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An efficient method for solving the nonuniqueness problem in acoustic scattering

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SUMMARY

The problem of acoustic wave scattering by closed objects via second kind integral equations, is considered. Based on, combined Helmholtz integral equation formulation (CHIEF) method, an efficient method for choosing and utilizing interior field relations is suggested for solving the nonuniqueness problem at the characteristic frequencies. The implementation of the algorithm fully utilizes previous computation and thus significantly reduces the CPU time compared to the usual least-squares treatment. The method is tested for acoustic wave scattering by both acoustically hard and soft spheres. Accurate results compared to the known exact solutions are obtained. Copyright © 2006 John Wiley & Sons, Ltd.

KEY WORDS: acoustic scattering; integral equations; Helmholtz equation; nonuniqueness

1. INTRODUCTION

The solution of acoustic wave scattering problems by a closed body placed in a homogeneous medium reduces to the solution of the Helmholtz equation. The integral formulations of the problem are very attractive as they eliminate the need to consider the unbounded domain associated with scattering problems. Integral equations are also very useful in deriving approximate solutions, e.g. at low frequency [1] and at high frequency [2]. The approach is quite useful in the provision of a useful viewpoint on the scattering mechanism.

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The surface integral equation (SIE) formulation of the problem may be deduced via applying Green's theorem or alternatively, by assuming that the solution can be represented in terms of auxiliary single or double layers. Although the formulation is very simple and has been widely used, it has the defect that the computed solutions may include some interior resonant solutions which lead to the nonuniqueness of the solution. In other words, the SIE does not have a unique solution at the natural frequencies of an associated Dirichlet problem. Such complications arise from the method of solution and not from the nature of the problem itself. A recent work discussed the mechanism by which the modal participation factor dominates the numerical instability at characteristic frequencies [3].

One of the main methods used to overcome nonuniqueness is the addition of the constraints on internal fields at a finite number (M) of points ($M \ll N$ where N is the number of unknowns). This technique is known as combined Helmholtz integral equation formulation (CHIEF) method. Applying the discretization of the scattering surface to the CHIEF equations leads to an overdetermined system of equations for the surface field. A potential problem with this approach is the choice of appropriate interior points. The interior point must not be a nodal point of the corresponding interior eigenmode at the considered natural frequency.

Copley [4] developed an approach based entirely on the Helmholtz integral relation evaluated at interior points. He showed that a unique solution is obtained if the integral relation is satisfied at all interior points. This method involves the solution of an integral equation of the first kind which is less stable numerically [5].

Another approach for solving the problem was introduced by Burton and Miller [6]. They formed a linear combination of the Helmholtz integral equation used by CHIEF and its normal derivative. This formulation is valid for all wave numbers [5]. A major problem with this approach is the evaluation of the hyper-singular integrals involving a double normal derivative of the free space Green's function.

The over-determined system of CHIEF method was treated by Mohsen and Abdelmageed [7] using a simplified Lagrange multiplier approach. However, the selected interior points for CHIEF method needs to be effective (nonnodal) and the number of points should be as minimum as possible to save the computational load. For the two-dimensional case, Chen *et al.* [8], proved that no more than two points are needed if the points are properly chosen.

In the present work, we, choose only those points where the field deviates mostly from zero (within a preset accuracy limit). This approach can reduce the number of interior points greatly while their nonnodal condition is guaranteed. We first get the LU decomposition of the unmodified system and we, next, add the modified M equations at the end of the system. Consequently, only the last M rows of L and columns of U are modified. This significantly reduces both the computation time and memory storage compared to usual least-square treatment. The suggestion, for simplicity, is applied to the problem of plane wave scattered by a sphere whose exact solution is known.

The organization of the paper is as follows. In Section 2, a brief introduction of the Helmholtz integral equation is presented. The nonuniqueness problem of the Helmholtz integral equation is discussed in Section 3 and methods for solving such a problem is considered. Section 4 is concerned with the use of interior points in solving the nonuniqueness problem. The proposed method, is then, presented in Section 5. In Section 6, the obtained numerical results using the proposed method are given. A discussion of the method advantages and savings is presented in Section 7.

2. INTEGRAL REPRESENTATIONS OF THE SOLUTION

Let V_i denote a bounded domain in $\mathfrak{R}^3$ with a boundary Σ which is a closed Lyapunov surface. The usual integral equation formulation is obtained either by assuming that the solution may be represented by auxiliary single or double layer or via using Helmholtz integral representation. It is convenient to introduce the following notations:

$$S\{\phi\} \equiv \int_\Sigma \phi(q)G(p,q)\,\mathrm{d}s_q \tag{1}$$

$$D\{\phi\} \equiv \int_\Sigma \phi(q)\partial_{nq}G(p,q)\,\mathrm{d}s_q \tag{2}$$

$$K\{\phi\} \equiv \partial_{np}S\{\phi\} \tag{3}$$

$$N\{\phi\} \equiv \partial_{np}D\{\phi\} \tag{4}$$

where ∂_n denotes the derivative with respect to n, n is the outward normal to Σ and $G = \exp(\mathrm{i}kR)/(4\pi R)$ is the free space Green's function, $R = |p - q|$ and (p,q) denote a field point and an integration point, respectively. $S\{.\}$ and $D\{.\}$ are the single and double layer operators, respectively. We denote surface values and points by lower case. A field velocity, U^i is incident on the scatterer bounded by Σ. Applying Green's second identity, we obtain

$$U^i + D\{u\} - S\{v\} = \begin{cases} u(P), & P \in V_o \\ u(p)/2, & P \in \Sigma \\ 0, & P \in V_i \end{cases} \tag{5}$$

where $v = \partial_n u$. This is usually known as Helmholtz integral formula [9]. One may also derive another relation at the surface of the boundary via taking the normal derivative of Helmholtz representation and taking the proper limit as the surface is approached. The legitimacy of this process was discussed by Goodrich [10] and Lin [11].

The properties of the single and double layer operators were considered by Colton and Kress [12] and Kirsch [13]. The modifications for surface points at edges and vertices were discussed by Terai [14]. An alternative formulation may be based on the representation in terms of a single (σ) or a double (δ) layer potentials [15, 16], and accordingly taking the limits as the surface is approached for both U and its normal derivative, we obtain

$$[1/2\mathbf{I} + D]\{\sigma\} - S\{\delta\} = u - u^i \tag{6a}$$

$$[1/2\mathbf{I} + K]\{\sigma\} + N\{\sigma\} = v - v^i \tag{6b}$$

where $\mathbf{I}$ is the identity operator. The representation derived from Green's formula may be obtained upon putting $\sigma \to u$ and $\delta \to v$.

The representation in terms of single and double layers can provide more regular integral equations [16]. On the other hand, in the Helmholtz representation one does not need intermediate variables and it is useful for the direct prediction of the field when the surface values are measured. The relations between the different representations were discussed in Reference [17].

Upon invoking the appropriate boundary conditions, we obtain the corresponding integral equations for the surface field, its derivative or the layer potentials. Thus for the Dirichlet boundary condition (soft scatterer), we may deduce the integral equations:

$$S\{v\} = u^i \tag{7a}$$

$$[1/2\mathbf{I} + K].\{v\} = v^i \tag{7b}$$

$$[1/2\mathbf{I} + D]\{\delta\} = -u^i \tag{7c}$$

For the Neumann boundary condition (hard scatterer) we obtain

$$[1/2\mathbf{I} - D]\{u\} = u^i \tag{8a}$$

$$N\{u\} = -v^i \tag{8b}$$

$$[1/2\mathbf{I} - K].\{\sigma\} = -v^i \tag{8c}$$

The study of the above equations reveals that one can deal simultaneously with both Dirichlet and Nuemann problems via Fredholm integral equations of the second kind using one computer code. In particular, the solution of both problems can be written in a standard program in the form

$$[1/2\mathbf{I} \pm K]\{w\} = v^i \tag{9a}$$

$$U = U^i \mp S\{w\} \tag{9b}$$

where the upper (lower) signs corresponds to Dirichlet (Neumann) boundary condition. An alternative formulation is

$$[1/2\mathbf{I} \pm D]\{w\} = u^i \tag{10a}$$

$$U = U^i \mp D(w) \tag{10b}$$

The existence and uniqueness of the solution to the resulting equations were discussed by Burton [18] and more recently by Benthien and Schenck [5].

3. THE NONUNIQUENESS PROBLEM

The general numerical treatment of the integral equations is usually based on the moment method [19, 20]. The method expands the unknown surface field in terms of a set of basis (trial) functions and requires the resulting equations to be satisfied in a weighted sense using a set of suitable weighting (test) functions. The conditions on the choice of these auxiliary functions are discussed in References [21, 22]. The proper choice enhances the stability and the convergence rate of the solution. Recent years witnessed phenomenal growth in computer technology coupled with the development of fast solvers with reduced computational complexity and memory requirements [23, 24]. This made a rigorous numerical solution of the problem of scattering by large objects feasible.

The numerical treatment may be tested via checking the validity of other SIE on the surface or the interior integral relations at interior points. The results may also be compared with known solutions using other accurate analytical, numerical or experimental results. One may also check whether the symmetry properties, if any, the reciprocity relations and the energy conservation requirements are satisfied.

While the original boundary value problem has a unique solution, the corresponding boundary integral equations may not be uniquely solvable at certain wavenumbers (characteristic frequencies) corresponding to the adjoint interior problem. This gives rise to analytical complications and considerable difficulty in the numerical treatment of the problem. The frequency range around resonance over which the numerical solution is incorrect depends greatly on the computational error. It is generally recommended to use accurate quadrature schemes to reduce this range [25]. Besides, the matrix solution algorithm should be properly chosen.

Several methods have been devised for surmounting the nonuniqueness problem. A survey of the literature on the subject up to 1976 has been undertaken by Burton [18, 26] and later surveys include the papers by Kleinman and Roach [27], and the book by Colton and Kress [12] and more recently considered by Benthien and Schenck [5]. Previous methods proposed to overcome this difficulty include the combined source (mixed potential) method, the composite (combined) field formulation, the use of interior Helmholtz integral relation, the modified Green's function and the source simulation technique.

This work is mainly concerned with the efficient implementation of the method which augments the SIE with additional interior integral relations.

4. THE USE OF INTERIOR HELMHOLTZ INTEGRAL RELATIONS

While the SIEs derived directly from Helmholtz formula suffer from nonuniqueness, the interior integral relation:

$$D\{u\} - S\{v\} + U^i = 0 \quad \forall P \in V_i \tag{11}$$

has a unique solution. This is called the extended integral equation. Copley [4, 28] reported that for axisymmetric bodies, it is sufficient to apply the above relation at some points along the axis of symmetry in V_i to obtain a unique solution.

A primary difficulty is the lack of a formalized method for the selection of the interior points to guarantee uniqueness. For an arbitrary geometry, the nodal surfaces are not known *a priori*, so selection of points becomes critical. More than one interior point would probably be needed to assure that at least one point does not lie on a nodal surface. Seybert and Rengarajan [29] demonstrated that it only takes one 'good' point to establish a unique solution. Since the density of nodal surfaces increases with increasing frequencies, more interior points may have to be used. However, Chen *et al.* [18] proved that no more than two points are needed if the points are properly chosen. It is generally recommended to choose these points close to the original surface to avoid nodal surfaces [30]. In the dual surface formulation [30], the interior points lie on a second interior surface parallel to the object's surface and close to it. In this work, the selection of the interior points is based on the large departure of their field values from zero.

5. THE PROPOSED METHOD

The main steps in the proposed method can be summarized in the following:

1. Solve the SIE using LU decomposition, if resonance is detected.
2. Calculate the field at some regularly spaced interior points.
3. Choose only those points where the field deviates mostly from zero (within a preset accuracy limit).
4. The interior field relations at these points are taken as constraints in the numerical solution of moment method.

Augmenting the N moment equations by M interior equations results in an overdetermined system of equations which is usually solved by least squares [31], a Lagrange multiplier's approach [29] or using optimization techniques [32]. A simplified Lagrange multiplier approach was implemented in Reference [7]. A new improved procedure using the predetermined LU decomposition is adopted next.

In order to demonstrate the application of the method, we consider the scattering of a plane wave $\exp(ikz)$ incident from $+\infty$ on a sphere of radius 'a' centred at the origin. We consider the sphere problem for simplicity and because it has an exact solution. We discuss, first, the scattering on an acoustically hard sphere. Due to symmetry, only the upper half of the sphere is considered. For simplicity, we divide the contour into a number of equal subintervals and the unknown field is assumed to be constant over each subinterval. Then the integral equation is enforced at the mid-points of the subintervals to yield a square system of equations as in the following equation:

$$\mathbf{A}\{u\} = [1/2\mathbf{I} - D]\{u\} = U^i \tag{12}$$

Equations are arranged to start from the illuminated part and ends at the shadow part of the object. We perform LU decomposition for the solution matrix. Taking the diagonal elements of $\mathbf{L}$ as unity, resonance is then detected by a large increase in the diagonal of $\mathbf{U}$ matrix. Then we test the interior field at some interior points along the axis of symmetry.

Resonance is characterized by a relatively sharp increase in internal field. If the field value is sufficiently far from zero, we then identify a nonuniqueness problem and the interior field equations at points of maximum increase are taken as our constraint. If one point is chosen, a factor (0.1–0.9) of the equation is added to the last equation in $\mathbf{A}$ to yield the matrix $\mathbf{A}^{\sim}$. $\mathbf{A}^{\sim'}$ then, has the same LU decomposition as $\mathbf{A}$ except the last row of $\mathbf{L}$ and the last column of $\mathbf{U}$ which can be easily computed. The procedure can be extended in a similar fashion when more than one interior constraint is used.

6. RESULTS

The test case we examine is the scattering of a plane wave from both hard and soft spheres. The incoming plane wave is in the negative z direction. Due to the axissymmetric nature of the sphere, the one half circle is discretized into 49 points, expressed in the integral equation by cylindrical coordinates, and the integrals singularity are solved elliptically as described in Reference [33].

The results are computed for $ka = 4.4934$ which is a resonance frequency for Helmholtz integral equations for both hard and soft acoustic scattering boundary conditions [34]. The nonuniqueness problem for acoustically hard sphere scattering is numerically treated by the proposed correcting method taking an addition factor of (0.9). Figure 1 shows a comparison between; the exact solution of the problem and the numerical solution of the problem with and without correction by the proposed method. Here, two selected interior points according to the proposed criterion at $z = -0.43a$ and $-0.62a$ located on the axis of symmetry (shadow side) is used.

The nonuniqueness problem of acoustically soft sphere scattering was also treated using the proposed method. Comparison of the results is shown in Figure 2 where the corrected results taking an addition factor of (0.5) are compared with both the exact and direct solutions of the scattering problem on an acoustically soft sphere. Two selected interior points according to the proposed criterion at $z = -0.3a$ and $-0.2a$ located on the axis of symmetry (shadow range) is used. The surface field is obtained and drawn in Figures 1 and 2 in polar form.

The exact solution is computed, in each case, using the well known series expansion with sufficient number of terms to ensure the accuracy of the results.

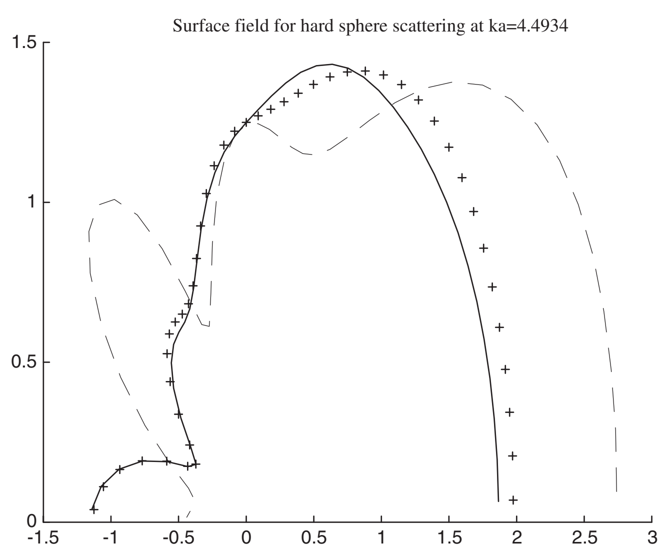

Figure 1. A comparison between analytical (−) and numerical solutions. (-.-) Numerical without correction, (+) numerical with interior point correction.

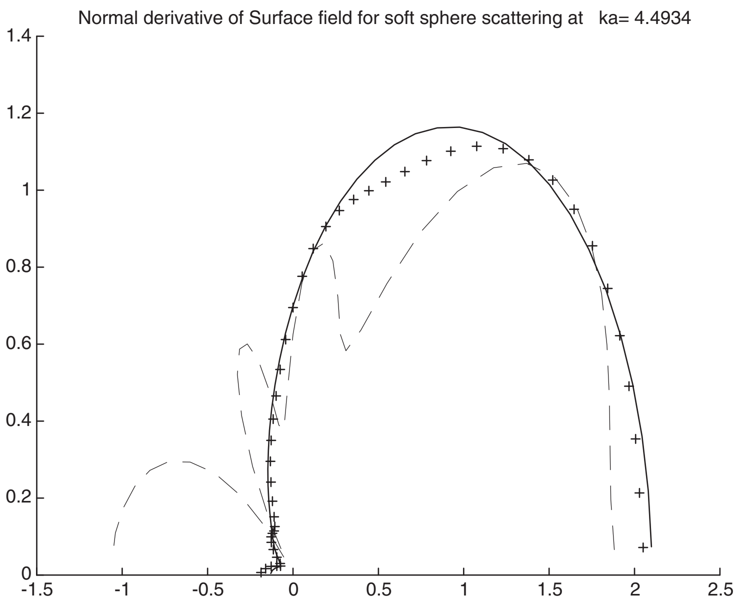

Figure 2. A comparison between analytical ($-$) and numerical solutions. (-.-) Numerical without correction, ($+$) numerical with interior point correction.

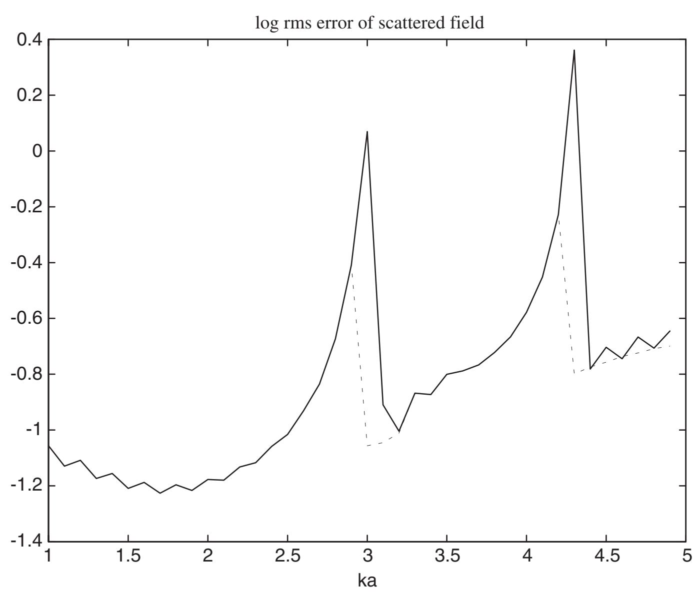

Figure 3. Log rms error between the solution using the proposed method and analytical solution with ka (interior points used around $ka=\pi$ and 4.4934). ($-$) Without correction, (...) with interior point correction.

To demonstrate the numerical instability and the changes in the solution around resonance, the program is used for the hard sphere case for a range of $ka = 1$–5 with a 0.1 step, while the axisymmetric half circle is divided into $10ka$ points. Figure 3 shows the solution behaviour versus the wave number. The vertical axis of the plot is the root-mean square error in the computed field relative to the analytical solution. The solutions around the natural frequencies (i.e $ka = \pi$ and 4.4934) are obtained with the help of interior fields as described above. Solid line in the figure shows the error without treatment while the dotted one shows the error after correcting the solution by the proposed method.

7. DISCUSSION

Different formulations of the solution of acoustic scattering by a closed object are considered. It is shown that one may deal with the hard and soft boundary conditions in one program using Fredholm integral equations of the second kind. The integrands are then similar. However, this advantage diminishes as the frequency increases since the solution time becomes more dominated by the matrix inversion and not by its filling. Besides, solving both problems simultaneously requires doubling the storage.

The problem of nonuniqueness of the numerical solution to the integral equation formulation of acoustic scattering by a closed object is considered. Recommendations on how to select the interior points and how to check their suitability are presented. In order to avoid the excessive cost of treating the overdetermined system, a simplified procedure is introduced. We add interior field equations to SIE in the shadow region. In this case the previously computed matrix and its LU decomposition are fully utilized and minor additional work is required to solve the new system.

In this method, the interior equations act as constraints (adding to original equations at the surface with a factor in the range 0.1–0.9) which bring the solution to a unique value. The results show that it is sufficient to use few interior equations ($M = 2$) even at $ka = 4.4934$. Only, M new rows of $\mathbf{L}$ and M new columns of $\mathbf{U}$ are to be calculated. Since $M \ll N$ typically, our implementation does not significantly add to the solution time. The method can be added to existing codes with insignificant additional computational burden.

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