PERGAMON

Computers and Structures 69 (1998) 421–431

Computers & Structures

An effective method for finding values on and near boundaries in the elastic BEM

H.B. Chen[a], P. Lu[a, 1], M.G. Huang[a], F.W. Williams[b, *]

[a]*Department of Modern Mechanics, University of Science and Technology of China, Hefei, Anhui 230026, People's Republic of China*
[b]*Division of Structural Engineering, Cardiff School of Engineering, Cardiff University, Cardiff CF2 3TB, U.K.*

Received 17 December 1996; accepted 28 April 1998

Abstract

Recent research has paid great attention to the exact calculation of physical values on and near the boundaries of engineering problems. In the boundary element method (BEM), this problem demands special attention because various kinds of singularity may occur when these values are calculated. Despite extensive research, this problem has not yet been solved well. In the present paper, the elimination of various kinds of singular integrals in displacement boundary integral equations and their derivative forms is studied in detail. Firstly, a modified treatment which uses rigid body displacement solutions is introduced to calculate displacements near the boundary. Then derivative formulations of the displacement boundary integral equations are deduced, which form the basis of an investigation of the related hypersingular and nearly singular integrals. Similarly, a modified treatment, which uses the unit displacement derivative solutions together with some general numerical techniques, is proposed for the calculation of displacement derivatives on and near the boundary considered. Finally, strains and stresses are obtained from the calculated displacement derivatives by the use of the compatibility and constitutive equations. Numerical examples show that the proposed method is simple, regular and accurate in treating most singular or nearly singular integrals in the elastic BEM.

Keywords: Boundary element method; Displacement derivatives; Hypersingular integrals; Elasticity

1. Introduction

In the boundary element method (BEM) analysis, effective calculation of the values on or near considered boundaries has received increased attention because of practical engineering requirements. In the conventional BEM, interior values are usually calculated directly according to proper physical relationships, after the boundary unknowns have been obtained. However, as points of interest, or source points, approach a boundary, the integral kernels will fluctuate in the boundary elements close to the points, indicating a significant singularity effect. In such cases, conventional treatment causes calculation errors. Furthermore, some physical variables on the boundary are also difficult to calculate directly by the conventional BEM. For example, stresses on boundaries in elasticity generally have to be calculated indirectly, which also causes considerable errors. To a certain degree, these shortcomings restrict the application of the BEM.

Many researchers have contributed to overcoming these difficulties. Ghosh et al. [1] proposed a boundary element formulation in which, instead of boundary displacements, the displacement derivatives along a boundary are chosen as the basic unknowns, in order to reduce the singularity order of the boundary integrals. Later, Hong and Chen [2] developed a kind of dual BEM, which contains two types of boundary integral equations, these being boundary integral equations

* Corresponding author.
[1] Currently visiting academic in the Division of Structural Engineering, Cardiff School of Engineering, Cardiff University, Cardiff CF2 3TB, UK.

0045-7949/98/$ - see front matter
PII: S0045-7949(98)00122-9

for displacement and traction, respectively. This kind of treatment provides a good approach for analysing potential problems (e.g. fluid and crack problems) [3, 4], three-dimensional (3D) problems [5] and fracture problems [6, 7]. Zhang and Lou [8] provided a treatment for calculating stresses close to boundaries. Wang et al. [9, 10] proposed a particular solution method to calculate stresses and displacements on and near boundaries, which was developed further by Chen et al. [11] to treat elastoplastic problems. Guiggiani and co-workers [12–16] presented several methods for the direct evaluation of hypersingular integrals in the 2 and 3D BEMs. To summarise, these authors have improved the calculation of values on or near considered boundaries in boundary element analysis by attacking the problem from several different directions. However, the problems have not been addressed completely. In the dual BEM, only smooth boundaries are considered for the traction boundary integral equations. The direct calculation approaches [8, 12–16] are relatively complicated to implement. The indirect approach [9, 10] is inaccurate when calculating the stresses for stress concentration problems, because extreme inaccuracy occurs when calculated points are very close to the boundary.

In the present paper, the situations of singular and nearly singular integrals in displacement boundary integral equations and their derivative forms are studied carefully. A systematic treatment of these singular and nearly singular integrals is presented for elastic problems based on an indirect approach. As a preliminary step, the displacements near a boundary are discussed first, and the relevant nearly singular integrals are determined indirectly by use of the rigid body displacement solutions. Then the derivative formulations of displacement boundary integral equations are derived rigorously in a concise form, and corresponding modifications are given to avoid the hypersingular integrals which appear in the derivative boundary integral equations. The modified method gives good accuracy when calculating the higher order variables, i.e. displacement derivatives, strains and stresses, on and near the boundary. Compared with other direct or indirect approaches, e.g. those in the previous paragraph, the method in this paper is simple, regular and efficient in treating most singular and nearly singular integrals in the BEM. Hence, the method presented provides an alternative effective treatment for the calculation of values on and near boundaries, which could be used in boundary element analysis.

2. Calculation of displacements close to the boundary

Once boundary unknowns, i.e. either the surface displacements u_j or the surface tractions p_j, are obtained for a domain Ω that has surface Γ, the interior values of displacements u_i in elasticity can be calculated from [17]:

$$u_i(\xi) = \int_\Gamma u_{ij}^*(\xi, x)p_j(x)d\Gamma - \int_\Gamma p_{ij}^*(\xi, x)u_j(x)d\Gamma + \int_\Omega u_{ij}^*(\xi, x)b_j(x)\,d\Omega, \tag{1}$$

where u_{ij}^* and p_{ij}^* are Kelvin's fundamental solutions given by:

$$u_{ij}^*(\xi, x) = \frac{-1}{16\pi(1-\nu)Gr}[(3-4\nu)\delta_{ij} + r_{,j}r_{,i}],$$

$$p_{ij}^*(\xi, x) = \frac{-1}{8\pi(1-\nu)r^2}\{(1-2\nu)(r_{,i}n_j - r_{,j}n_i) - [(1-2\nu)\delta_{ij} + 3r_{,i}r_{,j}]r_{,l}n_l\}$$

for 3D problems, and by

$$u_{ij}^*(\xi, x) = \frac{-1}{8\pi(1-\nu)G}[(3-4\nu)\delta_{ij}\ln r - r_{,j}r_{,i}],$$

$$p_{ij}^*(\xi, x) = \frac{1}{4\pi(1-\nu)r}\{(1-2\nu)(r_{,i}n_j - r_{,j}n_i) - [(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}]r_{,l}n_l\}$$

for plane strain problems. Here, $b_j(x)$ are body forces; ξ and x indicate source and field points, respectively; n is the unit outward normal vector to the boundary; G is the shear modulus; ν is Poisson's ratio; δ_{ij} is the Kronecker delta; and

$$r_i = x_i(x) - x_i(\xi), \quad r = \sqrt{r_i r_i}, \quad r_{,i} = \frac{\partial r}{\partial x_i(x)} = \frac{r_i}{r}.$$

If the $b_j(x)$ are due to gravity, the body force integral terms in Eq. (1) can be converted into surface integrals to give:

$$\int_\Omega u_{ij}^*(\xi, x)b_j(x)\,d\Omega = \int_\Gamma f_i(\xi, x)\,d\Gamma,$$

where

$$f_i(\xi, x) = \frac{r}{8\pi G}(2\ln(r^{-1}) - 1)\left[b_i r_{,j}n_j - \frac{1}{2(1-\nu)}n_i r_{,j}b_j\right]$$

for plane strain problems and

$$f_i(\xi, x) = \frac{1}{8\pi G}\left[b_i r_{,j}n_j - \frac{1}{2(1-\nu)}n_i r_{,j}b_j\right]$$

for 3D problems. It can easily be seen from the above that the gravitational integral terms do not have singularities whether the source point ξ is in the domain Ω or on the boundary Γ.

Numerical analysis requires discretization of the integral Eq. (1), such that it can be written in a matrix notation as [17]

$$\mathbf{u}(\xi) = \mathbf{GP} - \mathbf{HU} + \mathbf{b}, \tag{2}$$

where $\mathbf{u} = [u_1, u_2]^{\mathrm{T}}$ or $[u_1, u_2, u_3]^{\mathrm{T}}$ for 2- or 3D problems, respectively; **U** and **P** are boundary displacements and tractions, respectively, taken at the boundary nodes; **b** are the known body force integrals; $\mathbf{G} = [\mathbf{g}_{\xi 1}, \ldots, \mathbf{g}_{\xi q}, \ldots, \mathbf{g}_{\xi N}]$ and $\mathbf{H} = [\mathbf{h}_{\xi 1}, \ldots, \mathbf{h}_{\xi q}, \ldots, \mathbf{h}_{\xi N}]$; the sub-matrices $\mathbf{g}_{\xi q}$ and $\mathbf{h}_{\xi q}$ $(q = 1, 2, \ldots, N)$ are both of order $\beta \times \beta$; β is the dimension of domain Ω; and N is the total number of boundary nodes.

When ξ is an interior point, the numerical integrals in matrices **G** and **H** are all non-singular and can be treated simply by Gaussian quadrature. However, when ξ approaches a boundary, singularity effects appear gradually in **G** and **H**. This is why values calculated near the boundary by the BEM are generally worse than values obtained at interior points. Therefore, Eq. (2) should be modified for calculation of values near a boundary. Analysis of the singularities of the matrices **G** and **H** shows that the kernels to be integrated in **G** have logarithmic singularities, which can be eliminated even if ξ is on the boundary. However, the kernel functions in **H** are Cauchy singularities, which have to be treated specially. Therefore, if Eq. (2) is used to calculate displacements near the boundary, the errors will be caused mainly by singular integrals related to the matrix **H**. In particular, as the source point ξ in Eq. (2) approaches any boundary node, say node p, the integral errors in **H** will be contributed mainly by the sub-matrix related to node p, i.e. $\mathbf{h}_{\xi p}$. In this case, the sub-matrix $\mathbf{h}_{\xi p}$ could be determined indirectly by applying the rigid body displacement condition [17] to avoid the errors, i.e. if **I** is the unit matrix,

$$\mathbf{h}_{\xi p} = -\left(\mathbf{I} + \sum_{\substack{q=1\\q\neq p}}^{N} \mathbf{h}_{\xi q}\right). \tag{3}$$

Numerical examples show that this treatment is better than using direct integration for $\mathbf{h}_{\xi p}$.

For problems with unbounded area, an additional term

$$\lim_{\rho\to\infty} \int_{\Gamma_\rho} (\mathbf{U}^*\mathbf{P} - \mathbf{P}^*\mathbf{U})\,\mathrm{d}\Gamma$$

should be added to the right-hand side of Eq. (2), in which $\mathbf{U}^*$ and $\mathbf{P}^*$ are matrices composed of the fundamental solutions u^*_{ij} and p^*_{ij}, see below Eq. (1). It can be verified that this additional term becomes zero at the limit for general load conditions. Therefore, Eq. (2) is still valid for unbounded problems, except that the rigid body displacement condition of Eq. (3) should be modified to

$$\mathbf{h}_{\xi p} = -\sum_{\substack{q=1\\q\neq p}}^{N} \mathbf{h}_{\xi q}. \tag{4}$$

3. Derivative formulation of boundary integral equations

To calculate the stresses at interior or boundary points, derivatives of the displacements u_i should first be obtained. Consideration is now given to the partial differentiation of Eq. (1) with respect to the interior source point ξ along coordinate x_k, i.e.

$$\frac{\partial u_i(\xi)}{\partial x_k(\xi)} = \int_\Gamma \frac{\partial u^*_{ij}(\xi, x)}{\partial x_k(\xi)} p_j(x)\,\mathrm{d}\Gamma - \int_\Gamma \frac{\partial p^*_{ij}(\xi, x)}{\partial x_k(\xi)} u_j(x)\,\mathrm{d}\Gamma + \int_\Omega \frac{\partial u^*_{ij}(\xi, x)}{\partial x_k(\xi)} b_j(x)\,\mathrm{d}\Omega,$$

or, in simplified form,

$$u_{i,k} = \int_\Gamma u^*_{ij,k} p_j\,\mathrm{d}\Gamma - \int_\Gamma p^*_{ij,k} u_j\,\mathrm{d}\Gamma + \int_\Gamma f_{i,k}\,\mathrm{d}\Gamma, \tag{5}$$

where

$$u^*_{ij,k} = \frac{1}{8\alpha\pi G(1-\nu)r^\alpha}[(3-4\nu)r_{,k}\delta_{ij} - r_{,i}\delta_{jk} - r_{,j}\delta_{ki} + \beta r_{,i}r_{,j}r_{,k}],$$

$$p^*_{ij,k} = \frac{1}{4\alpha\pi(1-\nu)r^\beta}\Big\{\beta\frac{\partial r}{\partial n}[(r_{,i}\delta_{jk} + r_{,j}\delta_{ki}) - (1-2\nu)r_{,k}\delta_{ij} - \gamma r_{,i}r_{,j}r_{,k}] + (1-2\nu)(\delta_{jk}n_i + \delta_{ij}n_k - \delta_{ki}n_j) + \beta r_{,j}r_{,i}n_k + \beta(1-2\nu)(r_{,i}r_{,k}n_j - r_{,j}r_{,k}n_i)\Big\},$$

in which $\alpha = 1$, $\beta = 2$ and $\gamma = 4$ for plane strain problems, and $\alpha = 2$, $\beta = 3$ and $\gamma = 5$ for 3D problems. The body force terms $f_{i,k}$ in Eq. (5) is given by:

$$f_{i,k} = \frac{1}{8\pi G}\left\{2\frac{\partial r}{\partial n}b_i r_{,k} - \frac{1}{1-\nu}r_{,i}r_{,k}b_l r_{,l} + [1 - 2\ln(r^{-1})]\left[b_i n_k - \frac{1}{2(1-\nu)}b_k n_i\right]\right\}$$

for plane strain problems, and by

$$f_{i,k} = \frac{1}{8\pi G r}\left[\frac{1}{2(1-\nu)}n_i(b_k - r_{,k}b_l r_{,l}) - b_i(n_k - r_{,k}n_l r_{,l})\right]$$

for 3D problems.

When the source point ξ is on the boundary, integral Eq. (5) should be modified because of singularities. Therefore, consideration is now given to a small indentation consisting of a hemispherical (or semicircular for 2D problems) region with boundary point ξ as its centre and ϵ as its radius. The region intersects boundary Γ on $\Delta\Gamma$ and makes a cutting surface Γ_ϵ, as shown in Fig. 1.

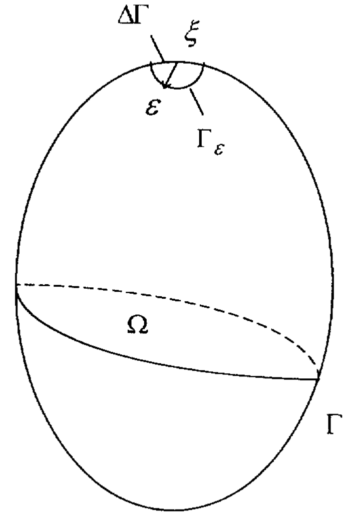

Fig. 1. Sperical (or circular) region around ξ.

By taking integrals along $\Gamma - \Delta\Gamma + \Gamma_\epsilon$, with the point ξ as an out-of-domain point, Eq. (5) becomes:

$$\lim_{\epsilon\to 0}\int_{\Gamma-\Delta\Gamma+\Gamma_\epsilon}(u^*_{ij,k}p_j - p^*_{ij,k}u_j + f_{i,k})\,\mathrm{d}\Gamma = 0 \tag{6}$$

where $p^*_{ij,k} = O(r^{-\beta})$ as $\epsilon \to 0$, and the corresponding integrals are hypersingular, and so are difficult to treat in numerical calculations. To reduce the order of the hypersingular integrals of the second term of Eq. (6) to that of the Cauchy principal value integrals of the first term, Eq. (6) can be written as:

$$\begin{aligned}
&\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}\{p^*_{ij,k}[u_j - u_j(\xi) \\
&\quad - u_{j,l}(\xi)r_l] - u^*_{ij,k}[p_j - \sigma_{jl}(\xi)n_l]\}\,\mathrm{d}\Gamma \\
&\quad + \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}[p^*_{ij,k}u_{j,l}(\xi)r_l - u^*_{ij,k}\sigma_{jl}(\xi)n_l]\,\mathrm{d}\Gamma \\
&\quad + \lim_{\epsilon\to 0}\int_{\Gamma-\Delta\Gamma+\Gamma_\epsilon}p^*_{ij,k}u_j(\xi)\,\mathrm{d}\Gamma - \lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}f_{i,k}\,\mathrm{d}\Gamma \\
&= \lim_{\epsilon\to 0}\int_{\Gamma-\Delta\Gamma}\{u^*_{ij,k}p_j - p^*_{ij,k}[u_j - u_j(\xi)]\}\,\mathrm{d}\Gamma \\
&\quad + \lim_{\epsilon\to 0}\int_{\Gamma-\Delta\Gamma}f_{i,k}\,\mathrm{d}\Gamma,
\end{aligned} \tag{7}$$

where $\sigma_{ij}(\xi)$ is the stress components at ξ. Since $r = \epsilon$ on Γ_ϵ, $\mathrm{d}\Gamma = O(r^\alpha)$, $u_j - u_j(\xi) - u_{j,l}(\xi)r_l = O(r_j^2)$ and $p_j - \sigma_{jl}(\xi)n_l = O(r_j)$. Therefore, the first integral on the left-hand side in Eq. (7) becomes zero as $\epsilon \to 0$. The third integral on the left-hand side is also zero because the surface traction resultants of the fundamental solutions are zero on any enclosed boundary that excludes the singular point ξ. The body force integrals in Eq. (7) gives:

$$\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}f_{i,k}\,\mathrm{d}\Gamma = 0, \quad \lim_{\epsilon\to 0}\int_{\Gamma-\Delta\Gamma}f_{i,k}\,\mathrm{d}\Gamma = \int_\Gamma f_{i,k}\,\mathrm{d}\Gamma.$$

Since $p^*_{ij,k}u_{j,l}(\xi)r_l - u^*_{ij,k}\sigma_{jl}(\xi)n_l = O(r^{-\alpha})$ on Γ_ϵ, the second integral on the left-hand side of Eq. (7) can be written as:

$$\lim_{\epsilon\to 0}\int_{\Gamma_\epsilon}[p^*_{ij,k}u_{j,l}(\xi)r_l - u^*_{ij,k}\sigma_{jl}(\xi)n_l]\mathrm{d}\Gamma \equiv c'_{ikjl}(\xi)u_{j,l}(\xi), \tag{8}$$

where the c'_{ikjl} will be called derivative free coefficients, and are given in the Appendix for the case when the source point ξ is at corners on a boundary. For a smooth boundary, they are simply

$$c'_{ikjl}(\xi) = \frac{1}{2}\delta_{(ik)(jl)} = \begin{cases}\frac{1}{2} & ik = jl \\ 0 & ik \neq jl\end{cases} \tag{9}$$

The only integral in Eq. (7) remaining to be determined is the first integral on the right-hand side, I say, which is now shown to exist. Let

$$u^*_{ij,k} = \frac{1}{r^\alpha}\varphi^*_{ij,k}, \quad p^*_{ij,k} = \frac{1}{r^\beta}\bar{\varphi}^*_{ij,k}.$$

and let I be separated into the two parts

$$I = I_1 + I_2,$$

where

$$\begin{aligned}
I_1 &= \lim_{\epsilon\to 0}\int_{\Gamma-\Delta\Gamma}\frac{1}{r^\alpha}\Big\{\varphi^*_{ij,k}[p_j - p_j(\xi)] \\
&\quad - \frac{1}{r}\bar{\varphi}^*_{ij,k}[u_j - u_j(\xi) - u_{j,l}(\xi)r_l]\Big\}\,\mathrm{d}\Gamma, \\
I_2 &= \lim_{\epsilon\to 0}\int_{\Gamma-\Delta\Gamma}\frac{1}{r^\alpha}[\varphi^*_{ij,k}\sigma_{jl}(\xi)n_l - \bar{\varphi}^*_{ij,k}u_{j,l}(\xi)r_{,l}]\,\mathrm{d}\Gamma.
\end{aligned}$$

Since $p_j - p_j(\xi) = O(r_j)$ and $u_j - u_j(\xi) - u_{j,l}(\xi)r_l = O(r_j^2)$, I_1 is a weakly singular integral, which is absolutely convergent, and equals $-\int_\Gamma f_{i,k}\mathrm{d}\Gamma$ at the limit $\epsilon \to 0$. Furthermore, the integral I_2 also exists and is bounded because the $\sigma_{jl}(\xi)$ and $u_{j,l}(\xi)$ are constants on the boundary $\Gamma - \Delta\Gamma$. To obtain I_2, a uniform stress field with the constant stress components $\sigma_{jl}(\xi)$ is considered. Under this condition, the following identity equation exists

$$\lim_{\epsilon\to 0}\int_{\Gamma-\Delta\Gamma+\Gamma_\epsilon}\frac{1}{r^\alpha}[\bar{\varphi}^*_{ij,k}u_{j,l}(\xi)r_{,l} - \varphi^*_{ij,k}\sigma_{jl}(\xi)n_l]\,\mathrm{d}\Gamma = 0.$$

Comparing the above equation with Eq. (8) gives $I_2 = c'_{ikjl}(\xi)u_{j,l}(\xi)$. Hence, the existence of the integral I, i.e. of I_1 and I_2, has been verified. Therefore, the derivatives of the boundary integral equations when the source point ξ is on the boundary can finally be written, by using Eq. (6) and all of the equations between it and here, as:

$$c'_{ikjl}(\xi)u_{j,l}(\xi) = \int_{\Gamma} u^*_{ij,k}p_j\,\mathrm{d}\Gamma - \int_{\Gamma} p^*_{ij,k}[u_j - u_j(\xi)]\,\mathrm{d}\Gamma + \int_{\Gamma} f_{i,k}\,\mathrm{d}\Gamma. \quad (10)$$

4. Numerical treatment of singular integrals in derivative boundary integral equations

After discretization and numerical interpolations, the derivative boundary integrals of Eq. (10) can be written in matrix notation as:

$$\mathbf{C}'(p)\mathbf{u}'(p) = \mathbf{G}'\mathbf{P} - \mathbf{H}'\mathbf{U} + \mathbf{b}', \quad (11)$$

where, $\mathbf{U}$ and $\mathbf{P}$ are the vectors of surface node displacements and tractions of Eq. (2); $\mathbf{u}' = [u_{1,1}, u_{2,2}, u_{2,1}, u_{1,2}]^{\mathrm{T}}$ for 2D problems ($\beta = 2$), or $\mathbf{u}' = [u_{1,1}, u_{2,2}, u_{3,3}, u_{2,3}, u_{3,2}, u_{3,1}, u_{1,3}, u_{2,1}, u_{1,2}]^{\mathrm{T}}$ for 3D ones ($\beta = 3$); $\mathbf{C}'(p)$ is the singular free coefficient matrix of the derivative BIE with order $\beta^2\times\beta^2$; $\mathbf{G}' = [\mathbf{g}'_{p1}, \ldots \mathbf{g}'_{pq}, \ldots \mathbf{g}'_{pN}]$ and $\mathbf{H}' = [\mathbf{h}'_{p1}, \ldots \mathbf{h}'_{pq}, \ldots \mathbf{h}'_{pN}]$ are derivatives of the matrices $\mathbf{G}$ and $\mathbf{H}$ of Eq. (2), so that sub-matrices $\mathbf{g}'_{pq}$ and $\mathbf{h}'_{pq}$ are all of order $\beta^2\times\beta$; p; indicates the boundary node at which the source point ξ is located, $q(=1, \ldots, N)$ is the number of a certain boundary node; and $\mathbf{b}'$ is the known derivative of the surface vector converted from the body force integral. It can be seen from Eq. (10) that the principal sub-matrix $\mathbf{h}'_{pp}$ in $\mathbf{H}'$, which involves hypersingular integrals, can be expressed as

$$\mathbf{h}'_{pp} = -\sum_{\substack{q=1\\ q\neq p}}^{N} \mathbf{h}'_{pq}, \quad (12)$$

which is simply the rigid body displacement condition often used in boundary element analysis. Thus, the direct numerical integrals of the hypersingular terms in Eq. (11) can be avoided.

Matrix $\mathbf{C}'$ in Eq. (11) can also be obtained indirectly by applying a set of unit derivative displacement fields to Eq. (11). By observing the linear transformation relationships that exist between the derivatives of displacements and the surface tractions, the principal block $\mathbf{g}'_{pp}\mathbf{P}'_p$ in $\mathbf{G}'\mathbf{P}$ can then be absorbed into $\mathbf{C}'\mathbf{u}'(p)$, which avoids the direct numerical calculations of the Cauchy principal integrals in $\mathbf{G}'$. In this way, the derivative free coefficients can be expressed as:

$$\mathbf{C}' = \sum_{\substack{q=1\\ q\neq p}}^{N} (\mathbf{g}'_{pq}\mathbf{P}'_q - \mathbf{h}'_{pq}\mathbf{U}'_q), \quad (13)$$

where $\mathbf{P}'_q$ and $\mathbf{U}'_q$ are the traction and non-rigid movement displacement matrices at the boundary node q, respectively, under the action of a set of unit displacement derivative fields, and are given by:

$$\mathbf{P}'_q = 2G\begin{bmatrix} \frac{1-v}{1-2v}n_1 & \frac{v}{1-2v}n_1 & \frac{1}{2}n_2 & \frac{1}{2}n_2 \\ \frac{v}{1-2v}n_2 & \frac{1-v}{1-2v}n_2 & \frac{1}{2}n_1 & \frac{1}{2}n_1 \end{bmatrix}_q,$$

$$\mathbf{U}'_q = \begin{bmatrix} r_1 & 0 & 0 & r_2 \\ 0 & r_2 & r_1 & 0 \end{bmatrix}_q$$

for plane stain problems, and by

$$\mathbf{P}'_q = 2G \begin{bmatrix} \frac{1-v}{1-2v}n_1 & \frac{v}{1-2v}n_1 & \frac{v}{1-2v}n_1 & 0 & 0 & \frac{1}{2}n_3 & \frac{1}{2}n_3 & \frac{1}{2}n_2 & \frac{1}{2}n_2 \\ \frac{v}{1-2v}n_2 & \frac{1-v}{1-2v}n_2 & \frac{v}{1-2v}n_2 & \frac{1}{2}n_3 & \frac{1}{2}n_3 & 0 & 0 & \frac{1}{2}n_1 & \frac{1}{2}n_1 \\ \frac{v}{1-2v}n_3 & \frac{v}{1-2v}n_2 & \frac{1-v}{1-2v}n_3 & \frac{1}{2}n_2 & \frac{1}{2}n_2 & \frac{1}{2}n_1 & \frac{1}{2}n_1 & 0 & 0 \end{bmatrix}_q,$$

$$\mathbf{U}'_q = \begin{bmatrix} r_1 & 0 & 0 & 0 & 0 & 0 & r_3 & 0 & r_2 \\ 0 & r_2 & 0 & r_3 & 0 & 0 & 0 & r_1 & 0 \\ 0 & 0 & r_3 & 0 & r_2 & r_1 & 0 & 0 & 0 \end{bmatrix}_q$$

for 3D problems, where $r_i = x_i(q) - x_i(h)$, and q is the boundary node at which the field point x is located.

To obtain the off-principal sub-matrices $\mathbf{g}'_{pq}$ and $\mathbf{h}'_{pq}$ in $\mathbf{G}'$ and $\mathbf{H}'$, singular integrals must be evaluated numerically. To do so, a polar coordinate transformation technique should be introduced among the boundary elements around the singular point p. For 3D problems, for instance, the general distorted quadrilateral boundary elements connected to the singular point p can be firstly transformed into simple square elements with unit side length, and then should be further imaginarily divided into several triangular elements, as shown in Fig. 2. Each triangular element, say $p12$ in Fig. 2, in the Cartesian coordinate system (η, ζ) can then be transformed into a sector in a polar coordinate system (ρ, θ) by using proper transformation relationships, i.e. $\eta = \rho$ and $\zeta = \rho\theta$ for triangle $p12$. Since

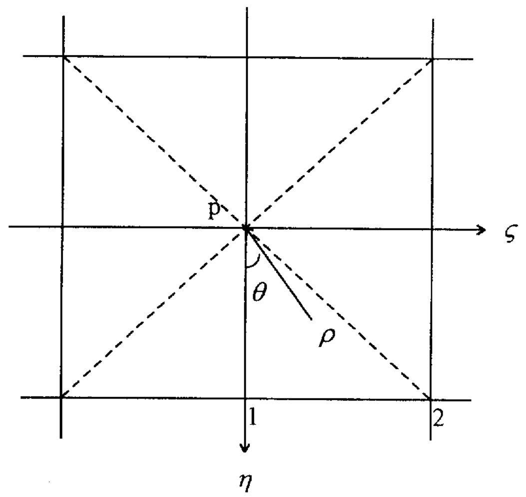

Fig. 2. Transformations of the boundary elements connected to singularity point p.

both η and ζ vary between 0 and 1, the variables ρ and θ must also vary in this range. With this coordinate transformation, the corresponding Jacobian determinants, among the elements connected to node p, are $|J| = \partial(\eta,\zeta)/\partial(\rho,\theta) = \rho = O(r)$, which cancel the first-order singularities which previously existed in the boundary element integrals connected to the singular point p.

With the general treatments given above, all the singular integrals in the integral Eq. (11) can be evaluated effectively, and the errors caused by direct numerical calculation of singular integrals can be avoided. This is particularly significant when calculating the derivative BIEs, which contain integrals of higher order singularity.

For problems with infinite boundaries, an additional term should be added in Eq. (11), as indicated between Eqs. (3) and (4) in the second section. The limit of this additional term is zero under general load or rigid movement fields, but is equal to the unit displacement derivative under a unit displacement derivative field. Therefore, Eqs. (11) and (12) are still valid for infinite problems, with Eq. (13) modified to become:

$$\mathbf{C}' = \sum_{\substack{q=1\\q\neq p}}^{N}(\mathbf{g}'_{pq}\mathbf{P}'_q - \mathbf{h}'_{pq}\mathbf{U}'_q) + \mathbf{I}, \tag{14}$$

5. Numerical treatment of displacement derivatives at a point close to a boundary

When the source point ξ is in the domain, the discrete derivative boundary integral Eq. (11) can be reduced to

$$\mathbf{u}'(\xi) = \mathbf{G}'\mathbf{P} - \mathbf{H}'\mathbf{U} + \mathbf{b}', \tag{15}$$

in which all numerical integrals are non-singular and can be treated simply by ordinary Gaussian quadrature. However, as ξ approaches the boundary, the integral kernels in Eq. (15) will fluctuate violently in the boundary elements close to point ξ, demonstrating a significant singularity effect. In this case, the direct use of Eq. (15) will cause a large calculation error. To reduce the singularity effect, Eq. (15) must be modified according to the previous section to give:

$$\mathbf{C}'(\xi)\mathbf{u}'(\xi) = \mathbf{G}'\mathbf{P} - \mathbf{H}'\mathbf{U} + \mathbf{b}', \tag{16}$$

which has the same form as Eq. (11). However, here ξ is an interior source point close to the boundary and $\mathbf{C}'(\xi)$ could be called a singularity effect matrix, which is introduced to adjust the deviation of the coefficient matrices, i.e. $\mathbf{G}'$ and $\mathbf{H}'$, caused by the singularity effect of the boundary layer. Matrix $\mathbf{C}'(\xi)$ can be calculated from Eq. (13), but with its point p replaced by the interior point ξ, and with the term with subscript $q = \xi$ included in the summation, i.e.

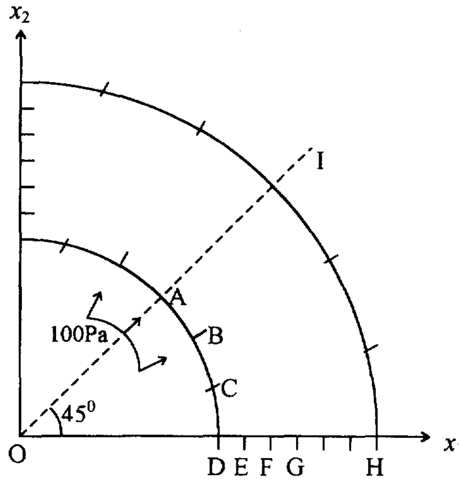

Fig. 3. Thick-walled cylinder subjected to internal pressure.

Table 1
Boundary node stresses for thick-walled cylinder of Example 1 calculated by the method presented, with the exact results of Ref. [18] shown in brackets

	Solution		
node	σ_{11}	σ_{22}	σ_{12}
A	19.472 (19.048)	19.472 (19.048)	−118.830 (−119.048)
B	−39.831 (−40.476)	78.905 (78.571)	−102.960 (−103.098)
C	−83.325 (−84.051)	123.590 (122.146)	−59.084 (−59.524)
D	−100.430 (−100.000)	137.150 (138.095)	−1.599 (0.000)
E	−57.623 (−57.143)	94.577 (95.238)	−0.441 (0.000)
F	−33.967 (−33.862)	71.491 (71.958)	−0.081 (0.000)
G	−19.895 (−19.825)	57.539 (57.920)	−0.013 (0.000)
H	0.048 (0.000)	38.013 (38.095)	0.174 (0.000)
I	18.918 (19.048)	18.918 (19.048)	−18.938 (−19.048)

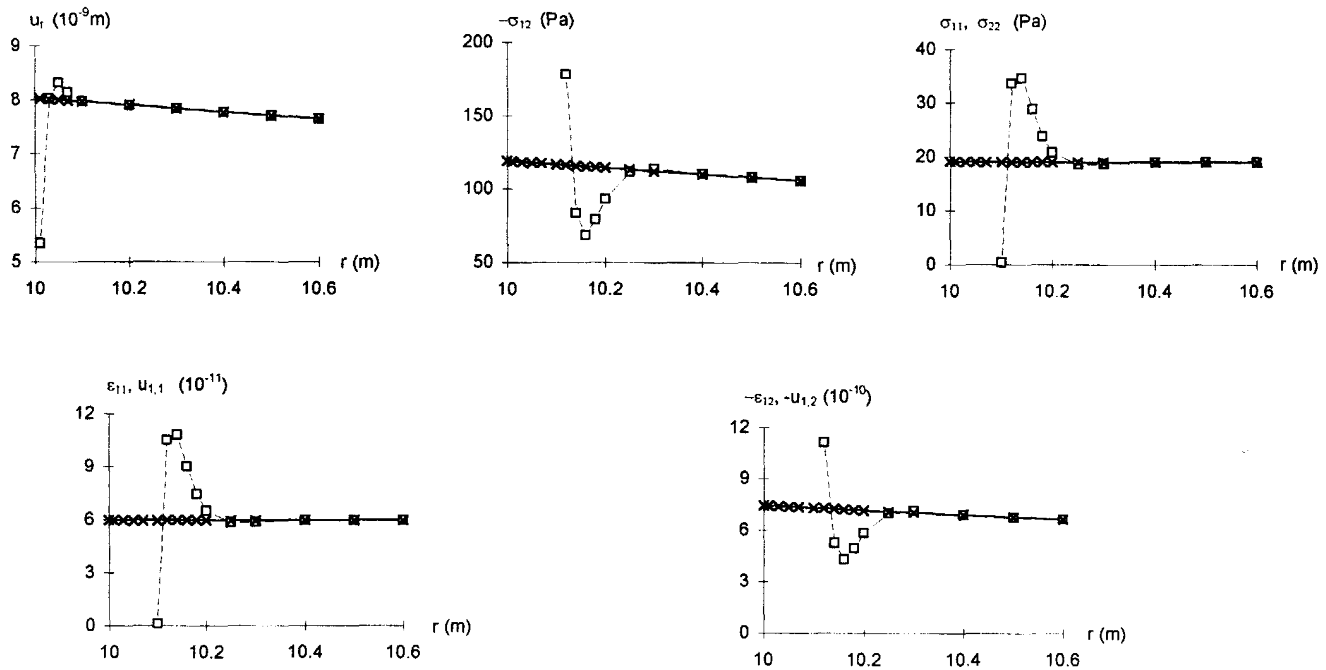

Fig. 4. Radial displacement, shear stress $-\sigma_{12}$, normal stress σ_{11}, normal strain ϵ_{11} and shear strain $-\epsilon_{12}$, on and near inner boundary along line *OI* of Example 1, see Fig. 3. Key: —exact; □ calculated directly; × from method presented.

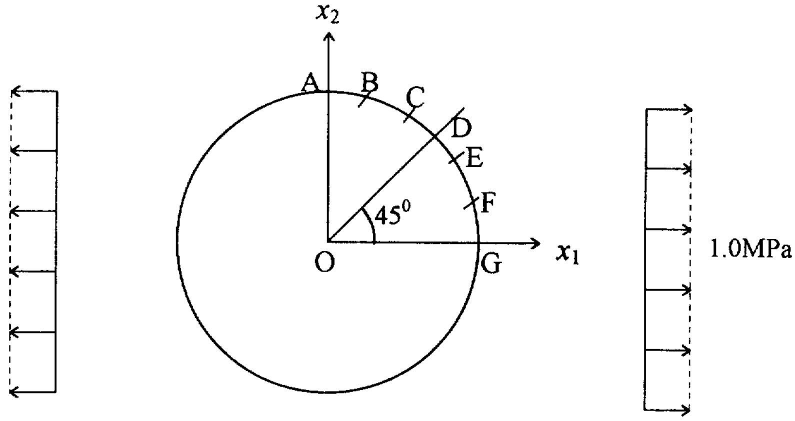

Fig. 5. An infinite plate with a hole subjected to uniform tensile stress at infinity, showing only those nodes at which results are given.

$$\mathbf{C}'(\xi) = \sum_{q=1}^{N} (\mathbf{g}'_{\xi q} \mathbf{P}'_q - \mathbf{h}'_{\xi q} \mathbf{U}'_q). \tag{17}$$

Therefore, the corresponding sub-matrix in $\mathbf{G'P}$ of Eq. (16) no longer needs to be separated. Other techniques for eliminating singular effects used in the previous section should still be used here.

For infinite problems, the above treatments are also available, except that Eq. (17) should be modified to:

$$\mathbf{C}'(\xi) = \sum_{q=1}^{N} (\mathbf{g}'_{\xi q} \mathbf{P}'_q - \mathbf{h}'_{\xi q} \mathbf{U}'_q) + \mathbf{I}. \tag{18}$$

As soon as the derivatives of the displacements at any point in the domain or on its boundary are obtained by the method suggested above, the strains at any point can be calculated according to the displacement–strain relationships

$$\epsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}), \tag{19}$$

and stresses can then be obtained from the stress–strain relationships

$$\sigma_{ij} = G(u_{i,j} + u_{j,i}) + \frac{2Gv}{1-2v} u_{k,k}\delta_{ij}. \tag{20}$$

6. Numerical examples

Two examples taken from Ref. [18] are given to show the validity of the treatments given above, for which the elastic modulus E = 200 GPa and ν = 0.25. The cubic spline interpolation is introduced on the boundary elements and the impractical numbers in the examples (very large size with low loads) are unimportant because the problems are both linear.

6.1. Example 1

A long thick-walled circular cylinder has an inner radius of 10 m, and an outer radius of 25 m. It undergoes plane strain deformation due to an internal pressure, 100 Pa. Fig. 3 shows the problem and its discretization.

Table 2
Hole boundary node stresses for infinite plate with hole of Example 2 calculated by the method presented, with the exact results of Ref. [18] shown in brackets

	Solution		
node	σ_{11}	σ_{22}	σ_{12}
A	2.997 (3.000)	0.004 (0.000)	0.000 (0.000)
B	2.549 (2.549)	0.185 (0.183)	−0.680 (−0.683)
C	1.503 (1.500)	0.501 (0.500)	−0.862 (−0.866)
D	0.507 (0.500)	0.500 (0.500)	−0.497 (−0.500)
E	0.009 (0.000)	0.001 (0.000)	0.002 (0.000)
F	−0.039 (−0.049)	−0.679 (−0.683)	0.184 (0.183)
G	0.010 (0.000)	−0.997 (−1.000)	0.000 (0.000)

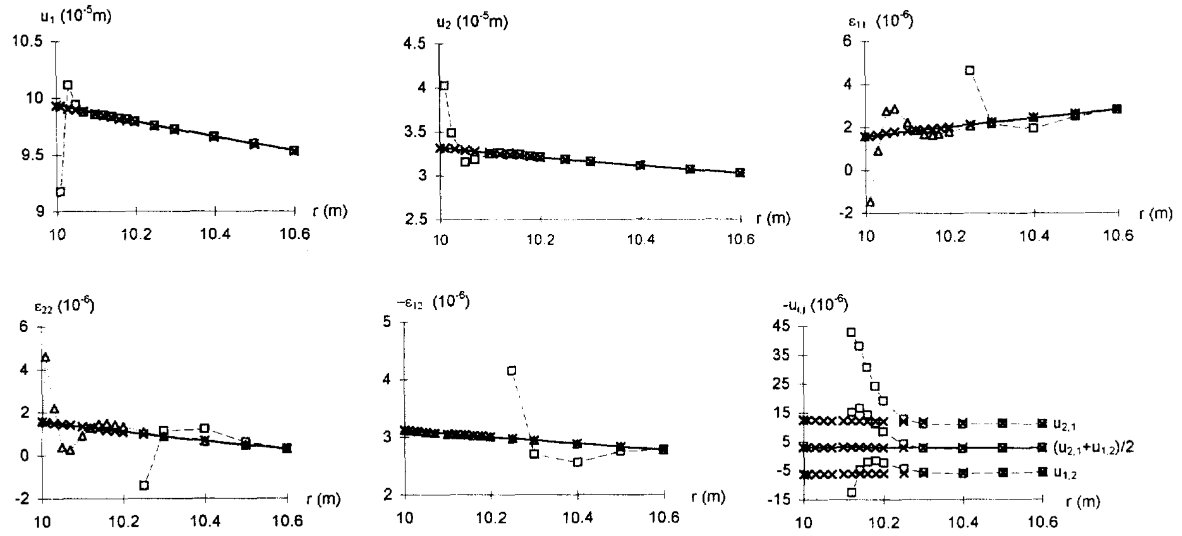

Fig. 6. Displacements u_1 ans u_2, strains ϵ_{11}, ϵ_{22} and $-\epsilon_{12}$ and $-u_{i,j}$ on and near the boundary of the hole, measured along line OD of Example 2, see Fig. 5. Key: — exact [18]; □ calculated directly; △ from Ref. [10]; and × method presented.

Table 1 gives stresses calculated by the modified method given above at the boundary points shown in Fig. 3. The results compare well with the exact ones [18] also given in Table 1. Fig. 4 shows a set of comparable results for displacements, stresses and strains near and on the inner boundary along the line OI shown in Fig. 3, for which, because of symmetry, $\sigma_{11} = \sigma_{22}$ and $\epsilon_{12} = \frac{1}{2}(u_{1,2} + u_{2,1}) = u_{1,2}$.

It can be seen that there is excellent agreement between the results obtained by the proposed method and the exact ones, whereas the results obtained by the conventional BEM treatment, i.e. by direct calculation with inner point formulations without any modification, are very inaccurate as the point considered gets close to the inner boundary. It can also be seen that the strains and stresses, which involve derivatives of the displacements, have much larger errors than do the displacements.

6.2. Example 2

An infinite plate with a hole of radius 10 m is subjected to a uniform tensile stress of 1.0 MPa at infinity, see Fig. 5.

Table 2 gives the stresses calculated by the method presented and by the exact method [18], which can be seen to be in good agreement. Fig. 6 shows a set of comparable results given by four different methods. Again, it can be seen that there is excellent agreement between the results of the proposed method and the exact ones, even close to the boundary considered (indeed, in the limit the proposed method gives good results even at the boundary itself). Since stresses and strains both involve first derivatives of the displacements and can be expressed in terms of each other by the constitutive relations, for simplicity results for stresses are not plotted. Three of the graphs of Fig. 6 give the results calculated by the method of Ref. [10], which can be seen to have considerable errors for ϵ_{11} and ϵ_{22} close to the boundary. The curve for $-\epsilon_{12}$ can be obtained as $\frac{1}{2}(u_{1,2} + u_{2,1})$. The two parts of the shear strain, $u_{1,2}$ and $u_{2,1}$, can be separated by the present treatment and this may be quite important when calculating the J integral of fracture mechanics.

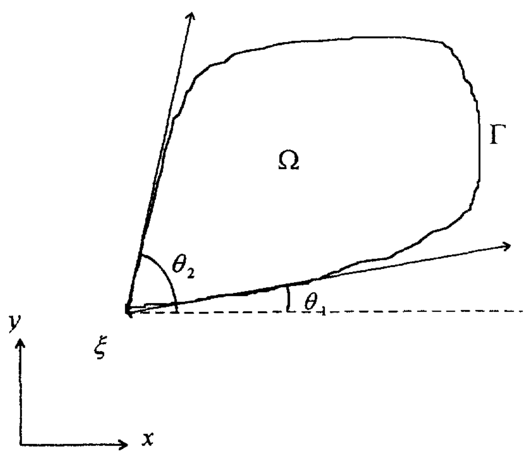

Fig. 7. Tangents at a corner node.

Table 3
Parameters d_{ikjl} for derivative free term coefficients

ik/jl	11	22	21	12
11	$2\theta(1-\nu)+\frac{3-7\nu+4\nu^2}{2(1-2\nu)}\sin 2\theta+\frac{1}{4}\sin 4\theta$	$\frac{1+\nu-4\nu^2}{2(1-2\nu)}\sin 2\theta-\frac{1}{4}\sin 4\theta$	$\frac{7-4\nu}{2}\sin^2\theta-2\sin^4\theta$	$\frac{5-4\nu}{2}\sin^2\theta-2\sin^4\theta$
22	$-\frac{1+\nu-4\nu^2}{2(1-2\nu)}\sin 2\theta-\frac{1}{4}\sin 4\theta$	$2\theta(1-\nu)+\frac{3-7\nu+4\nu^2}{2(1-2\nu)}\sin 2\theta+\frac{1}{4}\sin 4\theta$	$-\frac{3+4\nu}{2}\sin^2\theta+2\sin^4\theta$	$-\frac{1+4\nu}{2}\sin^2\theta+2\sin^4\theta$
21	$\frac{2-\nu-4\nu^2}{1-2\nu}\sin^2\theta-2\sin^4\theta$	$\frac{2-5\nu+4\nu^2}{1-2\nu}\sin^2\theta+2\sin^4\theta$	$2\theta(1-\nu)+\frac{3-4\nu}{4}\sin 2\theta-\frac{1}{4}\sin 4\theta$	$\frac{5-4\nu}{2}\sin 2\theta-\frac{1}{4}\sin 4\theta$
12	$\frac{6-13\nu+4\nu^2}{1-2\nu}\sin^2\theta-2\sin^4\theta$	$\frac{2+7\nu-4\nu^2}{1-2\nu}\sin^2\theta+2\sin^4\theta$	$\frac{-5+4\nu}{2}\sin 2\theta-\frac{1}{4}\sin 4\theta$	$2\theta(1-\nu)-\frac{3-4\nu}{4}\sin 2\theta-\frac{1}{4}\sin 4\theta$

7. Conclusions

The present paper has given a derivation of the derivative formulations of displacement boundary integral equations, including the corresponding treatments needed to avoid the hypersingular integrals usually present in the derivative boundary integral equations. The resulting method is an effective way to calculate displacements, strains and stresses on and near boundaries. Hence, problems often encountered in the conventional BEM calculations are overcome. Numerical examples show that this modified method has a higher computational accuracy when compared with other numerical methods for calculating the variables on and near the boundary considered. Therefore, the method proposed in the paper provides an important modification to the conventional BEM. It makes the calculation of variables on and near boundaries both easy and accurate, which is of significance for practical applications of the BEM.

Acknowledgements

The authors are grateful for support provided by the National Natural Science Foundation of China (No. 195 72061) and the Cardiff Advanced Chinese Engineering Centre of Cardiff University, UK.

Appendix A

A.1. Appendix: derivative free term coefficients c'_{ikjl}

In this Appendix, the free term coefficients of the derivative boundary integral equations at corners are given for plane strain problems. Let ξ be a general corner node on a boundary and let θ_1 and θ_2 be the angles between the x-axis and tangent vectors at the corner, respectively, as shown in Fig. 7. The considered free term coefficients can be expressed as:

$$c'_{ikjl}=\frac{1}{4\pi(1-\nu)}d'_{ikjl}\Big|_{\theta_1}^{\theta_2},$$

in which the d'_{ikjl} are functions of θ given in Table 3.

References

[1] Ghosh N, Rajiyah H, Mukherjee S. A new boundary element method formulation for linear elasticity. J. Appl. Mech. 1986;53:69–76.

[2] Hong H, Chen J. Derivations of integral equations of elasticity. J. Eng. Mech. 1988;6:1028–44.

[3] Gray LJ. Boundary element method for regions with thin internal cavities. Report, IBM Bergen Scientific Centre, 1987.

[4] Rudolphi TJ, Krishnasamy G, Schmerr LW, Rizzo FJ. On the use of strongly singular integral equations for crack problems. In: Brebbia CA, editors. Boundary elements X. Southampton: Computational Mechanics Publications, 1988.

[5] Gray LJ, Mattha LF, Ingraffea AR. Hypersingular integrals in boundary element fracture analysis. Int. J. Numer. Meth. Engng 1990;29:1135–58.

[6] Portela A, Aliabadi MH, Rooke DP. The dual boundary element method: effective implementation for crack problems. Int. J. Numer. Meth. Engng 1992;33:1269–87.

[7] Chen WH, Chen TC. An efficient dual boundary element method technique for a two-dimensional fracture problem with multiple cracks. Int. J. Numer. Meth. Engng 1995;38:1739–56.

[8] Zhang GH, Lou ZW. Formulation for stress calculation of boundary layer point in BEM. In: Tanaka M, Du Q, editors. Boundary element methods. New York: Pergamon Press, 1990:73–82.

[9] Wang Y, Qiao R, Zhu H. Particular solution method adjusted by singularity for calculating boundary stress. In: Du Q, Tanaka M, editors. Proceedings of the Fourth China-Japan Symposium on BEM, 1991:91–102.

[10] Wang YC, Li HQ, Chen HB, Wu Y. Particular solution method to adjust singularity for the calculation of stress and displacement at arbitrary point. Acta Mech. Sinica 1994;26:222–32 (in Chinese).

[11] Chen HB, Wang YC, Lu P. Stress rate integral equations of elastoplasticity. Acta Mech. Sinica 1996;12:55–64 (English Series).

[12] Guiggiani M, Casalini P. Direct computation of Cauchy principal value integrals in advanced boundary elements. Int. J. Numer. Meth. Engng 1987;24:1711–20.

[13] Guiggiani M, Gigante A. A general algorithm for multidimensional Cauchy principal value integrals in the boundary element method. J. Appl. Mech 1990;57:906–15.

[14] Guiggiani M, Krishnasamy G, Rudolphi TJ, Rizzo EJ. General algorithm for the numerical solution of hypersingular boundary integral equation. J. Appl. Mech 1992;59:604–14.

[15] Guiggiani M. Computing principal-value integrals in 3-D BEM for time-hamonic elastodynamics—a direct approach. Comm. Appl. Num. Meth. 1992;8:141–9.

[16] Guiggiani M. Hypersingular formulation for boundary stress evaluation. Engng Anal. Bound. Elem. 1994;13:169–79.

[17] Brebbia CA, Telles JCF, Wrobel LC. Boundary element techniques. Berlin: Springer, 1984.

[18] Du Q, Yu S, Yao Z. Theory of elasticity. Beijing: Science Press (in Chinese), 1986.