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The natural boundary integral equation in potential problems and regularization of the hypersingular integral

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Abstract

In general, one of the dual boundary integral equations (BIE) is always a kind of derivative BIE with the hyper-singular integral. The treatment of the hyper-singularities is difficult in the boundary element methods (BEM). This paper focuses on the BIE in the two-dimensional potential problems. A series of transformations are manipulated on the conventional potential derivative BIE in order to eliminate the hyper-singularity. It leads to a new natural BIE in the two-dimensional potential problems. The natural BIE also belongs to the derivative BIE, but only contains the strongly singular integral. The evaluation for the strongly singular integral is given by the subtraction method. As a result, another boundary element analysis according to the natural BIE can obtain more accurate potential derivatives on the boundary in comparison with the conventional BEM. Furthermore, the natural BIE can also be applied to calculating the potential derivatives at the interior points very close to the boundary. Some comparisons with exact solutions are done to illustrate the application and efficiency of the natural BIE.

Keywords: BEM; Potential; Hypersingular integral; Natural BIE; Principal value integral

1. Introduction

Boundary element methods were proposed from solving the boundary integral equations of the potential problems at the beginning. The governing equations of potential problems are a kind of elliptic differential equations, such as the Laplace equation and Poisson equation, which can describe steady-state heat transfer, electrostatic fields, steady-flow fields etc. The initial contributions to the boundary element methods should be owed to Jaswon [1], Symm [2], Hess and Smith [3], Massonnet [4] in 1960s. Then, Rizzo [5] and Cruse [6] established the displacement boundary integral equations of elasticity problems. With the development of the last four decades, the application of boundary element methods has shown great success for the analyses of engineering structures, especially for the solid mechanics and potential problems, such as acoutics, elasticity, plasticity, creep and crack problems etc.

The boundary integral equations of the primary fields have sufficient conditions to determine the solutions for the regular boundary problems, which are usually termed the conventional BIE. The conventional potential BIE can be applied to obtaining boundary potential function u and its normal derivative $\partial u/\partial n$. If the boundary potential derivative $\partial u/\partial x_i$ is required, one way is directly to differentiate the potential function along the boundary. However, it reduces the accuracy by one order. Another way seeks to establish the derivative boundary integral equation for the secondary field [7–10]. On the other hand, as is well known, the conventional BIE is not sufficient to solve the boundary problems with narrow and thin domains, such as the crack problems [10], seepage flow problems with sheet-piles [11]. It is because the BIE is not independent on the close two surfaces of a degenerate boundary. Thus, some

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researchers have proposed to add the derivative BIE on the degenerate boundaries. It constructs a variety of dual BIE, e.g., including the potential derivative BIE and traction BIE [7,10,12] in elasticity problems, so that the degenerate boundary problems can be solved by the use of the dual BEM. Unfortunately, the hypersingular integral arises from the derivative BIE. For the case, the concept of Hadamard finite part integral is introduced [13].

Some methods have been proposed for treating the hypersingular integral. Gray et al. [18] transform the hypersingular surface integral into the line integral along the contour of the element, and then the numerical quadrature is employed to compute the line integral. The subtraction method [14,15,19] is used by subtracting and adding back a term for the hypersingular integrand at a load point. As a result, the hypersingular integrals are transformed into the strongly singular integrals by the Stokes' theorem. Detailed reviews about treating the hypersingular integrals can be found in the papers [16,17]. Recently, Chen et al. [20] study the hypersingular integrals in the potential normal derivative BIE, and verify that the contributions from both the hypersingular integrals and strongly singular integrals for the free terms of the BIE are, respectively, half and half for the two-dimensional case, one third and two thirds for three-dimensions. Using the above methods seems to gain the expectant results to evaluate the hypersingular integrals in BEM. However, according to the authors' best knowledge, the convinced results to compute the hypersingular integrals in a corner or discontinuous point of the physical quantities have not been found. Moreover, those strongly singular integrals from the transformation of the hypersingular integrals have not also been evaluated efficiently. In fact, if one separately computes both the hypersingular and strongly singular integrals, the existence of the solutions is doubtful at the discontinuous points. Of course, the sum of the two singular integrals is generally existent. However, it is difficult to find a way to determine it.

The authors [21] have studied the displacement derivative BIE in the two-dimensional elasticity problems. By introducing the natural boundary variables, the hypersingular integral on any boundary can be transformed into one strongly singular integral. Then, the resulting strongly singular integrand is treated by the combination with another strongly singular integrand. By a series of the deductions, it produces a natural BIE with the natural boundary variables and primary variables, where the corresponding Cauchy principal value integral can be calculated perfectly.

In this paper, an analogous idea is employed in the two-dimensional potential derivative BIE. A new natural BIE about two-dimensional potential theory is established by the regularization of the hypersingular integrals. There only exists the strong singularity in the natural BIE, which can be easily evaluated, instead of the original hyper-singularity.

2. The singularities of the conventional BIE of the potential problems

The potential at any point $\boldsymbol{y}$ in the domain Ω is represented by the potential BIE

$$u(\boldsymbol{y}) = \int_\Gamma U^*(\boldsymbol{x},\boldsymbol{y})q(\boldsymbol{x})\,\mathrm{d}\Gamma(\boldsymbol{x}) - \int_\Gamma Q^*(\boldsymbol{x},\boldsymbol{y})u(\boldsymbol{x})\,\mathrm{d}\Gamma(\boldsymbol{x}) + \int_\Omega U^*(\boldsymbol{x},\boldsymbol{y})b(\boldsymbol{x})\,\mathrm{d}\Omega(\boldsymbol{x}), \quad \boldsymbol{y} \text{ in } \Omega \tag{1}$$

where $\Gamma = \partial\Omega$, u is the potential function, $q = \partial u/\partial n$ the normal derivative of u on the boundary. $b(\boldsymbol{x})$ is the body source. $\boldsymbol{y}$ is the source or load point and $\boldsymbol{x}$ the field point. U^* and Q^* are the fundamental solutions as follows

$$[U^*]_{2D} = \frac{1}{2\pi}\ln\frac{1}{r}, \quad [U^*]_{3D} = \frac{1}{4\pi r} \tag{2a}$$

$$Q^* = -\frac{1}{2\pi\alpha r^\alpha} r_{,n} \tag{2b}$$

where $\alpha = 1$ for two-dimensions, $\alpha = 2$ for three-dimensions. Let y_i and x_i denote the Cartesian coordinate components of $\boldsymbol{y}$ and $\boldsymbol{x}$, respectively. r is the distance from $\boldsymbol{y}$ to $\boldsymbol{x}$. n_i denotes the components of the outward normal vector of the boundary. We can write

$$\left.\begin{aligned} r_i &= x_i - y_i, \quad r = \sqrt{r_i r_i} \\ r_{,i} &= \partial r/\partial x_i, \quad r_{,n} = \partial r/\partial n = r_{,i} n_i \end{aligned}\right\} \tag{3}$$

Let $\boldsymbol{y}$ approach the boundary Γ in Eq. (1). The conventional potential BIE is given as

$$C(\boldsymbol{y})u(\boldsymbol{y}) = \int_\Gamma U^* q\,\mathrm{d}\Gamma - \fint_\Gamma Q^* u\,\mathrm{d}\Gamma + \int_\Omega U^* b\,\mathrm{d}\Omega, \quad \boldsymbol{y} \in \Gamma \tag{4}$$

where $\fint_\Gamma(\cdots)\mathrm{d}\Gamma$ denotes the Cauchy principal value integral, $C(\boldsymbol{y})$ is defined as the potential singular coefficients. $C(\boldsymbol{y})$ equals $\varphi/2\pi$ for two-dimensional case where φ denotes the interior angle at $\boldsymbol{y}$ on the boundary Γ.

By differentiating the two sides in Eq. (1) with respect to y_i, it results in the potential derivative BIE

$$u_{,i}(\boldsymbol{y}) = \int_\Gamma U^*_{,i(y)} q\,\mathrm{d}\Gamma - \int_\Gamma Q^*_{,i(y)} u\,\mathrm{d}\Gamma + \int_\Omega U^*_{,i(y)} b\,\mathrm{d}\Omega, \quad i = 1,2, \ \boldsymbol{y} \mathrm{in} \Omega \tag{5}$$

Table 1
The singularities of the fundamental solutions in boundary integral equations

Fundamental solutions	2-D case	3-D case	Definition	
			$r=0$	$r\to 0,\ r\neq 0$
U^*	$\ln r$	$1/r$	Weak singularity	Nearly weak singularity
Q^*, $U^*_{,i}$	$1/r$	$1/r^2$	Strong singularity	Nearly strong singularity
$Q^*_{,i}$	$1/r^2$	$1/r^3$	Hyper-singularity	Nearly hyper-singularity

where

$$U^*_{,i(y)} = \frac{1}{2\pi\alpha r^{\alpha}} r_{,i(x)} \tag{6a}$$

$$Q^*_{,i(y)} = \frac{1}{2\pi\alpha r^{\alpha+1}}[n_i - (\alpha+1)r_{,i}r_{,n}] \tag{6b}$$

In the boundary element analyses for the potential problems, we obtain the boundary potential $u(\boldsymbol{x})$ and its normal derivative $q(\boldsymbol{x})$ on the boundary by solving BIE (4). Then the u and $u_{,i}$ at interior point $\boldsymbol{y}$ are calculated from Eqs. (1) and (5), respectively. When the source point $\boldsymbol{y}$ approaches to the boundary, the singular integrals occur in the BIE on the boundary of $\boldsymbol{x}\to\boldsymbol{y}$. The singularities in BIE are shown in Table 1.

As $\boldsymbol{y}$ is on the boundary, by doing the limit analysis for Eq. (5), the derivative BIE is derived as

$$D_{ij}(\boldsymbol{y})u_{,j}(\boldsymbol{y}) = \fint_\Gamma U^*_{,i} q\,\mathrm{d}\Gamma - \fint_\Gamma Q^*_{,i} u\,\mathrm{d}\Gamma + \int_\Omega U^*_{,i} b\,\mathrm{d}\Omega, \quad \boldsymbol{y} \text{ on } \Omega \tag{7}$$

where D_{ij} is termed singular coefficient in correspondence with $u_{,j}(\boldsymbol{y})$. $\fint(\cdots)\,\mathrm{d}\Gamma$ denotes the Hadamard finitepart of the integral. It seems to obtain $u_{,j}$ from Eq. (7). However, BIE (7) contains both the strongly singular and hypersingular integrals. The two integrals exist the Cauchy principal value (CPV) and Hadamard finite part (HFP) senses, respectively. In some literatures [12,19], the hypersingular integral is reduced to the strongly singular integral. However, the resulting integral has not evaluated satisfactorily, especially at the discontinuous point, although the CPV integral in the conventional potential BIE is achieved perfectly.

3. The natural boundary integral equation in the potential theory

For treating the hypersingular integrals in the two-dimensional derivative BIE, a transformation of the integration by parts is used for Eq. (5) in the section.

The body integral in Eq. (5) shows the weak singularity, which can be calculated without difficulty. Specially, when $b(\boldsymbol{x})$ is constant, the body integrals in both Eqs. (5) and (7) can be transformed to the boundary integrals as

$$\frac{1}{2\pi}\int_\Omega \frac{r_{,j}}{r} b\,\mathrm{d}\Omega = \frac{b}{2\pi}\int_\Omega (\ln r)_{,j}\,\mathrm{d}\Omega = \frac{b}{2\pi}\int_\Gamma \ln r n_j\,\mathrm{d}\Gamma \tag{8}$$

Therefore, without loss of generality, we shall neglect the body source $b(\boldsymbol{x})$ in the following.

Here, consider the two-dimensional potential problems. The source point $\boldsymbol{y}$ is in domain Ω. Draw the line $\bar{x}_1$ through $\boldsymbol{y}$ in parallel with the global x_1-axis. Let $\hat{\boldsymbol{y}}$ denote the point where $\bar{x}_1$ intersects with the boundary Γ (see Fig. 1), ϕ be the angle between the x_1-axis and the vector $\boldsymbol{r}$ [22]. Note that ϕ has a discontinuity across $\hat{\boldsymbol{y}}$ with a jump of 2π.

Introduce the symbol tensor δ_{ij} and permutation tensor ϵ_{ij} ($\epsilon_{11}=\epsilon_{22}=0,\ \epsilon_{12}=-\epsilon_{21}=1$). One can find

$$n_j\epsilon_{jl} = \tau_l, \quad \tau_j\epsilon_{jl} = -n_l, \quad j,l=1,2 \tag{9}$$

Here and in what follows, the summation convention is used. It is seen in Fig. 1 that we have

$$r_{,1} = \frac{\partial r}{\partial x_1} = \cos\phi, \quad r_{,2} = \frac{\partial r}{\partial x_2} = \sin\phi$$

on Γ. Differentiating the above expressions with respect to x_i produces

$$-\sin\phi\phi_{,i} = \frac{1}{r}(\delta_{1i} - r_{,1}r_{,i}), \quad \cos\phi\phi_{,i} = \frac{1}{r}(\delta_{2i} - r_{,2}r_{,i})$$

Thus, there is

$$\phi_{,i} = \frac{1}{r}(r_{,1}\delta_{2i} - r_{,2}\delta_{1i}) = \frac{1}{r}\epsilon_{mi}r_{,m(\boldsymbol{x})}$$

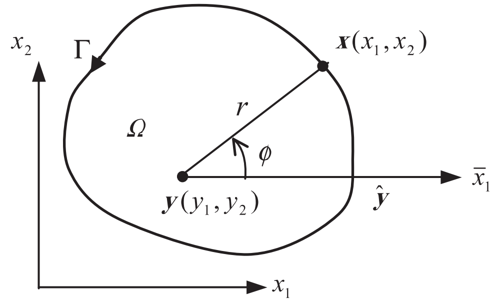

Fig. 1. Definition of angle ϕ.

Multiplying the above expression with τ_i and noting Eq. (9), this is

$$\frac{\partial \phi}{\partial s} = \phi_{,i}\tau_i = \frac{1}{r} r_{,n} \tag{10}$$

It can be easily verified that there exists the geometric relation

$$\frac{1}{r^2}(n_j - r_{,j} r_{,n}) = \frac{1}{r}\frac{\partial}{\partial x_j} r_{,n} = \frac{1}{r} r_{,jn} = \frac{\partial}{\partial x_j}\left(\frac{1}{r} r_{,n}\right) + r_{,n}\frac{r_{,j}}{r^2}$$

on Γ. Hence one can obtain

$$\frac{1}{r^2}(2r_{,j}r_{,n} - n_j) = -\frac{\partial}{\partial x_j}\left(\frac{1}{r} r_{,n}\right) = -\frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial s}\right) = -\phi_{,js} \tag{11}$$

After substituting Eq. (11) into Eq. (6b), we observe the integral of the second term in Eq. (5) on an arbitrary segment $\Gamma_1 = \widehat{\boldsymbol{x}_1 \boldsymbol{x}_2}$ of Γ, as is shown in Fig. 2. By partial integration, it produces

$$\begin{aligned} \int_{\Gamma_1} \frac{1}{r^2}(2r_{,j}r_{,n} - n_j)u(\boldsymbol{x})\,\mathrm{d}\Gamma(\boldsymbol{x}) &= -\int_{\Gamma_1} \phi_{,js} u(\boldsymbol{x})\,\mathrm{d}\Gamma \\ &= -\int_{\Gamma_1} u\,\mathrm{d}\phi_{,j} = \int_{\Gamma_1} \phi_{,j}\frac{\partial u}{\partial s}\,\mathrm{d}\Gamma - u\phi_{,j}\Big|_{x=x_1}^{x_2} \\ &= \int_{\Gamma_1} \frac{1}{r} r_{,m}\epsilon_{mj}\frac{\partial u}{\partial s}\,\mathrm{d}\Gamma - \frac{1}{r} r_{,m}\epsilon_{mj} u\Big|_{x=x_1}^{x_2} \end{aligned} \tag{12}$$

If $\Gamma = \Gamma_1$, the last term of Eq. (12) equals 0. Let $\Gamma = \Gamma_1 + \Gamma_2$. Substituting Eqs. (6) and (12) into Eq. (5) has

$$\begin{aligned} \frac{\partial u(\boldsymbol{y})}{\partial y_i} = {} & \int_{\Gamma} \frac{1}{2\pi r} r_{,i} q\,\mathrm{d}\Gamma + \int_{\Gamma_1} \frac{1}{2\pi r} r_{,m}\epsilon_{mi}\frac{\partial u}{\partial s}\,\mathrm{d}\Gamma \\ & - \frac{1}{2\pi r} r_{,m}\epsilon_{mi} u\Big|_{x=x_1}^{x_2} - \int_{\Gamma_2} \frac{1}{2\pi r^2}(n_i - 2r_{,i}r_{,n})u\,\mathrm{d}\Gamma \end{aligned} \tag{13}$$

The first and second integrals of the right-hand side in Eq. (13) have the strong singularities if $\boldsymbol{y}$ tends to Γ_1. Usually the two integrals are separately computed in many literatures, but those efforts have not got satisfactory results. Here, we consider the combination of the two integrals on Γ_1 and make one reorganization for it. We find

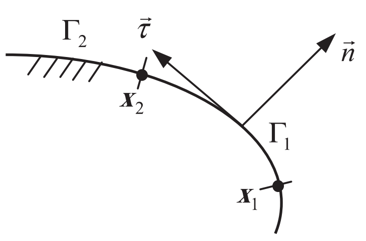

Fig. 2. Making the boundary Γ_1.

$$\begin{aligned} I_i(\boldsymbol{y}) &= \frac{1}{2\pi}\int_{\Gamma_1} \frac{r_{,i}}{r} q\,\mathrm{d}\Gamma + \frac{1}{2\pi}\int_{\Gamma_1} \frac{r_{,m}}{r}\epsilon_{mi}\frac{\partial u}{\partial s}\,\mathrm{d}\Gamma \\ &= \frac{1}{2\pi}\int_{\Gamma_1} \frac{1}{r}\left(r_{,i}n_j\frac{\partial u}{\partial x_j} + r_{,m}\epsilon_{mi}\tau_j\frac{\partial u}{\partial x_j}\right)\mathrm{d}\Gamma \\ &= \frac{1}{2\pi}\int_{\Gamma_1} \frac{1}{r}\left[(r_{,i}n_1 + r_{,m}\epsilon_{mi}\tau_1)\frac{\partial u}{\partial x_1}\right. \\ &\qquad \left. + (r_{,i}n_2 + r_{,m}\epsilon_{mi}\tau_2)\frac{\partial u}{\partial x_2}\right]\mathrm{d}\Gamma \\ &= \frac{1}{2\pi}\int_{\Gamma_1} \frac{1}{r}\left(r_{,n}\delta_{ij}\frac{\partial u}{\partial x_j} - r_{,s}\epsilon_{ij}\frac{\partial u}{\partial x_j}\right)\mathrm{d}\Gamma, \quad \boldsymbol{y} \text{ in } \Omega \end{aligned} \tag{14}$$

Let $\boldsymbol{y}$ be on Γ_1. The next step is to do the limit analysis at $\boldsymbol{y}$ in Eq. (14). Referring to Fig. 3, the domain Ω is subtracted by a small sector of radius ε around $\boldsymbol{y}$. The remaining domain is enveloped by the new boundary $\Gamma' = \Gamma_1 + \Gamma_2 - \Delta\Gamma + \Gamma_\varepsilon$, so that the source point $\boldsymbol{y}$ lies outside Γ'. There is evidently

$$\lim_{\varepsilon\to 0} \Gamma' = \Gamma$$

At this time, Eq. (14) is written as

$$I_i = \frac{1}{2\pi}\lim_{\varepsilon\to 0}\left\{\int_{\Gamma_1 - \Delta\Gamma} + \int_{\Gamma_\varepsilon} \frac{1}{r}\left(r_{,n}\delta_{ij}\frac{\partial u}{\partial x_j} - r_{,s}\epsilon_{ij}\frac{\partial u}{\partial x_j}\right)\mathrm{d}\Gamma\right\} \tag{15}$$

Use the notations

$$\int_{\Gamma_1} (\cdots)\,\mathrm{d}\Gamma = \lim_{\varepsilon\to 0}\int_{\Gamma_1 - \Delta\Gamma} (\cdots)\,\mathrm{d}\Gamma \tag{16}$$

$$I_i^\varepsilon = \lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \frac{1}{r}\left(r_{,n}\delta_{ij}\frac{\partial u}{\partial x_j} - r_{,s}\epsilon_{ij}\frac{\partial u}{\partial x_j}\right)\mathrm{d}\Gamma \tag{17}$$

There are

$$r_{,s} = 0, \quad r_{,n} = -1, \quad r = \varepsilon, \quad \mathrm{d}\Gamma = \varepsilon\,\mathrm{d}\theta$$

$$I_i^\varepsilon(\boldsymbol{y}) = -\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \frac{1}{r}\delta_{ij}\frac{\partial u}{\partial x_j}\,\mathrm{d}\Gamma = -\lim_{\varepsilon\to 0}\int_{\Gamma_\varepsilon} \frac{\partial u}{\partial x_i}\,\mathrm{d}\theta$$

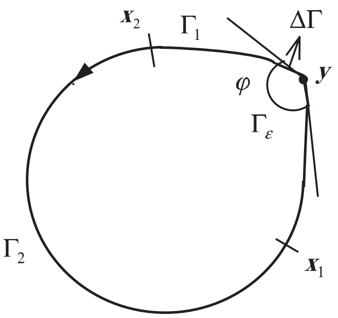

Fig. 3. Limit analysis at $\boldsymbol{y}$.

on Γ_ε. Since $\partial u/\partial x_i$ is continuous along the boundary, $\partial u/\partial x_i$ can be considered to be constant on Γ_ε when $\varepsilon \to 0$. Hence

$$I_i^\varepsilon = -\frac{\partial u}{\partial x_i}(\boldsymbol{y}) \lim_{\varepsilon\to 0} \int_{\Gamma_\varepsilon} \mathrm{d}\theta = -\varphi \frac{\partial u}{\partial y_i}(\boldsymbol{y}) \tag{18}$$

where φ is the interior angle at $\boldsymbol{y}$. Eq. (15), together with Eqs. (18) and (16), is substituted into Eq. (13). It gives

$$\begin{aligned}\varphi \frac{\partial u(\boldsymbol{y})}{\partial y_i} = & \int_{\Gamma_1} \frac{1}{r}\left(r_{,n}\delta_{ij}\frac{\partial u}{\partial x_j} - r_{,s}\epsilon_{ij}\frac{\partial u}{\partial x_j}\right) \mathrm{d}\Gamma \\ & - \frac{1}{r} r_{,m}\epsilon_{mi} u \Big|_{x=x_1}^{x_2} + \int_{\Gamma_2} \frac{1}{r} r_{,i} q \,\mathrm{d}\Gamma \\ & - \int_{\Gamma_2} \frac{1}{r^2}(n_i - 2r_{,i}r_{,n}) u \,\mathrm{d}\Gamma, \\ & \boldsymbol{y} \text{ on } \Gamma, \quad i = 1,2 \end{aligned} \tag{19}$$

It can be seen that there only exists the Cauchy principal value integral which is easily calculated (see Section 4). Here, the above equation is termed the natural boundary integral equation of the potential theory, in which the potential derivatives $q_j = \partial u/\partial x_j$ $(j = 1,2)$ are two new boundary variables on Γ_1, the variables on Γ_2 are still u and q.

The conventional BEM obtains the potential u and its normal derivative q by solving the potential BIE (4). Then substituting them into Eq. (19), the BEM based on the above BIE is again implemented directly to compute any potential derivative on the boundary instead of differentiating the potential. In fact, the second boundary element analysis can be done only on the local boundary Γ_1.

4. The Cauchy principal value integral in the natural BIE

In the natural BIE (19), assume that the source point $\boldsymbol{y}$ lies at the intersection node of the elements Γ_a and Γ_b, as is shown in Fig. 4. For the case, there exist the strongly singular integrals only on Γ_a and Γ_b. Make $\Gamma_1 = \Gamma_a + \Gamma_b$. Without loss of generality, consider Γ_a

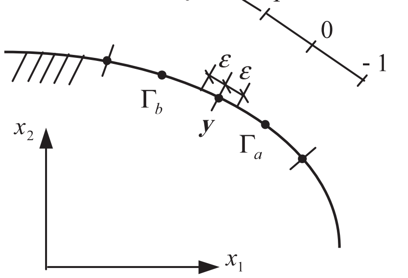

Fig. 4. CPV integral.

and Γ_b as the quadratic elements with three nodes. The Cauchy principal value integrals in Γ_a and Γ_b are evaluated in the following.

On the element Γ_e, the global coordinate system is transformed into the local coordinate system $o\xi$. The geometry of Γ_e is described by the quadratic shape functions as

$$x_i(\xi) = \sum_{s=1}^{3} N^{(s)}(\xi) x_i^{(s)}, \quad i = 1,2,3 \tag{20}$$

$$\mathrm{d}s = J\,\mathrm{d}\xi, \quad J(\xi) = \sqrt{x_i' \cdot x_i'} \tag{a}$$

where

$$N^{(1)} = \tfrac{1}{2}\xi(\xi - 1), \quad N^{(2)} = 1 - \xi^2, \quad N^{(3)} = \tfrac{1}{2}\xi(\xi + 1) \tag{b}$$

As $\boldsymbol{y}$ is on Γ_e, its coordinate value in $o\xi$ system is ξ_0 where $\xi_0 \in [-1, 1]$. Introduce

$$\delta = \xi - \xi_0, \quad A = \sqrt{x_i' x_i'}\Big|_{\xi=\xi_0} = J(\xi_0) \tag{c}$$

Any kernel function on Γ_e in Eq. (19) can be expressed by the Laurent series with respect to δ as

$$\begin{aligned} & x_i - y_i = x_i'(\xi_0)\delta + 0(\delta^2) \\ & r = \sqrt{r_i r_i} = \pm A\delta + 0(\delta^2), \quad \frac{1}{r} = \pm\frac{1}{A\delta} + 0(1) \\ & r_{,i} = \pm\frac{x_i'(\xi_0)}{A} + 0(\delta), \quad \tau_i = \pm\frac{x_i'(\xi_0)}{A} + 0(\delta) \end{aligned} \tag{d}$$

where "+" is taken for $\delta \geqslant 0$, "−" for $\delta < 0$. The CPV integral in Eq. (19) is written as

$$\begin{aligned} J_i = & \int_{\Gamma_1} \frac{1}{r}\left(r_{,n}\delta_{ij}\frac{\partial u}{\partial x_j} - r_{,s}\epsilon_{ij}\frac{\partial u}{\partial x_j}\right)\mathrm{d}\Gamma \\ = & \int_{\Gamma_1} \frac{1}{r}(r_{,n}\delta_{ij} - r_{,s}\epsilon_{ij}) q_j \,\mathrm{d}\Gamma = \int_{\Gamma_a+\Gamma_b} M_{ij} q_j \,\mathrm{d}\Gamma \end{aligned} \tag{21}$$

where

$$M_{ij} = \frac{1}{r}(r_{,n}\delta_{ij} - r_{,s}\epsilon_{ij})$$

Noting Eq. (d), the above can be rewritten as

$$\begin{aligned} M_{ik}(\xi) &= \pm\left(\frac{1}{A\delta} + 0(1)\right)(r_{,n}\delta_{ik} - r_{,s}\epsilon_{ik}) =: \frac{1}{\delta} f_{ik}(\xi) \\ &= \frac{1}{\delta}[f_{ik}(\xi_0) + 0(\delta)] \end{aligned} \tag{22}$$

Making $\xi \to \xi_0$, there are $r_{,n} = 0$, $r_{,s} = \pm 1$ from Eq. (d), so

$$f_{ik}(\xi_0) = -\epsilon_{ik}/J(\xi_0) \tag{e}$$

in Eq. (22). If Γ_e is a quadratic isoparameter element with three nodes, the potential derivatives on Γ_e are described approximately by the quadratic shape function as

$$q_k^{(e)}[\boldsymbol{x}(\xi)] = \sum_{s=1}^{3} N^{(s)}(\xi) q_k^{(es)}, \quad e = a, b, \ s = 1, 2, 3 \tag{23}$$

where s is the local number of each node on the quadratic element according to the positive direction of the boundary, $q_k^{(es)}$ is the potential derivative with respect to x_k at node s on Γ_e. In the convention, the symbols $b1$ and $a3$ are to denote the same one node (see Fig. 4), i.e., the intersection node $\boldsymbol{y}$ of Γ_a and Γ_b. Introducing Eqs. (22) and (23) into (21) produces

$$J_i = \sum_{s=1}^{3} \left[\int_{\Gamma_a} F_{ik}^{(as)}(\xi) q_k^{(as)} \, \mathrm{d}\xi + \int_{\Gamma_b} F_{ik}^{(bs)}(\xi) q_k^{(bs)} \, \mathrm{d}\xi \right] \tag{f}$$

where

$$F_{ik}^{(es)}(\xi) = M_{ik}^{(e)}(\xi) N^{(s)} J^{(e)}(\xi) =: \frac{1}{\delta} g_{ik}^{(es)}(\xi) = \frac{1}{\delta} [g_{ik}^{(es)}(\xi_0) + 0(\delta)] \tag{g}$$

$$g_{ik}^{(es)}(\xi) = f_{ik}^{(e)}(\xi) N^{(s)}(\xi) J^{(e)}(\xi) \tag{h}$$

The source point is always positioned at the three nodes on the element in BEM, in turn, noting Eqs. (b) and (e), so the above expression has

$$g_{ik}^{(es)}(\xi_0) = f_{ik}^{(e)}(\xi_0) N^{(s)}(\xi_0) J^{(e)}(\xi_0) = -\epsilon_{ik} \tag{24}$$

Analyzing Eqs. (g), (h), (b) and (24), it can be known that the integrand $F_{ik}^{(es)}(\xi)$ in Eq. (f) has $1/\delta$ singularity at ξ_0 in the cases, i.e., when $\xi_0 = 1$, $s = 3$ for Γ_a and $\xi_0 = -1$, $s = 1$ for Γ_b. The free coefficient in the singular term is $-\epsilon_{ik}$. Substituting Eq. (g) into (f) and using the subtraction method, the result is

$$J_i = \lim_{\varepsilon \to 0} \left[q_k^{(a3)}(\boldsymbol{y}) I_{ik}^{(a3)} + q_k^{(b1)}(\boldsymbol{y}) I_{ik}^{(b1)} + \sum_{s=1}^{2} I_{ik}^{(as)} q_k^{(as)} + \sum_{s=2}^{3} I_{ik}^{(bs)} q_k^{(bs)} \right] \tag{25}$$

where

$$\lim_{\varepsilon \to 0} I_{ik}^{(a3)} = \int_{-1}^{1} \left[F_{ik}^{(a3)} + \frac{\epsilon_{ik}}{\xi - 1} \right] \mathrm{d}\xi - \lim_{\varepsilon \to 0} \int_{-1}^{1-\varepsilon/J^{(a)}(1)} \frac{\epsilon_{ik}}{\xi - 1} \, \mathrm{d}\xi = \int_{-1}^{1} \left(F_{ik}^{(a3)} + \frac{\epsilon_{ik}}{\xi - 1} \right) \mathrm{d}\xi + \epsilon_{ik} \ln[2J^{(a)}(1)] - \epsilon_{ik} \lim_{\varepsilon \to 0} \ln \varepsilon \tag{26a}$$

$$\lim_{\varepsilon \to 0} I_{ik}^{(b1)} = \int_{-1}^{1} \left(F_{ik}^{(b1)} + \frac{\epsilon_{ik}}{\xi + 1} \right) \mathrm{d}\xi - \epsilon_{ik} \ln[2J^{(b)}(-1)] + \epsilon_{ik} \lim_{\varepsilon \to 0} \ln \varepsilon \tag{26b}$$

$$\lim_{\varepsilon \to 0} I_{ik}^{(as)} = \int_{-1}^{1} F_{ik}^{(as)} \, \mathrm{d}\xi \quad (s = 1, 2) \tag{26c}$$

$$\lim_{\varepsilon \to 0} I_{ik}^{(bs)} = \int_{-1}^{1} F_{ik}^{(bs)} \, \mathrm{d}\xi \quad (s = 2, 3) \tag{26d}$$

In the above, Eqs. (26c) and (26d) are regular integrals. The integral of every first term of the right-hand sides in Eqs. (26a) and (26b) is also regular integral, but $\ln \varepsilon$ in every last term is singular. Note that q_k is continuous at $\boldsymbol{y}$. There is

$$q_k^{(a3)}(\boldsymbol{y}) = q_k^{(b1)}(\boldsymbol{y}) = q_k(\boldsymbol{y}) \tag{i}$$

After introducing Eqs. (26a) and (26b) into (25), the singularities are eliminated by calculating the sum $\lim_{\varepsilon \to 0}(I_{ik}^{(a3)} + I_{ik}^{(b1)})$. Finally, the computational formulation of the Cauchy principal value integrals on the elements Γ_a and Γ_b becomes

$$J_i = q_k(\boldsymbol{y}) \int_{-1}^{1} \left[F_{ik}^{(a3)} + \frac{\epsilon_{ik}}{\xi - 1} \right] \mathrm{d}\xi + q_k(\boldsymbol{y}) \int_{-1}^{1} \left[F_{ik}^{(b1)} + \frac{\epsilon_{ik}}{\xi + 1} \right] \mathrm{d}\xi - \epsilon_{ik} q_k(\boldsymbol{y}) \ln \frac{J^{(b)}(-1)}{J^{(a)}(1)} + \sum_{s=1}^{2} q_k^{(as)} \int_{-1}^{1} F_{ik}^{(as)} \, \mathrm{d}\xi + \sum_{s=2}^{3} q_k^{(bs)} \int_{-1}^{1} F_{ik}^{(bs)} \, \mathrm{d}\xi \tag{27}$$

When a source point lies in the interior on the element Γ_a, there is $\xi_0 \in (-1, 1)$. For this case, the strongly singular integral only arises on Γ_a in the natural BIE. By using the same analysis as above, the CPV integral is evaluated as below

$$J_i^{(a)} = \int_{\Gamma_a} M_{ik} q_k(\boldsymbol{x}) \, \mathrm{d}\Gamma = \sum_{s=1}^{3} I_{ik}^{(as)} q_k^{(as)} \tag{28}$$

$$I_{ik}^{(as)} = \int_{-1}^{1} \left[F_{ik}^{(as)} - \frac{1}{\xi - \xi_0} g_{ik}^{(as)}(\xi_0) \right] \mathrm{d}\xi + g_{ik}^{(as)}(\xi_0) \ln \frac{1 - \xi_0}{1 + \xi_0} \tag{29}$$

There are no singularities in Eqs. (27) and (29). Therefore, the existence of the CPV integrals in the natural BIE has been verified and can be computed by Eqs. (27) and (28).

5. The natural BEM in the potential problems

Here, the boundary element analysis based on the natural BIE is termed the natural BEM. In the section, we solve the potential and its normal derivative on the boundary from the conventional BIE. Then the boundary potential derivatives can be obtained by using the natural BEM. Furthermore, by the introduction of Eq. (14) into (13), it produces

$$\begin{aligned}\frac{\partial u(\boldsymbol{y})}{\partial y_i} = & \int_{\Gamma_1} \frac{1}{r}\left(r_{,n}\delta_{ij}\frac{\partial u}{\partial x_j} - r_{,s}\epsilon_{ij}\frac{\partial u}{\partial x_j}\right)\mathrm{d}\Gamma \\ & - \frac{1}{r} r_{,m}\epsilon_{mi} u \Big|_{x=x_1}^{x_2} + \int_{\Gamma_2}\frac{1}{r} r_{,i} q\,\mathrm{d}\Gamma \\ & - \int_{\Gamma_2}\frac{1}{r^2}(n_i - 2r_{,i}r_{,n})u\,\mathrm{d}\Gamma, \quad \boldsymbol{y} \text{ in } \Omega,\ i=1,2\end{aligned} \tag{30}$$

The above equation can replace Eq. (5) to evaluate the potential derivatives in the domain. It is important that Eq. (30) only contains the nearly strongly singular integrals instead of the nearly hypersingular integrals in Eq. (5). Thus, in general, it is expected that the use of Eq. (30) should achieve more accurate results than Eq. (5).

Example 1. The Saint–Venant torsion of a straight bar with elliptical cross-section.

The length of the semi-axis in x_1-direction is $a = 10$ and $b = 5$ in x_2-direction on the elliptical section. Here and in what follows, assume that the relative units are all compatible. The warping function $u(x_1, x_2)$ satisfies the Laplace equation $\nabla^2 u = 0$. Due to symmetry, only one quarter of the elliptic cross-section is considered. The boundary conditions are shown in Fig. 5. The exact solutions of the problem are

$$u = -0.6x_1x_2, \quad q_1 = -0.6x_2, \quad q_2 = -0.6x_1$$

The warping function's gradients of the boundary nodes are calculated by the conventional boundary integral equation (CBIE) together with difference of the potential u and natural boundary integral equation (NBIE), respectively. Divide the boundary elements as follows:

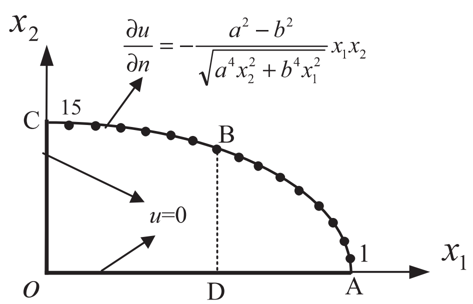

Fig. 5. The Saint–Venant torsion of a straight bar with elliptical cross-section.

there are 4 linear elements on side $\overline{OC}$, 8 linear elements on side $\overline{OA}$ and 16 linear elements on the $\widehat{ABC}$.

The results of the gradient q_1 on elliptical contour are shown in Fig. 6. The errors of the two methods in comparison with the exact solution are plotted in Fig. 7, which shows that the results obtained by NBIE are more accurate than those by CBIE.

As is well known, when the source point $\boldsymbol{y}$ is close to the boundary but not on the boundary, the nearly singular integrals occur in Eqs. (1) and (5). The conventional numerical quadrature is invalid to calculate the nearly singular integrals, especially for the nearly hypersingular integrals in Eq. (5).

Many efforts for the evaluation of the nearly singular integrals have proposed some computational strategies, e.g., in the literatures [19,23–25]. Niu et al. [26] propose a regularization algorithm to calculate the nearly strongly singular and hypersingular integrals in

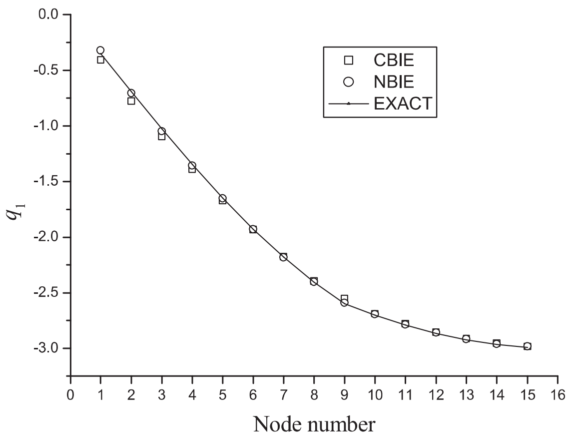

Fig. 6. The warping function's gradient q_1 on the arc $\widehat{ABC}$.

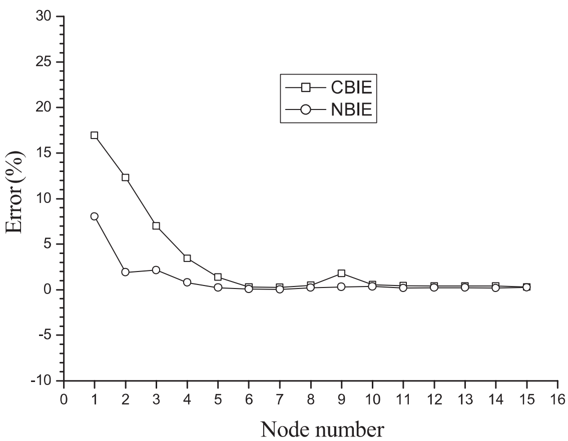

Fig. 7. Errors of the warping function's gradient q_1 on the arc $\widehat{ABC}$.

two-dimensional BEM. After the boundary integral equations are discretized on the boundary, the concept of a relative distance, which is the ratio of the minimum distance between the source point and the closest element to the length of the element, is introduced to describe the influence of the nearly singular integrals. The relative distance is separated from the singular integrands by the use of integration by parts so that the nearly singular integrals are transformed to non-singular representations.

Here, we make it a comparison to calculate the potential gradients of the interior points according to the BIE (5) and (30), as well as with or without the regularization algorithm, respectively. Using the same mesh as above, the results of the potential gradients along $\overrightarrow{DB}$ direction close to the boundary node B (5, 4.3301267) in Fig. 5 are shown in Fig. 8. The relative distance means

$$e = 2\delta/L$$

where L is the length of the element Γ_e, δ is the minimum distance from $\boldsymbol{y}$ to Γ_e. The ways of CBIE and NBIE without the regularization mean that the nearly singular integrals in Eqs. (5) and (30) are directly calculated by the Gauss numerical quadrature. The CBIE and NBIE with the regularization are to use the regularization algorithm [26] for Eqs. (5) and (30), respectively.

Example 2. Thermal transfer in a circular ring.

The inner and outer radii are 1 and 2, respectively. Because of symmetry, one half of the ring is considered, as shown in Fig. 9. The corresponding boundary conditions are described in Fig. 9. The CBIE and NBIE with or without the regularization are applied to calculating the fluxes. The inner and outer circular arcs are divided into 18 and 34 uniform linear elements, respectively. The

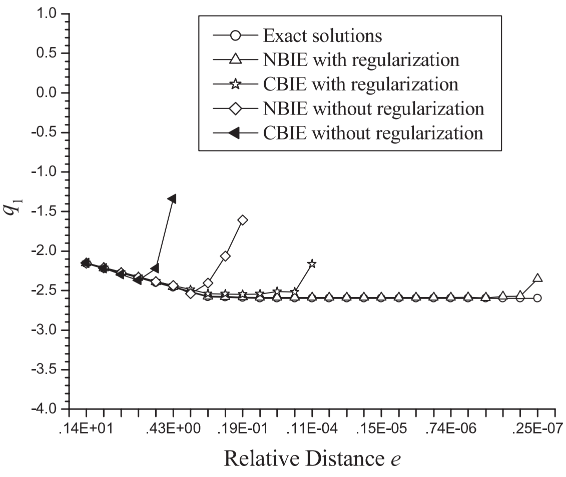

Fig. 8. The warping function's gradient q_1 at the interior points.

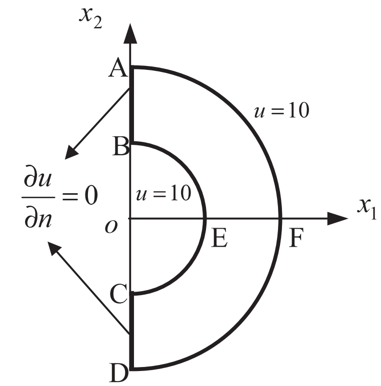

Fig. 9. Thermal transfer in a circular ring.

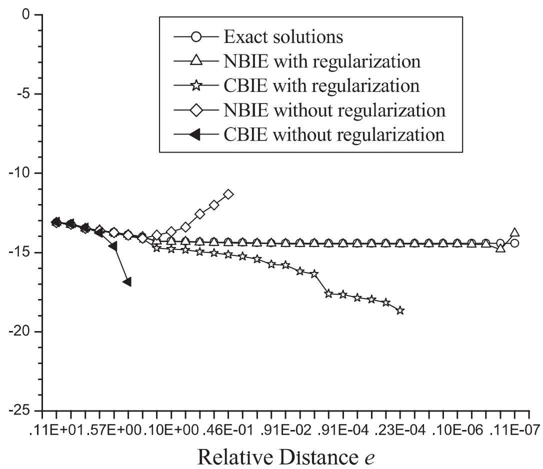

Fig. 10. The flux q_1 of the interior points.

segments AB and CD are divided into four uniform linear elements, respectively. The flux q_1 at the interior points along $\overrightarrow{FE}$ direction close to the node E (1, 0) are shown in Fig. 10.

In Figs. 8 and 10, it can be seen that the CBIE with and without the regularization lead to bad deterioration when the relative distance equals 0.43E+00 and 0.11E−04, respectively. Alternatively, the results of the NBIE without and with the regularization are very accurate until the relative distance reduces to 0.19E−01 and 0.25E−07, respectively. It illustrates that the NBIE also achieve success to calculate the potential gradients of the interior points close to the boundary in comparison with the CBIE.

6. Conclusions

For avoiding the difficulty of the evaluation of the hypersingular integrals in the derivative BIE, based on the conventional potential derivative BIE, the present paper establishes the natural BIE in two-dimensional potential theory. The natural BIE is a kind of new derivative BIE, but only contains the strongly singular integral which is easily determined by the limit analysis. After the use of the conventional BEM, the boundary element analysis according to the new BIE can obtain more accurate boundary potential derivatives in comparison with the conventional BIE. It is interesting that the natural BEM can only be implemented in a local boundary if desirable. Moreover, the natural BIE can also calculate the potential derivatives very close to the boundary, especially together with the regularization algorithm. The technique is important when the thin-walled structures are analyzed by the BEM.

The natural BIE is first established in elasticity theory [21]. In the paper, it is verified that the potential theory also exists the natural BIE similar to the elasticity problem. Furthermore, it is found that there are the same properties in the two natural BIEs. Therefore, it is naturally imagined that there exist the corresponding natural BIE in the other field problems.

It is expected that the natural BIE and conventional BIE can construct a group of dual BIE in order to analyze the potential problems with the degenerate boundaries. The dual BIE, after discretizing, should have better condition about the system of equations to ensure the solutions, because of no hypersingular integral.

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