Comput. Methods Appl. Mech. Engrg. 197 (2008) 4858–4865

Contents lists available at ScienceDirect

Comput. Methods Appl. Mech. Engrg.

journal homepage: www.elsevier.com/locate/cma

Investigation of rotating hollow cylinders of strain-hardening viscoplastic materials by sequential limit analysis

S.-Y. Leu *

Department of Aviation Mechanical Engineering, China Institute of Technology, No. 200, Jhonghua Street, Hengshan Township, Hsinchu County 312, Taiwan, ROC

ARTICLE INFO

Article history:
Received 5 June 2007
Received in revised form 5 July 2008
Accepted 17 July 2008
Available online 5 August 2008

Keywords:
Limit analysis
Sequential limit analysis
Plastic limit angular velocity
Rotating hollow cylinder
Nonlinear strain hardening
Strain-rate sensitivity

ABSTRACT

The paper employs sequential limit analysis to investigate rotating hollow cylinders of nonlinear isotropic strain-hardening viscoplastic materials. A computational optimization procedure was developed to account for hardening material properties and weakening behavior corresponding to the strain-rate sensitivity and widening deformation. A sequence of limit analysis problems was then conducted to seek the corresponding plastic limit angular velocities sequentially by a general algorithm incorporated with an inner and outer iterative sequence. Particularly, analytical solutions of plastic limit angular velocities and the onset of instability were derived for rigorous comparisons. The corresponding stability condition was also obtained explicitly. Specifically, the implicit form of the onset of instability was solved by the fixed point iteration. It is found that the computed limit angular velocities are rigorous upper bounds and match well with analytical solutions.

1. Introduction

Limit analysis is a conventional but yet convenient and comparable tool [1–13], especially in conjunction with finite element methods [14] and computational optimization techniques [15]. Providing efficiently the plastic limit loads with simple input data, limit analysis plays the role of a snapshot look at the structural performance. Furthermore, sequential limit analysis is to conduct a sequence of limit analysis problems with updating local yield criteria in addition to the configuration of the deforming structures. Accordingly, it has been illustrated widely that sequential limit analysis is an accurate and efficient tool for the large deformation analysis [16–26].

Especially, a general algorithm featuring a combined smoothing and successive approximation (CSSA) presented by Yang [27] has been utilized successfully with satisfactory results at a modest cost in certain problems of limit analysis [8] and sequential limit analysis [18–26]. Particularly, its numerical efficiency has been revealed explicitly through some quantitative comparisons with elasto-plastic analysis by Kim and Huh [21]. On the other hand, its convergence analysis was originally performed and validation was also conducted rigorously [22,23,25] by considering sequential limit analysis of pressurized hollow cylinders of viscoplastic materials [22], involving materials with nonlinear isotropic hardening [23], or involving materials with viscoplastic nonlinear isotropic hardening [25].

Plastic limit angular velocities of cylinders are useful information requested frequently for structure optimal design and safety evaluation. For investigating such problems of optimization features, the author and his coworker [24] have applied successfully sequential limit analysis based on the CSSA algorithm to deal with the rotating problems involving nonlinear isotropic hardening materials. Much effort [28–31] has been made to such interesting topics by investigating the elastic–plastic behavior and the fully plastic state. Similar attention [32–36] is also paid to the limit angular velocities of disks.

Based on the previously successful applications [22–25], the paper aims to extend further the above-mentioned CSSA algorithm to upper-bound limit analysis of rotating problems considering the combination effect of strain hardening and strain-rate dependence. The paper will focus on the rotating cylinders problems involving nonlinear isotropic hardening viscoplastic materials. Contrary to the normalization on stress filed or velocity field along some edge boundaries (e.g. [8,18–23,25]) adopted usually in the CSSA algorithm, the rotating problem formulation to be concerned features the velocity control on the whole domain [24]. It is also noted that such problems feature in involving hardening material properties and weakening behavior corresponding to the strain-rate sensitivity in addition to widening deformation. Thus, the applicability of the CSSA algorithm is to be validated by numerical and analytical studies of thick-walled cylinders involving materials of the von Mises model with viscoplastic nonlinear isotropic hardening. Particularly,

* Fax: +886 3 5936297.
E-mail address: syleu@cc.chit.edu.tw

0045-7825/$ - see front matter © 2008 Elsevier B.V. All rights reserved.
doi:10.1016/j.cma.2008.07.006

corresponding to the specific normalization condition adopted in the paper, the onset of instability and the existence of hardening phenomena before the weakening behavior are to be investigated analytically and explicitly. And the limiting cases of the current work are to be converted to the previous results [24].

In the following sections, the paper employs sequential limit analysis to deal with the rotating hollow cylinders of the von Mises materials with viscoplastic nonlinear isotropic strain hardening. By sequential limit analysis, the paper is to treat the plasticity problems as a sequence of limit analysis problems seeking the least upper bound of plastic angular velocities sequentially. A computational optimization procedure is to be developed to appropriately account for the interaction between hardening material properties and weakening behavior corresponding to the strain-rate sensitivity and widening deformation. Particularly, analytical solutions of plastic limit angular velocity, the onset of instability and the stability condition are to be derived for rigorous comparisons and validation.

2. Problem formulation

2.1. Lower bound formulation

The hollow cylinder is considered to rotate about its axis at a constant angular velocity ω. It is assumed that the angular acceleration is negligible. In the beginning, we consider a general plane-strain problem with the domain D consisting of the static boundary, ∂D_s, and the kinematic boundary, ∂D_k. The problem is then to search for the maximum allowable angular velocity factor, $\rho\omega^2(\sigma)$, subjected to constraints of static and constitutive admissibility such that

$$\begin{aligned} &\text{maximize} \quad \rho\omega^2(\sigma) \\ &\text{subject to} \quad \nabla\cdot\sigma + \rho\omega^2\,\vec{r} = 0 \quad \text{in } D \\ &\qquad\qquad \|\sigma\|_\vee \leqslant \sigma_Y \quad \text{in } D \end{aligned} \tag{1}$$

where ρ is the constant material density of the rotating hollow cylinders, ω is the angular velocity, $\rho\omega^2\,\vec{r}$ is the centrifugal force with $\vec{r}$ the position vector, $\|\sigma\|_\vee$ denotes the von Mises primal norm on stress tensor σ and σ_Y is a material constant. Therefore, this constrained problem is simply to maximize the angular velocity factor $\rho\omega^2(\sigma)$ representing the magnitude of the driving load.

Obviously, the problem statement leads naturally to the lower bound formulation seeking the extreme solution under constraints of static and constitutive admissibility. The statically admissible solutions satisfy the equilibrium equation and the static boundary condition. The constitutive admissibility is stated by the yield criterion in an inequality form. As to the existence of a unique extreme solution, we can further interpret the solutions as sets as shown in the work of Huh and Yang [8], Yang [26]. Note that, the equilibrium equation is linear and the constitutive inequality is convex and bounded. Accordingly, the intersection of statically admissible set and constitutively admissible set is convex and bounded. Thus, the existence of a unique maximum to the convex programming problem can be confirmed.

2.2. Upper bound formulation

Now we intend to transform the lower bound formulation to the upper bound formulation as similar to the previous work of Huh and Yang [8], Leu and Chen [24]. Equilibrium equations can be restated weakly in the form as

$$\int_D \vec{u}\cdot(\nabla\cdot\sigma + \rho\omega^2\,\vec{r})\mathrm{d}A = 0, \tag{2}$$

where $\vec{u}$ is a kinematically admissible velocity field. Integrating by parts, using the divergence theorem and imposing static boundary conditions, we may rewrite Eq. (2) to give an expression for $\rho\omega^2(\sigma)$ as

$$\int_D \vec{u}\cdot(\rho\omega^2\,\vec{r})\mathrm{d}A = \rho\omega^2(\sigma)\int_D \vec{u}\cdot\vec{r}\,\mathrm{d}A = \int_D \sigma:\dot{\varepsilon}\mathrm{d}A, \tag{3}$$

where $\dot{\varepsilon}$ is the strain rate tensor.

Since the power $\sigma:\dot{\varepsilon}$ is nonnegative. It is clear that $\sigma:\dot{\varepsilon} = |\sigma:\dot{\varepsilon}|$. Further, according to a generalized Hölder inequality [37], and the normality condition in plasticity [38], it results in

$$\sigma:\dot{\varepsilon} = |\sigma:\dot{\varepsilon}| \leqslant \|\sigma\|_\vee\|\dot{\varepsilon}\|_{-\vee} = \bar{\sigma}\dot{\bar{\varepsilon}}, \tag{4}$$

where $\|\dot{\varepsilon}\|_{-\vee}$ is the dual norm [8] of $\|\sigma\|_\vee$ based on the flow rule associated with the von Mises yield criterion. Therefore, we have

$$\begin{aligned} \rho\omega^2(\sigma)\int_D \vec{u}\cdot\vec{r}\,\mathrm{d}A &= \int_D \sigma:\dot{\varepsilon}\mathrm{d}A \leqslant \int_D \|\sigma\|_\vee\|\dot{\varepsilon}\|_{-\vee}\mathrm{d}A \\ &\leqslant \sigma_Y\int_D \|\dot{\varepsilon}\|_{-\vee}\mathrm{d}A. \end{aligned} \tag{5}$$

Since $\vec{u}$ appears homogeneously and linearly in Eq. (3) and inequality (5), we can normalize the relationship by setting the following normalization:

$$\int_D \vec{u}\cdot\vec{r}\,\mathrm{d}A = 1, \tag{6}$$

which is to be treated as one of constraints. Note that, the normalization condition involving the velocity field is imposed on the whole domain. In the previous works (e.g. [8,22,23,25]), the normalization condition was, in stead, simply related to the stress field [8] or the velocity filed [22,23,25] control along some boundaries.

Accordingly, $\rho\omega^2(\sigma)$ can be bounded above by $\rho\bar{\omega}^2(\vec{u})$ as

$$\rho\omega^2(\sigma) = \int_D \sigma:\dot{\varepsilon}\mathrm{d}A \leqslant \int_D \|\sigma\|_\vee\|\dot{\varepsilon}\|_{-\vee}\mathrm{d}A \leqslant \sigma_Y\int_D \|\dot{\varepsilon}\|_{-\vee}\mathrm{d}A = \rho\bar{\omega}^2(\vec{u}). \tag{7}$$

Thus, the upper bound formulation is stated in the form of a constrained minimization problem as

$$\begin{aligned} &\text{minimize} \quad \rho\bar{\omega}^2(\vec{u}) \\ &\text{subject to} \quad \rho\bar{\omega}^2(\vec{u}) = \sigma_Y\int_D \|\dot{\varepsilon}\|_{-\vee}\mathrm{d}A \\ &\qquad\qquad \int_D \vec{u}\cdot\vec{r}\,\mathrm{d}A = 1 \text{ in } D \\ &\qquad\qquad \nabla\cdot\vec{u} = 0 \quad \text{in } D \\ &\qquad\qquad \text{kinematic boundary conditions on } \partial D_k \end{aligned} \tag{8}$$

where $\nabla\cdot\vec{u} = 0$ is the incompressibility constraint inherent in the von Mises model.

Therefore, the upper bound formulation seeks sequentially the least upper bound for each step on kinematically admissible solutions. Accordingly, the primal–dual problems (1) and (8) are convex programming problems following the work of Huh and Yang [8] and Yang [26] and as demonstrated by Yang [11,12,39]. Thus, for each step, there exist a unique maximum and minimum to problems (1) and (8), respectively.

Thus, the extreme values of the lower bound functional $\rho\omega^2(\sigma)$ and its corresponding upper bound functional $\rho\bar{\omega}^2(\vec{u})$ are equal to the unique, exact solution $\rho\omega^{*2}$ for each step in a process. Namely

$$\text{maximize} \quad \rho\omega^2(\sigma) = \rho\omega^{*2} = \text{minimize} \quad \rho\bar{\omega}^2(\vec{u}). \tag{9}$$

2.3. Discretized and augmented functional

To discretize the continuous domain and surface boundary, we adopt four-node quadrilateral isoparametric elements [14]. Applying finite-element discretization, the original functional in the problem Eq. (8) is approximated by a new one in a finite-dimensional space of the vector $\{U\}$, the discrete approximation of the velocity field. We restate the problem as

$$\begin{aligned} &\text{minimize} \quad \rho\tilde{\omega}^2(\{U\}) = \sum_{e=1}^{N_e} \sigma_Y \sqrt{\{U\}^t[K_{e1}]\{U\}} \\ &\text{subject to} \quad \{U\}^t\{R\} = 1 \\ &\qquad\qquad\quad \{U\}^t\{C\} = 0 \end{aligned} \tag{10}$$

where N_e denotes the numbers of elements used to discretize the domain; the superscript t denotes transposition; $[K_{e1}]$ is the element stiffness matrix, $\{C\}$ and $\{R\}$ are vectors.

To deal with the constrained minimization problem Eq. (10), we utilize the penalty function method [40] and the Lagrangian multiplier method [40] to relax the incompressibility constraint and to impose the normalization condition. The corresponding unconstrained minimization problem is then expressed as

$$\begin{aligned} &\text{minimize} \quad \rho\tilde{\omega}^2(\{U\}) + \frac{\beta}{2}p(\{U\}) - \lambda(\{U\}^t\{R\} - 1) \\ &\text{with} \quad p(\{U\}) = \sum_{e=1}^{N_e} \{U\}^t[K_{e2}]\{U\} \end{aligned} \tag{11}$$

where the penalty parameter β is a sufficiently large positive constant, λ is the Lagrangian multiplier, and $[K_{e2}]$ is the coefficient matrix corresponding to the incompressibility constraint. It is noted that the element stiffness matrix $[K_{e1}]$ is positive semi-definite such that the objective functional is non-smooth over some rigid regions. The resulting numerical difficulty is to be overcome in the next section.

3. Computations

In the paper, the behavior of viscoplastic, nonlinear isotropic hardening is as adopted by Haghi and Anand [41]

$$\sigma_Y = [\sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon})]\left(\frac{\dot{\bar{\varepsilon}}}{\dot{\bar{\varepsilon}}_0}\right)^m, \tag{12}$$

where σ_0 is the initial yield strength, σ_∞ is the saturation value of σ_0 and h is the hardening exponent. $\bar{\varepsilon}$ is the equivalent strain and $\dot{\bar{\varepsilon}}$ the equivalent strain rate. $\dot{\bar{\varepsilon}}_0$ and m are positive valued material parameters called the reference strain rate and strain-rate sensitivity, respectively.

While conducting a sequence of limit analysis problems sequentially, we need to update the current yield criterion in addition to the configuration of the deforming structures. At the first step, we have the equivalent strain rate $\bar{\varepsilon}^1 = 0$. For the current step $n \geqslant 2$, the value of $\bar{\varepsilon}^n$ is obtained as the following expression:

$$\bar{\varepsilon}^n = \sum_{i=1}^{n-1} \dot{\bar{\varepsilon}}_i \Delta t_i, \tag{13}$$

where Δt_i is the step size.

Further, we then update the yield stress as

$$(\sigma_Y)^n_{j+1} = [\sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon}^n)]\left(\frac{\dot{\bar{\varepsilon}}^n_{j+1}}{\dot{\bar{\varepsilon}}_0}\right)^m, \tag{14}$$

where $(\sigma_Y)^n_{j+1}$ is the yield stress obtained for the current iteration, $\dot{\bar{\varepsilon}}^n_{j+1}$ is calculated with the velocity vector $\{U\}_{j+1}$

To solve the minimization problem Eq. (11), we apply the necessary condition for the minimum of $\rho\tilde{\omega}^2(\{U\}) + \frac{\beta}{2}p(\{U\}) - \lambda(\{U\}^t\{R\} - 1)$, namely taking its first derivative with respect to $\{U\}$, and the Lagrangian multiplier λ, respectively. Moreover, the objective functional is smoothed by a small real number δ to overcome the numerical difficulty resulting from non-smoothness over some rigid regions as detailed by Huh and Yang [8]. Reorganizing the nonlinear equations, linear matrix-vector equations are then produced as

$$[K]\{U\} = \lambda\{R\}, \tag{15}$$

$$\{U\}^t\{R\} - 1 = 0, \tag{16}$$

with

$$[K]\{U\} = \sum_{e=1}^{N_e} (\sigma_Y)^n_{j+1} \frac{[K_{e1}]\{U\}_{j+1}}{\sqrt{\{U^*\}^t_j[K_{e1}]\{U^*\}_j + \delta^2}} + \beta\sum_{e=1}^{N_e}[K_{e2}]\{U\}_{j+1}, \tag{17}$$

where subscriptions j, $(j+1)$ indicate quantities corresponding to any successive iterations.

Combining Eqs. (15) and (16), we express λ, $\{U\}$ in each step as follows

$$\lambda = \frac{1}{\{R\}^t[K]^{-1}\{R\}}, \tag{18}$$

$$\{U\} = \lambda[K]^{-1}\{R\}, \tag{19}$$

where $[K]^{-1}$ is the inverse of $[K]$.

As expressed in Eqs. (17) and (18), the current value of λ_{j+1} is based on the value of $\{U^*\}_j$ obtained at the preceding iteration j. With the acquired λ_{j+1}, the other unknown $\{U\}_{j+1}$ is then calculated as expressed in Eq. (19). Initially, an arbitrary $\{U\}_0$ is assumed as the first estimate. A convergent sequence of $\lambda(\{U^*\}_j)$ is then generated iteratively and converges to the plastic limit angular velocity.

Computationally, an inner and outer iterative sequence is conducted to solve the minimization problem. From one outer iteration to the next, the smoothing parameter δ used in the inner iteration is allowed to decrease and then convergence to zero finally. Stopping criterion based on the ratio of Euclidean norms $E_u = \|\{U^*\}_j - \{U^*\}_{j-1}\|_2/\|\{U^*\}_{j-1}\|_2$ is applied to check the convergence of each step. All the above-mentioned procedures have been summarized as the flowchart shown in the previous work by Leu and Chen [24].

4. Numerical examples

The paper employs sequential limit analysis to investigate the plastic limit angular velocity of hollow cylinders involving strain-hardening viscoplastic materials in plane-strain conditions. In the

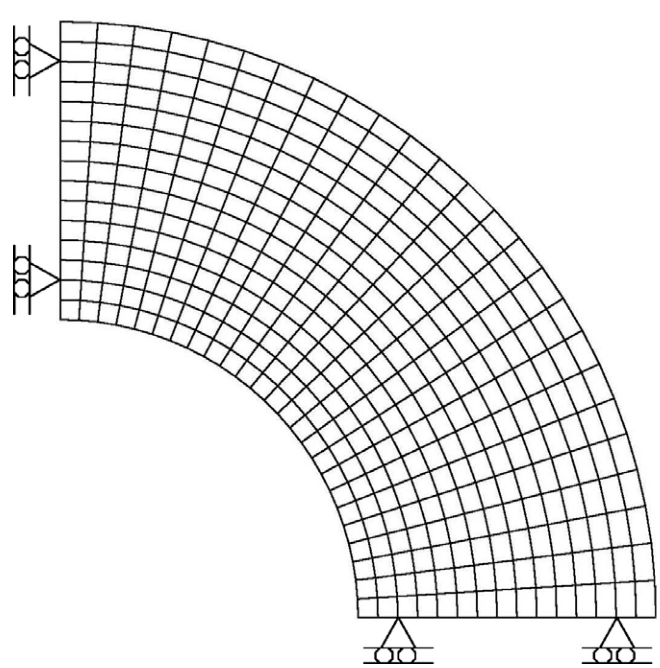

Fig. 1. Finite-element model of a rotating hollow cylinder.

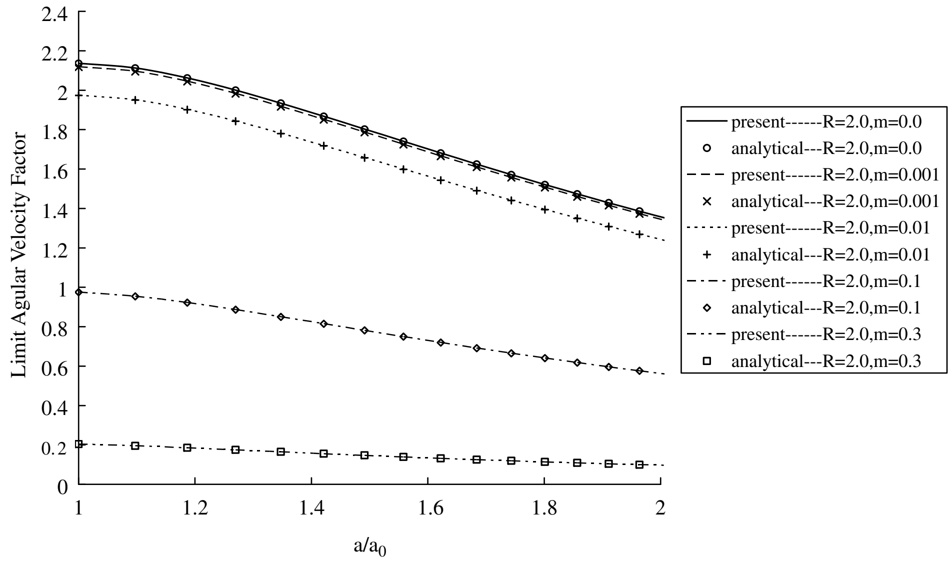

Fig. 2. Effect of strain-rate sensitivity m on the normalized plastic limit angular velocity factor $\rho\omega^2 b_0^2/\sigma_0$ with yield strength ratio $R = \sigma_\infty/\sigma_0 = 2$.

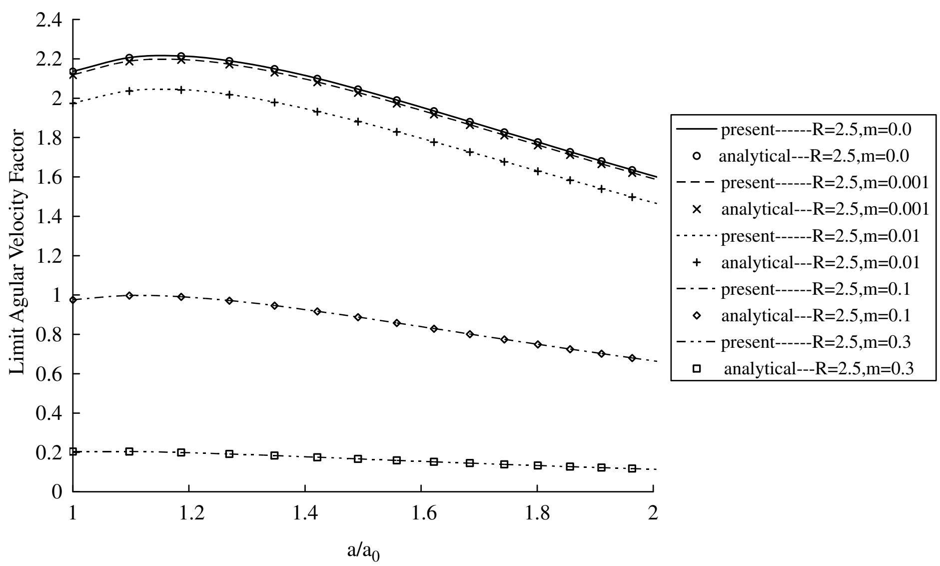

Fig. 3. Effect of strain-rate sensitivity m on the normalized plastic limit angular velocity factor $\rho\omega^2 b_0^2/\sigma_0$ with yield strength ratio $R = \sigma_\infty/\sigma_0 = 2.5$.

formulation, the centrifugal force associated with the angular velocity is the driving load to cause the rotating cylinders fully plastic. In the computations, the behavior of viscoplastic, nonlinear isotropic hardening is as adopted by Haghi and Anand [41] as shown in Eq. (12).

For the sake of rigorous validation, analytical solutions for the limit angular velocity are derived as detailed in Appendix A.1. As shown in Appendix A.1, the key point that makes analytical solutions possible is the consideration of limit analysis theorem, axisymmetric problems in plane-strain conditions and the assumption of a certain value of the hardening exponent. To be complete, the onset of the instability and the stability condition are also investigated analytically in Appendix A.2. Comparisons between numerical results and analytical solutions are then made as to assure the reliable applications.

In the numerical examples, the initial inner and outer radii are denoted as a_0 and b_0, respectively. The angular velocity required to keep the deforming cylinder fully plastic is then computed sequentially by using the CSSA algorithm. In the following case studies, we adopt the following parameters of consistent dimensions: $a_0 = 5.0$, $b_0 = 10.0$, $h = \sqrt{3}$, $\dot{\bar{\varepsilon}}_0 = 1.0$ and a constant step size $\Delta t = 1.0$. The problem concerned is axisymmetric and in plane strain conditions. Without effort made to the development of the corresponding element type in the paper, the problem is simulated in an equivalent way without loss of accuracy as in the previous work [22–25]. One quarter of the axisymmetric structure is simulated as shown in Fig. 1 due to geometric and loading symmetry. Four-node bilinear quadrilateral isoparametric elements are utilized to discretize the problem domain. The finite element mesh of 15×25 elements shown in Fig. 1 is adopted in the following computations. In the beginning, the first-step limit angular velocity is obtained. The first-step solution is the limit value of the angular velocity causing the cylinder of dimensions a_0 and b_0 fully plastic. Following the first step, each step in sequential limit analysis gets started with the result obtained in the preceding step. A sequence of limit analysis problems is then solved to obtain sequentially numerical solutions of the rotating problem.

Firstly, we consider the rotating cylinder with various values of the strain-rate sensitivity m. For the case with $m = 0$, the problem is then reduced to a rate independent plasticity problem involving strain-hardening materials as investigated in the previous work [24]. Parametric studies are performed with various values of the strain-rate sensitivity m together with the values of the yield strength ratio $R = \sigma_\infty/\sigma_0 = 2.0$ and $R = \sigma_\infty/\sigma_0 = 2.5$, respectively. The results of the normalized plastic limit angular velocity factor $\rho\omega^2 b_0^2/\sigma_0$ are summarized in Figs. 2 and 3. All the computed upper bounds agree very well with the analytical solutions.

Secondly, parametric studies are performed with various values of the yield strength ratio $R = \sigma_\infty/\sigma_0$ associated with the values of the strain-rate sensitivity $m = 0.1$, $m = 0.3$, respectively. The results of the normalized plastic limit angular velocity factor $\rho\omega^2 b_0^2/\sigma_0$ are summarized in Figs. 4 and 5. All the computed upper bounds match very well with the analytical solutions.

On the other hand, there may be a strengthening phenomenon before the weakening phenomenon as shown in Figs. 2–5 depending on the value of the yield strength ratio $R = \sigma_\infty/\sigma_0$ relative to the value of the strain-rate sensitivity m. For some values of the yield strength ratio $R = \sigma_\infty/\sigma_0$ and the strain-rate sensitivity m, rotating hollow cylinders are strengthened due to the strain hardening until the onset of instability. Following that, however, the weakening phenomenon is observed while the effect of strain-rate sensitivity and widening deformation counteracts that of the strain hardening. Note that, the simulation of the plastic deformation localization is beyond the scope of the paper. The onset of instability concerned is about the plastic instability marked by the rotating speed maximum while dealing with thick-walled cylinders, see the work by Rimrott [31], Chakrabarty [42]. Namely, the strengthening due to material hardening is exceeded by the weakening resulting from the widening deformation and the strain-rate sensitivity.

As detailed in Appendix A.2, the onset of instability can be calculated by the following mathematical condition:

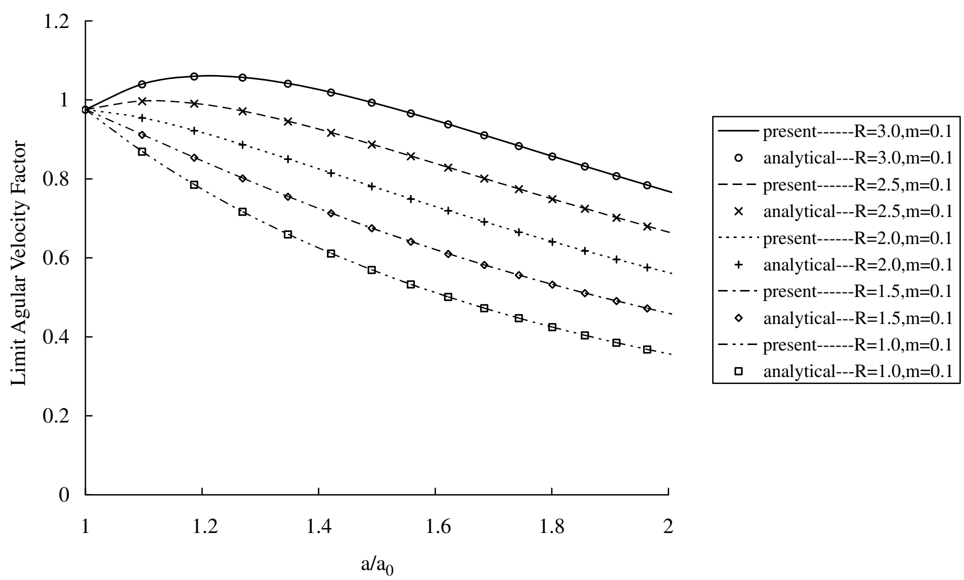

Fig. 4. Effect of yield strength ratio $R = \sigma_\infty/\sigma_0$ on the normalized plastic limit angular velocity factor $\rho\omega^2 b_0^2/\sigma_0$ with strain-rate sensitivity $m = 0.1$.

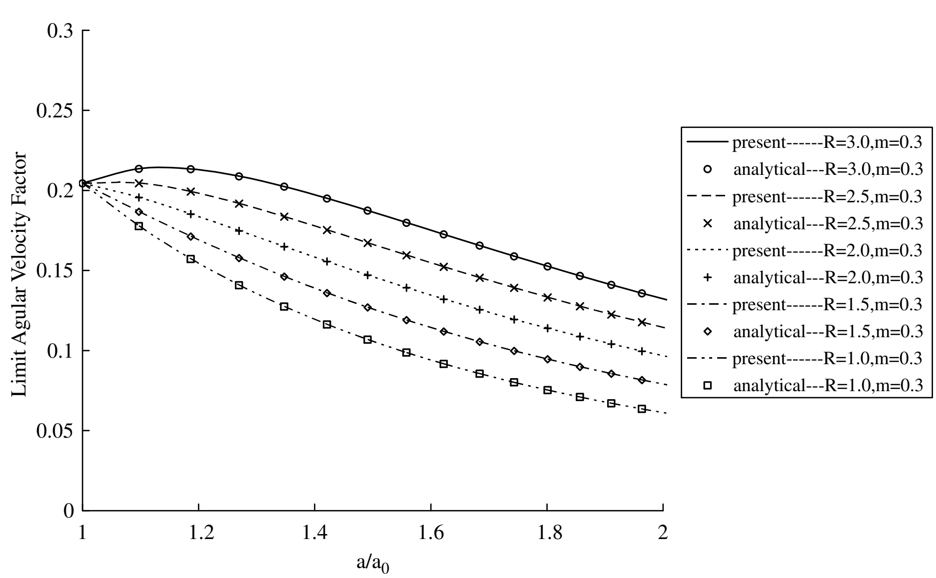

Fig. 5. Effect of yield strength ratio $R = \sigma_\infty/\sigma_0$ on the normalized plastic limit angular velocity factor $\rho\omega^2 b_0^2/\sigma_0$ with strain-rate sensitivity $m = 0.3$.

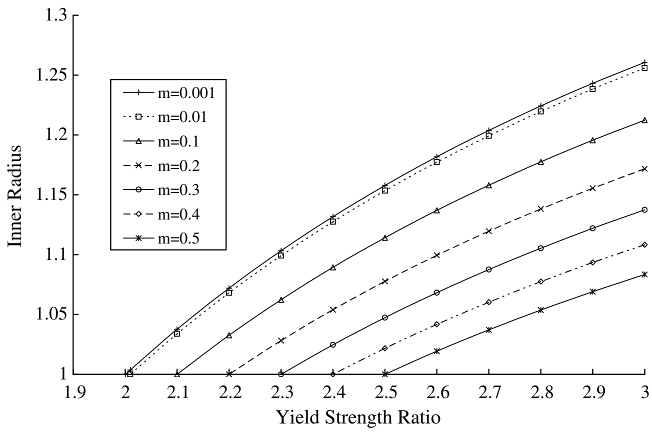

Fig. 6. Effect of strain-rate sensitivity m and yield strength ratio $R = \sigma_\infty/\sigma_0$ on the onset of instability in terms of the inner radius a/a_0.

$$\frac{\partial(\rho\omega^2 b_0^2/\sigma_0)}{\partial a} = 0. \tag{20}$$

Thus, corresponding to the initial inner and outer radii a_0, b_0 and the viscoplastic strain-hardening behavior adopted by Haghi and Anand [41] together with the hardening exponent $h = \sqrt{3}$, the onset of instability can be obtained implicitly as

$$\frac{(m\sigma_\infty/\sigma_0 + 1)}{m+1}\frac{(b^{2m+2} - a^{2m+2})}{b^{2m+2}} = (\sigma_\infty/\sigma_0 - 1)\frac{a_0^2}{a^2} - (\sigma_\infty/\sigma_0 - 1)\frac{a^{2m+2}b_0^2}{b^{2m+4}}. \tag{21}$$

For the sake of completeness, we then solve this nonlinear equation for the onset of instability by means of the fixed point iteration [43]. Namely, we recognize Eq. (21) as

$$\frac{a}{a_0} = \sqrt{\frac{(\sigma_\infty/\sigma_0 - 1)b^{2m+4}}{\frac{m\sigma_\infty/\sigma_0+1}{m+1}(b^{2m+4} - a^{2m+2}b^2) + (\sigma_\infty/\sigma_0 - 1)a^{2m+2}b_0^2}}. \tag{22}$$

The onset of instability is then acquired in terms of the inner radius a/a_0 by using fixed point iteration.

Fig. 6 shows the relationship between the onset of instability and the yield strength ratio $R = \sigma_\infty/\sigma_0$ with various values of the strain-rate sensitivity m. Again, the computed results for the onset of instability are in good agreement with the analytical solutions as shown in Figs. 2–6. On the other hand, it is found that the strengthening phenomena exist only for the cases with $\sigma_\infty/\sigma_0 > m + 2$. Note that, considering the viscoplastic strain-hardening behavior with the hardening exponent $h = \sqrt{3}$, the stability condition for the widening problem of rotating hollow cylinders is obtained analytically as $\sigma_\infty/\sigma_0 > m + 2$ as detailed in Appendix A.2.

5. Conclusions

Plastic limit angular velocities of cylinders are useful information for structure optimal design or safety evaluation. The paper employs sequential limit analysis to investigate the plastic limit angular velocity of rotating hollow cylinders made of nonlinear isotropic strain-hardening viscoplastic materials. The plasticity problem was formulated as a sequence of limit analysis problems stated in the upper bound formulation with the angular velocity factor as the objective function. Specifically, the corresponding normalization condition was imposed on the whole domain. A computational optimization procedure was developed for seeking the corresponding plastic limit angular velocities sequentially.

Rigorous upper bounds are then computed sequentially and effectively based on a combined smoothing and successive approximation (CSSA) algorithm incorporated with an inner and outer iterative sequence. The CSSA algorithm is comparable for its simple implementation and unconditional convergence. Particularly, analytical solutions of the plastic limit angular velocity as well as the onset of instability and the stability condition corresponding to the hardening exponent $h = \sqrt{3}$ were also derived in the paper for rigorous comparisons. The onset of instability was specifically acquired in terms of the inner radius a/a_0 by using fixed point iteration. Especially, it is found numerically and analytically that the strengthening phenomena exist only for the cases with $\sigma_\infty/\sigma_0 >$ $m + 2$ considering the viscoplastic strain-hardening behavior with the hardening exponent $h = \sqrt{3}$ and the strain-rate sensitivity m.

Numerical and analytically studies of rotating hollow cylinders have demonstrated the accuracy of the computational optimization procedure presented here. The computed limit angular velocities are in good agreement with analytical solutions and are rigorous upper bounds.

Appendix A.1

We consider a plane-strain problem with a rotating hollow cylinder made of strain-hardening viscoplastic materials simulated by the von Mises model. The initial interior and exterior radii of the cylinder are denoted by a_0 and b_0. Also, its current interior and exterior radii are denoted by a and b. As shown in the Eq. (1), we consider a problem of widening deformation with the centrifugal force being the driving load. The behavior of viscoplastic, nonlinear isotropic hardening is as adopted by Haghi and Anand [41]

$$\sigma_Y = [\sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon})]\left(\frac{\dot{\bar{\varepsilon}}}{\dot{\bar{\varepsilon}}_0}\right)^m, \tag{A.1}$$

where σ_0 is the initial yield strength, σ_∞ is the saturation stress and h is the hardening exponent, $\bar{\varepsilon}$ is the equivalent strain and $\dot{\bar{\varepsilon}}$ the equivalent strain rate. $\dot{\bar{\varepsilon}}_0$ and m are positive valued material parameters called the reference strain rate and strain-rate sensitivity, respectively.

Similar to the procedures adopted by the previous work of Leu [22,23,25], Leu and Chen [24], we derive the analytical solutions as follows.

In the cylindrical coordinate system, the incompressibility condition requires that

$$\frac{\partial v}{\partial r} + \frac{v}{r} = 0, \tag{A.2}$$

where v is the radial velocity at a point (r,θ). Accordingly, the radial velocity can be expressed as

$$v = \frac{a\dot{a}}{r}, \tag{A.3}$$

where a, $\dot{a}$ are the interior radius and interior velocity, respectively. Accordingly, we can express the strain rates as

$$\dot{\varepsilon}_r = \frac{\partial v}{\partial r} = -\frac{a\dot{a}}{r^2}, \tag{A.4}$$

$$\dot{\varepsilon}_\theta = \frac{v}{r} = \frac{a\dot{a}}{r^2}, \tag{A.5}$$

$$\dot{\varepsilon}_z = 0 \tag{A.6}$$

and from Eqs. (A.4)–(A.6) we obtain the equivalent strain rate

$$\dot{\bar{\varepsilon}} = \sqrt{\frac{2}{3}(\dot{\varepsilon}_r^2 + \dot{\varepsilon}_\theta^2 + \dot{\varepsilon}_z^2)} = \frac{2}{\sqrt{3}}\frac{a\dot{a}}{r^2}. \tag{A.7}$$

Accordingly, the equivalent strain is obtained as

$$\bar{\varepsilon} = \int \dot{\bar{\varepsilon}}\,\mathrm{d}t = \frac{1}{\sqrt{3}}\ln\frac{r^2}{r_0^2}, \tag{A.8}$$

where r_0 is the initial radius to the location concerned.

The components of the stress deviator, s_r, s_θ, s_z, can be obtained by considering the flow rule and satisfying the yield condition. Thus, we obtain

$$s_r = -\frac{1}{\sqrt{3}}[\sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon})]\left(\frac{\dot{\bar{\varepsilon}}}{\dot{\bar{\varepsilon}}_0}\right)^m, \tag{A.9}$$

$$s_\theta = \frac{1}{\sqrt{3}}[\sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon})]\left(\frac{\dot{\bar{\varepsilon}}}{\dot{\bar{\varepsilon}}_0}\right)^m, \tag{A.10}$$

$$s_z = 0. \tag{A.11}$$

Thus, the stresses are given as

$$\sigma_r = s + s_r, \tag{A.12}$$

$$\sigma_\theta = s + s_\theta, \tag{A.13}$$

$$\sigma_z = s + s_z, \tag{A.14}$$

where s is the mean normal stress.

Substituting Eqs. (A.12)–(A.14) into the following equilibrium equation:

$$\frac{\partial \sigma_r}{\partial r} + \frac{\sigma_r - \sigma_\theta}{r} = -\rho\omega^2 r. \tag{A.15}$$

Therefore, we obtain

$$\frac{\partial \sigma_r}{\partial r} = -\frac{\sigma_r - \sigma_\theta}{r} - \rho\omega^2 r = \frac{2}{\sqrt{3}r}[\sigma_\infty - (\sigma_\infty - \sigma_0)\exp(-h\bar{\varepsilon})]\left(\frac{\dot{\bar{\varepsilon}}}{\dot{\bar{\varepsilon}}_0}\right)^m - \rho\omega^2 r. \tag{A.16}$$

Note that $h = \sqrt{3}$ is used in the derivations. Thus, with the boundary conditions $\sigma_r(r = a) = 0$ and $\sigma_r(r = b) = 0$, the limit value of the angular velocity factor $\rho\omega^2$ at the current radii a, b is given by

$$\rho\omega^2 = \frac{2}{b^2 - a^2}\left(\frac{1}{\sqrt{3}}\right)^{m+1}\left(\frac{2a\dot{a}}{\dot{\bar{\varepsilon}}_0}\right)^m\left\{\frac{\sigma_0}{m}\left(\frac{1}{a^{2m}} - \frac{1}{b^{2m}}\right) - \frac{(\sigma_\infty - \sigma_0)}{m+1}(a_0^2 - a^2)\left(\frac{1}{a^{2m+2}} - \frac{1}{b^{2m+2}}\right)\right\}. \tag{A.17}$$

If the angular velocity factor $\rho\omega^2$ is normalized by σ_0/b_0^2, then we have the normalized angular velocity factor $\rho\omega^2 b_0^2/\sigma_0$ in the form as

$$\frac{\rho\omega^2 b_0^2}{\sigma_0} = \frac{2b_0^2}{b^2 - a^2}\left(\frac{1}{\sqrt{3}}\right)^{m+1}\left(\frac{2a\dot{a}}{\dot{\bar{\varepsilon}}_0}\right)^m\left\{\frac{1}{m}\left(\frac{1}{a^{2m}} - \frac{1}{b^{2m}}\right) - \frac{(\sigma_\infty/\sigma_0 - 1)}{m+1}(a_0^2 - a^2)\left(\frac{1}{a^{2m+2}} - \frac{1}{b^{2m+2}}\right)\right\}. \tag{A.18}$$

For the case with $m = 0$

$$\lim_{m\to 0}\frac{a^{-m} - a^m b^{-2m}}{m} = \ln\left(\frac{b^2}{a^2}\right). \tag{A.19}$$

Thus, we reduce the viscoplasticity problems to rate independent plasticity problems [24] with the strain-rate sensitivity $m = 0$, such that

$$\frac{\rho\omega^2 b_0^2}{\sigma_0} = \frac{2b_0^2}{b^2 - a^2}\left\{\frac{1}{\sqrt{3}}\ln\frac{b^2}{a^2} - \frac{(\sigma_\infty/\sigma_0 - 1)}{\sqrt{3}}\left(\frac{a_0^2}{a^2} - \frac{b_0^2}{b^2}\right)\right\}. \tag{A.20}$$

For the case with $\sigma_\infty = \sigma_0$, we reduce to non-hardening power-law viscoplasticity problems such that

$$\frac{\rho\omega^2 b_0^2}{\sigma_0} = \frac{2b_0^2}{b^2 - a^2}\left(\frac{1}{\sqrt{3}}\right)^{m+1}\left(\frac{2a\dot{a}}{\dot{\bar{\varepsilon}}_0}\right)^m\left\{\frac{1}{m}\left(\frac{1}{a^{2m}} - \frac{1}{b^{2m}}\right)\right\}. \tag{A.21}$$

Appendix A.2

To consider instability and then the existence of the maximum value of the limit angular velocity during the whole widening process, we apply the necessary condition for the maximum of $\rho\omega^2 b_0^2/\sigma_0$, namely the following mathematical expression with the current interior radius a:

$$\frac{\partial(\rho\omega^2 b_0^2/\sigma_0)}{\partial a} = 0. \tag{A.22}$$

Note that, we have $a^2 - a_0^2 = b^2 - b_0^2 = r^2 - r_0^2$, namely $a\dot{a} = b\dot{b} = r\dot{r}$, due to the incompressibility condition inherent in the von Mises model. Particularly, we recall the normalization condition $\int_D \vec{u}\cdot\vec{r}\,dA = 1$ adopted in the computational procedure as detailed in Eq. (6). Accordingly, it implies that $a\dot{a} = b\dot{b} = r\dot{r}$ is a constant in the computations in the paper. Thus, $\partial(\rho\omega^2 b_0^2/\sigma_0)/\partial a = 0$ we have

$$\frac{-1}{a^{2m+1}} + \frac{a}{b^{2m+2}} + \frac{\sigma_\infty/\sigma_0 - 1}{m+1}\left(\frac{1}{a^{2m+1}} - \frac{a}{b^{2m+2}}\right) + (\sigma_\infty/\sigma_0 - 1)\frac{a_0^2 - a^2}{a^{2m+3}} - (\sigma_\infty/\sigma_0 - 1)\frac{a_0^2 a - a^3}{b^{2m+4}} = 0. \tag{A.23}$$

We can reorganize the equation in the form as

$$\frac{(m\sigma_\infty/\sigma_0 + 1)}{m+1}\frac{(b^{2m+2} - a^{2m+2})}{b^{2m+2}} = (\sigma_\infty/\sigma_0 - 1)\frac{a_0^2}{a^2} - (\sigma_\infty/\sigma_0 - 1)\frac{a^{2m+2}b_0^2}{b^{2m+4}}. \tag{A.24}$$

To solve the nonlinear equation, we apply the method of fixed point iteration [43] to acquire the onset of instability in terms of a/a_0. Thus, the nonlinear equation is reorganized as

$$\frac{a^2}{a_0^2} = \frac{(\sigma_\infty/\sigma_0 - 1)b^{2m+4}}{\frac{m\sigma_\infty/\sigma_0 + 1}{m+1}(b^{2m+4} - a^{2m+2}b^2) + (\sigma_\infty/\sigma_0 - 1)a^{2m+2}b_0^2}. \tag{A.25}$$

And get the solution of a/a_0 in the form ready for the method of fixed point iteration [43]

$$\frac{a}{a_0} = \sqrt{\frac{(\sigma_\infty/\sigma_0 - 1)b^{2m+4}}{\frac{m\sigma_\infty/\sigma_0 + 1}{m+1}(b^{2m+4} - a^{2m+2}b^2) + (\sigma_\infty/\sigma_0 - 1)a^{2m+2}b_0^2}}. \tag{A.26}$$

Finally, we come to consider the condition of stability, namely the existence of hardening phenomena before the weakening behavior. Mathematically, it is to consider the case expressed in the form

$$\frac{\partial(\rho\omega^2 b_0^2/\sigma_0)}{\partial a} > 0. \tag{A.27}$$

Certainly, the condition expressed by Eq. (A.27) is equivalent to see if there is the solution $a/a_0 > 1$ to Eq. (A.26). Therefore, corresponding to the viscoplastic strain-hardening behavior with the hardening exponent $h = \sqrt{3}$, we can get the stability condition by Eq. (A.26) as

$$\frac{\sigma_\infty}{\sigma_0} > m + 2. \tag{A.28}$$

Therefore, if the viscoplastic strain-hardening behavior adopted by Haghi and Anand [41] with the hardening exponent $h = \sqrt{3}$, then there exists strengthening phenomenon if the rotating cylinders are made of hardening materials with $\sigma_\infty/\sigma_0 > m + 2$.

References

[1] E. Anderheggen, H. Knöpfel, Finite element limit analysis using linear programming, Int. J. Solids Struct. 8 (1972) 1413–1431.
[2] A. Bottero, R. Negre, J. Pastor, S. Turgeman, Finite element method and limit analysis theory for soil mechanics problems, Comput. Methods Appl. Mech. Engrg. 22 (1980) 131–149.

[3] A. Capsoni, L. Corradi, A finite element formulation of the rigid-plastic limit analysis problem, Int. J. Numer. Methods Engrg. 40 (1997) 2063–2086.
[4] E. Christiansen, Computation of limit loads, Int. J. Numer. Methods Engrg. 17 (1981) 1547–1570.
[5] L. Corradi, L. Luzzi, P. Vena, Finite element limit analysis of anisotropic structures, Comput. Methods Appl. Mech. Engrg. 195 (2006) 5422–5436.
[6] N. Dang Hung, Direct limit analysis via rigid-plastic finite elements, Comput. Methods Appl. Mech. Engrg. 8 (1976) 81–116.
[7] P.G. Hodge, T. Belytschko, Numerical methods for the limit analysis of plates, J. Appl. Mech. 35 (1968) 796–802.
[8] H. Huh, W.H. Yang, A general algorithm for limit solutions of plane stress problems, Int. J. Solids Struct. 28 (1991) 727–738.
[9] J. Pastor, T-H. Thai, P. Francescato, New bounds for the height limit of a vertical slope, Int. J. Numer. Anal. Methods Geomech. 24 (2000) 165–182.
[10] S.W. Sloan, Upper bound limit analysis using finite elements and linear programming, Int. J. Numer. Anal. Methods Geomech. 13 (1989) 263–282.
[11] W.H. Yang, Minimization approach to limit solutions of plates, Comput. Methods Appl. Mech. Engrg. 28 (1981) 265–274.
[12] W.H. Yang, Admissibility of discontinuous solutions in mathematical models of plasticity, in: W.H. Yang (Ed.), Topics in Plasticity, AM Press, MI, 1991.
[13] Y.G. Zhang, P. Zhang, W.M. Xue, Limit analysis considering initial constant loadings and proportional loadings, Comput. Mech. 14 (1994) 229–234.
[14] J.N. Reddy, An Introduction to the Finite Element Method, McGraw-Hill, New York, 1993.
[15] D.G. Luenberger, Linear and Nonlinear Programming, Addison-Wesley, MA, 1984.
[16] L. Corradi, N. Panzeri, C. Poggi, Post-critical behavior of moderately thick axisymmetric shells: a sequential limit analysis approach, Int. J. Struct. Stabil. Dynam. 1 (2001) 293–311.
[17] L. Corradi, N. Panzeri, A triangular finite element for sequential limit analysis of shells, Adv. Engrg. Software 35 (2004) 633–643.
[18] H. Huh, C.H. Lee, Eulerian finite-element modeling of the extrusion process for work-hardening materials with the extended concept of limit analysis, J. Mater. Process. Technol. 38 (1993) 51–61.
[19] H. Huh, C.H. Lee, W.H. Yang, A general algorithm for plastic flow simulation by finite element limit analysis, Int. J. Solids Struct. 36 (1999) 1193–1207.
[20] C.L. Hwan, An upper bound finite element procedure for solving large plane strain deformation, Int. J. Numer. Methods Engrg. 40 (1997) 1909–1922.
[21] K.P. Kim, H. Huh, Dynamic limit analysis formulation for impact simulation of structural members, Int. J. Solids Struct. 43 (2006) 6488–6501.
[22] S.-Y. Leu, Limit analysis of viscoplastic flows using an extended general algorithm sequentially: convergence analysis and validation, Comput. Mech. 30 (2003) 421–427.
[23] S.-Y. Leu, Convergence analysis and validation of sequential limit analysis of plane-strain problems of the von Mises model with nonlinear isotropic hardening, Int. J. Numer. Methods Engrg. 64 (2005) 322–334.
[24] S.-Y. Leu, J.T. Chen, Sequential limit analysis of rotating hollow cylinders of nonlinear isotropic hardening, CMES-Comput. Model. Engrg. Sci. 14 (2006) 129–140.
[25] S.-Y. Leu, Analytical and numerical investigation of strain-hardening viscoplastic thick-walled cylinders under internal pressure by using sequential limit analysis, Comput. Methods Appl. Mech. Engrg. 196 (2007) 2713–2722.
[26] W.H. Yang, Large deformation of structures by sequential limit analysis, Int. J. Solids Struct. 30 (1993) 1001–1013.
[27] W.H. Yang, A variational principle and an algorithm for limit analysis of beams and plates, Comput. Methods Appl. Mech. Engrg. 33 (1982) 575–582.
[28] E.A. Davis, F.M. Connelly, Stress distribution and plastic deformation in rotating cylinders of strain-hardening material, J. Appl. Mech. 26 (1959) 25–30.
[29] J. Lenard, J.B. Haddow, Plastic collapse speeds for rotating cylinders, Int. J. Mech. Sci. 14 (1972) 285–292.
[30] A. Nadai, Theory of Flow and Fracture of Solids, McGraw-Hill, New York, 1950.
[31] F.P.J. Rimrott, On the plastic behavior of rotating cylinders, J. Appl. Mech. 27 (1960) 309–315.
[32] N. Alexandrova, S. Alexandrov, Elastic–plastic distribution in a rotating annular disk, Mech. Based Des. Struct. Mech. 32 (2004) 1–15.
[33] A.N. Eraslan, H. Argeso, Limit angular velocities of variable thickness rotating disks, Int. J. Solids Struct. 39 (2002) 3109–3130.
[34] U. Güven, The fully plastic rotating disk with rigid inclusion, ZAMM 77 (1997) 714–716.
[35] G. Ma, H. Hao, Y. Miyamoto, Limit angular velocity of rotating disk with unified yield criterion, Int. J. Mech. Sci. 43 (2001) 1137–1153.
[36] Y. Orcan, A.N. Eraslan, Elastic–plastic stresses in linearly hardening rotating solid disks of variable thickness, Mech. Res. Commun. 29 (2002) 269–281.
[37] W.H. Yang, On generalized Hölder inequality, Nonlin. Anal. Theory Methods Appl. 16 (1991) 489–498.
[38] D.C. Drucker, A definition of stable inelastic material in the mechanics of continua, J. Appl. Mech. 81 (1959) 101–106.
[39] W.H. Yang, A duality theorem for plastic plates, Acta Mech. 69 (1987) 177–193.
[40] J.N. Reddy, Applied Functional Analysis and Variational Methods in Engineering, McGraw-Hill, New York, 1986.
[41] M. Haghi, L. Anand, Analysis of strain-hardening viscoplastic thick-walled sphere and cylinder under external pressure, Int. J. Plasticity 7 (1991) 123–140.
[42] J. Chakrabarty, Theory of Plasticity, McGraw-Hill, New York, 1987.
[43] P.G. Ciarlet, Introduction to Numerical Linear Algebra and Optimisation, Cambridge University Press, Cambridge, 1989.