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Development of the Fast Multipole Boundary Element Method for Acoustic Wave Problems

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Abstract In this chapter, we review some recent development of the fast multipole boundary element method (BEM) for solving large-scale acoustic wave problems in both 2-D and 3-D domains. First, we review the boundary integral equation (BIE) formulations for acoustic wave problems. The Burton-Miller BIE formulation is emphasized, which uses a linear combination of the conventional BIE and hypersingular BIE. Next, the fast multipole formulations for solving the BEM equations are provided for both 2-D and 3-D problems. Several numerical examples are presented to demonstrate the effectiveness and efficiency of the developed fast multipole BEM for solving large-scale acoustic wave problems, including scattering and radiation problems, and half-space problems.

1 Introduction

Solving acoustic wave problems is one of the most important applications of the BEM, which can be used in analyzing sound fields for noise controls in automobiles, airplanes, and many other consumer products. Acoustic waves often exist in an infinite medium outside a structure which is in vibration (a radiation problem) or impinged upon by an incident wave (a scattering problem). With the BEM, only the boundary of the structure needs to be discretized. In addition, the boundary conditions at infinity can be taken into account analytically in the boundary integral equation formulations and thus these conditions can be satisfied exactly. The governing equation for acoustic wave problems is the Helmholtz equation, which has been solved using the BIE/BEM for more than four decades (see, e.g., some of the early work in Schenck 1968; Burton and Miller 1971; Ursell 1973; Kleinman and Roach 1974; Jones 1974; Meyer et al. 1978; Seybert et al. 1985; Kress 1985; Seybert and Rengarajan 1987; Cunefare and Koopmann 1989, 1991; Everstine and Henderson 1990; Martinez 1991; Cunefare and Koopmann 1991)). Especially, the

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work by Burton and Miller (1971) has been regarded as a classical one which provides a very elegant way to overcome the so called fictitious frequency difficulties existing in the conventional BIE for exterior acoustic wave problems. The Burton and Miller's BIE formulation has been used by many others in their research on the BEM for acoustic problems (e.g., Krishnasamy et al. 1990; Amini 1990; Wu et al. 1991; Liu and Rizzo 1992; Liu 1992; Yang 1997; Liu and Chen 1999).

The fast multipole method (FMM) developed by Rokhlin and Greengard (Rokhlin 1985; Greengard and Rokhlin 1987; Greengard 1988) has been extended to solving Helmholtz equation for quite some time (see, e.g., Rokhlin 1990; Rokhlin 1993; Coifman et al. 1993; Engheta et al. 1992; Lu and Chew 1993; Wagner and Chew 1994; Epton and Dembart 1995; Koc and Chew 1998; Gyure and Stalzer 1998; Greengard *et al.* 1998; Tournour and Atalla 1999; Gumerov and Duraiswami 2003; Darve and Havé 2004; Fischer et al. 2004; Chen and Chen 2004; Shen and Liu 2007). Most of these works are good for solving acoustic wave problems at either low frequencies or high frequencies. For example, Greengard *et al.*, (1998) suggested a diagonal translation in the FMM for low frequency range. Rokhlin (1993) and Lu and Chew (1993) proposed diagonal form of the translation matrices for high frequency range for the Helmholtz equation. Wagner and Chew (1994) used ray propagation approach to further accelerate the FMM for high frequency range. A new adaptive fast multipole BEM for 3-D acoustic wave problems was given in Shen and Liu (2007) and large acoustic models with degrees of freedom (in complex variables) above 200,000 have been solved successfully on laptop PCs (Shen and Liu 2007).

2 Basic Equations for Acoustic Wave Problems

Consider the Helmholtz equation governing time-harmonic acoustic wave fields:

$$\nabla^2\phi + k^2\phi = 0 \qquad \forall \mathbf{x} \in E, \tag{1}$$

where $\phi = \phi(\mathbf{x}, \omega)$ is the complex acoustic pressure, $k = \omega/c$ the wavenumber, ω the circular frequency, c the speed of sound, and $\nabla^2(\,) = \partial^2(\,)/\partial x_k \partial x_k = (\,)_{,kk}$. The acoustic domain E can be an *infinite* domain exterior to a body V (Fig. 1) or a *finite* domain interior to a closed surface.

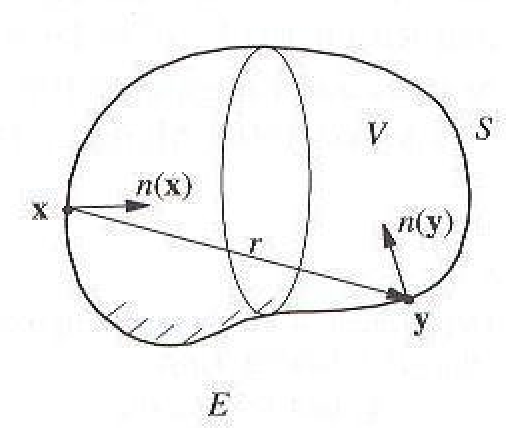


Fig. 1 The acoustic medium E, body V and boundary S

The boundary conditions for acoustic wave problems can be classified as follows:

$$(a)\ Pressure\ is\ given: \quad \phi = \overline{\phi}, \qquad \forall \mathbf{x} \in S; \tag{2}$$

$$(b)\ Velocity\ is\ given: q \equiv \frac{\partial \phi}{\partial n} = \overline{q} = i\omega\rho v_n, \quad \forall \mathbf{x} \in S; \tag{3}$$

$$(c)\ Impedance\ is\ given: \phi = Z v_n, \quad \forall \mathbf{x} \in S; \tag{4}$$

in which ρ is the mass density, v_n the normal velocity, Z the specific impedance, and the barred quantities indicate given values. For exterior (infinite domain) acoustic wave problems, the field at infinity must also satisfy the Sommerfeld radiation condition.

For 2-D problems, the fundamental solution is given by:

$$G(\mathbf{x}, \mathbf{y}, \omega) = \frac{i}{4} H_0^{(1)}(kr), \tag{5}$$

$$F(\mathbf{x}, \mathbf{y}, \omega) \equiv \frac{\partial G(\mathbf{x}, \mathbf{y}, \omega)}{\partial n(\mathbf{y})} = -\frac{ik}{4} H_1^{(1)}(kr) r_{,l} n_l(\mathbf{y}), \tag{6}$$

where r is the distance between $\mathbf{x}$ and $\mathbf{y}$, and $H_n^{(1)}()$ denotes the Hankel function of the first kind (Abramowitz and Stegun 1972). For 3-D problems, the fundamental solution is given by:

$$G(\mathbf{x}, \mathbf{y}, \omega) = \frac{1}{4\pi r} e^{ikr}, \tag{7}$$

$$F(\mathbf{x}, \mathbf{y}, \omega) \equiv \frac{\partial G(\mathbf{x}, \mathbf{y}, \omega)}{\partial n(\mathbf{y})} = \frac{1}{4\pi r^2}(ikr - 1) r_{,j} n_j(\mathbf{y}) e^{ikr}. \tag{8}$$

3 BIE Formulations

The solution of the Helmholtz equation is given by the representation integral:

$$\phi(\mathbf{x}) = \int_S [G(\mathbf{x}, \mathbf{y}, \omega) q(\mathbf{y}) - F(\mathbf{x}, \mathbf{y}, \omega)\phi(\mathbf{y})]\, dS(\mathbf{y}) + \phi^I(\mathbf{x}), \quad \forall \mathbf{x} \in E, \tag{9}$$

where $q = \partial\phi/\partial n$ and $\phi^I(\mathbf{x})$ is an incident wave. Equation (9) is the representation integral of the solution ϕ inside the domain E for Helmholtz equation (1) for both exterior and interior domain problems. Once the values of both ϕ and q are known on S, Eq. (9) can be applied to calculate ϕ everywhere in E, if needed.

Let the source point $\mathbf{x}$ approach the boundary S. We obtain the following conventional boundary integral equation (CBIE) for acoustic wave problems:

$$c(\mathbf{x})\phi(\mathbf{x}) = \int_S [G(\mathbf{x}, \mathbf{y}, \omega)q(\mathbf{y}) - F(\mathbf{x}, \mathbf{y}, \omega)\phi(\mathbf{y})]\, dS(\mathbf{y}) + \phi^I(\mathbf{x}), \ \forall \mathbf{x} \in S, \quad (10)$$

where the constant $c(\mathbf{x}) = 1/2$, if S is smooth around $\mathbf{x}$. The integral with the G kernel is a weakly-singular integral, while the one with the F kernel is a strongly-singular (CPV) integral. It is well known that this CBIE has a major defect for exterior domain problems, that is, it has nonunique solutions at a set of fictitious eigenfrequencies associated with the resonate frequencies of the corresponding interior problems (Burton and Miller 1971). This difficulty is referred to as the *fictitious eigenfrequency difficulty*. A remedy to this problem is to use the normal derivative BIE in conjunction with this CBIE. Taking the derivative of integral representation (9) with respect to the normal at the point $\mathbf{x}$ and letting $\mathbf{x}$ approach S, we obtain the following hypersingular boundary integral equation (HBIE):

$$\tilde{c}(\mathbf{x})q(\mathbf{x}) = \int_S [K(\mathbf{x}, \mathbf{y}, \omega)q(\mathbf{y}) - H(\mathbf{x}, \mathbf{y}, \omega)\phi(\mathbf{y})]\, dS(\mathbf{y}) + q^I(\mathbf{x}), \ \forall \mathbf{x} \in S \quad (11)$$

where $\tilde{c}(\mathbf{x}) = 1/2$ if S is smooth. For 2-D problems, the two new kernels are:

$$K(\mathbf{x}, \mathbf{y}, \omega) \equiv \frac{\partial G(\mathbf{x}, \mathbf{y}, \omega)}{\partial n(\mathbf{x})} = \frac{ik}{4} H_1^{(1)}(kr) r_{,j}\, n_j(\mathbf{x}), \quad (12)$$

$$\begin{aligned} H(\mathbf{x}, \mathbf{y}, \omega) \equiv \frac{\partial F(\mathbf{x}, \mathbf{y}, \omega)}{\partial n(\mathbf{x})} &= \frac{ik}{4r} H_1^{(1)}(kr) n_j(\mathbf{x}) n_j(\mathbf{y}) \\ &\quad - \frac{ik^2}{4} H_2^{(1)}(kr) r_{,j}\, n_j(\mathbf{x}) r_{,l}\, n_l(\mathbf{y}). \end{aligned} \quad (13)$$

For 3-D problems, the two new kernels are:

$$K(\mathbf{x}, \mathbf{y}, \omega) \equiv \frac{\partial G(\mathbf{x}, \mathbf{y}, \omega)}{\partial n(\mathbf{x})} = -\frac{1}{4\pi r^2}(ikr - 1) r_{,j}\, n_j(\mathbf{x}) e^{ikr}, \quad (14)$$

$$\begin{aligned} H(\mathbf{x}, \mathbf{y}, \omega) \equiv \frac{\partial F(\mathbf{x}, \mathbf{y}, \omega)}{\partial n(\mathbf{x})} &= \frac{1}{4\pi r^3} \{(1 - ikr) n_j(\mathbf{y}) \\ &\quad + \left[k^2 r^2 - 3(1 - ikr)\right] r_{,j}\, r_{,l}\, n_l(\mathbf{y})\}\, n_j(\mathbf{x}) e^{ikr}, \end{aligned} \quad (15)$$

In HBIE (11), the integral with the kernel K is a strongly-singular (CPV) integral, while the one with the H kernel is a hypersingular (HFP) integral. For exterior acoustic wave problems, a dual BIE (CHBIE, or composite BIE (Liu and Rizzo 1992)) formulation using a linear combination of the CBIE (10) and HBIE (11) can be written as:

$$\text{CBIE} + \beta\text{HBIE} = 0, \tag{16}$$

where β is the coupling constant. This dual BIE formulation is called the Burton-Miller formulation (Burton and Miller 1971) for acoustic wave problems and has been shown by Burton and Miller to yield unique solutions at all frequencies, if β is a complex number (which, for example, can be chosen as $\beta = i/k$ (Kress 1985)).

CBIE (10) and HBIE (11) contain singular integrals that are difficult to evaluate analytically even on constant elements. Numerical integration can be employed to compute all the singular integrals with proper care, but it has been found not very efficient computationally with higher-order elements. As in all the other problems using the BIE/BEM, the best approach in such cases is to use the weakly-singular forms of these BIEs, which are obtained analytically and do not introduce any approximations. The weakly-ssingular forms of the BIEs for acoustic wave problems can be found in Liu and Rizzo (1992) and Liu and Chen (1999).

The discretized equations of the CBIE, HBIE, or the Burton-Miller's BIE formulation, in either singular or weakly-singular forms, can be written as:

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1N} \\ a_{21} & a_{22} & \cdots & a_{2N} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N1} & a_{N2} & \cdots & a_{NN} \end{bmatrix} \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_N \end{Bmatrix} = \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{Bmatrix}, \quad \text{or} \quad \mathbf{A}\boldsymbol{\lambda} = \mathbf{b}, \tag{17}$$

where $\mathbf{A}$ is the system matrix, $\boldsymbol{\lambda}$ the vector of unknown boundary variables at the nodes, $\mathbf{b}$ the known vector, and N the number of nodes on the boundary. For acoustic wave problems, this system of equations is in complex numbers, that is, all the coefficients and variables are complex numbers and thus the memory requirement is four times as large as its counterpart in potential problems. As a result of this, only small models have been solved using the conventional BEM.

4 Fast Multipole Formulation for 2-D Acoustic Wave Problems

We first discuss the fast multipole BEM formulation for 2-D acoustic wave problems (Nishimura 2002). Iterative solver GMRES will be used to solve the system of equations (17) in which the far field contributions will be evaluated using the fast multipole method.

The 2-D formulation is based on Graf's equation (Abramowitz and Stegun 1972) (page 363, equation (9.1.79)) for the kernel, that is, the far field expansion for the G

kernel can be represented as the following:

$$G(\mathbf{x}, \mathbf{y}, \omega) = \frac{i}{4} \sum_{n=-\infty}^{\infty} O_n(\mathbf{y}_c, \mathbf{x}) I_{-n}(\mathbf{y}_c, \mathbf{y}), \quad |\mathbf{x} - \mathbf{y}_c| > |\mathbf{y} - \mathbf{y}_c|, \tag{18}$$

where $\mathbf{y}_c$ is the expansion point close to $\mathbf{y}$ and the two auxiliary functions O and I are given by:

$$O_n(\mathbf{x}, \mathbf{y}) = i^n H_n^{(1)}(kr) e^{in\alpha}, \tag{19}$$

$$I_n(\mathbf{x}, \mathbf{y}) = (-i)^n J_n(kr) e^{in\alpha}. \tag{20}$$

In the above two expressions, $J_n()$ denotes the Bessel-J function (Abramowitz and Stegun 1972) and α is the polar angle of the vector $\vec{r}$ from $\mathbf{x}$ to $\mathbf{y}$. Using Eq. (18), the far field expansion for the F kernel is given by:

$$F(\mathbf{x}, \mathbf{y}, \omega) = \frac{i}{4} \sum_{n=-\infty}^{\infty} O_n(\mathbf{y}_c, \mathbf{x}) \frac{\partial I_{-n}(\mathbf{y}_c, \mathbf{y})}{\partial n(\mathbf{y})}, \quad |\mathbf{x} - \mathbf{y}_c| > |\mathbf{y} - \mathbf{y}_c|, \tag{21}$$

in which the derivative is obtained by the formula:

$$\frac{\partial I_n(\mathbf{x}, \mathbf{y})}{\partial n(\mathbf{y})} = \frac{(-i)^n k}{2} [J_{n+1}(kr) e^{i\delta} - J_{n-1}(kr) e^{-i\delta}] e^{in\alpha}, \tag{22}$$

with δ being the angle between the vector $\vec{r}$ from $\mathbf{x}$ to $\mathbf{y}$ and the outward normal.

Applying expansions in Eqs. (18) and (21), one can evaluate the G and Fintegrals in CBIE (10) on S_c (a subset of S that is away from the source point $\mathbf{x}$) with the following *multipole expansions*:

$$\int_{S_c} G(\mathbf{x}, \mathbf{y}, \omega) q(\mathbf{y}) dS(\mathbf{y}) = \sum_{n=-\infty}^{\infty} O_n(\mathbf{y}_c, \mathbf{x}) M_n(\mathbf{y}_c), |\mathbf{x} - \mathbf{y}_c| > |\mathbf{y} - \mathbf{y}_c|, \tag{23}$$

$$\int_{S_c} F(\mathbf{x}, \mathbf{y}, \omega) \phi(\mathbf{y}) dS(\mathbf{y}) = \sum_{n=-\infty}^{\infty} O_n(\mathbf{y}_c, \mathbf{x}) \widetilde{M}_n(\mathbf{y}_c), |\mathbf{x} - \mathbf{y}_c| > |\mathbf{y} - \mathbf{y}_c|, \tag{24}$$

where M_n and $\widetilde{M}_n$ are the *multipole moments* centered at $\mathbf{y}_c$ and given by:

$$M_n(\mathbf{y}_c) = \frac{i}{4} \int_{S_c} I_{-n}(\mathbf{y}_c, \mathbf{y}) q(\mathbf{y}) dS(\mathbf{y}), \tag{25}$$

$$\widetilde{M}_n(\mathbf{y}_c) = \frac{i}{4} \int_{S_c} \frac{\partial I_{-n}(\mathbf{y}_c, \mathbf{y})}{\partial n(\mathbf{y})} \phi(\mathbf{y}) dS(\mathbf{y}). \tag{26}$$

When the multipole expansion center is moved from $\mathbf{y}_c$ to $\mathbf{y}_{c'}$, we have the following *M2M translations* for both M_n and $\widetilde{M}_n$:

$$M_n(\mathbf{y}_{c'}) = \sum_{m=-\infty}^{\infty} I_{n-m}(\mathbf{y}_{c'}, \mathbf{y}_c) M_m(\mathbf{y}_c), \tag{27}$$

which is derived using the following identity:

$$I_n(\mathbf{y}_{c'}, \mathbf{y}) = \sum_{m=-\infty}^{\infty} I_{n-m}(\mathbf{y}_{c'}, \mathbf{y}_c) I_m(\mathbf{y}_c, \mathbf{y}). \tag{28}$$

The *local expansion* for the G kernel integral in CBIE (10) is given as follows:

$$\int_{S_c} G(\mathbf{x}, \mathbf{y}, \omega) q(\mathbf{y}) dS = \sum_{n=-\infty}^{\infty} I_{-n}(\mathbf{x}_L, \mathbf{x}) L_n(\mathbf{x}_L), \tag{29}$$

where $\mathbf{x}_L$ is the local expansion point close to $\mathbf{x}$ ($|\mathbf{x} - \mathbf{x}_L| < |\mathbf{y} - \mathbf{x}_L|$) and the expansion coefficients are given by the following *M2L translation*:

$$L_n(\mathbf{x}_L) = \sum_{m=-\infty}^{\infty} (-1)^m O_{n-m}(\mathbf{x}_L, \mathbf{y}_c) M_m(\mathbf{y}_c). \tag{30}$$

This result, which is different from that given in (Nishimura 2002), is derived based on the following identity:

$$O_n(\mathbf{x}_L, \mathbf{y}) = \sum_{m=-\infty}^{\infty} (-1)^m O_{n-m}(\mathbf{x}_L, \mathbf{y}_c) I_m(\mathbf{y}_c, \mathbf{y}). \tag{31}$$

Similarly, the local expansion for the F kernel integral in CBIE (10) is given by:

$$\int_{S_c} F(\mathbf{x}, \mathbf{y}, \omega) \phi(\mathbf{y}) dS = \sum_{n=-\infty}^{\infty} I_{-n}(\mathbf{x}_L, \mathbf{x}) L_n(\mathbf{x}_L), \tag{32}$$

with $\widetilde{M}_n$ replacing M_n in the M2L translation (30).

The local expansion center in expansion (29) can be shifted from $\mathbf{x}_L$ to $\mathbf{x}_{L'}$ using the following *L2L translations*:

$$L_n(\mathbf{x}_{L'}) = \sum_{m=-\infty}^{\infty} I_m(\mathbf{x}_L, \mathbf{x}_{L'}) L_{n-m}(\mathbf{x}_L), \tag{33}$$

which is derived using the following identity:

$$O_n(\mathbf{x}_{L'}, \mathbf{y}) = \sum_{m=-\infty}^{\infty} I_m(\mathbf{x}_L, \mathbf{x}_{L'}) O_{n-m}(\mathbf{x}_L, \mathbf{y}). \tag{34}$$

For the HBIE (11), the local expansion of the K kernel integral can be written as:

$$\int_{S_c} K(\mathbf{x}, \mathbf{y}, \omega) q(\mathbf{y}) dS = \sum_{n=-\infty}^{\infty} \frac{\partial I_{-n}(\mathbf{x}_L, \mathbf{x})}{\partial n(\mathbf{x})} L_n(\mathbf{x}_L), \tag{35}$$

with the same local expansion coefficient $L_n(\mathbf{x}_L)$ given by Eq. (30). Similarly, the local expansion for the H kernel integral is given by:

$$\int_{S_c} H(\mathbf{x}, \mathbf{y}, \omega) \phi(\mathbf{y}) dS = \sum_{n=-\infty}^{\infty} \frac{\partial I_{-n}(\mathbf{x}_L, \mathbf{x})}{\partial n(\mathbf{x})} L_n(\mathbf{x}_L), \tag{36}$$

with $\widetilde{M}_n$ replacing M_n in Eq. (30) for evaluating $L_n(\mathbf{x}_L)$. Therefore, the same moments, M2M, M2L and L2L translations as used for the G and F integrals in the CBIE are used for the K and H integrals in the HBIE, respectively.

5 Fast Multipole Formulation for 3-D Acoustic Wave Problems

The fast multipole method for solving the Burton-Miller's BIE (16) is discussed in this section for the 3-D cases (Shen and Liu 2007). The fundamental solution $G(\mathbf{x}, \mathbf{y}, \omega)$ for Helmholtz equations in 3-D can be expanded as (see, e.g., Epton and Dembart 1995; Yoshida 2001):

$$G(\mathbf{x}, \mathbf{y}, \omega) = \frac{ik}{4\pi} \sum_{n=0}^{\infty} (2n+1) \sum_{m=-n}^{n} O_n^m(k, \mathbf{x} - \mathbf{y}_c) \bar{I}_n^m(k, \mathbf{y} - \mathbf{y}_c), \quad |\mathbf{x} - \mathbf{y}_c| > |\mathbf{y} - \mathbf{y}_c|, \tag{37}$$

where k is the wavenumber, $\mathbf{y}_c$ an expansion point near $\mathbf{y}$, O_n^m the outer function and I_n^m the inner function. Similarly, the kernel $F(\mathbf{x}, \mathbf{y}, \omega)$ can be expanded as:

$$F(\mathbf{x}, \mathbf{y}, \omega) = \frac{ik}{4\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1) O_n^m(k, \mathbf{x} - \mathbf{y}_c) \frac{\partial \bar{I}_n^m(k, \mathbf{y} - \mathbf{y}_c)}{\partial n(\mathbf{y})}, \quad |\mathbf{x} - \mathbf{y}_c| > |\mathbf{y} - \mathbf{y}_c|. \tag{38}$$

Using Eqs. (37) and (38), we can evaluate the G and F integrals in CBIE (10) on S_c with the following *multipole expansions*:

$$\int_{S_c} G(\mathbf{x}, \mathbf{y}, \omega) q(\mathbf{y}) dS(\mathbf{y}) = \frac{ik}{4\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1) O_n^m(k, \mathbf{x} - \mathbf{y}_c) M_{n,m}(k, \mathbf{y}_c), \quad |\mathbf{x} - \mathbf{y}_c| > |\mathbf{y} - \mathbf{y}_c|, \tag{39}$$

$$\int_{S_c} F(\mathbf{x}, \mathbf{y}, \omega) \phi(\mathbf{y}) dS(\mathbf{y}) = \frac{ik}{4\pi} \sum_{n=0}^{\infty} \sum_{m=-n}^{n} (2n+1) O_n^m(k, \mathbf{x} - \mathbf{y}_c) \widetilde{M}_{n,m}(k, \mathbf{y}_c), \quad |\mathbf{x} - \mathbf{y}_c| > |\mathbf{y} - \mathbf{y}_c|, \tag{40}$$

where $M_{n,m}$ and $\widetilde{M}_{n,m}$ are the *multipole moments* centered at $\mathbf{y}_c$ and given by:

$$M_{n,m}(k, \mathbf{y}_c) = \int_{S_c} \bar{I}_n^m(k, \mathbf{y} - \mathbf{y}_c) q(\mathbf{y}) dS(\mathbf{y}), \tag{41}$$

$$\widetilde{M}_{n,m}(k, \mathbf{y}_c) = \int_{S_c} \frac{\partial \bar{I}_n^m(k, \mathbf{y} - \mathbf{y}_c)}{\partial n(\mathbf{y})} \phi(\mathbf{y}) dS(\mathbf{y}). \tag{42}$$

The *M2M, M2L and L2L translations* for 3-D Helmholtz BIEs can be found in Shen and Liu (2007) and Yoshida (2001). Adaptive fast multipole algorithms (Shen and Liu 2007) have also been employed to further accelerate the solutions of the fast multipole BEM.

6 Numerical Examples

Several 2-D and 3-D examples of acoustic wave problems are presented in this section. Constant triangular elements are used in all these examples, for which one can use singularity subtraction approach to analytically evaluate the singular and hypersingular integrals involving the static kernels. In all the 3-D examples, the maximum number of elements in a leaf is set to 100, the number of multipole and local expansion terms set to 10 and the tolerance to 10^{-3}. All the computations for the 3-D examples were done on a laptop PC with an Intel 1.6 GHz Centrino processor and 512 MB memory.

6.1 Scattering from Cylinders in 2-D Medium

A 2-D scattering problem with a rigid cylinder and the incident wave coming from the right is considered first (Fig. 2). The cylinder has a radius $a = 1$ and is discretized with line elements. A relative error of 0.01% is achieved with 1,000 elements for $ka = 1$. Figure 2 shows the magnitude of the scattered pressure field outside the cylinder in a square region. Figure 3 shows the computed scattered field by an array of multiple cylinders with $ka = 0.1$.

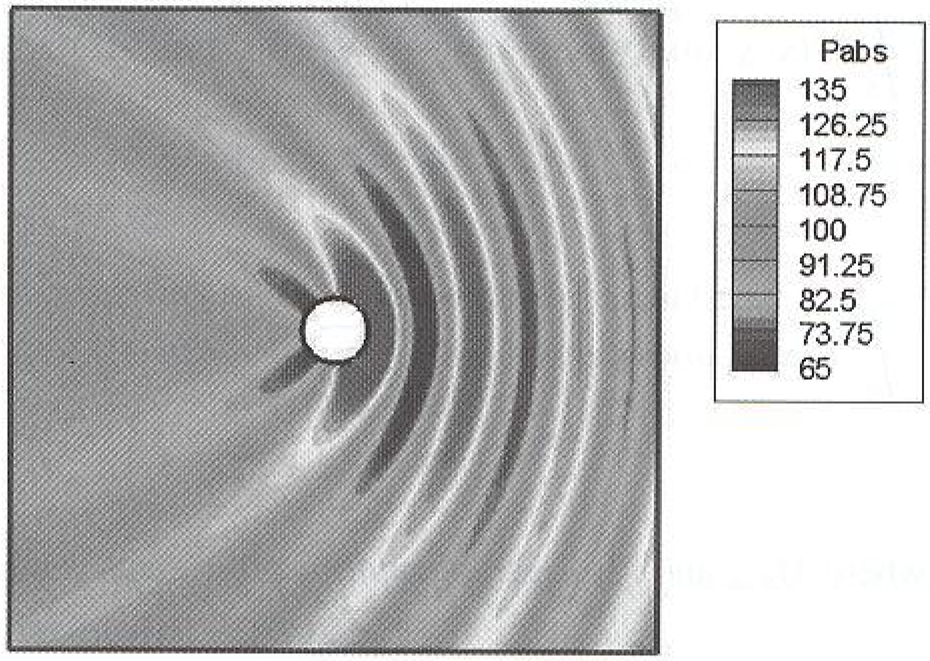


Fig. 2 Scattering from a single cylinder

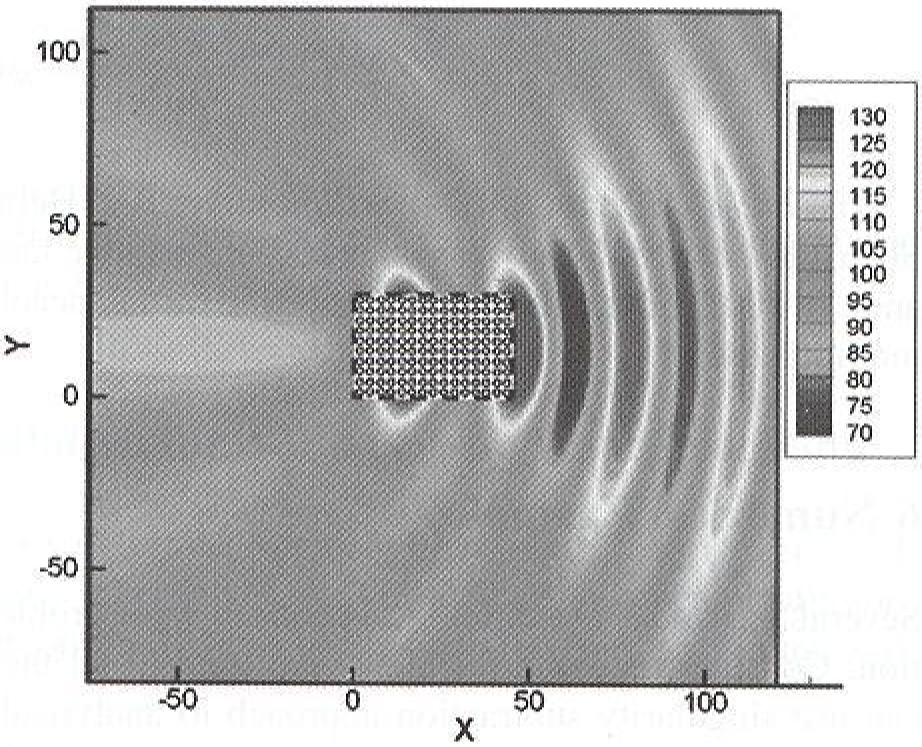


Fig. 3 Scattering from multiple cylinders

6.2 Radiation from a Pulsating Sphere

A pulsating sphere with radius $a = 1$ m is used to verify the fast multipole BEM code for 3-D radiation problems. The normalized wave number ka varies from 1 to 10. The total number of elements is 1,200. The velocity potentials at $(5a, 0, 0)$ are plotted in Fig. 4, which shows that the conventional BEM with the CBIE fails to predict the surface velocity potential at the fictitious frequencies ($ka = \pi, 2\pi, \ldots,$ for this case). The results using the conventional BEM with the Burton-Miller's (CHBIE) formulation agree well with the analytical solution at all wavenumbers. The fast multipole BEM with the CHBIE also yields very close results to those of the conventional BEM with the CHBIE, which suggests that the truncation error introduced in multipole expansions is very small for problems with ka ranging from 1 to 10.

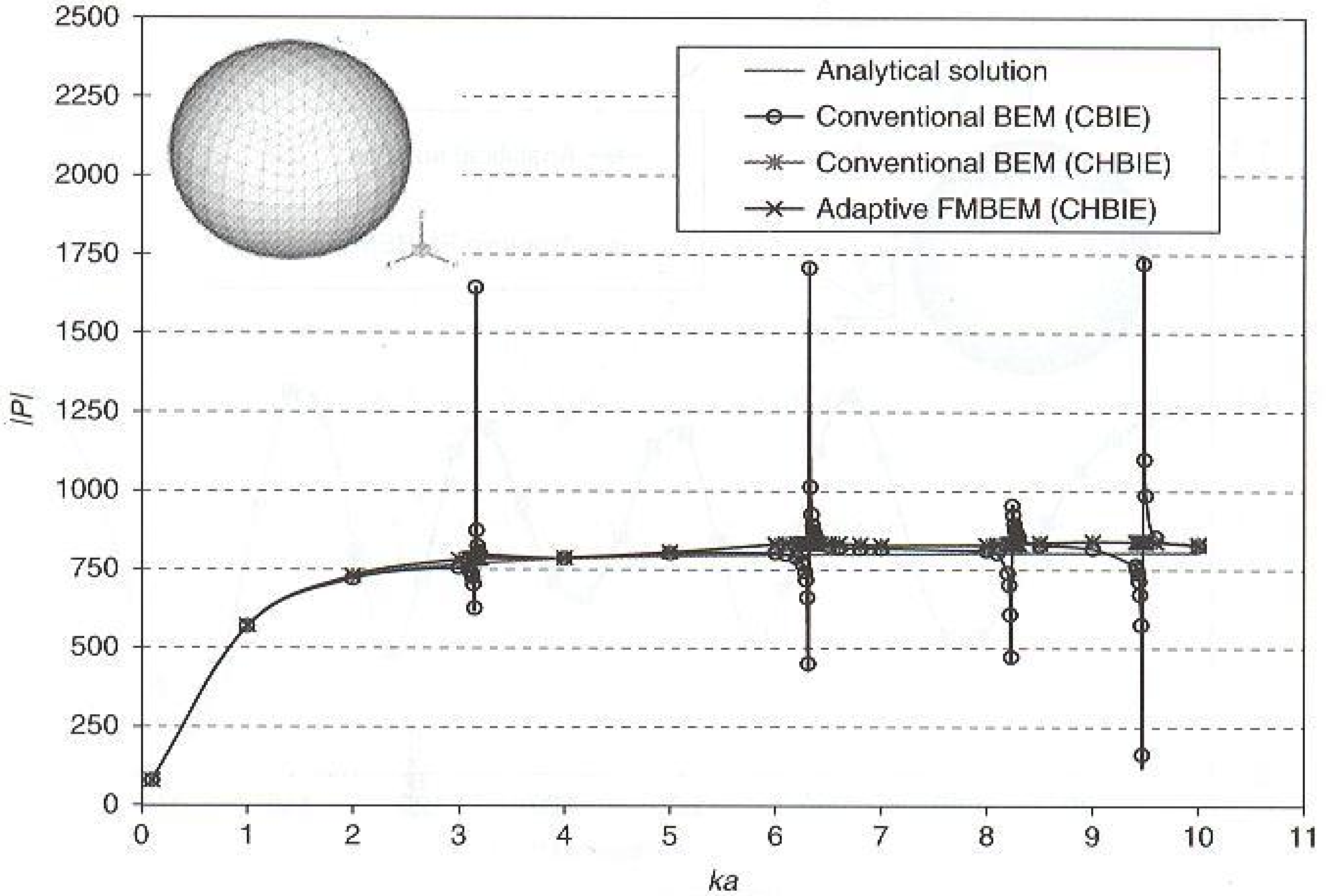


Fig. 4 Frequency sweep plot for the pulsating sphere model

6.3 Scattering from a Rigid Sphere

A rigid sphere with radius $a = 1$ m centered at (0, 0, 0) is used to test the fast multipole BEM code for scattering problems. The sphere is meshed with 1,200 elements and impinged upon by an incident wave of unit amplitude $\phi^I(x, y, z) = e^{-ikz}$, with $ka = \pi$, one of the fictitious eigenfrequencies for the CBIE, and traveling along the negative z axis. Sample field points are evenly distributed on a semicircle of $r = 5a$, centered at (0, 0, 0). The velocity potential curves plotted in Fig. 5 shows that the adaptive FMBEM using Burton-Miller formulation successfully overcomes the non-uniqueness difficulties at this fictitious frequency and yields very accurate results.

6.4 Scattering from Multiple Objects

A multi-scatterer model containing 1,000 randomly distributed capsule-like rigid scatterers in a $2 \times 2 \times 2$ m domain is studied next. Each scatterer is meshed with 200 boundary elements, with a total of 200,000 elements for the entire model. The incident wave is e^{-ikx} with $k = 1$. Sample points are taken at an annular data collection surface with inner and outer radius equal to 5 and 10, respectively. The computed velocity potential distribution contour is shown in Fig. 6 for this discretization. Total CPU time used to solve this large model is 3,352 s using the laptop PC.

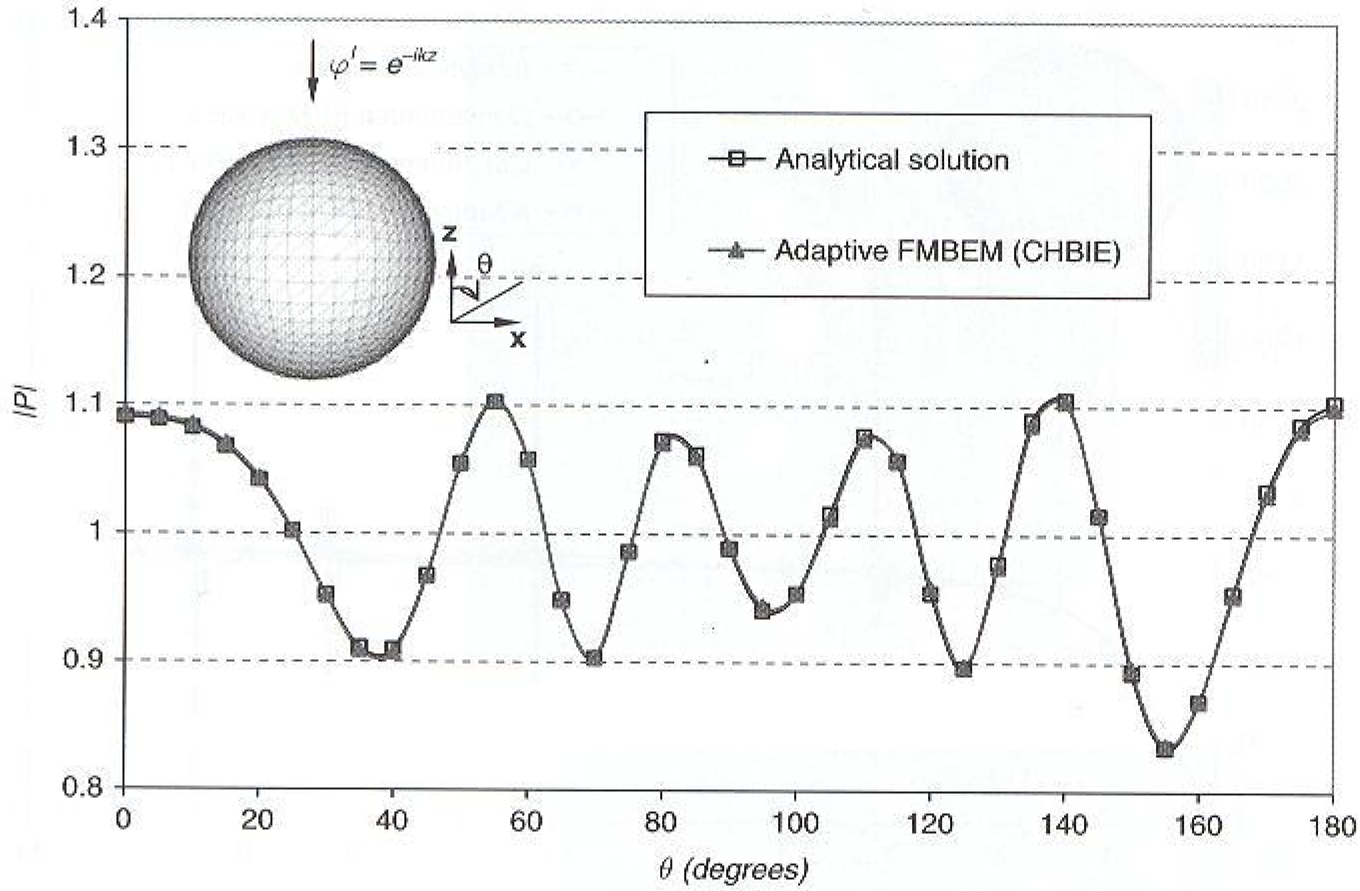
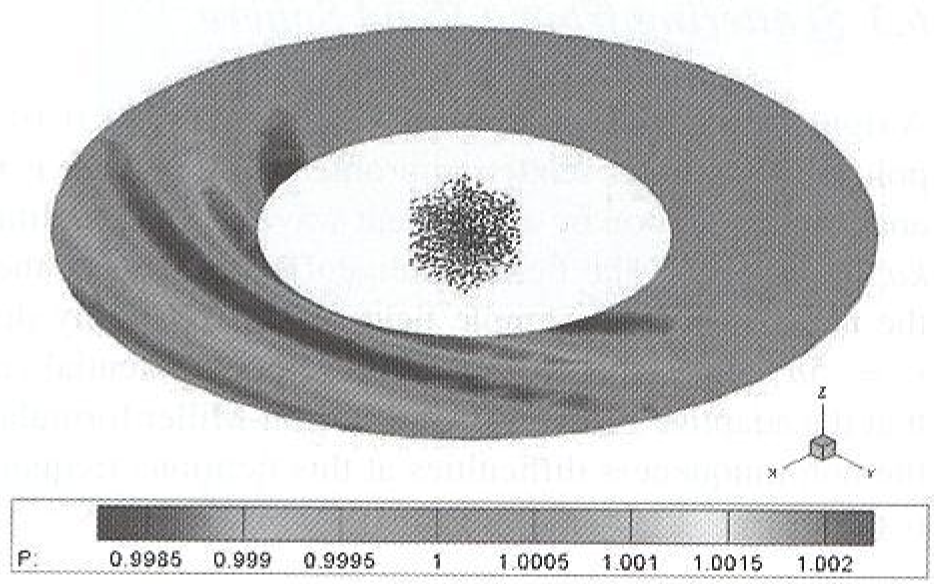


Fig. 5 Scattering from the rigid sphere at the fictitious eigenfrequency $ka = \pi$

Fig. 6 Computed velocity potential for the multiple scatterer model



To study the computational efficiency of the fast multipole BEM, the BEM model is rerun with an increasing number of scatterers in the model. The numbers of elements are increased from 1,600 to 200,000, corresponding to 8 to 1,000 scatterers in the model. The total CPU time used to solve these multiple scatterer problems on the laptop PC is shown in Fig. 7, which exhibits a linear behavior and thus suggests the $O(N)$ efficiency of the developed fast multipole BEM code.

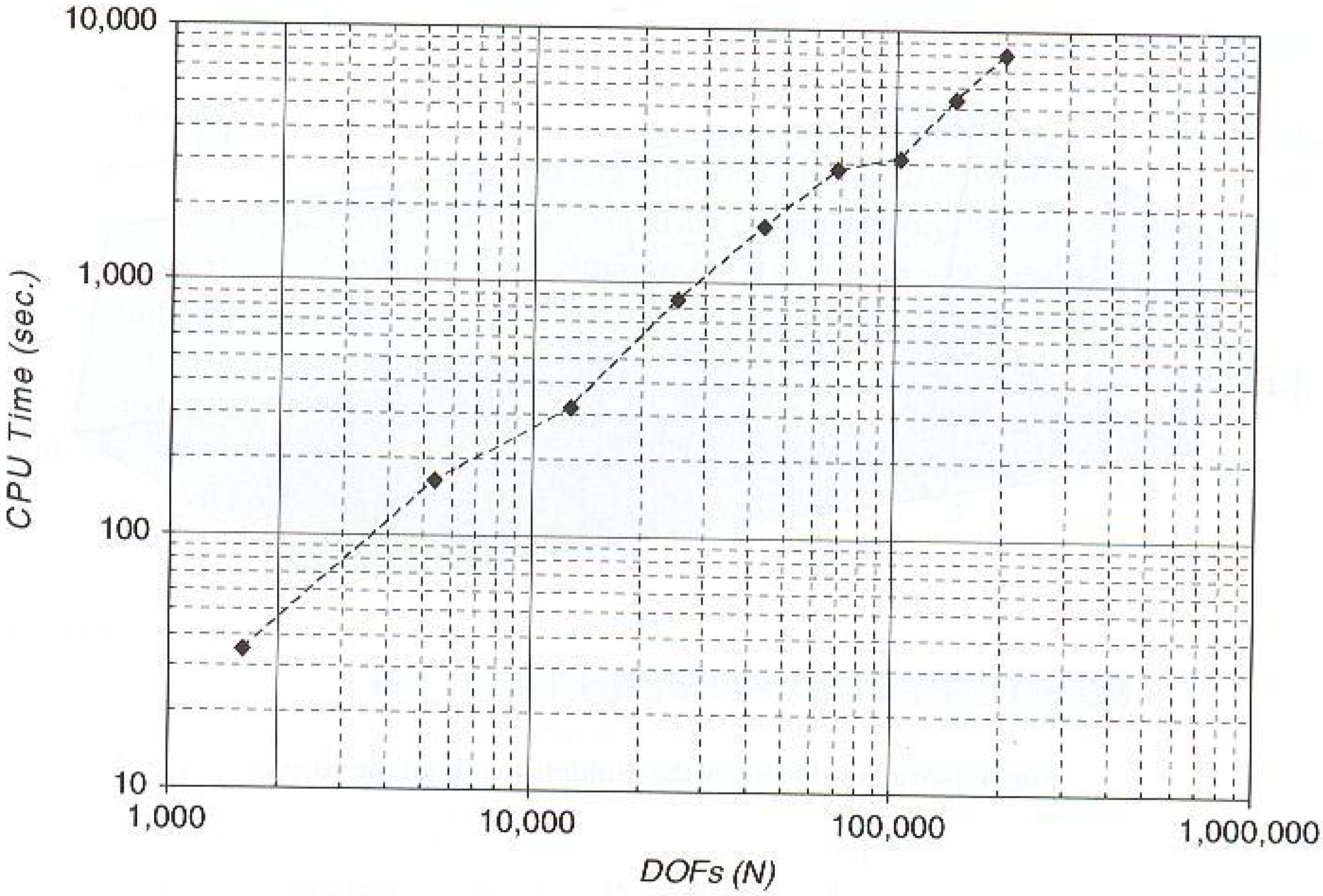


Fig. 7 Total CPU time used to solve the multiple scatterer problem

6.5 Analysis of Sound Barriers – A Half-Space Acoustic Wave Problem

Many of the acoustic problems are present in half spaces, such as noise control problems due to airplane takeoff or landing near an airport, or due to traffic on highway near a residential area. Using the BEM, these half-space acoustic wave problems can also be modeled readily. In these cases, half-space Green's function need to be employed and the same adaptive algorithm for the full-space problems can be employed. Detailed formulations of the adaptive fast multipole BEM for 3-D half-space acoustic wave problems can be found in Shen and Liu (2009).

Figures 8 and 9 show the evaluated sound levels (in dB) for a BEM model of three buildings near a highway without and with a sound barrier, respectively, using the fast multipole BEM for half-space acoustic wave problems (Shen and Liu 2009). The dimensions ($L \times W \times H$) of the three buildings are $30 \times 10 \times 20$, $20 \times 12 \times 15$ and $9.5 \times 9 \times 8$ (in m), respectively. The barrier has a height of 6 m and length of 255.94 m. One source point load with 20 Hz frequency is located 13 m away from the middle point of the barrier and 1 m above the ground. The BEM model contains 56,465 triangular elements. In the case with no sound barrier, the surface of the larger building closest to the source has the maximum sound level of 94 dB, while the smaller building that is furthest away from the source registers the smallest dB, as shown in Fig. 8. After inserting the barrier in the model, the maximum sound level on the surfaces of the buildings is reduced to 90 dB, as shown in Fig. 9. The

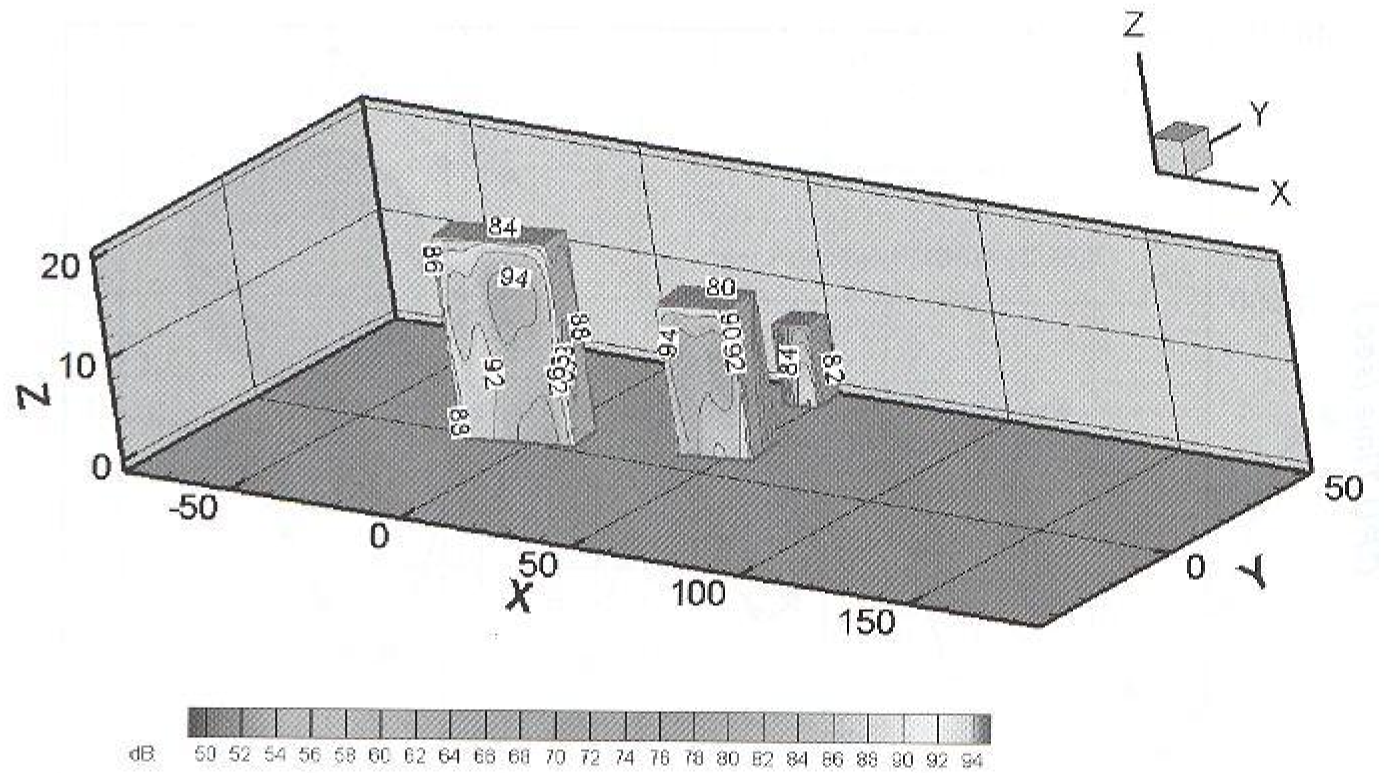


Fig. 8 Noise level (dB) on the buildings without the barrier

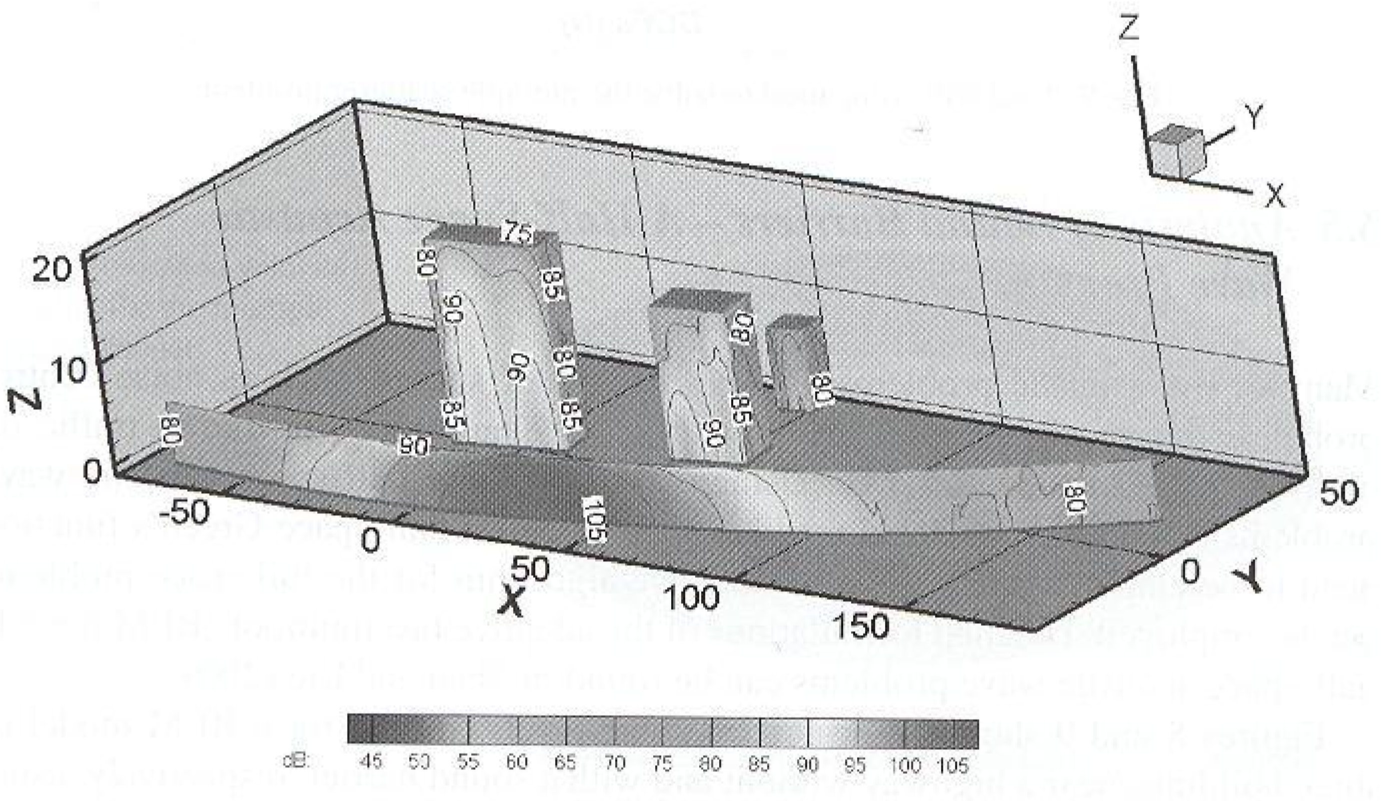


Fig. 9 Noise level (dB) on the buildings with the barrier

effect of the sound barriers in reducing the noise levels near highways is evident from this BEM simulation.

7 Conclusions

Some of the recent development of the fast multipole BEM for both 2-D and 3-D acoustic wave problems are reviewed in this paper. The basic formulations are provided and the numerical examples clearly demonstrate the potentials of the fast multipole BEM for solving large-scale acoustic wave problems. Improvements are still need be made to the fast multipole BEM discussed in this chapter. For example, adaptive tree structures can be implemented which can handle slender structure more effectively. For M2L translations, the recursive relations of the translation operators and use rotation-coaxial translation decomposition of the translation operators given by Gumerov and Duraiswami (2003) can be applied to reduce the computational complexity. The developed fast multipole BEM can also be extended to solve many other coupled acoustic problems, such as acoustic waves interacting with elastic structures (Chen and Liu 1999; Chen et al. 2000), and multi-domain acoustic wave problems as in biological applications.

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