 CMES, vol.1429, no.1, pp.1-43, 2009

Error Analysis of Trefftz Methods for Laplace's Equations and Its Applications[1]

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Abstract: For Laplace's equation and other homogeneous elliptic equations, when the particular and fundamental solutions can be found, we may choose their linear combination as the admissible functions, and obtain the expansion coefficients by satisfying the boundary conditions only. This is known as the Trefftz method (TM) (or boundary approximation methods). Since the TM is a meshless method, it has drawn great attention of researchers in recent years, and Inter. Workshops of TM and MFS (i.e., the method of fundamental solutions). A number of efficient algorithms, such the collocation algorithms, Lagrange multiplier methods, etc., have been developed in computation. However, there still exists a gap of convergence and errors between computation and theory. In this paper, convergence analysis and error estimates are explored for Laplace's equations with the solution $u \in H^k(k > \frac{1}{2})$, to achieve polynomial convergence rates. Such a basic theory is important for TM and MFS and their further developments. Numerical experiments are provided to support the analysis and to display the significance of its applications.

Keywords: Meshless method, collocation Trefftz method, Trefftz method, method of fundamental solutions, Lagrange multiplier, singularity problem, error analysis, Laplace's equation, Motz's problem.

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[1] Partial results were presented at the Minisymposium on Meshfree and Generalized/Extended Finite Element Methods in the 9th US National Congress on Computational Mechanics (USNCCM9), San Francisco, California, USA, July 23-26, 2007.

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1 Introduction

For solving the elliptic equations, when the particular solutions (PS) and fundamental solutions (FS) satisfying elliptic equations are found, we may choose their linear combination as the admissible functions, and their expansion coefficients can be sought by satisfying the exterior and the interior boundary conditions. This is called the Trefftz method (TM) [Trefftz (1926)] (or boundary approximation methods in [Li (1998); Li, Mathon and Sermer (1987)]). We may establish the collocation equations directly based on the boundary conditions, called the collocation Trefftz methods (CTM), or employ multipliers coupling Dirichlet and Neumann boundary conditions. When particular solutions and fundamental solutions are used in TM (or CTM), they are called the method of particular solutions (MPS) and the method of fundamental solutions (MFS). The TM, MFS and MPS are the meshless methods. In recent years, the meshless methods, in particular the Meshless Local Petrov-Galerkin (MLPG), have interested great attention in the scientific community (see [Atluri and Shen (2002); Atluri, Han, Shen (2003); Atluri, Han, Rajendran (2004); Wordelman, Aluru and Ravaioli (2002); Chen, Karageorghis and Smyrlis (2009)]). The detailed references on MLPG can be found from the monograph [Atluri (2004)].

The MFS was first used in [Kupradze (1963)] and in its modern numerical version by [Mathon and Johnston (1977)]. Because the MFS is a meshless method and has exponential convergence property for smooth solutions, it has been used in engineering computations, for examples, [Cho, Golberg, Muleshkov and Li (2004); Hon and Wei (2005); Smyrlis and Karageorghis (2003); Young and Ruan (2005); Young, Tsai, Lin and Chen (2006)]. The ill-conditioning (ill-posed) is a severe issue of the MFS. For Dirichlet problems, the exponential growth of the traditional condition number was provided in [Kitagawa (1991); Smyrilis and Karageorghis (2001); Smyrilis and Karageorghis (2004)] for the disk domains, where both source and collocation points are located uniformly on circles. In order to release of the ill-posed of the MFS, [Jin (2004)] has proposed a new numerical scheme for solving the Laplace and biharmonic equations subjected to noisy boundary data. Recently, [Young, et al. (2007); Liu (2007)] also have proposed a modified method of fundamental solutions (MMFS) for solving the Laplace problems. In [Bogomolny (1985); Li (2009)], an error analysis of the MFS is established, based on the errors of PS and the extra-errors between PS and FS. Hence error analysis for the TM using PS is essential, and this is the goal of this paper.

Take the MPS for example. The harmonic polynomials with degree N are chosen as the basis functions, and the numerical solutions u_N are obtained to satisfy the boundary conditions as best as possible, which can be realized by minimizing the boundary errors in the Sobolev norm $\|\varepsilon\|_B = \{\|\varepsilon\|^2_{0,\Gamma_D} + w^2\|\varepsilon_\nu\|^2_{0,\Gamma_N}\}^{\frac{1}{2}}$, where

$\varepsilon = u - u_N$, u is the true solution, Γ_D and Γ_N are the Dirichlet and the Neumann boundaries respectively, and ν is the outward normal to Γ_N. Let N denote the number of harmonic functions. When the weight constant $w = \frac{1}{N}$, we will prove that if $u \in H^k(S)$, the errors of the solutions by TM and CTM have the bound, $\|\varepsilon\|_B = O(\frac{1}{N^{(k-1/2)-\delta}})$ where $k(> \frac{1}{2})$ is not necessarily integer, and $0 < \delta \ll 1$. Numerical experiments are carried to support the error analysis made.

Numerical experiments are also reported for the MFS using the fundamental solutions $\ln \overline{PQ}$, where P and Q are the collocation and resource points, respectively. We choose three methods, (1) multipliers coupling Neumann conditions, also called the hybrid Trefftz method (HTM) (see [Jirousek and Wroblewski (1996); Jirousek (1978); Jirousek and Venkstesh (1992); Freitas and Wang (1998); Qin (2000)]), (2) multipliers coupling Dirichlet conditions as the traditional multiplier methods (see [Babuška (1973); Babuška, Oden and Lee (1978); Pitkäranta (1979)], and (3) collocation equations in the CTM. The numerical results display that the CTM, as multiplier-free methods, is superior due to less unknowns and the simplicity of algorithms (see [Li, Lu, Hu and Cheng (2008)]). The advantages of multiplier-free methods also coincide with the conclusions made in [Herrera and Yates (2009)] for domain decomposition methods.

Here we mention some important references related to this paper; an extensive literature of the TM and the CTM can be found in [Li, Lu, Huang and Cheng (2007); Li, Lu, Hu and Cheng (2008)]. For exponential convergence rates, the error analysis was given for the TM in [Li, Mathon and Sermer (1987)], and for the CTM in [Lu, Hu and Li (2004)]. The TM and the CTM using piecewise particular (singular or smooth) solutions are developed in [Li, Mathon and Sermer (1987); Li, Lu, Hu and Cheng (2005)]. New developments of the TM and the CTM are summarized in the recent book [Li, Lu, Hu and Cheng (2008)]. Besides, an error analysis of the TM for biharmonic equations is given in [Comodi and Mathon (1991)], but only the error bounds in L^2 norm were derived.

This paper is organized as follows. In the next section, the TM and the CTM are described. In Section 3, error bounds are derived for the solutions by the TM. In Section 4, numerical experiments by the CTM are carried out to support the analysis made. In Section 5, applications of the TM are given for the MFS by using multipliers and collocation techniques. In the last section, discussions and remarks are addressed.

2 Trefftz Methods and Collocation Trefftz Methods

Consider Laplace's equation with the mixed type of Dirichlet and Neumann conditions

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in } S, \tag{1}$$

$$u\Big|_{\Gamma_D} = f, \;\; u_\nu\Big|_{\Gamma_N} = g, \tag{2}$$

where S is a bounded and simply connected domain, $\partial S = \Gamma_D \cup \Gamma_N$, $u_\nu = \frac{\partial u}{\partial \nu}$ is the exterior normal derivative, and f and g are the functions smooth enough. For easy exposition we first begin with the simple TM and CTM, and discuss the multiplier methods in Section 5.

Suppose that the harmonic functions $\{\phi_i\}$, such as particular solutions or fundamental solutions, are explicit and known, and the solution can be expanded as

$$u = \sum_{i=1}^{\infty} \bar{c}_i \phi_i, \tag{3}$$

where $\bar{c}_i$ are the true expansion coefficients. Hence we may choose the finite term of (3) as

$$u_N = \sum_{i=1}^{N} c_i \phi_i, \tag{4}$$

as the admissible functions, where c_i are the unknown coefficients to be sought by the TM. The coefficients can be found by satisfying the boundary conditions in (2). Denote the boundary functional

$$I(v) = \int_{\Gamma_D} (v-f)^2 + w^2 \int_{\Gamma_N} (v_\nu - g)^2, \tag{5}$$

where w is a weight to balance the Dirichlet and the Neumann boundary conditions. In computation, we choose $w = \frac{1}{N}$ based on the analysis in [Li (1998); Li, Mathon and Sermer (1987)]. The Trefftz method (i.e., the boundary approximation method in [Li (1998); Li, Mathon and Sermer (1987)]) reads: to seek $u_N \in V_N$ such that

$$I(u_N) = \min_{v \in V_N} I(v), \tag{6}$$

where V_N is the set of the functions in (4).

We may compute the integrals in $I(v)$ by quadrature rule, such as the central or the Gaussian rule. Denote

$$\hat{I}(v) = \widehat{\int}_{\Gamma_D} (v-f)^2 + w^2 \widehat{\int}_{\Gamma_N} (v_\nu - g)^2, \tag{7}$$

where $\widehat{\int}_{\Gamma_D}$ and $\widehat{\int}_{\Gamma_N}$ are the integration approximations of $\int_{\Gamma_D}$ and $\int_{\Gamma_N}$, respectively. Hence, the collocation Trefftz method (CTM) is designed to seek $u_N \in V_N$ such that

$$\hat{I}(u_N) = \min_{v \in V_N} \hat{I}(v). \tag{8}$$

On the other hand, we may establish the collocation equations directly from (2). Let Γ_D and Γ_N be divided into small subsections, Q_i denote their central nodes, and Δh_i denote their meshspacings. The collocation equations on the boundary nodes Q_i are obtained straightforwardly, as

$$\sqrt{\Delta h_i} u_N(Q_i) = \sqrt{\Delta h_i} f(Q_i), \quad Q_i \in \Gamma_D, \tag{9}$$
$$w\sqrt{\Delta h_i} (u_N)_\nu(Q_i) = w\sqrt{\Delta h_i} g(Q_i), \quad Q_i \in \Gamma_N. \tag{10}$$

In computation, the total number m of the nodes Q_i is chosen always larger than N. Equations (9) and (10) form an over-determined system, which can be solved by the least squares method using the QR method. The equivalence of (9) and (10) with (8) is proven in [Lu, Hu and Li (2004)], and is called the collocation Trefftz method (CTM).

Denote $\varepsilon = u - u_N$, where u is the true solution, and u_N is the solution by the TM or the CTM. Define the boundary errors,

$$\|\varepsilon\|_B = \{\|\varepsilon\|^2_{0,\Gamma_D} + w^2 \|\varepsilon_\nu\|^2_{0,\Gamma_N}\}^{\frac{1}{2}}, \tag{11}$$
$$\overline{\|\varepsilon\|}_B = \{\overline{\|\varepsilon\|}^2_{0,\Gamma_D} + w^2 \overline{\|\varepsilon_\nu\|}^2_{0,\Gamma_N}\}^{\frac{1}{2}}, \tag{12}$$

where $\overline{\|v\|}_{0,\Gamma_D} = \{\widehat{\int}_{\Gamma_D} v^2\}^{\frac{1}{2}}$ and $\overline{\|v_\nu\|}_{0,\Gamma_N} = \{\widehat{\int}_{\Gamma_N} v_\nu^2\}^{\frac{1}{2}}$. Hence the TM in (6) and the CTM in (8) can be expressed as

$$\|u - u_N\|_B = \min_{v \in V_N} \|u - v\|_B, \tag{13}$$
$$\overline{\|u - u_N\|}_B = \min_{v \in V_N} \overline{\|u - v\|}_B. \tag{14}$$

Under a very rough approximation of quadrature rule, there exist the norm bounds

$$C_0 \|v\| \le \overline{\|v\|}_B \le C_1 \|v\|_B, \quad \forall v \in V_N, \tag{15}$$

where C_0 and C_1 are two positive constants independent of N. In [Lu, Hu and Li (2004)] we prove that the convergence rates of the solutions by the TM and the CTM are the same. Hence, below we discuss only the error analysis of $\|\varepsilon\|_B$ for the TM, since the same error bounds of $\overline{\|\varepsilon\|}_B$ for the CTM can be derived similarly, based on the arguments in [Lu, Hu and Li (2004)].

3 Error Analysis

3.1 Basic Theorems

First we cite the results of [Babuška, Szabo and Suri (1981), p. 518], as a lemma.

Lemma 3.1 *Let $u \in H^k(S)$, there exists a sequence of polynomials z_N with degree N such that*

$$\|u - z_N\|_{l,S} \le CN^{-(k-l)}\|u\|_{k,S}, \tag{16}$$

where $0 \le l \le k$, l and k are not necessarily integers, and C is a constant independent of N and u.

In the following, C is a constant independent of N and u, but its values may be different in different occurrences. We obtain the following theorem.

Theorem 3.1 *Let $u \in H^k(S)$ be harmonic, there exists a sequence of harmonic polynomials z_N with degree N such that*

$$\|u - z_N\|_{l,S} \le CN^{-(k-l)}, \tag{17}$$

where $0 \le l \le k$, l and k are not necessarily integers.

Proof : Differences of Theorem 3.1 from Lemma 3.1 are that the solution u and the polynomials are harmonic. We will follow the approaches of proof in [Eisenstat (1974), p. 672]. In fact, Lemma 3.1 holds for the analytic function ϕ and the polynomial w_N with degree N in complex,

$$\|\phi - w_N\|_{l,S} \le CN^{-(k-l)}\|\phi\|_{k,S}. \tag{18}$$

Denote

$$w_N = \sum_{j=1}^{N} \beta_j z^j, \tag{19}$$

where $z = r\exp(i\theta), \beta_j = a_j + ib_j, i = \sqrt{-1}$, and a_j and b_j are real. Let the real part be

$$u = Re(\phi), \ \ z_N = Re(w_N), \tag{20}$$

to give

$$z_N = \sum_{j=1}^{N} r^j (a_j \cos j\theta - b_j \sin j\theta) = P_N(x,y) = \sum_{j+k=0}^{N} a_{jk} x^j y^k, \tag{21}$$

where $x = r\cos\theta$ and $y = r\sin\theta$. Hence we have from (18) and (20)

$$\begin{aligned} \|z - z_N\|_{l,S} &= \|Re(\phi - w_N)\|_{l,S} \\ &\leq \|\phi - w_N\|_{l,S} \leq CN^{-(k-l)}\|\phi\|_{k,S} \leq C_1 N^{-(k-l)}, \end{aligned} \tag{22}$$

where C_1 is a constant independent of N and u. This is the desired result (17). ∎

We cite a lemma from [Babuška and Aziz (1972), p. 32].

Lemma 3.2 *For $\Delta u = 0$, we have for any $s \in R$ and every integer $j \geq 0$,*

$$\|\frac{\partial^j u}{\partial \nu^j}\|_{s,\partial S} \leq C\|u\|_{s+\frac{1}{2}+j,S}. \tag{23}$$

Now we give a main theorem.

Theorem 3.2 *Let $u \in H^k(S)(k > \frac{1}{2})$ be the solution of (1) and (2). For harmonic polynomials u_N with degree N obtained from the TM, there exists the bound,*

$$\|u - u_N\|_B \leq CN^{-(k-\frac{1}{2})}, \tag{24}$$

the boundary norm $\|\varepsilon\|_B = \|u - u_N\|_B$ is defined in (11).

Proof : Since $\varepsilon = u - u_N$ is harmonic, we have from Lemma 3.2

$$\|\varepsilon_\nu\|_{0,\Gamma_N} \leq \|\varepsilon_\nu\|_{0,\partial S} \leq C\|\varepsilon\|_{\frac{3}{2},S}, \tag{25}$$

and from [Ciarlet (1991)]

$$\|\varepsilon\|_{0,\Gamma_D} \leq \|\varepsilon\|_{0,\partial S} \leq C\|\varepsilon\|_{\frac{1}{2},S}. \tag{26}$$

Hence we have from (11), (25) and (26)

$$\|\varepsilon\|_B = \{\|\varepsilon\|^2_{0,\Gamma_D} + w^2\|\varepsilon_\nu\|^2_{0,\Gamma_N}\}^{\frac{1}{2}}$$
$$\leq \|\varepsilon\|_{0,\Gamma_D} + w\|\varepsilon\|_{0,\Gamma_N} \leq C\{\|\varepsilon\|_{\frac{1}{2},S} + w\|\varepsilon\|_{\frac{3}{2},S}\}. \quad (27)$$

Based on Theorem 3.1, there exists a sequence of harmonic polynomials z_N with degree N such that

$$\|u - z_N\|_{\frac{1}{2},S} \leq CN^{-(k-\frac{1}{2})}, \quad (28)$$

$$\|u - z_N\|_{\frac{3}{2},S} \leq CN^{-(k-\frac{3}{2})}. \quad (29)$$

By noting $w = \frac{1}{N}$, we have from (27) – (29)

$$\|u - z_N\|_B \leq C\{\|u - z_N\|_{\frac{1}{2},S} + w\|u - z_N\|_{\frac{3}{2},S}\} \leq CN^{-(k-\frac{1}{2})}. \quad (30)$$

On the other hand, we obtain from (13)

$$\|u - u_N\|_B = \min_{v \in V_N} \|u - v\|_B \leq \|u - z_N\|_B \leq CN^{-(k-\frac{1}{2})}. \quad (31)$$

This is the desired result (24), and completes the proof of Theorem 3.2. ∎

3.2 Error Bounds of TM for Solution Singularities

The Sobolev norm with fractional degree is defined by [Adams (1975), p. 214],

$$\|u\|_{p,S} = \left\{\|u\|^2_{m,S} + \sum_{|\alpha|=m} \int_S \int_S \frac{|D^\alpha u(\mathbf{x}) - D^\alpha u(\mathbf{y})|^2}{|\mathbf{x}-\mathbf{y}|^{2+2\sigma}} d\mathbf{x} d\mathbf{y}\right\}^{\frac{1}{2}}, \quad (32)$$

where $p = m + \sigma$, m is an integer and $0 < \sigma \leq 1$. In (32), $\mathbf{x}$ and $\mathbf{y}$ denote the variables in 2D. Moreover, there exists the other definition in [Adams (1975)p. 225],
,

$$\|u\|_{p,S} = \left\{\|u\|^2_{m,S} + \sum_{|\alpha|=m} \sup_{0<|\mathbf{h}|<\eta} \int_{S_h} \frac{|\Delta_h D^\alpha u(\mathbf{x})|^2}{|\mathbf{h}|^{2\sigma}}\right\}^{\frac{1}{2}}. \quad (33)$$

In (33), $\mathbf{h} \in R^2, \eta > 0$, $S_h = \{\mathbf{x} \in S, dist(\mathbf{x}, \partial S) \geq 2\eta\}$, and the notation is given by

$$\Delta_h f(\mathbf{x}) = f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}). \quad (34)$$

We may define the interpolation of derivatives by

$$D^{m+\sigma}(u) = \{D^{m+1}(u)\}^\sigma \{D^m(u)\}^{1-\sigma}. \quad (35)$$

We have the following lemma.

Lemma 3.3 *Let $p = m+\sigma$, m is an integer, and $0 < \sigma \leq 1$. When $D^{m+\sigma}(u) \in L^2(S)$, then $u \in H^{m+\sigma}(S)$.*

Proof : We may prove the conclusion of Lemma 3.3 for the case $m = 0$ without loss of generality. From (33) and (35)

$$D^{\sigma}(u) = \{D(u)\}^{\sigma}\{u\}^{1-\sigma} \in L^2(S_h), \tag{36}$$

then

$$\|u\|_{\sigma,S} = \left\{\|u\|_{0,S}^2 + \int_{S_h} \frac{|\Delta_h u(\mathbf{x})|^2}{|\mathbf{h}|^{2\sigma}}\right\}^{\frac{1}{2}} \leq C. \tag{37}$$

It suffices to show

$$\int_{S_h} \frac{|\Delta_h u(\mathbf{x})|^2}{|\mathbf{h}|^{2\sigma}} \leq C \int_{S_h} \{D(u)\}^{2\sigma}\{u\}^{2(1-\sigma)} d\mathbf{x}. \tag{38}$$

In fact, we have

$$\frac{|\Delta_h u(\mathbf{x})|^2}{|\mathbf{h}|^{2\sigma}} = \left\{\frac{|\Delta_h u(\mathbf{x})|}{|\mathbf{h}|}\right\}^{2\sigma} \times \{\Delta_h u(\mathbf{x})\}^{2-2\sigma}. \tag{39}$$

By noting that

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{|\Delta_h u(\mathbf{x})|}{|\mathbf{h}|} = Df(\mathbf{x}), \tag{40}$$

we have

$$\frac{|\Delta_h u(\mathbf{x})|}{|\mathbf{h}|} \leq C|Df(\mathbf{x})|. \tag{41}$$

Moreover, there exists bound,

$$|\Delta_h u(\mathbf{x})| = |u(\mathbf{x}+\mathbf{h}) - u(\mathbf{x})| \leq |u(\mathbf{x}+\mathbf{h})| + |u(\mathbf{x})| \leq C|u(\mathbf{x})|. \tag{42}$$

Combining (39), (41) and (42) gives

$$\frac{|\Delta_h u(\mathbf{x})|^2}{|\mathbf{h}|^{2\sigma}} \leq C|\{Du(\mathbf{x})\}^{2\sigma}\{u(\mathbf{x})\}^{2(1-\sigma)}|. \tag{43}$$

Then the desired result (38) follows, and completes the proof of Lemma 3.3. ∎

Below, we apply Theorem 3.3 for the singularities often occurring in Laplace's equation on a polygon. First, we consider the angular singularities with

$$u = O(r^{\alpha}), \tag{44}$$

where the non-integer $\alpha > 0$. For the multiple interior angles with different solutions $u = O(r_i^{\alpha})$, the strongest singularity as (44) is considered by $\alpha = \min_i \alpha_i$. We have the following lemma.

Lemma 3.4 *For the angular singularities as (44), then $u \in H^{\alpha+1-\delta}(S)$, where* $0 < \delta \ll 1$

Proof : We only prove the case of $0 < \alpha < 1$ without loss of generality. For the solution (44), we have

$$\frac{\partial u}{\partial r} = O(r^{\alpha-1}), \;\; \frac{\partial^2 u}{\partial r^2} = O(r^{\alpha-2}). \tag{45}$$

Denote the domain $S_h = \{(r,\theta)\Big|0 \le r \le R, 0 \le \theta \le \Theta\}$. Obviously we have

$$\iint_{S_h} (\frac{\partial u}{\partial r})^2 \le C\int_0^{\Theta} d\theta \int_0^R (r^{\alpha-1})^2 r dr = C_1 \int_0^R r^{2\alpha-1} dr = C_1 R^{2\alpha}, \tag{46}$$

and

$$\iint_{S_h} (\frac{\partial^2 u}{\partial r^2})^2 \le C\int_0^R (r^{\alpha-2})^2 r dr = C\int_0^R r^{2\alpha-3} dr = C_1 r^{2(\alpha-1)}\Big|_0^R = \infty. \tag{47}$$

We conclude that

$$u = O(r^{\alpha}) \in H^1(S), \;\; u \notin H^2(S). \tag{48}$$

Consider the interpolation of derivatives,

$$\begin{aligned} D^{1+\beta}(u) &= (\frac{\partial^2 u}{\partial r^2})^{\beta} (\frac{\partial u}{\partial r})^{1-\beta} \\ &= O(r^{\beta(\alpha-2)} r^{(1-\beta)(\alpha-1)}) = O(r^{(\alpha-\beta)-1}). \end{aligned} \tag{49}$$

Hence the integral has a bound,

$$\iint_{S_h} (D^{1+\beta}(u))^2 \le C\int_0^R r^{2(\alpha-\beta)-2} r dr = C\frac{R^{2(\alpha-\beta)}}{2(\alpha-\beta)}, \tag{50}$$

provided that $\beta < \alpha < 1$. This is valid if let

$$\beta = \alpha - \delta, \ 0 < \delta \ll 1. \tag{51}$$

Hence we conclude that

$$u = O(r^{\alpha}) \in H^{\alpha+1-\delta}(S). \tag{52}$$

This completes the proof of Lemma 19. ∎

Next, there exists the discontinuity of the solution with $u = O(\frac{\theta}{\Theta})$, based on Laplace's solutions on a polygon with the Dirichlet and the Neumann boundary conditions in [Li, Lu, Hu and Cheng (2005)]. We have the following lemma.

Lemma 3.5 *For the solution with* $u = O(\frac{\theta}{\Theta})$*,* $u \in H^{1-\delta}(S)$*, where* $0 < \delta \ll 1$.

Proof : We have the integrals

$$\int\int_{S_h} u^2 \le C, \tag{53}$$

$$|u|^2_{1,S_h} \le C\int_{S_h} (\frac{\partial u}{r\partial\theta})^2 \le C_1 \int_0^R (\frac{1}{r})^2 r dr = C_1 \lim_{r\to 0} \ln r = \infty. \tag{54}$$

Hence we conclude that

$$u = O(\frac{\theta}{\Theta}) \in L^2(S), \ \ u \notin H^1(S). \tag{55}$$

Denote the interpolation of derivatives

$$D^{\beta}(u) = (\frac{\partial u}{r\partial\theta})^{\beta} (\frac{\theta}{\Theta})^{1-\beta} = O((\frac{1}{r})^{\beta}\theta^{1-\beta}), \ 0 < \beta < 1. \tag{56}$$

Then we have

$$\int\int_{S_h} (D^{\beta}(u))^2 \le C\int_0^{\Theta}\int_0^R (\frac{1}{r})^{2\beta}\theta^{2(1-\beta)} r dr d\theta \tag{57}$$

$$\le C\int_0^R (\frac{1}{r})^{2\beta} r dr = C r^{2(1-\beta)}\Big|_0^R < C,$$

provided that $\beta < 1$. We may let $\beta = 1-\delta$, $0 < \delta \ll 1$. Based on Lemma 3.3, we conclude that $u = O(\frac{\theta}{\Theta}) \in H^{1-\delta}(S)$, and completes the proof of Lemma 3.5. ∎

Below, let us consider the mild singularities with $u = O(r^k \ln r), k = 1, 2...$, to have the following lemma.

Lemma 3.6 *For the solution with* $u = O(r^k \ln r), k = 1, 2, ..., u \in H^{k+1-\delta}(S)$, *where* $0 < \delta \ll 1$.

Proof : Since

$$\frac{\partial^k u}{\partial r^k} = k! \ln r + c, \quad \frac{\partial^{k+1} u}{\partial r^{k+1}} = k! \frac{1}{r}, \tag{58}$$

where c is a constant, we have

$$\iint_{S_h} (\frac{\partial^k u}{\partial r^k})^2 = O(\int_0^R (k! \ln r + c)^2 r dr) \leq C, \tag{59}$$
$$\iint_{S_h} (\frac{\partial^{k+1} u}{\partial r^{k+1}})^2 = O(\int_0^R (\frac{1}{r})^2 r dr) = \infty.$$

Hence we conclude that

$$u = O(r^k \ln r) \in H^{k+\sigma}(S), \quad 0 < \sigma < 1. \tag{60}$$

Define the interpolation of derivatives,

$$D^{k+\sigma} u = \left\{ \frac{d^{k+1} u}{dr^{k+1}} \right\}^{\sigma} \times \left\{ \frac{d^k u}{dr^k} \right\}^{1-\sigma}, \quad 0 < \sigma < 1. \tag{61}$$

We have from (58)

$$D^{k+\sigma} u = O(\frac{1}{r^{\sigma}} (\ln r)^{1-\sigma}). \tag{62}$$

The integral leads to

$$\iint_{S_h} (D^{k+\sigma} u)^2 \leq C \int_0^R \frac{1}{r^{2\sigma}} (\ln r)^{2-2\sigma} r dr. \tag{63}$$

Since when $r \to 0$,

$$|\ln r|^{2-2\sigma} \leq C \frac{1}{r^{\mu}}, \tag{64}$$

for any $\mu > 0$, we have

$$\int_0^R \frac{1}{r^{2\sigma}} (\ln r)^{2-2\sigma} r dr \leq C \int_0^R r^{1-2\sigma-\mu} dr = C \frac{R^{2-2\sigma-\mu}}{2 - 2\sigma - \mu}, \tag{65}$$

provide that $2 - 2\sigma - \mu > 0$. This gives $\sigma < 1 - \frac{\mu}{2}$. Let $\sigma = 1 - \frac{\mu}{2} - \delta_1 = 1 - \delta$, where $0 < \delta_1 \ll 1$. Hence $\delta = \frac{\mu}{2} + \delta_1$ with $0 < \delta \ll 1$. We conclude that $u \in H^{k+1-\delta}(S)$, and completes the proof of Lemma 3.6. ∎

From Lemmas 3.4 – 3.6, based on Theorem 3.2 we have the following theorem.

Theorem 3.3 *Let $u \in H^k(S)(k > \frac{1}{2})$ be the solution of (1) and (2). Then the harmonic polynomials u_N of degree N are obtained by the TM, which have the following errors.*
Case A: For angular singularities with $u = O(r^\alpha), \alpha \geq \frac{1}{4}$ [1], there exists the error bound,

$$\|\varepsilon\|_B = \|u - u_N\|_B \leq C\frac{1}{N^{\alpha+\frac{1}{2}-\delta}}, \quad 0 < \delta \ll 1. \tag{66}$$

Case B: For the discontinuity with $u = O(\frac{\theta}{\Theta})$ there exists the error bound,

$$\|\varepsilon\|_B \leq C\frac{1}{N^{\frac{1}{2}-\delta}}, \quad 0 < \delta \ll 1. \tag{67}$$

Case C: For the mild singularity with $u = O(r^k \ln r), k = 1, 2, ...,$ there exists the error bound,

$$\|\varepsilon\|_B \leq C\frac{1}{N^{k+\frac{1}{2}-\delta}}, \quad 0 < \delta \ll 1. \tag{68}$$

Proof : We only prove the conclusion for Case A, since the proof of Cases B and C is similar. From Lemma 3.4, we have $u \in H^{\alpha+1-\delta}(S)$. The desired result (66) follows from Theorem 3.2. This completes the proof of Theorem 3.3. ∎

3.3 Other Error Bounds

In this subsection, we will derive the errors in H^1 norm in the entire domain S, and the errors of the coefficients, in particular the errors of leading coefficients.
First we have a lemma.

Lemma 3.7 *Suppose that there exists a constant $\mu > 0$ independent of N such that*

$$\|\varepsilon\|_{1,\Gamma_D} \leq CN^\mu \|\varepsilon\|_{0,\Gamma_D}, \tag{69}$$

where $\varepsilon = u - u_N$, u and u_N are the true solution and the TM solution of harmonic polynomial of degree N, respectively. Then there exists the bound,

$$\|\varepsilon\|_{1,S} \leq C(N^{\frac{\mu}{2}} + \frac{1}{w}\}\|\varepsilon\|_B. \tag{70}$$

[1] For the mixed type of Dirichlet and the Neumann boundary conditions on a polygon, $\alpha \geq \frac{1}{4}$ based on the analysis in [Li, Lu, Hu and Cheng (2005); Li, Lu, Hu and Cheng (2008)].

Proof : For the harmonic function $\varepsilon = u - u_N$, there exists the bound from [Oden and Reddy (1976), p. 192]

$$\|\varepsilon\|_{1,S} \leq C\{\|\varepsilon\|_{\frac{1}{2},\Gamma_D} + \|\varepsilon_\nu\|_{-\frac{1}{2},\Gamma_N}\}. \tag{71}$$

From the interpolation of the Sobolev norm in [Babuška and Aziz (1972)] and the assumption (69) we have

$$\|\varepsilon\|_{\frac{1}{2},\Gamma_D} \leq C\{\|\varepsilon\|_{1,\Gamma_D}\|\varepsilon\|_{0,\Gamma_D}\}^{\frac{1}{2}} \leq CN^{\frac{\mu}{2}}\|\varepsilon\|_{0,\Gamma_D}. \tag{72}$$

Also there exists the bound,

$$\|\varepsilon\|_{-\frac{1}{2},\Gamma_N} \leq C\|\varepsilon\|_{0,\Gamma_N}. \tag{73}$$

Hence we have from (71) – (73)

$$\|\varepsilon\|^2_{1,S} \leq C\{N^\mu\|\varepsilon\|^2_{0,\Gamma_D} + \|\varepsilon_\nu\|^2_{0,\Gamma_N}\} \tag{74}$$
$$\leq C(N^\mu + \frac{1}{w^2})\{\|\varepsilon\|^2_{0,\Gamma_D} + w^2\|\varepsilon_\nu\|^2_{0,\Gamma_N}\} = C(N^\mu + \frac{1}{w^2})\|\varepsilon\|^2_B.$$

This gives the desired result (70), and completes the proof of Lemma 3.7. ∎

In [Li (1998)], the power $\mu = 2$ in (69) is proved for the polynomials of degree N. We may assume $\mu = 2$. Hence since $w = \frac{1}{N}$ in the algorithms, we have the following theorem.

Theorem 3.4 *Let (69) and $\mu = 2$ hold. Then when $w = \frac{1}{N}$, there exists the bound,*

$$\|\varepsilon\|_{1,S} \leq CN\|\varepsilon\|_B. \tag{75}$$

Moreover for Laplace's equation on a polygon, there exist the following error bounds:
Case A: For angular singularities with $u = O(r^\alpha), \alpha \geq \frac{1}{4}$,

$$\|\varepsilon\|_{1,S} \leq C\frac{1}{N^{\alpha-\frac{1}{2}-\delta}},\ \ 0 < \delta \ll 1. \tag{76}$$

Case B: For the discontinuity with $u = O(\frac{\theta}{\Theta})$, there exists the error bound,

$$\|\varepsilon\|_{1,S} \leq CN^{\frac{1}{2}+\delta},\ \ 0 < \delta \ll 1. \tag{77}$$

Case C: For the mild singularity with $u = O(r^k \ln r), k = 1,2,...,$ there exists the error bound,

$$\|\varepsilon\|_{1,S} \leq C\frac{1}{N^{k-\frac{1}{2}-\delta}},\ \ 0 < \delta \ll 1. \tag{78}$$

Proof : When $m = 2$ and $w = \frac{1}{M}$, the first result (75) is obtained directly from Lemma 3.7, and the other results (76) – (78) follow from Theorem 3.3. This completes the proof of Theorem 3.4. ∎

Based on Theorem 3.4, for the discontinuity with $u = O(\frac{\theta}{\Theta})$, the derivatives diverge. Hence, we must deal with it carefully by choosing the singularity solutions in TM. Moreover, for the angular singularities with $u = O(r^{\alpha})$, $\alpha \geq \frac{1}{4}$, the singularity solutions ought to also be chosen in TM.

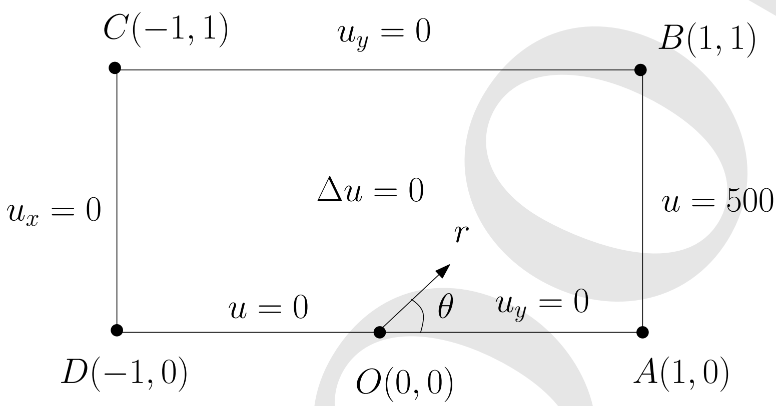

Figure 1: Motz's problem.

Let us consider Motz's problem, in which the solution satisfies Laplace's equation on the rectangle $S = \{(x,y)| -1 \leq x \leq 1, 0 \leq y \leq 1\}$ with the following boundary conditions (see Figure 1),

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad S, \tag{79}$$

$$u = 0 \quad \text{on } \overline{OD}, \;\; u_y = 0 \quad \text{on } \overline{OA}, \tag{80}$$

$$u = 500 \quad \text{on } \overline{AB}, \;\; u_y = 0 \quad \text{on } \overline{BC}, \;\; u_x = 0 \quad \text{on } \overline{CD}, \tag{81}$$

where $u_x = \frac{\partial u}{\partial x}$ and $u_y = \frac{\partial u}{\partial y}$. The singular solutions are known as

$$u = \sum_{k=0}^{\infty} d_k r^{k+\frac{1}{2}} \cos(k + \frac{1}{2})\theta, \tag{82}$$

where d_k are the true coefficients. We choose the finite term

$$u = \sum_{k=0}^{N} D_k r^{k+\frac{1}{2}} \cos(k + \frac{1}{2})\theta, \tag{83}$$

where D_k are the unknown coefficients. Based on Theorem 3.4, the singular solutions as (83) with $u = O(r^{\frac{1}{2}})$ must be chosen in TM and CTM. Note that the leading coefficient D_0 (i.e., the approximation of d_0), which represents the crack intensity factor, is important in the fracture mechanics. The numerical solutions and the leading coefficients of Motz's problem by the CTM are provided in [Lu, Hu and Li (2004)], to display the exponential convergence rates.

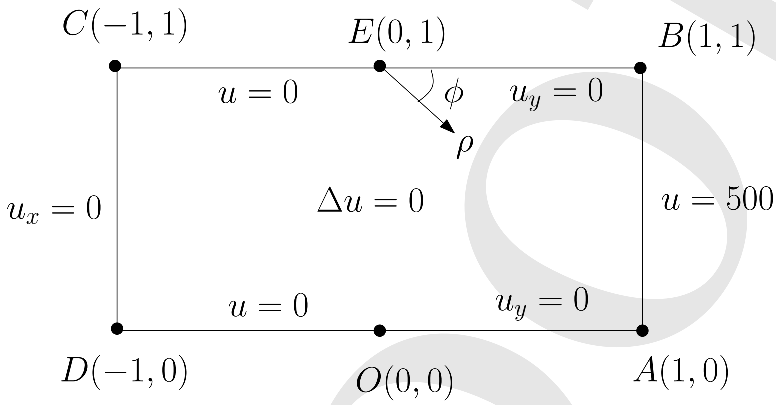

Figure 2: The angular singularity at point E in Model A.

Below, we consider variants of Motz's problems by changing the boundary conditions on $\overline{AB} \cup \overline{BC} \cup \overline{CD}$. Hence, besides the angular singularity $u = O(r^{\frac{1}{2}})$ at O, there may have other singularities. First consider Model A where only the boundary condition on $\overline{BC}$ is changed by (see Figure 2)

$$u = 0 \quad \text{on } \overline{CE}, \;\; u_y = 0 \quad \text{on } \overline{BE}, \tag{84}$$

the rest of boundary conditions retain the same as in Motz's problems. Model A has another angular singularity with $u = O(\rho^{\frac{1}{2}})$ at point E. Since the coefficients D_i in (83) are important in application, we will derive the errors of D_i, to have the following theorem.

Theorem 3.5 *For the coefficients D_k by the TM, there exists the bound,*

$$\sqrt{\sum_{k=0}^{N} (k+\frac{1}{2})(d_k - D_k)^2} \le C\|\varepsilon\|_{1,S}, \tag{85}$$

where $\varepsilon = u - u_N$, and d_i are the true coefficients in (82).

Proof : Denote $S_R = \{(r,\theta)|0 \le r \le R \le 1, 0 \le \theta \le \pi\}$ and $l_R = \{(r,\theta)|r = R < 1, 0 \le \theta \le \pi\}$. Since $\varepsilon = u - u_N$ is harmonic, we have from the Green formula

$$|\varepsilon|_{1,S}^2 = \iint_{S_R} ((\varepsilon_x)^2 + (\varepsilon_y)^2) = \int_{l_R} \frac{\partial \varepsilon}{\partial r}\varepsilon, \tag{86}$$

where

$$\varepsilon = u - u_N = \sum_{k=0}^{N} (d_i - D_i) r^{k+\frac{1}{2}} \cos(k+\frac{1}{2})\theta + \sum_{k=N+1}^{\infty} d_k r^{k+\frac{1}{2}} \cos(i+\frac{1}{2})\theta. \tag{87}$$

By using the orthogonality of $\cos(k+\frac{1}{2})\theta$, we have, after some manipulation,

$$\int_{l_R} \frac{\partial \varepsilon}{\partial r}\varepsilon = \frac{\pi}{2}\sum_{k=0}^{N} R^{2k+1}\{(k+\frac{1}{2})(d_k - D_k)^2 + \sum_{k=N+1}^{\infty}(k+\frac{1}{2})(d_k)^2\}. \tag{88}$$

By choosing $R = 1$, we have

$$\sum_{k=0}^{N}(k+\frac{1}{2})(d_k - D_k)^2 \le C\int_{l_R} \frac{\partial \varepsilon}{\partial r}\varepsilon = C|\varepsilon|_{1,S_R}^2 \le C\|\varepsilon\|_{1,S_R} \le C\|\varepsilon\|_{1,S}. \tag{89}$$

This is the desired result (85), and completes the proof of Theorem 3.5. ∎

From Theorem 3.5 we have the following corollary.

Corollary 3.1 *For the coefficients D_k by the TM, there exists the bound,*

$$|d_k - D_k| \le C\frac{1}{\sqrt{k+\frac{1}{2}}}\|\varepsilon\|_{1,S}. \tag{90}$$

In particular for the leading coefficient D_0,

$$|d_0 - D_0| \le C\|\varepsilon\|_{1,S}. \tag{91}$$

For Model A, we have the singular solutions near the point E,

$$u = \sum_{k=0}^{\infty} c_i \rho^{k+\frac{1}{2}} \cos(k+\frac{1}{2})\phi, \tag{92}$$

where (ρ,ϕ) are the polar coordinates with origin E in Figure 2, and c_i are the true coefficients. In fact, the true coefficients c_i are defined by

$$c_k = \frac{2}{\pi}\frac{1}{\rho^{k+\frac{1}{2}}}\int_0^{\pi} u(\rho,\phi)\cos(k+\frac{1}{2})\phi d\phi,\ k = 0,1,... \tag{93}$$

Once the coefficients D_i in (83) are obtained by the TM or the CTM, we may compute the approximate coefficients C_i by

$$C_k = \frac{2}{\pi}\frac{1}{\rho^{k+\frac{1}{2}}}\int_0^\pi u_N(\rho,\phi)\cos(k+\frac{1}{2})\phi d\phi,\ k=0,1,... \tag{94}$$

We are particularly interested in the leading coefficients,

$$C_0 = \frac{2}{\pi}\frac{1}{\rho^{\frac{1}{2}}}\int_0^\pi u_N(\rho,\phi)\cos(\frac{1}{2})\phi d\phi, \tag{95}$$

$$C_1 = \frac{2}{\pi}\frac{1}{\rho^{\frac{3}{2}}}\int_0^\pi u_N(\rho,\phi)\cos(\frac{3}{2})\phi d\phi. \tag{96}$$

We have the following corollary.

Corollary 3.2 *For the coefficients C_0 and C_1 from (95) and (96) respectively, there exist the bounds,*

$$|c_0 - C_0| \le C\|\varepsilon\|_{1,S}, \tag{97}$$

$$|c_1 - C_1| \le C\|\varepsilon\|_{1,S}. \tag{98}$$

Proof : We have from (93) and (94)

$$|c_0 - C_0| = \frac{2}{\pi}\Big|\frac{1}{\sqrt{\rho}}\int_0^\pi (u-u_N)\cos\frac{\phi}{2}\,d\phi\Big| = \frac{2}{\pi}\Big|\frac{1}{\rho^{\frac{3}{2}}}\int_0^\pi (u-u_N)\cos\frac{\phi}{2}\rho\,d\phi\Big| \tag{99}$$

$$\le C\Big|\int_{l_R^*}(u-u_N)\cos\frac{\phi}{2}\Big|,$$

where $l_R^* = \{(\rho,\phi)|\rho = R, 0\le\phi\le\pi\}$. Hence we obtain from the Schwarz inequality

$$|\int_{l_R^*}(u-u_N)\cos\frac{\phi}{2}| \le C\|u-u_N\|_{0,l_R^*}. \tag{100}$$

By the imbedding theorem [Ciarlet (1991)],

$$\|u-u_N\|_{0,l_R^*} \le C\|u-u_N\|_{1,S}, \tag{101}$$

the desired result (97) follows from (99) and (101). The proof for (98) is similar, and this completes the proof of Corollary 3.2. ∎

The criteria of TM and CTM are the errors of H^1 norm, because the derivative errors are also crucial in PDE solutions. For the variants of Motz's problem, if the

Dirichlet boundary conditions are all assigned on $\Gamma^* = \overline{AB} \cup \overline{BC} \cup \overline{CD}$, we may have better bounds of leading coefficients than those in Corollary 3.2. In this case, the boundary norm (11) is reduced to

$$\|\varepsilon\|_B = \|\varepsilon\|_{0,\Gamma^*}. \tag{102}$$

We have the following corollary.

Corollary 3.3 *For the Dirichlet boundary condition on Γ^*, the coefficients of variants of Motz's problem are obtained by the TM. Then there exist the bounds for the leading coefficients,*

$$|c_0 - C_0| \le C\|\varepsilon\|_B, \tag{103}$$
$$|c_1 - C_1| \le C\|\varepsilon\|_B, \tag{104}$$
$$|d_i - D_i| \le C\|\varepsilon\|_B, \;\; i = 0, 1, ... \tag{105}$$

where $\|\varepsilon\|_B$ is given in (102), and C_0 and C_1 from (95) and (96), and D_i are given in (83).

Proof : First we show (103). From [Ciarlet (1991)], we have

$$\|\varepsilon\|_{0,l_R^*} \le C\|\varepsilon\|_{\frac{1}{2},S}. \tag{106}$$

where $\varepsilon = u - u_N$ and $l_R^* = \{(\rho, \phi)|\rho = R, 0 \le \phi \le \pi\}$. Since $\Delta\varepsilon = 0$, there exists the bound from [Oden and Reddy (1976), p. 192],

$$\|\varepsilon\|_{\frac{1}{2},S} \le C\|\varepsilon\|_{0,\Gamma^*} = C\|\varepsilon\|_B. \tag{107}$$

Combining (106) and (107) gives

$$\|\varepsilon\|_{0,l_R^*} \le C\|\varepsilon\|_B. \tag{108}$$

On the other hand, we obtain from (99) and (100)

$$|c_0 - C_0| \le C\|\varepsilon\|_{0,l_R^*}. \tag{109}$$

The desired result (103) follows from (108) and (109). The proof for (104) is similar.

Next from the orthogonality, we have

$$D_k = \frac{2}{\pi} \frac{1}{r^{k+\frac{1}{2}}} \int_0^\pi u_N(r,\theta) \cos(k+\frac{1}{2})\theta \, d\theta, \tag{110}$$

where $u_N(r,\theta)$ is given in (83). Hence obtain for $r=1$

$$|d_k - D_k| = \frac{2}{\pi}\Big|\int_0^{\pi} (u - u_N(1,\theta))\cos(k+\frac{1}{2})\theta \, d\theta\Big| \tag{111}$$
$$= \frac{2}{\pi}\Big|\int_{l_R} (u - u_N)\Big| \le C\|u - u_N\|_{0,l_R},$$

where $R=1$. Eq. (108) also holds for l_R,

$$\|\varepsilon\|_{0,l_R} \le C\|\varepsilon\|_B, \tag{112}$$

to give the desired result (105). This completes the proof of Corollary 3.3. ∎

Remark 3.1. For the Dirichlet boundary condition on Γ^* (or on the entire boundary ∂S), without a need of (69), we may obtain better error estimates than those in Theorem 3.4 (also see Corollary 3.3). When $u \in H^k(S)$, we may obtain $\|\varepsilon\|_{1,S} = O(\frac{1}{N^{k-1}})$, and $\|\varepsilon\|_{0,S} = O(\frac{1}{N^k})$ by the approaches in [Comodi and Mathon (1991)].

3.4 Extensions of the Error Analysis to Elliptic Equations

To close this section, we may follow [Eisenstat (1974)], to extend the error analysis of the TM from Laplace's equation to the following elliptic equation,

$$\mathscr{L}u = -\Delta + au_x + bu_y + cu = 0 \quad \text{in } S, \tag{113}$$
$$u = f \quad \text{on } \partial S,$$

where a, b and c are smooth functions. The admissible function u_N in (4) is replaced by the integral representation

$$V_N = Re(V[P_N]), \tag{114}$$

where P_N denotes the polynomials of degree N, and V is the integral operator for the solution of $\mathscr{L}u = 0$, based on [Bergman (1961); Vekua (1967)]. When the solution $u \in H^k(S)$, the TM solutions also have $\|\varepsilon\|_B = O(\frac{1}{N^{k+\frac{1}{2}-\delta}})$, $0 < \delta \ll 1$.

4 Numerical Experiments

For Model A in Figure 2, we choose the singular solutions (83) near the point O as the admissible functions [2], but ignore the angular singularity $u = O(\rho^{\frac{1}{2}})$ at point

[2] Note that the admissible function $u_N \in V_N$ are the harmonic polynomials of degree $N+\frac{1}{2}$, not degree N in the polynomials claimed in Theorems 4.3. We may consider the polynomials $P_{N+\frac{1}{2}}(z) = \sum_{k=0}^N d_k z^{k+\frac{1}{2}}$ in complex, and $v_N = Re(P_{N+\frac{1}{2}}(z))$. Let $t = \sqrt{z}$, and denote $\bar{v}_{2N+1} = v_N = Re(\sum_{k=0}^N d_k t^{2k+1})$. Hence, v_N may be regarded the harmonic polynomials of degree $2N+1$, and Theorems 3.2 – 3.4 hold.

E. Since the singularity at the singular point O has been dealt very well by (83) already, the reduced convergence rate results only from the singular point E, as if the smooth harmonic polynomials were used for the non-singular point O. For simplicity, we use the central rule. Let M denote the number of uniform subsections along $\overline{AB}$, and the total number of collocation equations is $m = 4M$. In computation, we choose $m > N+1$. Then by the CTM we obtain an over-determined system,

$$\mathbf{F}\mathbf{x} = \mathbf{b}, \tag{115}$$

where $\mathbf{F} \in R^{m\times(N+1)}, \mathbf{x} \in R^{L+1}$ and $\mathbf{b} \in R^m$. We may use the least squares method such as the QR method to solve (115), to obtain the coefficients D_i.

Table 1: The errors, condition numbers and the leading coefficients by the CTM for Model A.

N	M	$\|\varepsilon\|_B$	Cond_eff	Cond	D_0	D_1	C_0	C_1
5	4	27.3	0.248	5.56	321.2169	170.3488	318.8424	181.3407
10	8	16.1	1.20	32.9	317.7989	172.8318	318.3530	182.9862
20	10	7.92	3.20	0.165(4)	318.8069	172.6584	318.8000	183.2311
30	15	5.27	8.62	0.495(5)	318.8240	172.6676	318.8061	183.2810
40	20	3.95	11.4	0.231(7)	318.8625	172.7155	318.7656	183.2532
60	30	2.61	14.6	0.882(9)	318.8838	172.7173	318.8341	183.3116

Table 2: The errors, condition numbers and the leading coefficients by the CTM for Model B.

N	M	$\|\varepsilon\|_B$	Cond_eff	Cond	D_0	D_1	B_1	B_2
5	4	136	1.56	3.89	434.5970	125.6139	-124.0731	103.6239
10	8	104	3.12	30.2	434.6689	130.8647	-121.3181	105.9984
20	10	72.6	5.91	0.154(4)	434.9257	130.0535	-120.8917	104.9199
30	15	59.8	8.68	0.460(5)	434.9868	130.2417	-121.0965	105.3198
40	20	52.5	11.4	0.218(7)	434.9712	130.1866	-121.1190	105.3246
60	30	43.2	16.4	0.722(9)	434.9721	130.3489	-120.9532	104.9868

The stability can be measured by the traditional condition number

$$\text{Cond} = \frac{\sigma_{\max}(\mathbf{F})}{\sigma_{\min}(\mathbf{F})}, \tag{116}$$

Table 3: The errors, condition numbers and the leading coefficients by the CTM for Model C.

N	M	$\|\|\varepsilon\|\|_B$	Cond_eff	Cond	D_0	D_1
5	4	27.6	1.06	3.36	152.2088	122.2682
10	8	11.8	1.06	14.2	153.2930	123.1254
20	10	3.94	1.07	315	153.4401	123.2767
30	15	2.24	1.07	0.826(4)	153.5305	123.3646
40	20	1.49	1.07	0.229(6)	153.5613	123.3949
60	30	0.830	1.07	0.191(9)	153.5855	123.4165
80	40	0.545	1.07	/	153.5932	123.4240
100	50	0.393	1.07	/	153.5973	123.4275

where $\sigma_{\max}$ and $\sigma_{\min}$ are the maximal and the minimal singular values of matrix $\mathbf{F}$, respectively. Based the recent study in [Li, Chien and Huang (2007)], we may use the following better estimate on stability by the effective condition number,

$$\text{Cond\_eff} = \frac{\|\mathbf{b}\|}{\sigma_{\min}(\mathbf{F})\|\mathbf{x}\|}, \tag{117}$$

where $\|\mathbf{x}\|$ is the Euclidian norm given by

$$\|\mathbf{x}\| = \sqrt{\sum_{k=0}^{N} D_i^2}. \tag{118}$$

Once the coefficients D_i have been obtained from the CTM, the norm $\|\mathbf{x}\|$ is known, and the value Cond_eff can be easily computed. Hence the Cond_eff is the a posteriori stability estimates.

Based on Theorems 3.3 and 3.4 and Corollary 3.1, the convergence rates of u_N by the CTM are given by

$$\|\varepsilon\|_B = O(\frac{1}{N^{1-\delta}}), \;\; \|\varepsilon\|_{1,S} \le CN^{\delta}, \tag{119}$$

$$|d_0 - D_0| \le CN^{\delta}, \;\; |d_1 - D_1| \le CN^{\delta}, \;\; 0 < \delta \ll 1. \tag{120}$$

The errors $\|\varepsilon\|_B$, the Cond, the Cond_eff and the leading coefficients are listed in Table 1. The numerical data in Tables 1 - 3 are computed by Fortran programs under double precision. Based on data in Table 1, the curve of $\|\varepsilon\|_B$ is drawn in

Figure 5, from which we can see that the numerical rate $\|\varepsilon\|_B = O(\frac{1}{N})$ coinciding with (119). Interestingly, the Cond_eff $= O(N)$ is small; but the Cond is huge. This clearly shows the advantage of Cond_eff over Cond.

The leading coefficients C_0 and C_1 are also computed by (95) and (96) respectively and listed in Table 1 as well. Because of symmetry, there exist the equalities,

$$C_0 = D_0, \;\; C_1 = D_1. \tag{121}$$

From Table 1, we find that for $N = 60$ the leading coefficients D_0 and C_0 have the same four decimal digits. However, there exists an obvious discrepancy between D_1 and C_1, to have the relative error

$$\frac{|D_1 - C_1|}{|D_1|} = \frac{|172.7173 - 183.3116|}{172.7173} = 6\%. \tag{122}$$

Evidently, the leading coefficient D_0 from the CTM has a better performance than the analysis in (120), which implies a divergence.

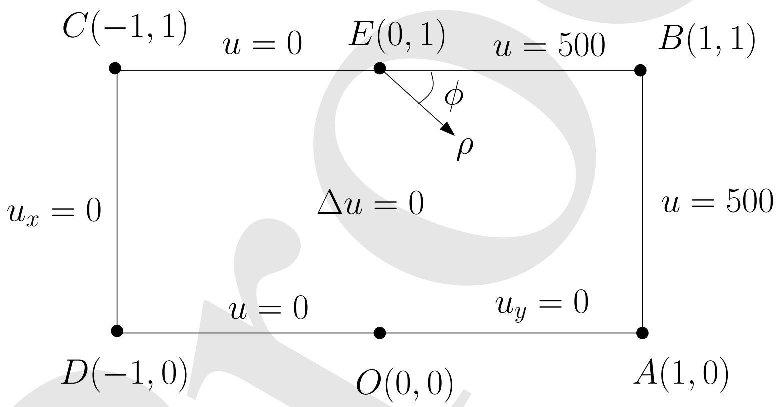

Figure 3: The discontinuity at point E in Model B.

Next, we deliberately design the other two variants of Motz's problems: Model B with the discontinuity solution at E, and Model C with $u = O(r\ln r)$ at point A, where the boundary conditions are shown in Figures 3 and 4. Only the singular solutions as in (83) at O are chosen in the CTM, to ignore the discontinuity at E and the mild singularity at A. For Models B with discontinuity at E, the convergence rates are given from Theorems 3.3 and 3.4 and Corollary 3.1,

$$\|\varepsilon\|_B = O(\frac{1}{N^{\frac{1}{2}-\delta}}), \;\; \|\varepsilon\|_{1,S} \le CN^{\frac{1}{2}+\delta}, \tag{123}$$

$$|d_0 - D_0| \le CN^{\frac{1}{2}+\delta}. \tag{124}$$

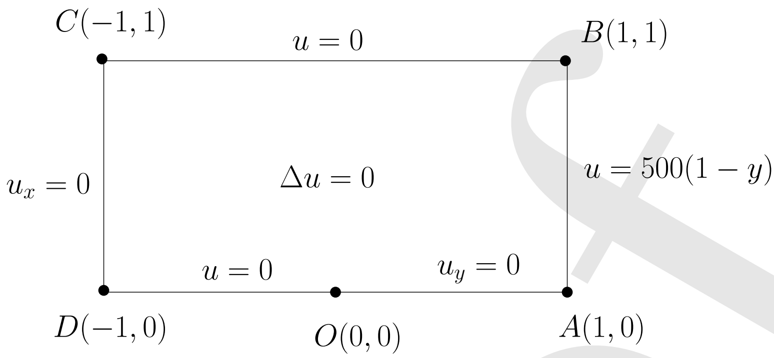

Figure 4: The mild singularity with $O(r\ln r)$ at point A in Model C.

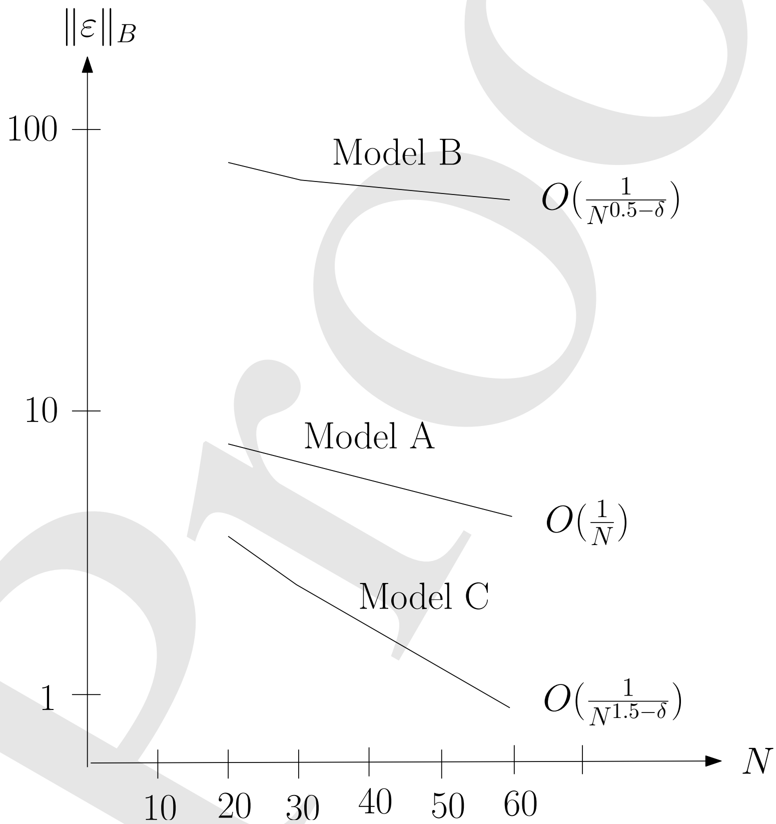

Figure 5: The curves of $\|\varepsilon\|_B$ by the CTM for Models A, B and C.

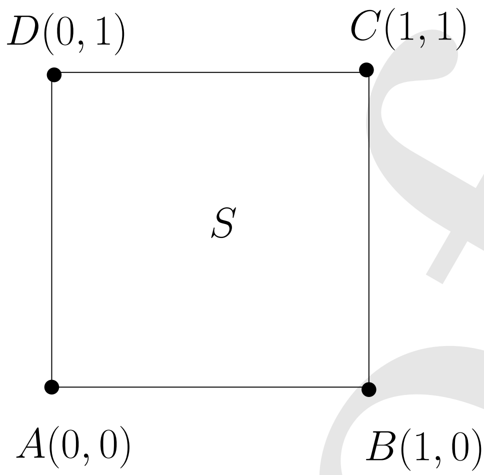

Figure 6: The solution domain of unit square.

For Model C with $O(r\ln r)$,

$$\|\varepsilon\|_B = O(\frac{1}{N^{\frac{3}{2}-\delta}}), \ \ \|\varepsilon\|_{1,S} \le O(\frac{1}{N^{\frac{1}{2}-\delta}}), \tag{125}$$

$$|d_0 - D_0| \le O(\frac{1}{N^{\frac{1}{2}-\delta}}). \tag{126}$$

The computed results for Models B and C are listed in Tables 2 and 3, respectively. From (123) and (124), the solution derivatives obtained from Model B may diverge, and the compute D_0 may be meaningless. Table 2 displays the satisfactory solutions. However, from (125) and (126), both the solution and its derivatives obtained from Model C are approximate, and the compute D_0 is trustworthy. In Table 3, when $N = 100$, the leading coefficients D_0 and D_1 may have five significant digits. On the other hand, the relative errors

$$\frac{\|\varepsilon\|_B}{500} = \frac{0.393}{500} = 7.86 \times 10^{-4}. \tag{127}$$

Hence, the errors of D_0 and D_1 have the smaller bound as $O(\|\varepsilon\|_B)$ shown in Corollary 3.3, although the boundary conditions are not purely the Dirichlet boundary conditions. All the above numerical results are consistent with the error analysis made.

To achieve high convergence rates, let $S = \cup S_i$, where S_i are disjoint subdomains, and each S_i has only one singularity. If the local singular solutions in S_i can be found, we may couple them along their common boundary by satisfying the continuity of the solution and its flux, as in [Li (1998)]. The collocation equations directly from the boundary conditions are one of coupling techniques, and the other techniques are the multiplier techniques explored in the next section.

Remark 4.1. The TM using FS leads to the method of fundamental solutions (MFS), and the TM using PS to the method of particular solutions (MPS). Since the MFS is one of TM, we may follow [Li (2009); Li, Lu, Hu and Cheng (2008)] and this paper, to also provide the algorithms and analysis of the MFS. Therefore, this paper is also important to the MFS. Since the MFS and the MPS are meshless, they have attracted a great attention of researchers. In [Li, Young, Huang, Liu and Cheng (2010)] numerical experiments of both MPS and MFS are provided for Laplace's and biharmonic equations, and comparisons are made in analysis and computation, to display that the MPS is superior to the MFS in accuracy and stability, the same conclusion as given in [Schaback (2003)]. More numerical experiments for MFS are given in the next section. In [Chen, Wu, Lee and Chen(2007); Chen, Lee, Yu and Shieh (2009)], comparisons are also made for MFS and MPS to solve Laplace's and biharmonic equations. Since their algorithms are similar, *"the equivalence"* of MFS and MPS is called. Note that their errors may be different, and the ill-conditioning of MFS is more severe. The equivalence may be interpreted as some similarity of MFS and MPS in algorithms.

5 Applications to Hybrid Trefftz Methods Using Fundamental Solutions

5.1 Hybrid Trefftz Methods Coupling Neumann Boundary Conditions

It is well known that the Lagrange multiplier is used for the Dirichlet condition for numerical partial differential equations (PDE) (see [Babuška (1973); Babuška, Oden and Lee (1978); Li (1998); Li, Lu, Huang and Cheng (2007); Li, Lu, Hu and Cheng (2008); Pitkäranta (1979)]). However, when the particular solutions satisfying PDE are chosen, the Neumann condition may be enforced a priori, and the Dirichlet condition is a consequence. This kind of multipliers was based on the minimum principle of complementary strain energy for elasticity problems, and given in [Jirousek (1978)], called the hybrid Trefftz method (HTM). Since then, the HTM becomes a very popular and competent method in engineering community, and reported in many papers. However, so far there seems to exist no analysis for such a kind of multipliers, to couple the Neumann condition, instead of the Dirichlet condition. To this end, the analysis of this paper also lays a basis of theory for HTM, and further analysis will appear elsewhere.

The Lagrange multiplier is used typically in minimization under constraints. We begin with the Dirichlet problem,

$$\begin{cases} -\Delta u + u = 0, & \text{in } S, \\ u = f, & \text{on } \partial S, \end{cases} \tag{128}$$

where S is a polygon, and Γ is its boundary $\Gamma = \partial S$. Define the energy

$$I_1(v) = \frac{1}{2}\iint_S (v_x^2 + v_y^2 + v^2) - \int_\Gamma f v_\nu, \tag{129}$$

where $u_x = \frac{\partial u}{\partial x}$, $u_y = \frac{\partial u}{\partial y}$ and $u_\nu = \frac{\partial u}{\partial \nu}$, and ν is the exterior normal of Γ. Suppose that the particular solutions are chosen in the TM, the admissible functions v in (129) also satisfy the equation in (128). The minimum of $I_1(v)$ gives

$$\iint_S (\nabla u \cdot \nabla v + uv) - \int_\Gamma f u_\nu = 0, \ \forall v \in H^1(S). \tag{130}$$

By using the Green formula and $-\triangle v + v = 0$, we have from (130)

$$\int_\Gamma u v_\nu - \int_\Gamma f v_\nu = \int_\Gamma (u - f) v_\nu = 0. \tag{131}$$

Since v and v_ν are arbitrary, the Dirichlet condition $u = f$ is obtained, as a consequence of minimization of $I_1(v)$.

Next, consider the mixed type of Dirichlet and Neumann conditions:

$$\begin{cases} -\triangle u + u = 0, & \text{in } S, \\ u = f, & \text{on } \Gamma_D, \\ u_\nu = g, & \text{on } \Gamma_N, \end{cases} \tag{132}$$

where $\Gamma = \partial S = \Gamma_D \cup \Gamma_N$. The Neumann condition on Γ_N is regarded as an a priori condition, which is dealt with by the Lagrange multiplier λ by adding $-\int_{\Gamma_N}(v_\nu - g)\lambda$. Then we define the other energy

$$I_2(v) = \frac{1}{2}\iint_S (v_x^2 + v_y^2 + v^2) - \int_{\Gamma_D} f v_\nu - \int_{\Gamma_N} (v_\nu - g)\lambda. \tag{133}$$

Similarly, the variational of $I_2(v)$ gives

$$\int_{\Gamma_N} (u_\nu - g)\mu = 0, \ \forall \mu \in H^{\frac{1}{2}}(\Gamma_N), \tag{134}$$

$$\int_\Gamma u v_\nu - \int_{\Gamma_D} f v_\nu - \int_{\Gamma_N} v_\nu \lambda = \int_{\Gamma_N} (u - \lambda) v_\nu + \int_{\Gamma_D} (u - f) v_\nu = 0, \ \forall v \in H^1(S). \tag{135}$$

Since v and v_ν are arbitrary, we have from (134) and (135)

$$u_\nu = g, \;\; \lambda = u \quad \text{on } \Gamma_N, \tag{136}$$

$$u = f \quad \text{on } \Gamma_D. \tag{137}$$

Note that based on (136), the true Lagrange multiplier λ is just the solution u on Γ_N. This multiplier is distinct from the traditional multiplier to couple the Dirichlet conditions well known in mathematics community, where $\lambda = u_\nu$. Besides, the equation $-\Delta u + u = 0$ in (128) and (132) may be replaced by Laplace's equation.

5.2 Numerical Results for Lagrange Multipliers for Neumann Condition

First, consider Laplace's equation with the Neumann condition,

$$\begin{aligned} &\Delta u = 0, \;\; (x,y) \in S, \\ &u_\nu = 0 \quad \text{on } \overline{AB} \cup \overline{BC} \cup \overline{AD} \\ &u_\nu = g = \pi \cos(\pi x)\sinh(\pi) \quad \text{on } \overline{CD}, \end{aligned} \tag{138}$$

where $S = [0,1]^2$, $\partial S = \overline{AB} \cup \overline{BC} \cup \overline{CD} \cup \overline{DA}$ (see Figure 6), and $u_\nu = \frac{\partial u}{\partial \nu}$ is the exterior normal derivatives to ∂S. To guarantee the existence of solutions, the Neumann boundary conditions must satisfy the consistent condition

$$\int_{\partial S} u_\nu = \int_{\overline{CD}} g = \pi \sinh \pi \int_0^1 \cos(\pi x) dx = 0.$$

However, the solutions of (138) are not unique, but with an arbitrary constant c. In fact, one true solution of (138) is given by

$$u(x,y) = \cos(\pi x)\cosh(\pi y) \quad \text{in } S. \tag{139}$$

In S, we choose the linear combination of fundamental solutions $\phi_i = \ln|\overline{PQ_i}|$:

$$v = v_N = \sum_{i=1}^{N} c_i \ln|\overline{PQ_i}|, \tag{140}$$

where c_i are coefficients, and $P \in S \cup \partial S$. The resource points Q_i are located outside of $S \cup \partial S$. Note that for the Neumann problems, a solution (140) plus any constant is also a solution.

Choose the $(\frac{1}{2}, \frac{1}{2})$ in Figure 6 as the origin of polar coordinates. The maximal radius of S is given by $r_{\max} = \max_S r = \frac{\sqrt{2}}{2}$. Then the resource points Q_i may be located uniformly on the circle ℓ_R with the radius $R > \frac{\sqrt{2}}{2}$ by

$$Q_i = (R\cos i\Delta\theta, R\sin i\Delta\theta), \;\; \Delta\theta = \frac{2\pi}{N}. \tag{141}$$

First, consider the HTM using multipliers to couple the Neumann conditions. and choose the following Lagrange multipliers:

$$\lambda_M = \begin{cases} \lambda_1 & = (\lambda_1)_M & = \sum_{i=0}^{M} a_i T_i(2x-1), & 0 \le x \le 1, & \text{on } \overline{AB}, \\ \lambda_2 & = (\lambda_2)_M & = \sum_{i=0}^{M} b_i T_i(2y-1), & 0 \le y \le 1, & \text{on } \overline{BC}, \\ \lambda_3 & = (\lambda_3)_M & = \sum_{i=0}^{M} d_i T_i(2x-1), & 0 \le y \le 1, & \text{on } \overline{CD}, \\ \lambda_4 & = (\lambda_4)_M & = \sum_{i=0}^{M} e_i T_i(2y-1), & 0 \le x \le 1, & \text{on } \overline{DA}, \end{cases} \tag{142}$$

where a_i, b_i, d_i and e_i are also unknown coefficients, and $T_i(x)$ are the Chebyshev polynomials of degree i, defined by

$$T_i(x) = \cos(i \cos^{-1}(x)), \quad -1 \le x \le 1. \tag{143}$$

Following Section 5.1, define the energy function

$$I(v) = \frac{1}{2} \iint_S |\nabla v|^2 - \int_{\partial S} \lambda (v_\nu - g) \tag{144}$$
$$= \frac{1}{2} \int_{\partial S} \frac{\partial v}{\partial \nu} v - \int_{\overline{AB}} \lambda_1 v_\nu - \int_{\overline{BC}} \lambda_2 v_\nu - \int_{\overline{AD}} \lambda_3 v_\nu - \int_{\overline{CD}} \lambda_4 (v_\nu - g).$$

When involving numerical integration, the integrals in (144) leads to

$$\tilde{I}(v) = \frac{1}{2} \widehat{\int}_{\partial S} \frac{\partial v}{\partial \nu} v - \widehat{\int}_{\overline{AB}} \lambda_1 v_\nu - \widehat{\int}_{\overline{BC}} \lambda_2 v_\nu - \widehat{\int}_{\overline{AD}} \lambda_3 v_\nu - \widehat{\int}_{\overline{CD}} \lambda_4 (v_\nu - g), \tag{145}$$

where $\widehat{\int}_{\partial S}$ and $\widehat{\int}_{\overline{AB}}$ are the approximations of $\int_{\partial S}$ and $\int_{\overline{AB}}$ by some quadrature, such as Gaussian rule. The variational of $\tilde{I}(v)$ yields the linear algebraic equations

$$\mathbf{A}\mathbf{x} = \mathbf{b}, \tag{146}$$

where the unknown vector $\mathbf{x}$ consists of the coefficients a_i, b_i, d_i, e_i, and the matrix $\mathbf{A}$ is nonsingular.

In fact, for the Neumann problem (138) the general solution is given by

$$\bar{v} = \bar{c} + v_N = \bar{c} + \sum_{i=1}^{N} c_i \ln |\overline{PQ_i}|, \tag{147}$$

where $\bar{c}$ is a constant, and v_N is given in (140). After the coefficients a_i, b_i, d_i, e_i have been obtained from (146), the constant $\bar{c}$ can be determined by the continuity: $\lambda = u$ on ∂S:

$$\int_{\partial S} (\bar{v} - \lambda) = 0, \tag{148}$$

to give

$$\bar{c} = -\frac{1}{|\partial S|} \int_{\partial S} (v_N - \lambda_M). \tag{149}$$

Hence the errors of solutions can be evaluated, and the condition number and the effective condition numbers are computed from (116) and (117).

The errors, condition numbers and the CPU time are listed in Tables 4 and 5. For the numerical data in Tables 4-9, 100 decimal working digits are used, and the CPU is counted in a computer with AMD Athlon 64 $\times 23600^+$. A good matching $N = 4M$ has been found by trial computation, and a good radius $R = 2.4$ can been seen from Table 5. From Table 4, there exist the numerical rates which are obtained by the least squares method,

$$\|u - v_N\|_{\infty,\Gamma} = O((0.299)^N), \;\; \|u_\nu - (v_N)_\nu\|_{\infty,\Gamma} = O((0.301)^N), \tag{150}$$

$$\|u - \lambda_M\|_{\infty,\Gamma} = O((0.515)^N), \;\; \|v_N - \lambda_M\|_{\infty,\Gamma} = O((0.515)^N),$$

$$\text{Cond} = O((3.45)^N), \;\; \text{Cond\_eff} = O((3.37)^N). \tag{151}$$

To provide a clear view of numerical rates, the curves of errors and condition numbers are drawn in Figures 7-9.

To close this subsection, let us briefly mention some theoretical work of HTM. The errors between the harmonic polynomials and fundamental solutions are derived in [Bogomolny (1985); Li (2009)]. By following the arguments for multipliers coupling the Dirichlet conditions in [Li (1998); Li, Lu, Hu and Cheng (2008)], under certain conditions, similar bounds as [Li (1998); Li, Lu, Hu and Cheng (2008)] of the errors by HTM can be obtained (see [Li, Huang, Lu and Hsu (2009)]).

5.3 Numerical Results for Lagrange Multipliers Coupling Dirichlet Condition

Next, consider Laplace's equation with the Dirichlet conditions (see Figure 6)

$$\Delta u = 0, \;\; (x,y) \in S,$$

$$u = 0 \quad \text{on } \overline{AB} \cup \overline{BC} \cup \overline{AD} \tag{152}$$

$$u = f = \sin(\pi x)\sinh(\pi y) \quad \text{on } \overline{CD}, \tag{153}$$

Table 4: The errors and condition numbers by the HTM for Neumann conditions with $R = 2.4$.

N,M	$\|u - v_N\|_{\infty,\Gamma}$	$\|u_\nu - (v_N)_\nu\|_{\infty,\Gamma}$	$\|u - \lambda_M\|_{\infty,\Gamma}$	$\|v_N - \lambda_M\|_{\infty,\Gamma}$	Cond	Cond_eff	CPU time
16,4	1.35(-2)	3.27(-1)	8.02(-2)	7.44(-2)	1.20(13)	1.10(10)	51.33
24,6	1.47(-5)	5.82(-4)	1.39(-3)	1.39(-3)	2.31(17)	1.07(14)	137.28
32,8	4.83(-9)	2.14(-7)	1.16(-5)	1.16(-5)	4.57(21)	1.79(18)	354.91
40,10	5.34(-13)	1.99(-11)	5.85(-8)	5.85(-8)	9.11(25)	3.19(22)	777.59
48,12	1.82(-17)	5.40(-16)	1.94(-10)	1.94(-10)	1.82(30)	5.83(26)	1003.74
56,14	9.22(-23)	3.87(-21)	6.38(-13)	6.38(-13)	3.64(34)	1.08(31)	1443.33
64,16	1.71(-27)	8.42(-26)	1.59(-15)	1.59(-15)	7.23(38)	2.00(35)	1904.89
ratio	0.299	0.301	0.515	0.515	3.45	3.37	1.08

Table 5: The errors and condition numbers by the HTM for Neumann conditions with $R = 2.4$ and $M = 8$.

R	$\|u - v_N\|_{\infty,\Gamma}$	$\|u_\nu - (v_N)_\nu\|_{\infty,\Gamma}$	$\|u - \lambda_M\|_{\infty,\Gamma}$	$\|v_N - \lambda_M\|_{\infty,\Gamma}$	Cond	Cond_eff	CPU time
0.8	6.83(-3)	6.12(-1)	1.02(-2)	1.13(-2)	8.86(5)	3.58(4)	423.53
1.2	4.96(-7)	3.22(-5)	1.20(-5)	1.18(-5)	1.52(11)	6.36(9)	426.28
1.6	4.52(-9)	2.26(-7)	1.19(-5)	1.19(-5)	4.82(15)	3.73(13)	378.23
2.0	5.20(-9)	1.50(-7)	1.17(-5)	1.17(-5)	1.02(19)	1.67(16)	364.20
2.4	4.83(-9)	2.14(-7)	1.16(-5)	1.16(-5)	4.57(21)	1.79(18)	355.89
2.8	4.52(-10)	2.09(-7)	1.16(-5)	1.16(-5)	7.45(23)	7.40(19)	356.44
3.2	3.86(-10)	1.96(-7)	1.16(-5)	1.16(-5)	5.93(25)	1.56(21)	352.95

with the solution $u(x,y) = \sin(\pi x)\sinh(\pi y)$ in S. The same fundamental solutions (140) in S and the multipliers in (142) are chosen. In order to couple the Dirichlet condition on ∂S, define the energy function

$$\tilde{I}_1(v) = \frac{1}{2}\widehat{\int}_{\partial S} \frac{\partial v}{\partial \nu} v - \widehat{\int}_{\partial S} \lambda(u - f) \tag{154}$$

where

$$\widehat{\int}_{\partial S} \lambda(u - f) = \widehat{\int}_{\overline{AB}} \lambda_1 v - \widehat{\int}_{\overline{BC}} \lambda_2 v - \widehat{\int}_{\overline{AD}} \lambda_3 v - \widehat{\int}_{\overline{CD}} \lambda_4 (v - f). \tag{155}$$

The variational of $\tilde{I}_1(v)$ also yields the linear algebraic equations (146). The multiplier to couple the Dirichlet condition is also simpler than that to couple the Neumann condition, because there is no arbitrary constant $\bar{c}$. The errors and condition numbers are listed in Table 6. Comparing Table 6 with Table 4, the errors are close to each other, but the condition numbers in Table 4 for the multiplier to couple the Neumann condition (i.e., the HTM) are much larger.

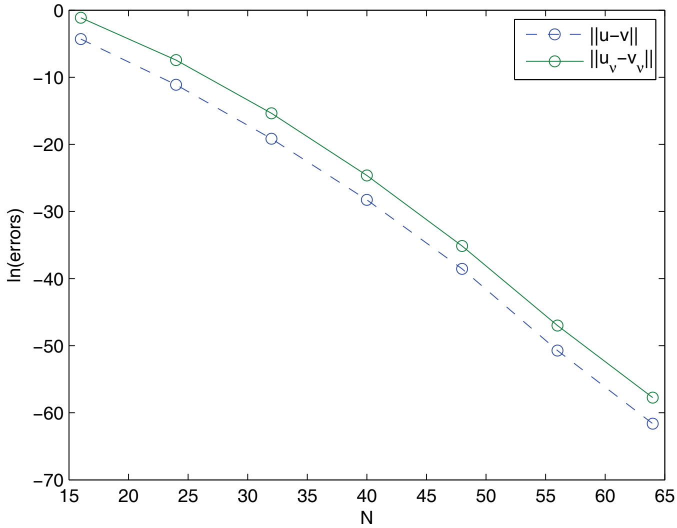

Figure 7: The curves of $\|u - v_N\|_{\infty,\Gamma}$ and $\|u_\nu - (v_N)_\nu\|_{\infty,\Gamma}$ of the HTM coupling the Neumann condition, based on Table 4.

5.4 Comparisons with the CTM

For (138) and (153), we only choose the fundamental solutions (140) without using multipliers, and use the collocation Trefftz methods (CTM). Errors and condition numbers are listed in Tables 7 and 8. For the Neumann problem (138), the constant is obtained by

$$\bar{c} = -\frac{1}{|\partial S|}\int_{\partial S}(\sin \pi x \sinh \pi y - v_N), \tag{156}$$

where v_N is given in (140).

In order to compare different methods, we collect in Table 9 the results at $N = 64$. From Table 9, we can see that the accuracy and stability of three methods are close to each other, but the algorithms of the CTM are much simpler without using the unknown multipliers. From Table 9, the ratios of CPU time are given by

$$\frac{1904.89}{9.67} = 197, \text{ for the Neumann problem}, \tag{157}$$

$$\frac{740.63}{13.75} = 53.8, \text{ for the Dirichlet conditions}. \tag{158}$$

The CTM needs much less CPU time than the multiplier methods do. Hence, the

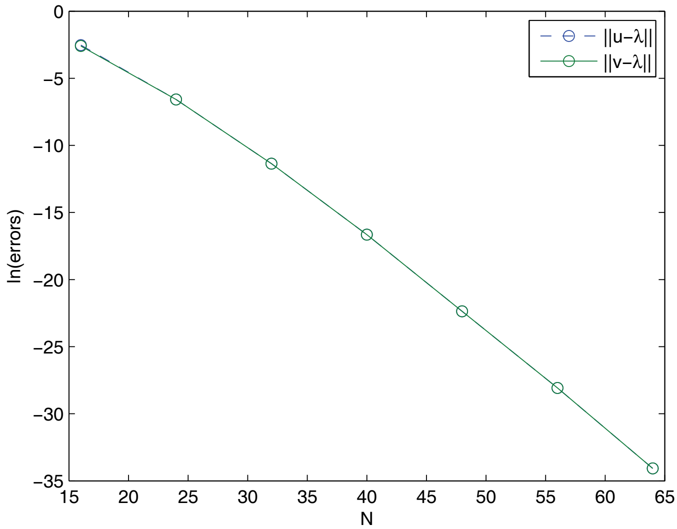

Figure 8: The curves of $\|u-\lambda_M\|_{\infty,\Gamma}$ and $\|v_N-\lambda_M\|_{\infty,\Gamma}$ the HTM coupling the Neumann condition, based on Table 4.

CTM is strongly recommended for application, since the algorithms and their programming are much simpler.

6 Concluding Remarks

To close this section, let us address the novelties in this paper.

1. In Theorems 3.2 and 3.3, when $u \in H^k(S)$, we derive the boundary errors $\|\varepsilon\|_B = O(\frac{1}{N^{k+\frac{1}{2}-\delta}})$ and the errors $\|\varepsilon\|_{1,S} = O(\frac{1}{N^{k-\frac{1}{2}-\delta}})$ in H^1 norm, where $0 < \delta \ll 1$. This analysis is new and important for the TM, compared to the existing literature (e.g., [Li, Mathon and Sermer (1987); Li, Lu, Hu and Cheng (2008); Lu, Hu and Li (2004)]).

2. In Theorem 3.3, for several important types of singularities of Laplace's solutions on polygons, the error bounds are also provided. Evidently, the error analysis in this paper provides a rather comprehensive and theoretical basis for TM, CTM and MFS. Also the numerical experiments in Section 4 have validated the error analysis in Section 3.

3. Numerical experiments for smooth solutions are also reported for the MFS using the fundamental solutions. Three methods are used: (1) multipliers cou-

Table 6: The errors and condition numbers by the HTM for Dirichlet conditions with $R = 2.4$.

N,M	$\|u-v_N\|_{\infty,\Gamma}$	$\|u_\nu-(v_N)_\nu\|_{\infty,\Gamma}$	$\|u_\nu-\lambda_M\|_{\infty,\Gamma}$	$\|(v_N)_\nu-\lambda_M\|_{\infty,\Gamma}$	Cond	Cond_eff	CPU time
16,4	5.40(-5)	1.76(-3)	8.78(-2)	8.61(-2)	1.52(6)	6.80(2)	35.88
24,6	2.61(-6)	1.66(-4)	1.39(-3)	1.28(-3)	8.32(8)	2.31(5)	91.66
32,8	1.37(-9)	1.20(-7)	9.44(-6)	9.54(-6)	4.14(11)	1.00(8)	166.97
40,10	1.96(-13)	1.81(-11)	4.65(-8)	4.65(-8)	1.93(14)	4.25(10)	375.34
48,12	1.03(-17)	6.34(-16)	1.51(-10)	1.51(-10)	8.66(16)	1.77(13)	420.81
56,14	1.10(-22)	4.42(-21)	4.91(-13)	4.91(-13)	3.84(19)	7.33(15)	562.44
64,16	7.31(-28)	1.67(-25)	3.20(-15)	3.20(-15)	1.70(22)	3.06(18)	740.63
ratio	0.325	0.331	0.518	0.518	2.16	2.13	1.06

Table 7: The errors and condition numbers by the CTM for Neumann conditions with $R = 2.4$.

N	$\|u-v_N\|_{\infty,\Gamma}$	$\|u_\nu-(v_N)_\nu\|_{\infty,\Gamma}$	Cond	Cond_eff	CPU time
16	1.70(-2)	2.55(-1)	4.63(8)	3.04(6)	1.39
24	1.66(-5)	4.26(-4)	1.55(13)	4.70(10)	1.97
32	4.74(-9)	1.60(-7)	2.69(17)	1.07(15)	3.73
40	4.56(-13)	1.64(-11)	6.08(21)	2.41(19)	6.17
48	1.43(-17)	4.42(-16)	1.35(26)	5.35(23)	6.39
56	9.95(-23)	4.08(-21)	2.98(30)	1.18(28)	8.06
64	1.42(-27)	7.65(-26)	6.52(34)	2.58(32)	9.67
ratio	0.297	0.312	3.49	3.48	1.04

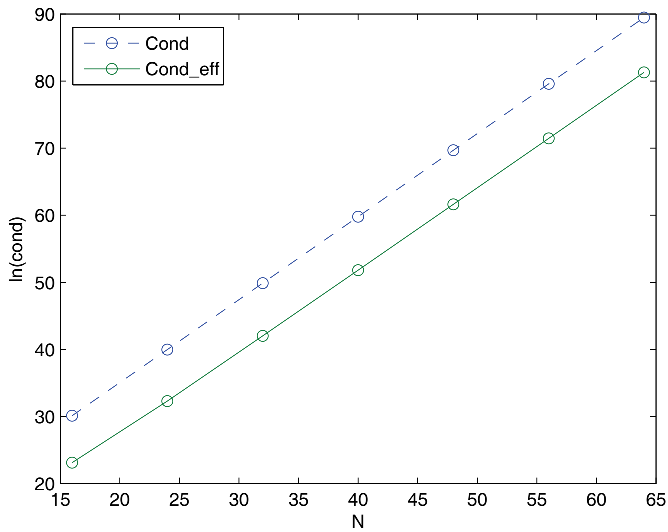

Figure 9: The curves of Cond and Cond_eff the HTM coupling the Neumann condition, based on Table 6.

pling Neumann conditions, called the hybrid Trefftz method (HTM) (see [Jirousek (1978); Jirousek and Venkstesh (1992); Freitas and Wang (1998); Qin (2000)]), (2) multipliers coupling Dirichlet conditions as the traditional multiplier methods, and (3) collocation equations as CTM. The numerical results display that the CTM, as multiplier-free methods, is best. Note that for two multiplier methods, more multiplier unknowns are needed, much more CPU time is needed (see (157) and 158)), and much more human efforts of programming must be taken [3]. The advantages of multiplier-free methods also coincide with the conclusions made in [Herrera and Yates (2009)] for domain decomposition methods.

4. In summary, the analysis in this paper is an important development of our recent book [Li, Lu, Hu and Cheng (2008)] for the solution $u \in H^k(S)(k > \frac{1}{2})$, to fill up the gap between the advanced computation and the existing theory. Moreover, the analysis in this paper is *essential* to the Trefftz method [Liu (2008b); Liu, Yeih and Atluri (2009); Pini, Mazzia and Sartoretto (2008); Rodriguez (2007); Sladek, Sladek, Tan and Atluri (2008); Song and Chen (2009)], the method of particular solutions [Tsai (2008)], the method of fundamental methods [Hu, Young and Fan

[3] The programming of the CTM is simple and easy due to its simplicity, but the programming of the multiplier methods is complicated and difficult. For the latter, W. C. Hsu spent dozen of debugging time as much as the former.

Table 8: The errors and condition numbers by the CTM for Dirichlet conditions with $R = 2.4$.

N	$\|u - v_N\|_{\infty,\Gamma}$	$\|u_\nu - (v_N)_\nu\|_{\infty,\Gamma}$	Cond	Cond_eff	CPU time
16	5.30(-5)	1.73(-3)	7.14(5)	3.16(2)	2.25
24	2.66(-6)	1.66(-4)	3.00(8)	9.16(4)	3.44
32	1.44(-9)	1.26(-7)	1.11(11)	3.33(7)	5.75
40	1.89(-13)	2.06(-11)	3.83(14)	1.15(10)	8.25
48	8.06(-18)	9.22(-16)	1.27(16)	3.80(12)	9.23
56	9.64(-23)	3.97(-21)	4.06(18)	1.22(15)	11.58
64	1.00(-27)	2.20(-25)	1.28(21)	3.83(17)	13.75
ratio	0.325	0.333	2.08	2.07	1.04

Table 9: The errors and condition numbers by different methods at $N = 64$.

N	$\|u - v_N\|_{\infty,\Gamma}$	$\|u_\nu - (v_N)_\nu\|_{\infty,\Gamma}$	Cond	Cond_eff	CPU time
Table 4	1.71(-27)	8.42(-26)	7.23(38)	2.00(35)	1904.89
Table 6	7.31(-28)	1.67(-25)	1.70(22)	3.06(18)	740.63
Table 7	1.42(-27)	7.65(-26)	6.52(34)	2.58(32)	9.67
Table 8	1.00(-27)	2.20(-25)	1.28(21)	3.83(17)	13.75

(2008); Liu (2008a)], the boundary method [He, Lim and Lim (2008);], as well as meshless methods [Haq, Siraj-Ul-Islam and Ali (2008); Haq, Siraj-Ul-Islam and Uddin (2009); Reutskiy (2008); Sageresan and Drathi (2008); Wen, Aliabadi and Liu (2008); Young et al. (2009); Zheng, et al. (2009)], because their error bounds can be derived based on the *basic* analysis in this paper.

Acknowledgements Authors are indebted to W. C. Hsu for the numerical examples in Section 5.

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