)$ (or t_o in Fig. 1b) is the source point.

Clearly, the BIE shown by Eq. (22) in real form can be easily obtained from its counterpart Eq. (21), which is expressed in the complex variable form.

3 The second complex variable BIE for the interior region

Previously, Eq. (12) represents the displacement at the domain point τ. Now we will find the stress at the domain point τ (Fig. 1). From Eqs. (10) and (12), we have the following result

$$\sigma_{n\tilde{z}}(\tau) = G\frac{\partial w(\tau)}{\partial n_\tau} = \frac{G}{4\pi}\int_\Gamma \frac{d}{dn_\tau}\{S_1(t,\tau)\}w(s)ds + \frac{1}{4\pi}\int_\Gamma \left(\frac{e^{i\gamma(\tau)}}{t-\tau} + \frac{e^{-i\gamma(\tau)}}{\bar{t}-\bar{\tau}}\right)p(s)ds, \quad (\tau \in S^+) \tag{26}$$

Eq. (26) can be rewritten as

$$p(\tau) = \sigma_{n\tilde{z}}(\tau) = G\frac{\partial w(\tau)}{\partial n_\tau} = \frac{G}{4\pi}\int_\Gamma S_2(t,\tau)w(s)ds + \frac{1}{4\pi}\int_\Gamma \left(\frac{e^{i\gamma(\tau)}}{t-\tau} + \frac{e^{-i\gamma(\tau)}}{\bar{t}-\bar{\tau}}\right)p(s)ds, \quad (\tau \in S^+) \tag{27}$$

where

$$S_2(t,\tau)=\frac{d}{dn_\tau}\{S_1(t,\tau)\}=\frac{e^{i(\beta(t)+\gamma(\tau))}}{(t-\tau)^2}+\frac{e^{-i(\beta(t)+\gamma(\tau))}}{(\bar{t}-\bar{\tau})^2}, \tag{28}$$

In Eq. (28), $\gamma(\tau)$ denotes the inclined angle for the component $\sigma_{n\tilde{z}}(\tau)$ in Fig. 1b.

In Eq. (27), letting $\tau \to t_o$, $\gamma(\tau) \to \gamma(t_o)(\tau \in S^+, t_o \in \Gamma)$ and using the Sokhotski-Plemelj's formulae (7) yields

$$\frac{p(t_o)}{2}=\frac{G}{4\pi}\int_\Gamma S_2(t,t_o)w(s)ds \\ +\frac{1}{4\pi}\int_\Gamma\left(\frac{e^{i\gamma(t_o)}}{t-t_o}+\frac{e^{-i\gamma(t_o)}}{\bar{t}-\bar{t}_o}\right)p(s)ds, \quad (t_o\in\Gamma) \tag{29}$$

where

$$S_2(t,t_o)=\frac{e^{i(\beta(t)+\gamma(t_o))}}{(t-t_o)^2}+\frac{e^{-i(\beta(t)+\gamma(t_o))}}{(\bar{t}-\bar{t}_o)^2}, \tag{30}$$

Clearly, $S_2(t,t_o)$ represents a kernel with the hypersingular integral.

It is noted here that, when taking the limit process $\tau \to t_o$ for the second integral at right hand side of Eq. (27), an additional term $p(t_o)/2$ was found. Thus, the left hand term in Eq. (29) becomes $p(t_o)/2$ (note that $p(t_o)/2 = p(t_o) - (p(t_o)/2)$). In the real variable BIE, this property has been obtained previously [6,7]. However, in this paper the property is obtained in a more explicit way, by using the Sokhotski-Plemelj's formulae (7).

At this stage, we may summarize two BIEs as follows. In the first BIE shown by Eq. (21), the boundary displacement $w(t_o)$ is related to two integrals, which contain the displacement $w(s)$ and traction $p(s)$ in the integrands. In addition, the kernels for $w(s)$ and $p(s)$ are Cauchy singular and weaker singular, respectively. In the second BIE shown by Eq. (29), the boundary displacement $p(t_o)$ is related to two integrals, which contain the displacement $w(s)$ and traction $p(s)$ in the integrands. In addition, the kernels for $w(s)$ and $p(s)$ are hypersingular and Cauchy singular, respectively. Since the two BIEs reflect the nature of elasticity, the first BIE and the second BIE are independent.

4 The first complex variable BIE for the exterior region

4.1 Formulation of the first complex variable BIE for the exterior region

In the exterior BVP, we may use the Somigliana identity on the region bounded by contours Σ_1 and Σ_2 (Fig. 2). The contour Σ_2 is placed at a remote place, which may be assumed as a sufficient large circle. Similar to the first complex variable BIE for the interior region, the use of the Somigliana identity will yield

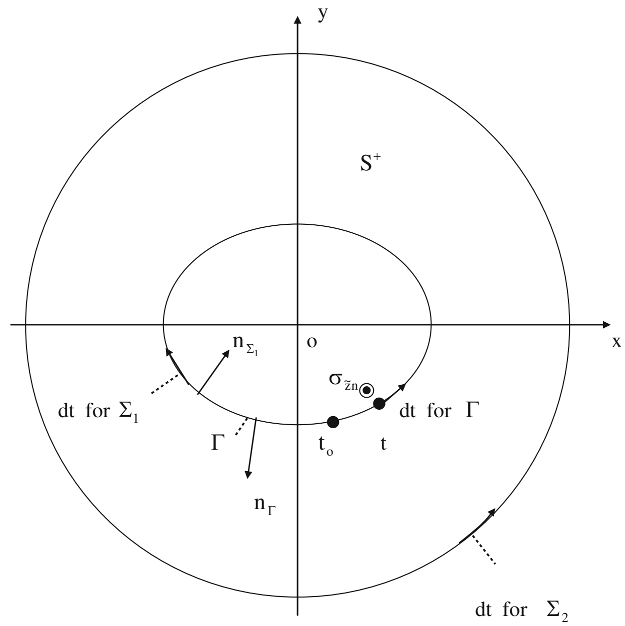


Fig. 2 Formulation of the Somigliana identity for exterior region

$$w(\tau)=\frac{1}{4\pi}\int_{\Sigma_1}S_1(t,\tau)w(s)ds-\frac{1}{4\pi G}\int_{\Sigma_1}(\ln(t-\tau) \\ +\ln(\bar{t}-\bar{\tau}))p(s)ds+E(\tau), \quad (\tau\in S^+) \tag{31}$$

where

$$E(\tau)=\frac{1}{4\pi}\int_{\Sigma_2}S_1(t,\tau)w(s)ds-\frac{1}{4\pi G}\int_{\Sigma_2}(\ln(t-\tau) \\ +\ln(\bar{t}-\bar{\tau}))p(s)ds, \quad (\tau\in S^+) \tag{32}$$

The term $E(\tau)$ in Eq. (31) denotes the contribution to the identity from the contour Σ_2 (Fig. 2).

In fact, we can rewrite "z" in Eq. (8) as "t" , where the variable "t" is a point on a sufficient large circle. Thus, the complex potential shown by Eq. (8) can be expressed

$$\phi(t)=-\frac{1}{2\pi}\ln(t-\tau)=-\frac{1}{2\pi}\left(\ln\ t-\sum_{k=1}^{\infty}\frac{1}{k}\frac{\tau^k}{t^k}\right), \\ \text{(with variable "}t\text{" on a large circle)} \tag{33}$$

Clearly, each component in Eq. (33), for example, $\phi(t) = -\ln t/2\pi$, or $\phi(t) = -\tau^3/6\pi t^3$ (here, τ-constant, t-the variable), causes non-vanishing stress anywhere. Thus, the complex potential shown by Eq. (33) is expressed in a pure deformable form. Alternatively, one can compare the expression for $\phi(t)$ shown by Eq. (33) with the expression Eq. (a1) in Appendix. Since there is no constant term in expression

(33) (refer to the constant a_o in Eq. (a1)), we see that the complex potential $\phi(t)$ shown by Eq. (33) has been expressed in the pure deformable form.

In fact, the term $E(\tau)$ is not always equal to zero. It is proved from the result in Appendix that if and only if the displacement in the physical field is expressed in a pure deformable form, the condition $E(\tau)=0$ will be satisfied. This is the regularity condition in the exterior boundary value problem (BVP). We assume that the displacement in the physical field has been expressed in a pure deformable form, and we have $E(\tau)=0$. It is noted here that the term $E(\tau)$ is a kind of mutual work difference integral (MWDI), which corresponds to an integral D(CR)/G in the Appendix.

Thus, from the satisfied condition $E(\tau)=0$, Eq. (31) can be rewritten as

$$w(\tau)=\frac{1}{4\pi}\int_{\Sigma_1} S_1(t,\tau)w(s)\mathrm{d}s-\frac{1}{4\pi G}\int_{\Sigma_1}(\ln(t-\tau)+\ln(\bar{t}-\bar{\tau}))p(s)\mathrm{d}s,\quad (\tau\in S^+) \tag{34}$$

Therefore, the expression for displacement $w(\tau)$ at the domain point in the exterior problem takes the same expression as shown by Eq. (12) in the interior problem.

In Eq. (34), letting $\tau\to t_o(\tau\in S^+, t_o\in\Sigma_1)$ (Fig. 2) and using the Sokhotski-Plemelj's formulae (7), we will find the first complex variable BIE for the exterior region as follows

$$\frac{w(t_o)}{2}=\frac{1}{4\pi}\int_{\Sigma_1} S_1(t,t_o)w(s)\mathrm{d}s-\frac{1}{4\pi G}\int_{\Sigma_1}(\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o))p(s)\mathrm{d}s,\quad (t_o\in\Sigma_1) \tag{35}$$

Equation (35) represents a Cauchy singular integral equation.

Clearly, the BIE shown by Eq. (35) in the exterior problem has the same form as shown by Eq. (21) in the interior problem. However, the two BIEs shown by Eqs. (35) and (21) have significant differences. Those differences are as follows.

An integration path Γ running in the anticlockwise direction is introduced below, which represents the inner boundary in the exterior BVP (Fig. 2). This notation, or Γ, will be used from here to the end of fifth section.

It is assumed that we consider the same contour configuration for the interior and the exterior problem (see Figs. 1, 2). In this case, the normal n_{Σ_1} to the clockwise path Σ_1 is just opposite to normal n_Γ to the anticlockwise path Γ. Thus, Eq. (35) can be rewritten as

$$\frac{w(t_o)}{2}=-\frac{1}{4\pi}\int_{\Gamma} S_1(t,t_o)w(s)\mathrm{d}s-\frac{1}{4\pi G}\int_{\Gamma}(\ln(t-t_o)+\ln(\bar{t}-\bar{t}_o))p(s)\mathrm{d}s,\quad \text{(for the exterior BVP)} \tag{36}$$

In the kernel $S_1(t,t_o)$, the component $\beta(t)$ must be related to n_Γ rather than n_{Σ_1} (Fig. 2).

The different structures of BIE in the interior problem and exterior problem must influence the properties of the BIE solution. For example, in the Neumann problem of interior problem, the input datum for $p(s)$, or the tractions on boundary in Eq. (21) must be in equilibrium. In the meantime, in the Neumann problem of exterior problem, the input datum for $p(s)$, or the tractions on boundary in Eq. (36) can be arbitrary.

4.2 First case of degenerate boundary for the first complex variable BIE for the exterior region

Below, we study two particular cases for the expression for $w(\tau)$ shown by Eq. (34) for the domain point. In the first case, it is assumed that the contour Σ_1 shrinks to a curve crack "L" (Fig. 3a), and the same magnitude traction with opposite direction are applied on the upper crack and the lower crack faces. Thus, we have

$$w^+(s)\neq w^-(s),\qquad (t(s)\in L) \tag{37a}$$

$$p^+(s)=-p^-(s),\qquad (t(s)\in L) \tag{37b}$$

In the meantime, we define

$$w_j(s)=w^+(s)-w^-(s),\qquad w_m(s)=w^+(s)+w^-(s) \tag{38}$$

In this case, the second term in the right hand of Eq. (34) disappears simply because of condition (37b). Thus, Eq. (34) is reduced to

$$w(\tau)=\frac{1}{4\pi}\int_L S_1(t,\tau)w_j(s)\mathrm{d}s,\quad (\tau\in S^+) \tag{39}$$

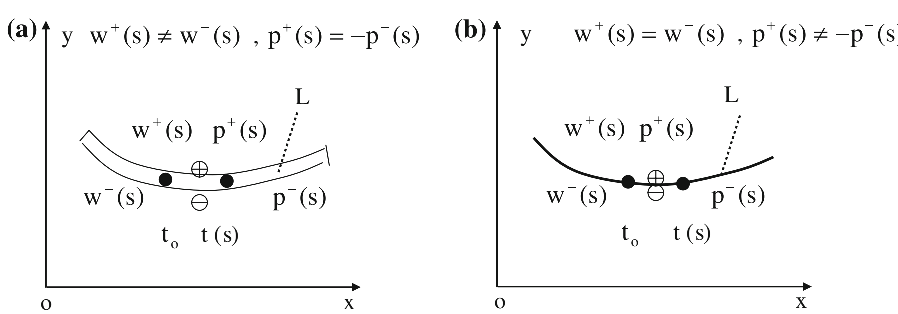


Fig. 3 **a** a curved crack, **b** a deformable curved line

In Eq. (39), letting $\tau \to t_o^+$ and $\tau \to t_o^-(t_o \in L)$ and using the Sokhotski-Plemelj's formulae (7) yields

$$w^+(t_o) = \frac{w_j(t_o)}{2} + \frac{1}{4\pi}\int_L S_1(t, t_o)w_j(s)\mathrm{d}s, \quad (t_o \in L) \tag{40}$$

$$w^-(t_o) = -\frac{w_j(t_o)}{2} + \frac{1}{4\pi}\int_L S_1(t, t_o)w_j(s)\mathrm{d}s, \quad (t_o \in L) \tag{41}$$

where $S_1(t, t_o)$ has been defined previously by Eq. (20).

From Eqs. (37), (40), (41), we have

$$\frac{w_m(t_o)}{2} = \frac{1}{4\pi}\int_L S_1(t, t_o)w_j(s)\mathrm{d}s, \quad (t_o \in L) \tag{42}$$

In the curved crack problem, the traction $p(s)$ is the input datum in the formulation, and the COD $w_j(s)$ (crack opening displacement function) is a value to be evaluated. From the solution for the COD, we can evaluate the stress intensity factor (SIF). It is a goal to obtain the COD and SIF in the curved crack problem. Clearly, Eq. (42) cannot provide any way to reach this goal.

In the meantime, Eq. (42) represents a relation between the average of displacements on the upper face and the lower face (or $w_m(s)/2$) and the subtraction of displacements on the upper face and the lower face (or $w_j(s)$, the crack opening displacement (COD) function).

4.3 Second case of degenerate boundary for the first complex variable BIE for the exterior region

In the second case, it is assumed that the contour Σ_1 shrinks to a curve line "L" (Fig. 3b), and the upper face and the lower face of curved line apply the same deformation. However, the tractions on the upper face and the lower face may not be the same. Thus, the deformable curved line problem is proposed (Fig. 3b). In this case, we have

$$w^+(s) = w^-(s), \quad (t(s) \in L) \tag{43a}$$

$$p^+(s) \neq -p^-(s), \quad (t(s) \in L) \tag{43b}$$

In the meantime, we define

$$p_j(s) = p^+(s) - p^-(s), \quad p_m(s) = p^+(s) + p^-(s), \quad (t(s) \in L) \tag{44}$$

In this case, the first term in the right hand of Eq. (34) disappears simply because of imposed condition (43a). Thus, Eq. (34) is reduced to

$$w(\tau) = -\frac{1}{4\pi G}\int_L (\ln(t-\tau) + \ln(\bar{t}-\bar{\tau}))p_m(s)\mathrm{d}s, \quad (\tau \in S^+) \tag{45}$$

In Eq. (45), letting $\tau \to t_o^+$ and $\tau \to t_o^-$ $(t_o \in L)$ yields

$$w^+(t_o) = w^-(t_o) = -\frac{1}{4\pi G}\int_L (\ln(t-t_o) + \ln(\bar{t}-\bar{t}_o))p_m(s)\mathrm{d}s, \quad (t_o \in L) \tag{46}$$

Equation (46) is the integral equation for the deformable curved line. Physically, the function $p_m(s)$ represents the body force applied on the curve. If function $p_m(s)$ is an input datum, the deformation $w^+(t_o) = w^-(t_o)$ will be obtained from Eq. (46). Alternatively, if function $w^+(t_o) = w^-(t_o)$ is an input datum, the body force $p_m(s)$ applied on curve will be obtained from the solution of Eq. (46).

4.4 Numerical examination

A numerical examination for the exterior BVP with the elliptic contour is presented below (Fig. 4). In computation, we assume that $a = 10$, $b/a = 0.25, 0.50, 0.75$ and 1.00, and $N = 120$ divisions are used in discretization. Constant displacement and traction elements are assumed along the boundary.

The previously introduced complex potential $\phi(z)$ shown by Eq. (1) is renamed by $\phi_1(z)$. The following mapping function is introduced

$$z = \omega(\varsigma) = R\left(\varsigma + \frac{m}{\varsigma}\right), \quad \left(\text{with } R = \frac{a+b}{2}, \; m = \frac{a-b}{a+b}\right) \tag{47}$$

Thus, we can define

$$\phi(\varsigma) = \phi_1(z)\mid_{z=\omega(\varsigma)} \tag{48}$$

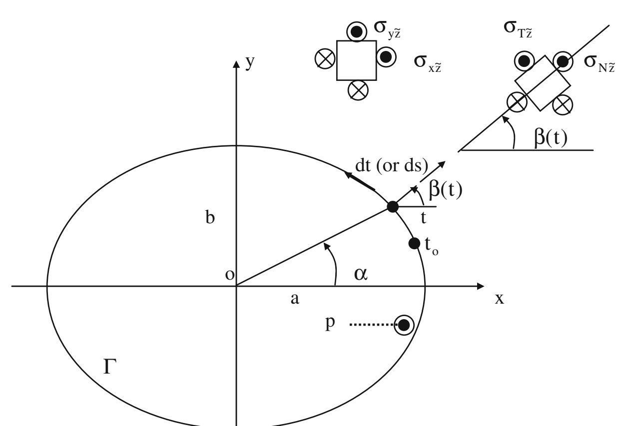


Fig. 4 An exterior BVP for an elliptic contour

Now we introduce the following complex potential

$$\phi(\varsigma) = i q_o a \frac{1}{\varsigma} \tag{49}$$

where q_o is a unit traction. It is easy to verify that the corresponding displacement and traction applied along the elliptic contour from $\phi(\varsigma)$ are as follows (Fig. 4)

$$w = \frac{q_o a}{G} \sin\theta, \quad \text{(at boundary point } x = a\cos\theta \;\; y = b\sin\theta) \tag{50}$$

$$p = q_o \sin\beta, \left(\text{at boundary point } x = a\cos\theta \;\; y = b\sin\theta, \text{ with } \tan\theta = \frac{a}{b}\tan\alpha = \frac{b}{a}\tan\beta\right) \tag{51}$$

In the first case, a Neumann BVP is studied. It assumed that the boundary traction takes the value $p = q_o \sin\beta$, or from the exact solution Eq. (51) (here $p = -\sigma_{N\tilde{z}}$). After solving the BIE shown by Eq. (36), the computed displacement "w" and the stress component $\sigma_{T\tilde{z}}$, are expressed as

$$w = \frac{q_o a}{G} f(b/a, \theta), \quad \text{(at boundary point } x = a\cos\theta \;\; y = b\sin\theta) \tag{52}$$

$$\sigma_{T\tilde{z}} = q_o g(b/a, \theta), \quad \text{(at boundary point } x = a\cos\theta \;\; y = b\sin\theta) \tag{53}$$

In the above-mentioned equations, the definition for $\sigma_{N\tilde{z}}$ and $\sigma_{T\tilde{z}}$ can be found from Fig. 4.

The computed results for $f(b/a, \theta)$ and $g(b/a, \theta)$ are plotted in Figs. 5 and 6, respectively. In fact, the exact results are plotted in a solid line, and numerical results in a dash line. Since higher accuracy is achieved in computation, the solid line curve and the dash line curve are merged into one curve. From the computed results, the stress concentration for the smaller case of b/a can be found. For example, in the place of $\theta = 0$, we have $g(b/a, \theta)\,|_{b/a=0.25} = 4.023$ (exact one 4.000) and $g(b/a, \theta)\,|_{b/a=1.00} = 1.000$.

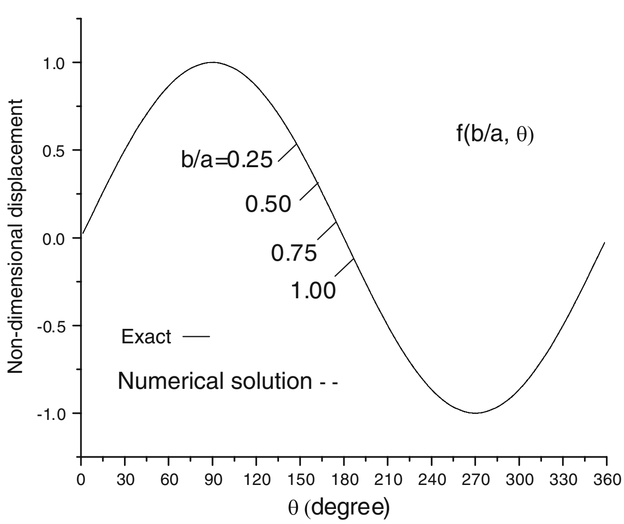


Fig. 5 Non-dimensional displacement $f(b/a, \theta)$ (see Eq. (52) and Fig. 4)

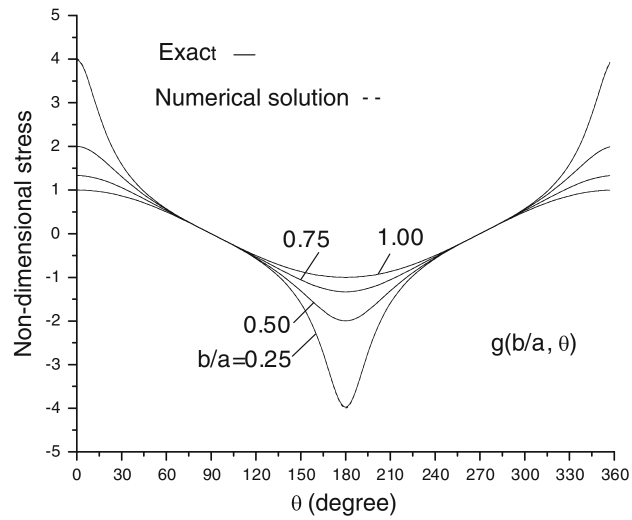


Fig. 6 Non-dimensional stress $g(b/a, \theta)$ (see Eq. (53) and Fig. 4)

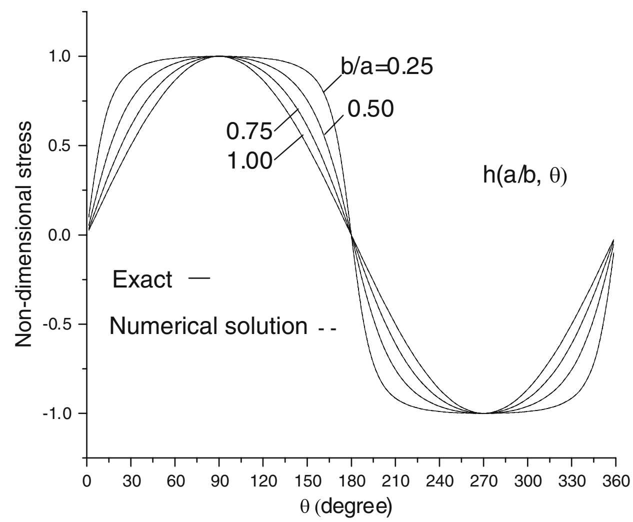


Fig. 7 Non-dimensional stress $h(b/a, \theta)$ (see Eq. (54) and Fig. 4)

In the second case, a Dirichlet BVP is studied. It assumed that the boundary displacement takes the value $w = \frac{q_o a}{G}\sin\theta$, or from the exct solution Eq. (50). After solving the BIE shown by Eq. (36), the computed stress "p" ($p = -\sigma_{N\tilde{z}}$) is expressed as

$$p = q_o h(b/a, \theta), \quad \text{(at boundary point } x = a\cos\theta \;\; y = b\sin\theta) \tag{54}$$

The computed results for $h(b/a, \theta)$ are plotted in Fig. 7. In fact, the exact results are plotted in a solid line, and numerical results in a dash line. Since higher accuracy is achieved in computation, the solid line curve and the dash line curve are merged into one curve. In addition, without regarding the ratio b/a, in the place of $\theta = \pi/2$, we have the maximum value $h(b/a, \theta)\mid_{\theta=\pi/2} = 1.000$.

5 The second complex variable BIE for the exterior region

5.1 Formulation of the second complex variable BIE for the exterior region

By the same reason mentioned in &4.1, or the condition $E(\tau) = 0$, from Eq. (34) and similar derivation in &3.1, the following traction expression for the domain point ($\tau \in S^+$) is obtained (Fig. 2)

$$p(\tau) = \sigma_{n\bar{z}}(\tau) = G\frac{\partial w(\tau)}{\partial n_\tau} = \frac{G}{4\pi}\int_{\Sigma_1} S_2(t,\tau)w(s)\mathrm{d}s + \frac{1}{4\pi}\int_{\Sigma_1}\left(\frac{\mathrm{e}^{i\gamma(\tau)}}{t-\tau} + \frac{\mathrm{e}^{-i\gamma(\tau)}}{\bar{t}-\bar{\tau}}\right)p(s)\mathrm{d}s, \quad (\tau \in S^+) \tag{55}$$

where $S_2(t,\tau)$ has been defined by Eq. (28) previously. Clearly, the process for obtaining Eq. (55) from Eq. (34) is similar to the process for obtaining Eq. (26) from Eq. (12), which is mentioned in &3.1.

In Eq. (55), letting $\tau \to t_o$ ($t_o \in \Sigma_1$) and using the Sokhotski-Plemelj's formulae (7) yields

$$\frac{p(t_o)}{2} = \frac{G}{4\pi}\int_{\Sigma_1} S_2(t,t_o)w(s)\mathrm{d}s + \frac{1}{4\pi}\int_{\Sigma_1}\left(\frac{\mathrm{e}^{i\gamma(\tau)}}{t-t_o} + \frac{\mathrm{e}^{-i\gamma(\tau)}}{\bar{t}-\bar{t}_o}\right)p(s)\mathrm{d}s, \quad (\text{here } t_o(s_o) \in \Sigma_1) \tag{56}$$

where $S_2(t,t_o)$ has been defined by Eq. (30) previously. Eq. (56) represents a hypersingular integral equation.

5.2 First case of degenerate boundary for the second complex variable BIE for the exterior region

Below, we study two particular cases for the expression for $p(\tau)$ shown by Eq. (55) for the domain point. In the first case, it is assumed that the contour Σ_1 shrinks to a curve crack "L" (Fig 3a), and the upper crack and the lower crack face are applied the same magnitude traction with opposite direction.

In this case, the property shown by Eq. (37a,b) are still valid, and Eq. (38) is still used. Thus, Eq. (55) can be reduced to

$$p(\tau) = \frac{G}{4\pi}\int_L S_2(t,\tau)w_j(s)\mathrm{d}s, \quad (\tau \in S^+) \tag{57}$$

In Eq. (57), letting $\tau \to t_o^+$ and $\gamma(\tau) \to \gamma(t_o)$ ($t_o \in L$) yields

$$p^+(t_o) = \frac{G}{4\pi}\int_L S_2(t,t_o)w_j(s)\mathrm{d}s, \quad (t_o \in L) \tag{58}$$

where $S_2(t,t_o)$ has been defined by Eq. (30) previously.

In Eq. (57), letting $\tau \to t_o^-$ and $\gamma(\tau) \to \pi + \gamma(t_o)$ ($t_o \in L$), a similar result to Eq. (58) will be obtained as follows

$$p^-(t_o) = -\frac{G}{4\pi}\int_L S_2(t,t_o)w_j(s)\mathrm{d}s = -p^+(t_o), \quad (t_o \in L) \tag{59}$$

where $S_2(t,t_o)$ has been defined by Eq. (30) previously.

Equation (58) or (59) represents a hypersingular integral equation for the curved crack problem. Generally, the traction p(s) is the input datum in the formulation, and the COD $w_j(s)$ (crack opening displacement function) is a value to be evaluated. Clearly, Eq. (58) or (59) provides the way to reach this goal. A real variable hypersingular integral equation similar to Eq. (58) was suggested in an earlier year [16].

5.3 Second case of degenerate boundary for the second complex variable BIE for the exterior region

In the second case, it is assumed that the contour Σ_1 shrinks to a curve line "L" (Fig 3b), and the upper face and the lower face of curve line are applied the same deformation. However, the tractions on the upper face and the lower face may not be the same. In this case, the properties shown by Eq. (43a,b) are still valid, and Eq. (44) is still used here.

In this case, after considering Eq. (43a), Eq. (55) is reduced to

$$p(\tau) = \frac{1}{4\pi}\int_L\left(\frac{\mathrm{e}^{i\gamma(\tau)}}{t-\tau} + \frac{\mathrm{e}^{-i\gamma(\tau)}}{\bar{t}-\bar{\tau}}\right)p_m(s)\mathrm{d}s, \quad (\tau \in S^+) \tag{60}$$

In Eq. (60), letting $\tau \to t_o^+$ and $\gamma(\tau) \to \gamma(t_o)$ ($t_o \in L$) and using the Sokhotski-Plemelj's formulae (7) yields

$$p^+(t_o) = \frac{p_m(t_o)}{2} + \frac{1}{4\pi}\int_L\left(\frac{\mathrm{e}^{i\gamma(t_o)}}{t-t_o} + \frac{\mathrm{e}^{-i\gamma(t_o)}}{\bar{t}-\bar{t}_o}\right)p_m(s)\mathrm{d}s, \quad (t_o \in L) \tag{61}$$

In Eq. (60), letting $\tau \to t_o^-$ and $\gamma(\tau) \to \pi + \gamma(t_o)$ ($t_o \in L$), a similar result to Eq. (61) will be obtained as follows

$$p^-(t_o) = \frac{p_m(t_o)}{2} - \frac{1}{4\pi}\int_L\left(\frac{\mathrm{e}^{i\gamma(t_o)}}{t-t_o} + \frac{\mathrm{e}^{-i\gamma(t_o)}}{\bar{t}-\bar{t}_o}\right)p_m(s)\mathrm{d}s, \quad (t_o \in L) \tag{62}$$

Thus, we have

$$\frac{p_j(t_o)}{2}=\frac{1}{4\pi}\int_L\left(\frac{e^{i\gamma(t_o)}}{t-t_o}+\frac{e^{-i\gamma(t_o)}}{\bar{t}-\bar{t}_o}\right)p_m(s)ds,\quad (t_o\in L) \tag{63}$$

Generally, the displacement w(s) is the input datum in the formulation and the body force density $p_m(s)$ is a value to be evaluated. Clearly, Eq. (63) cannot provide any way to reach this goal.

In the meantime, Eq. (63) represents a relation between the average of body force density on the upper face and the lower face ($p_m(s)$) and the subtraction of tractions on the upper face and the lower face (or $p_j(t_o)$).

6 Conclusions

From above-mentioned analysis we can see some advantages in BIE formulation if one uses the complex variable. For example, after using the Somigliana identity and the complex variable for the interior problem, a compact expression for the displacement at the domain point is obtained, which is shown by Eq. (12). In Eq. (12), there are two integrals

$$J_1(\tau)=\int_\Gamma S_1(t,\tau)w(s)ds,\quad \text{where}$$

$$S_1(t,\tau)=\frac{e^{i\beta(t)}}{t-\tau}+\frac{e^{-i\beta(t)}}{\bar{t}-\bar{\tau}},\quad (\tau\in S^+) \tag{64}$$

$$J_2(\tau)=\int_\Gamma (\ln(t-\tau)+\ln(\bar{t}-\bar{\tau}))p(s)ds,\quad (\tau\in S^+) \tag{65}$$

From the relation $dt = ie^{i\beta(t)}ds$ (see Fig. 1), it is easily seen that the integral $J_1(\tau)$ is a Cauchy type integral. A difficult point for the beginner of BIE is that there is a jump value in some integrals when a domain point approaches a boundary point, or $\tau\to t_o^+$, $\tau\to t_o^-$. Clearly, if the complex variable is used, the jump value can be easily seen from the property of the Cauchy type integral.

In the meantime, the property of $J_2(\tau)$ is also easy to see from the structure of the expression. Clearly, when $\tau\to t_o^+$ or $\tau\to t_o^-$, the integrand in $J_2(\tau)$ becomes $2\ln|t-t_o|\,p(s)$, and no jump value for the integral is found.

This paper clearly provides a displacement versus traction operator, as shown by Eq. (26). By using this operator, the traction expression for the domain point can be easily formulated which is shown by Eq. (27). In Eq. (27), there are two integrals

$$J_3(\tau)=\int_\Gamma S_2(t,\tau)w(s)ds,\quad \text{where}$$

$$S_2(t,\tau)=\frac{e^{i(\beta(t)+\gamma(\tau))}}{(t-\tau)^2}+\frac{e^{-i(\beta(t)+\gamma(\tau))}}{(\bar{t}-\bar{\tau})^2},\quad (\tau\in S^+) \tag{66}$$

$$J_4(\tau)=\int_\Gamma\left(\frac{e^{i\gamma(\tau)}}{t-\tau}+\frac{e^{-i\gamma(\tau)}}{\bar{t}-\bar{\tau}}\right)p(s)ds,\quad (\tau\in S^+) \tag{67}$$

From the relation $dt = ie^{i\beta(t)}ds$(see Fig. 1), it is easily seen that the integral $J_4(\tau)$ is a Cauchy type integral. In the meantime, the property of $J_3(\tau)$ is also easy to see from the structure of the expression. When $\tau\to t_o^+$ or $\tau\to t_o^-$, no jump value for the integral $J_3(\tau)$ is found.

For the exterior problem, similar derivations are suggested in the paper. Different to the interior problem, there are two degenerate boundary cases for the boundary, one is curved crack and the other is the deformable curved line. From the second complex BIE and the degenerate boundary case of a curved crack, the formulation of the hypersinular integral equation is obtained in a more explicit way, which is shown by Eq. (58).

In conclusion, this paper provides all possibilities, which happened in the BVP of antiplane elasticity. The formulation is based on a unified way with the usage of complex variable. The fundamental solution plays an important role in the formulation, which represents a particular solution caused by concentrated forces. Particularly, all the kernels in the BIEs are expressed in an explicit form. From the expressions of the kernels, one can easily determine the kinds of singularity for the involved kernels. Those singularities include the weaker singular with logarithmic function, the Cauchy singular, and hypersingular.

Appendix

Evaluation of a mutual work difference integral (MWDI) in antiplane elasticity.

In the following analysis, two stress fields, or the α-field and the β-field, for an infinite body containing many cracks, inclusions and holes are defined. In both fields, the remote stresses are assumed to be zero, or $\sigma_{zx}^{\infty}=0$, $\sigma_{zy}^{\infty}=0$ (Fig. 8).

The complex potentials for two fields can be expressed in the following form

$$\phi_{(\alpha)}(z)=A\ln z+a_0+\sum_{k=1}^{\infty}\frac{a_k}{z^k}\quad \text{(for the } \alpha\text{-field)} \tag{a1}$$

$$\phi_{(\beta)}(z)=B\ln z+b_0+\sum_{k=1}^{\infty}\frac{b_k}{z^k}\quad \text{(for the } \beta\text{-field)} \tag{a2}$$

From the single-valued condition of displacement, the constants "A" and "B" in Eqs. (a1) and (a2) must take a real value. In addition, the coefficients a_0, b_0, a_i, b_i (i =1,2,3,…) generally take complex values.

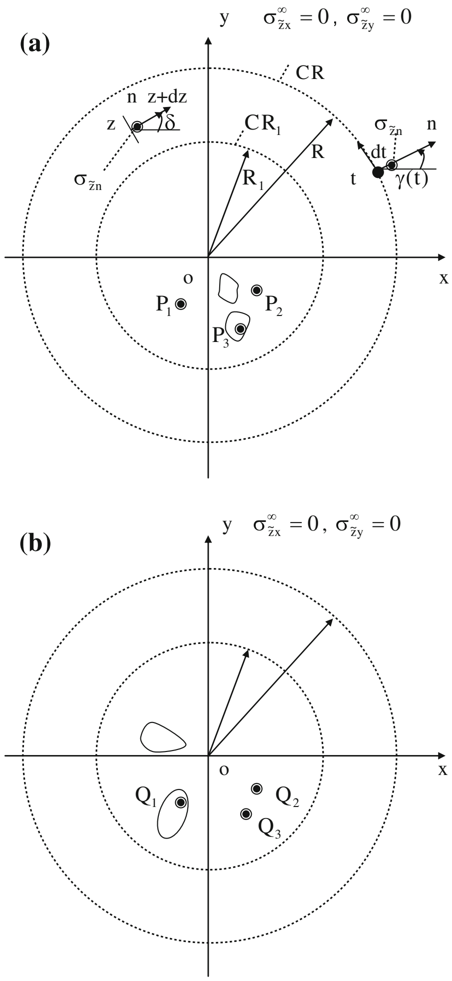


Fig. 8 Configurations and loadings for defining an integral: **a** the α- stress field, **b** the β-stress field

Note that the terms a_0, b_0 result in a rigid motion with no stress initiated. Since the terms a_0, b_0 are involved in Eqs. (a1) and (a2), the complex potentials shown by Eqs. (a1) and (a2) are said to be expressed in an impure deformable form.

Clearly, if one uses the Betti's reciprocal theorem in the ring area for the α-field and the β-field (Fig. 8), and obtains

$$D(\mathrm{CR}) - D(\mathrm{CR}_1) = 0, \quad \text{or } D(\mathrm{CR}) = D(\mathrm{CR}_1) \tag{a3}$$

where two integrals are defined as

$$D(\mathrm{CR}) = G \int_{\mathrm{CR}} (w_{(\alpha)}\sigma_{\bar{z}n(\beta)} - w_{(\beta)}\sigma_{\bar{z}n(\alpha)})\mathrm{d}s \tag{a4}$$

$$D(\mathrm{CR}_1) = G \int_{\mathrm{CR}_1} (w_{(\alpha)}\sigma_{\bar{z}n(\beta)} - w_{(\beta)}\sigma_{\bar{z}n(\alpha)})\mathrm{d}s \tag{a5}$$

In the integral $D(\mathrm{CR})$, CR denotes a sufficient large circle with radius "R". In addition, in the integral $D(\mathrm{CR}_1)$, CR_1 denotes a sufficient large circle with radius "R_1" (Fig. 8). Clearly, the integral $D(\mathrm{CR})$ is a path independent integral, which can be evaluated at an arbitrary circle with sufficient large value of "R".

From Eqs. (1)–(5), the expressions for the displacement and the stress are as follows (Fig. 8)

$$Gw(x, y) = \frac{1}{2}(\phi(z) + \overline{\phi(z)}) \tag{a6}$$

$$\sigma_{n\bar{z}} = G\frac{\partial w}{\partial n} = \frac{1}{2}\frac{\partial(\phi(z) + \overline{\phi(z)})}{\partial n}$$
$$= \frac{1}{2}(\phi'(z)\mathrm{e}^{i\delta} + \overline{\phi'(z)}\,\mathrm{e}^{-i\delta}) \tag{a7}$$

Since the integration is performed along the circle "CR" with radius "R"(Fig. 8), for a point "t" on the circular boundary "CR" we have

$$\bar{t} = \frac{R^2}{t}, \quad \mathrm{d}\bar{t} = -\frac{R^2}{t^2}\mathrm{d}t, \quad \mathrm{d}t = ie^{i\gamma(t)}\mathrm{d}s \tag{a8}$$

where $\gamma(t)$ represents an inclined angle of normal at the point "t" (Fig. 8).

Thus, for a point "t" on the boundary and "dt" along the boundary, from Eq. (a8) we have

$$Gw = \frac{1}{2}(\phi(t) + \overline{\phi(t)}) \tag{a9}$$

$$\sigma_{n\bar{z}}\mathrm{d}s = \left(G\frac{\partial w}{\partial n}\right)\mathrm{d}s = \frac{1}{2i}(\phi'(t)\mathrm{d}t - \overline{\phi'(t)}\,\mathrm{d}\bar{t})$$
$$= \frac{1}{2i}\left(\phi'(t)\mathrm{d}t + \frac{R^2}{t^2}\overline{\phi'(t)}\,\mathrm{d}t\right) \tag{a10}$$

After Substituting Eqs. (a1) and (a2) into Eq. (a4), the integral D(CR) is composed of many terms. Some typical terms are evaluated as follows.

In the first case, we assume

$$\phi_{(\alpha)1}(z) = A\ln z, \quad \text{(for the } \alpha\text{-field)} \tag{a11}$$
$$\phi_{(\beta)1}(z) = B\ln z, \quad \text{(for the } \beta\text{-field)} \tag{a12}$$

In this case, at a point "t" on the boundary and "dt" along the boundary we have

$$w_{(\alpha)1} = \frac{A}{B}w_{(\beta)1}, \quad \sigma_{\bar{z}n(\alpha)1} = \frac{A}{B}\sigma_{\bar{z}n(\beta)1} \tag{a13}$$

Substituting Eq. (a13) into Eq. (a4) yields

$$D(\mathrm{CR})_1 = G\int_{CR}(w_{(\alpha)1}\sigma_{\bar{z}n(\beta)1} - w_{(\beta)1}\sigma_{\bar{z}n(\alpha)1})\mathrm{d}s = 0 \tag{a14}$$

In the second case, we assume

$$\phi_{(\alpha)2}(z) = A\ln z \quad \text{(for the } \alpha\text{-field)} \tag{a15}$$
$$\phi_{(\beta)2}(z) = b_o \quad \text{(for the } \beta\text{-field)} \tag{a16}$$

In this case, at a point "t" on the boundary and "dt" along the boundary we have

$$Gw_{(\alpha)2} = A\ln R, \quad \sigma_{\bar{z}n(\alpha)2}ds = \frac{A}{i}\frac{dt}{t} \tag{a17}$$

$$Gw_{(\beta)2} = Re\, b_o, \quad \sigma_{\bar{z}n(\beta)2}ds = 0 \tag{a18}$$

Substituting Eqs. (a17) and (a18) into Eq. (a4) yields

$$D(\mathrm{CR})_2 = G\int_{\mathrm{CR}} (w_{(\alpha)2}\sigma_{\bar{z}n(\beta)2} - w_{(\beta)2}\sigma_{\bar{z}n(\alpha)2})ds$$

$$= -\frac{A(Re\, b_o)}{i}\int_{\mathrm{CR}} \frac{dt}{t} = -2\pi A(Re\, b_o) \tag{a19}$$

In the third case, we assume

$$\phi_{(\alpha)3}(z) = A\ln z, \quad (\text{for the } \alpha\text{-field}) \tag{a20}$$

$$\phi_{(\beta)3}(z) = \frac{b_k}{z^k}, \quad \phi'_{(\beta)3}(z) = -\frac{kb_k}{z^{k+1}}, \ (k \geq 1, \ \text{for the } \beta\text{-field}) \tag{a21}$$

In this case, at a point "t" on the boundary and "dt" along the boundary we have

$$Gw_{(\alpha)3} = A\ln R, \quad \sigma_{\bar{z}n(\alpha)3}ds = \frac{A}{i}\frac{dt}{t} \tag{a22}$$

$$Gw_{(\beta)3} = \frac{1}{2}\left(\frac{b_k}{t^k} + \frac{\bar{b}_k t^k}{R^{2k}}\right),$$

$$\sigma_{\bar{z}n(\beta)3}ds = -\frac{k}{2i}\left(\frac{b_k}{t^{k+1}} + \frac{\bar{b}_k t^{k-1}}{R^{2k}}\right)dt \tag{a23}$$

Substituting Eqs. (a22) and (a23) into Eq. (a4) yields

$$D(\mathrm{CR})_3 = G\int_{\mathrm{CR}} (w_{(\alpha)3}\sigma_{\bar{z}n(\beta)3} - w_{(\beta)3}\sigma_{\bar{z}n(\alpha)3})ds$$

$$= -\frac{kA\ln R}{2i}\int_{\mathrm{CR}} \left(\frac{b_k}{t^{k+1}} + \frac{\bar{b}_k t^{k-1}}{R^{2k}}\right)dt$$

$$-\frac{A}{2i}\int_{\mathrm{CR}} \left(\frac{b_k}{t^{k+1}} + \frac{\bar{b}_k t^{k-1}}{R^{2k}}\right)dt = 0 \tag{a24}$$

Clearly, for $k \geq 1$, there is no term dt/t involved in the integrand of (a24).

After substituting Eqs. (a1) and (a2) into Eq. (a4), and using the results shown in three examples and similar results, we will get the final result

$$D(\mathrm{CR}) = G\int_{\mathrm{CR}} (w_{(\alpha)}\sigma_{\bar{z}n(\beta)} - w_{(\beta)}\sigma_{\bar{z}n(\alpha)})ds$$

$$= -2\pi\{A(Re\, b_o) - B(Re\, a_o)\} \tag{a25}$$

Deleting the terms a_0, b_0 in Eqs. (a1) and (a2), the complex potentials for two fields can be expressed in the following form

$$\phi_{(\alpha)}(z) = A\ln z + \sum_{k=1}^{\infty}\frac{a_k}{z^k} \quad (\text{for the } \alpha\text{-field}) \tag{a26}$$

$$\phi_{(\beta)}(z) = B\ln z + \sum_{k=1}^{\infty}\frac{b_k}{z^k} \quad (\text{for the } \beta\text{-field}) \tag{a27}$$

Since no constant terms (or a_0, b_0) are involved in Eqs. (a26) and (a27), the complex potentials shown by Eqs. (a26) and (a27) are said to be expressed in a pure deformable form.

In this case, substituting $a_0 = 0, b_0 = 0$ into Eq. (a25) yields

$$D(\mathrm{CR}) = G\int_{\mathrm{CR}} (w_{(\alpha)}\sigma_{\bar{z}n(\beta)} - w_{(\beta)}\sigma_{\bar{z}n(\alpha)})ds = 0 \tag{a28}$$

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