Engineering Analysis with Boundary Elements 33 (2009) 522–528

Contents lists available at ScienceDirect

Engineering Analysis with Boundary Elements

journal homepage: www.elsevier.com/locate/enganabound

A numerical method based on the boundary integral equation and dual reciprocity methods for one-dimensional Cahn–Hilliard equation

Mehdi Dehghan *, Davoud Mirzaei

Department of Applied Mathematics, Faculty of Mathematics and Computer Science, Amirkabir University of Technology, No. 424, Hafez Ave., 15914, Tehran, Iran

ARTICLE INFO

Article history:
Received 7 March 2008
Accepted 14 August 2008
Available online 25 September 2008

Keywords:
Boundary integral equation
Dual reciprocity method
Cahn–Hilliard equation
Mass conservation
Energy dissipation

ABSTRACT

This paper describes a numerical method based on the boundary integral equation and dual reciprocity methods for solving the one-dimensional Cahn–Hilliard (C–H) equation. The idea behind this approach comes from the dual reciprocity boundary element method that introduced for higher order dimensional problems. A time-stepping method and a predictor–corrector scheme are employed to deal with the time derivative and the nonlinearity respectively. Numerical results are presented for some examples to demonstrate the usefulness and accuracy of this approach. For these problems the energy functional dissipation and the mass conservation properties are investigated.

1. Introduction

The Cahn–Hilliard (C–H) equation [1,2]

$$\frac{\partial u}{\partial t}+\gamma\Delta^2 u-\Delta\psi'(u)=0,\quad \gamma>0 \tag{1.1}$$

was originally introduced as a phenomenological model of phase separation in a binary systems, such as alloys, glass and polymer mixtures. This equation is very important in materials science (see [1–3] and the references therein). It is a conservation law, in the sense that the average of the order parameter, which corresponds to a density of atoms, is conserved [4]. Since the pioneering work of Cahn and Hilliard [2], the C–H equation has been extensively applied to the inter-diffusion in binary polymer mixtures [5], in interfaces of polymers with dissimilar properties [6] and in the formation of interfaces via the inter-diffusion of layers [7]. The viscous C–H equation, which arises as a singular limit of the phase field model of phase transition, has been studied in [8]. The C–H equation with a logarithmic free energy (ψ contains logarithmic terms) has been considered in [9,10], where they also studied finite element approximation of the model equation. Global existence and uniqueness of the solution for (1.1) have been shown in [11–14] and etc. Long time behaviour of the solution has been studied in [15–17] using the free energy method.

There have been many algorithms developed and simulations performed for the C–H equations, using Finite Element Methods [18–25], Discontinuous Galerkin Techniques [26–28], Finite Difference Schemes [29–36], Spectral Methods [37,38], Collocation Techniques [39–41], Adomian Decomposition Procedure [42], μ-transform [43] and etc.

Finite Galerkin approximate solutions have been obtained by Elliott and French [18] and French and Jensen [22] for one-dimensional problems. For multidimensional problems, Elliott and French [19] considered nonconforming finite element method. Elliott and Larsson [20] have obtained error estimates of finite Galerkin solutions for smooth data as well as nonsmooth initial data. Elliott et al. [21] have obtained optimal order error bounds using a second order splitting method. Mixed finite element method has been applied to obtain approximate solutions for (1.1) by Dean et al. [23]. A multigrid finite element solver has been presented by Kay and Welford [25], a discontinuous Galerkin finite element method has been developed by Wells et al. [27] and adaptive finite element methods in two and three dimensions were considered by Baňas and Nürnberg [24] for this problem. Xia et al. [28] have developed the local discontinuous Galerkin (LDG) methods for (1.1) and Abels and Wilke [14] have investigated the asymptotic behavior of the nonlinear C–H equation with a logarithmic free energy and similar singular free energies, etc.

Furihata et al. [29] have examined the boundedness of the solution of a finite difference scheme [55] using discretized Lyapunov functional. Furihata [30] has depicted the dissipation of energy and conservation of mass for this problem. Also this author [31] has proposed a stable and conservative finite difference scheme to solve numerically the C–H equation. Alikakos et al. [13] have established rigorously the existence of some extremely slowly evolving solutions. Numerical solutions for Eq. (1.1) have been considered by Choo and Chung [32] using the Crank–Nicolson type finite difference method for one-dimensional problem

* Corresponding author.
E-mail addresses: mdehghan@aut.ac.ir (M. Dehghan), d_mirzaei@aut.ac.ir (D. Mirzaei).

0955-7997/$ - see front matter © 2008 Elsevier Ltd. All rights reserved.
doi:10.1016/j.enganabound.2008.08.008

and by Choo et al. [33] using a nonlinear conservative difference scheme for two-dimensional problem. Another Crank–Nicolson type method has been developed by Khiari et al. [34] for (1.1) in one dimension. Mello et al. [36] have presented a stable and fast conservative finite difference scheme to solve (1.1) with two improvements. Kim [35] has considered a conservative nonlinear multigrid method for the C–H equation with a variable mobility.

Ye [39] has employed the Fourier collocation method and Ye and Cheng [37] have developed the Fourier spectral method to solve numerically the C–H equation with periodic boundary conditions. Ye [40] has also used the Legendre collocation method to study this problem with Neumann boundary conditions. Danumjaya and Nandakumaran [41] have employed the orthogonal cubic spline collocation method for this equation. He et al. [38] have analyzed a class of large time-stepping methods for this equation.

It is worth pointing out that as a semi-analytic approach the Adomian decomposition method has been applied for the solution of viscous C–H equation by Momani and Erturk [42]. Two applications of μ-transform to the C–H equation have been presented by Mitlin [43]. Miranville and Piétrus [4] have obtained a new formulation of the generalizations of the C–H equation based on constitutive equations proposed by Gurtin. Hongjun and Changchun [44] have studied the instability of the traveling waves of the convective–diffusive C–H equation.

In this research we propose an alternative approach based on the boundary integral equation and dual reciprocity method (DRM) for solving the one-dimensional C–H equation (1.1). The idea behind this approach comes from the DRM introduced by Brebbia and Nardini [45] and Partridge and Brebbia [46] for higher order dimensional problems and it is to expand the inhomogeneous and nonlinear terms in terms of their values at the nodes which lie in the domain of problem. The inhomogeneous and nonlinear terms are approximated by interpolation in terms of some well-known functions $\phi(r)$, called radial basis functions (RBFs), where r is the distance between a source point and the field point. There exist a large class of interpolating RBFs that can be used [47]. These include the linear $1+r$, the polynomial $P_j(r)$, the thin plate spline (TPS) $r^{2n}\log(r)$, the Gaussian $\exp(-r^2/c^2)$, and the multiquadrics $(c^2+r^2)^{1/2}$, where c is the constant parameter, etc. In this paper we shall use the linear RBF. It is worth to point out that, there exist another class of RBFs such as $C-r$ and $C^2-r^2\ln r$ which used in method of particular integrals, where C is a constant chosen to be the largest dimension of the problem domain. To study this approach, authors refer the reader to [48,49] and the references therein.

Some authors (for example [50–52]) employed a similar approach for one-dimensional problems. Also the authors of [53,54] combined the dual boundary element method (DBEM) and multiple reciprocity method (MRM) to solve one-dimensional eigenproblems. This paper is meaningful to study and develop its idea to solve some one-dimensional partial differential equations.

The remainder of the paper is organized as follows: In Section 2, brief discussions for C–H equation are presented. In Section 3 the discretized version of equation is obtained and a predictor–corrector scheme is described to deal with the nonlinearity. In Section 4, numerical results are considered for two test problems and the known properties of equation are tested. Section 5 ends this report with a brief conclusion.

2. C–H equation

Let us consider the following classical C–H equation (1.1) in one-dimensional:

$$\frac{\partial u}{\partial t}+\gamma\frac{\partial^4 u}{\partial x^4}-\frac{\partial^2\psi'(u)}{\partial x^2}=0,\quad x\in\Omega,\ \ t\geqslant 0, \tag{2.1}$$

with Neumann boundary conditions

$$\frac{\partial u}{\partial x}=\frac{\partial^3 u}{\partial x^3}=0,\quad x\in\partial\Omega, \tag{2.2}$$

and initial condition

$$u(x,0)=u_0(x),\quad x\in\Omega, \tag{2.3}$$

where

$$\psi'(u)=\phi(u)=r_2u^3+r_1u^2+r_0u, \tag{2.4}$$

and γ is a prescribed positive constant, $\Omega=[a,b]$ and r_0, r_1, r_2 are given constants.

It is known if the initial data $u_0\in H^2_E(\Omega)=\{f\in H^2(\Omega)$: $\partial f/\partial x=0$ on $\partial\Omega\}$ then the problem (2.1)–(2.3) has a unique solution for all times [11].

Let us introduce w, the chemical potential, defined by

$$w=\phi(u)-\gamma u_{xx}, \tag{2.5}$$

then (2.1) may be rewritten as

$$\begin{cases} u_t-w_{xx}=0,\\ w-\phi(u)+\gamma u_{xx}=0. \end{cases} \tag{2.6}$$

We have, in view of (2.2) and (2.5), the following boundary condition on w:

$$\frac{\partial w}{\partial x}=0,\quad x\in\partial\Omega,$$

therefore the boundary conditions

$$\frac{\partial u}{\partial x}=\frac{\partial w}{\partial x}=0,\quad x\in\partial\Omega \tag{2.7}$$

are equivalent to (2.2).

The initial conditions for u and w are

$$u(x,0)=u_0(x),\quad w(x,0)=\phi(u_0)-\gamma\frac{\partial^2 u_0}{\partial x^2},\quad x\in\Omega. \tag{2.8}$$

The important features of C–H equations are as follows:

(i) For the extended Ginzberg–Landau free energy functional

$$\mathscr{F}(u)=\int_\Omega\left[\frac{\gamma}{2}u_x^2+\psi(u)\right]\mathrm{d}x, \tag{2.9}$$

where $\psi(u)$ is a Lyapunov functional (see [11,18,20,39]), $(\mathrm{d}/\mathrm{d}t)\mathscr{F}(u)\leqslant 0$: i.e., the energy is dissipative as time passes.

(ii) The total mass remains constant, i.e.

$$\frac{1}{|\Omega|}\int_\Omega u(x,t)\,\mathrm{d}x=\frac{1}{|\Omega|}\int_\Omega u(x,0)\,\mathrm{d}x=\mathscr{M},\quad t>0. \tag{2.10}$$

These properties play an important role in analyzing the solution of C–H equation.

3. Mathematical formulation

We use mixed boundary integral equation and Dual Reciprocity Method (DRM) for the system (2.6)–(2.8) to compute the approximate solutions. In order to derive the integral equation formulations for the problem, we start with the following integral identities of Eq. (2.6):

$$\int_a^b\left[\frac{\partial^2 w}{\partial x^2}-\frac{\partial u}{\partial t}\right]G\,\mathrm{d}x=0 \tag{3.1}$$

and

$$\int_a^b\left[\gamma\frac{\partial^2 u}{\partial x^2}+w-\phi(u)\right]G\,\mathrm{d}x=0, \tag{3.2}$$

where the function G is the fundamental solution for the one-dimensional Laplace operator defined by the equation

$$\frac{\partial^2 G}{\partial x^2}(x,\xi)=\delta(x,\xi), \tag{3.3}$$

where x and ξ are a field point and a source point, respectively and δ is the Dirac delta function. The fundamental solution and its derivative are given as follows:

$$G(x,\xi)=\tfrac{1}{2}|x-\xi|, \tag{3.4}$$

$$G_x(x,\xi)=G'(x,\xi)=\tfrac{1}{2}\mathrm{sgn}(x-\xi) \tag{3.5}$$

in which the symbol "sgn" denotes the signum function.

For applying the method, N source points x_j, $j=1,2,\ldots,N$ are chosen in $[a,b]$ where $a=x_1<x_2<\cdots<x_{N-1}<x_N=b$ and the following approximations are made:

$$\frac{\partial u}{\partial t}=g(x,t)=\sum_{k=1}^{N}\mu_k(x)\alpha_k(t), \tag{3.6}$$

$$\frac{1}{\gamma}[\phi(u)-w]=h(x,t)=\sum_{k=1}^{N}\mu_k(x)\beta_k(t), \tag{3.7}$$

where $\mu_k(x)=1+r_k$ is a linear RBF and $r_k=|x-x_k|$ for $k=1,2,\ldots,N$. From (3.1) and (3.6), we have

$$\int_a^b \frac{\partial^2 w}{\partial x^2}G\,\mathrm{d}x=\sum_{k=1}^{N}\left[\int_a^b \mu_k(x)G\,\mathrm{d}x\right]\alpha_k(t) \tag{3.8}$$

and from (3.2) and (3.7), we have

$$\int_a^b \frac{\partial^2 u}{\partial x^2}G\,\mathrm{d}x=\sum_{k=1}^{N}\left[\int_a^b \mu_k(x)G\,\mathrm{d}x\right]\beta_k(t). \tag{3.9}$$

If we represent the value of the function μ_k at source point x_i by μ_{ik} for $i=1,2,\ldots,N$, and set F as a $N\times N$ matrix that $F(i,k)=\mu_{ik}$ and $E=F^{-1}$, then we have

$$\alpha_k(t)=\sum_{j=1}^{N}E_{kj}g_j(t) \tag{3.10}$$

and

$$\beta_k(t)=\sum_{j=1}^{N}E_{kj}h_j(t), \tag{3.11}$$

where $g_j(t)=g(x_j,t)$ and $h_j(t)=h(x_j,t)$. Now consider a set of functions f_k such that

$$\frac{\partial^2 f_k}{\partial x^2}=\mu_k, \tag{3.12}$$

the function f_k is easily determined as

$$f_k=\tfrac{1}{2}r_k^2+\tfrac{1}{6}r_k^3. \tag{3.13}$$

Substituting (3.12) into (3.8) and (3.9) and applying the integration by parts, we can get the following expressions:

$$[G_i(x)p(x,t)]_a^b-[w(x,t)G_i'(x)]_a^b+w_i(t)=\sum_{k=1}^{N}[[G_if_k']_a^b-[f_kG_i']_a^b+f_{ik}]\alpha_k(t) \tag{3.14}$$

and

$$[G_i(x)q(x,t)]_a^b-[u(x,t)G_i'(x)]_a^b+u_i(t)=\sum_{k=1}^{N}[[G_if_k']_a^b-[f_kG_i']_a^b+f_{ik}]\beta_k(t) \tag{3.15}$$

for $i=1,2,\ldots,N$, where $p(x,t)=\partial w(x,t)/\partial x$, $q(x,t)=\partial u(x,t)/\partial x$ and $G_i(x)=G(x,\xi)|_{\xi=x_i}$ are the weight functions at the points $i=1,2,\ldots,N$ which satisfy in Eq. (3.3). $u_i(t)=u(\xi,t)|_{\xi=x_i}$, $w_i(t)=w(\xi,t)|_{\xi=x_i}$, $f_{ik}=f_k(x_i)$ and f_k' is the derivative of f_k and can be expressed as

$$f_k'=r_k+\tfrac{1}{2}r_k^2. \tag{3.16}$$

If we set

$$S_{ik}=[G_if_k']_a^b-[f_kG_i']_a^b+f_{ik}, \tag{3.17}$$

then Eqs. (3.14) and (3.15) take the following forms:

$$[G_i(x)p(x,t)]_a^b-[w(x,t)G_i'(x)]_a^b+w_i(t)=\sum_{k=1}^{N}S_{ik}\alpha_k(t), \tag{3.18}$$

$$[G_i(x)q(x,t)]_a^b-[u(x,t)G_i'(x)]_a^b+u_i(t)=\sum_{k=1}^{N}S_{ik}\beta_k(t). \tag{3.19}$$

From (3.10), the right-hand side of Eq. (3.18) can be written as

$$\sum_{k=1}^{N}S_{ik}\alpha_k(t)=\sum_{k=1}^{N}S_{ik}\sum_{j=1}^{N}E_{kj}g_j(t)=\sum_{j=1}^{N}M_{ij}g_j(t), \tag{3.20}$$

and from (3.11), the right-hand side of Eq. (3.19) is

$$\sum_{k=1}^{N}S_{ik}\beta_k(t)=\sum_{k=1}^{N}S_{ik}\sum_{j=1}^{N}E_{kj}h_j(t)=\sum_{j=1}^{N}M_{ij}h_j(t), \tag{3.21}$$

where

$$M_{ij}=\sum_{k=1}^{N}S_{ik}E_{kj}. \tag{3.22}$$

In other hand if

$$L=\begin{bmatrix}-G_1(a) & G_1(b)\\ -G_2(a) & G_2(b)\\ \vdots & \vdots\\ -G_N(a) & G_N(b)\end{bmatrix}_{N\times 2},\quad H=\begin{bmatrix}-G_1'(a) & G_1'(b)\\ -G_2'(a) & G_2'(b)\\ \vdots & \vdots\\ -G_N'(a) & G_N'(b)\end{bmatrix}_{N\times 2}, \tag{3.23}$$

then Eqs. (3.18), (3.20) and (3.22) yield

$$[L]\begin{bmatrix}p_1(t)\\ p_N(t)\end{bmatrix}-[H]\begin{bmatrix}w_1(t)\\ w_N(t)\end{bmatrix}+[I]\begin{bmatrix}w_1(t)\\ w_2(t)\\ \vdots\\ w_N(t)\end{bmatrix}=[M]\begin{bmatrix}g_1(t)\\ g_2(t)\\ \vdots\\ g_N(t)\end{bmatrix} \tag{3.24}$$

and Eqs. (3.19), (3.21) and (3.22) give

$$[L]\begin{bmatrix}q_1(t)\\ q_N(t)\end{bmatrix}-[H]\begin{bmatrix}u_1(t)\\ u_N(t)\end{bmatrix}+[I]\begin{bmatrix}u_1(t)\\ u_2(t)\\ \vdots\\ u_N(t)\end{bmatrix}=[M]\begin{bmatrix}h_1(t)\\ h_2(t)\\ \vdots\\ h_N(t)\end{bmatrix}, \tag{3.25}$$

where I is the N by N identity matrix, $p_1(t)=p(a,t)$, $p_N(t)=p(b,t)$, $w_j(t)=w(x_j,t)$, $q_1(t)=q(a,t)$, $q_N(t)=q(b,t)$ and $u_j(t)=u(x_j,t)$. We note that

$$g_j(t)=\frac{\partial u}{\partial t}(x_j,t) \tag{3.26}$$

and

$$h_j(t)=\phi(u(x_j,t))-w(x_j,t). \tag{3.27}$$

Now, (3.24) and (3.25) constitute a system of $2N$ equations in $2N$ unknown functions of t. Since the Neumann boundary conditions are applied, the unknown variables are given by $u_j(t)$ and $w_j(t)$, for $j=1,2,\ldots,N$. This system is solved approximately using the iterative scheme which is described in the following.

We make the following approximations:

$$\left.\frac{\partial u}{\partial t}\right|_j \simeq \frac{u_j^{(n+1)} - u_j^{(n)}}{\Delta t}, \quad j = 1, 2, \ldots, N,$$
$$w_j(t) \simeq \tfrac{1}{2}[w_j^{(n+1)} + w_j^{(n)}], \quad j = 1, 2, \ldots, N,$$
$$u_j(t) \simeq \tfrac{1}{2}[u_j^{(n+1)} + u_j^{(n)}], \quad j = 1, 2, \ldots, N, \tag{3.28}$$

where $u_j^{(n)} = u(x_j, n\Delta t)$ and $w_j^{(n)} = w(x_j, n\Delta t)$. By assuming $\lambda = 1/\Delta t$, $\mathbf{u} = [u_1, u_2, \ldots, u_N]^{\mathrm{T}}$ and $\mathbf{w} = [w_1, w_2, \ldots, w_N]^{\mathrm{T}}$, (3.24)–(3.28) yield

$$-\frac{1}{2}[H]\left(\begin{bmatrix} w_1 \\ w_N \end{bmatrix}^{(n+1)} + \begin{bmatrix} w_1 \\ w_N \end{bmatrix}^{(n)}\right) + \frac{1}{2}[I]([\mathbf{w}]^{(n+1)} + [\mathbf{w}]^{(n)}) = \lambda[M]([\mathbf{u}]^{(n+1)} - [\mathbf{u}]^{(n)}) \tag{3.29}$$

and

$$-\frac{1}{2}[H]\left(\begin{bmatrix} u_1 \\ u_N \end{bmatrix}^{(n+1)} + \begin{bmatrix} u_1 \\ u_N \end{bmatrix}^{(n)}\right) + \frac{1}{2}[I]([\mathbf{u}]^{(n+1)} + [\mathbf{u}]^{(n)}) = -\frac{1}{2}[M]([\mathbf{w}]^{(n+1)} + [\mathbf{w}]^{(n)}) + [M][\Phi], \tag{3.30}$$

where $\Phi = [\phi(\tilde{u}_1), \phi(\tilde{u}_2), \ldots, \phi(\tilde{u}_N)]^{\mathrm{T}}$, $\tilde{u}_j$ is given by the known approximation of $u(x_j, t)$, as described earlier. We note that, due to the zero boundary conditions (2.7) the first terms of left-hand sides of Eqs. (3.24) and (3.25) do not appear in (3.29) and (3.30) respectively. At the first time level (when $n = 0$), the initial conditions (2.8) give

$$u_j^{(0)} = u(x_j, 0)$$

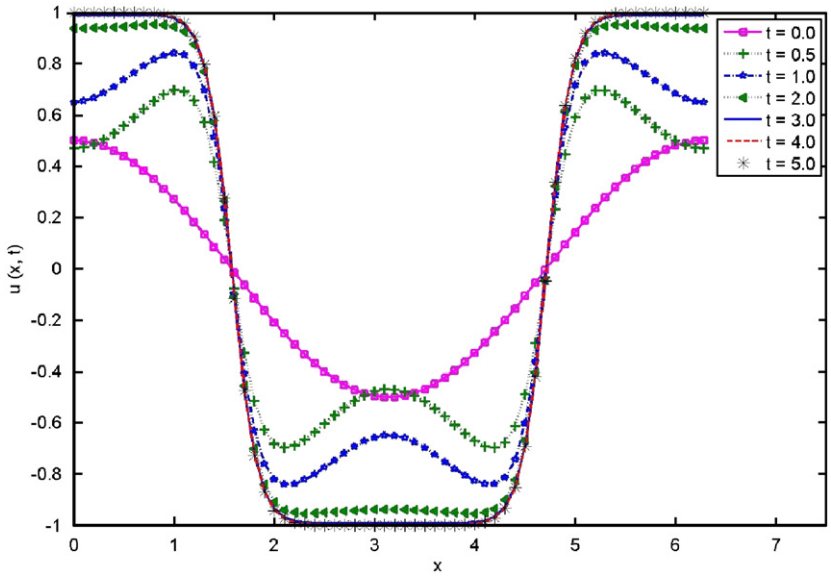

Fig. 1. The evolution from $t = 0$ to 5.

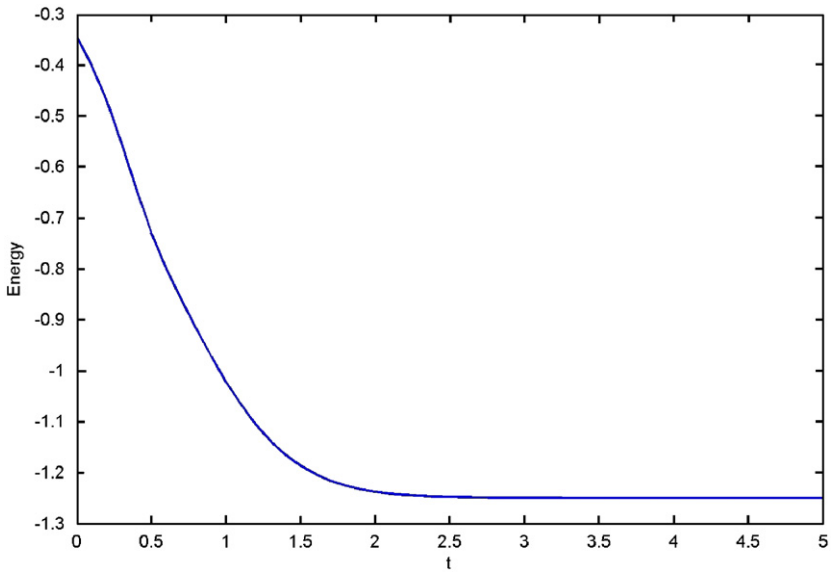

Fig. 2. The energy from $t = 0$ to 5.

and

$$w_j^{(0)} = w(x_j, 0).$$

In each time level (for example time level $n+1$) at first we set

$$\phi(\widetilde{u}_j) = \phi(u_j^{(n)}).$$

Having this, Eqs. (3.29) and (3.30) are solved as a system of linear algebraic equations for unknowns $u_j^{(n+1)}$ and $w_j^{(n+1)}$ for $j = 1, 2, \ldots, N$, assuming that $u_j^{(n)}$ and $w_j^{(n)}$ are known from the previous time level. Recompute

$$\phi(\widetilde{u}_j) = \phi(1/2u_j^{(n+1)} + 1/2u_j^{(n)}),$$

where $u_j^{(n+1)}$ is obtained recently from solving Eqs. (3.29) and (3.30). We iterate between calculating $\phi(\widetilde{u}_j)$ and computing the approximate values of the unknowns, until all the unknown quantities converge to within a prescribed number of significant figures, i.e. a predictor–corrector approach is adopted in each time level. Once the prescribed convergence is achieved, we can move on to the following time level. The resulting solutions from the current time level, provide the known values for the next iteration. This process is repeated, until reaching to the desirable time t.

4. Numerical results

To show the efficiency and accuracy of this method, it was applied to two examples. We need to iterate between finding an estimation for $\widetilde{\mathbf{u}}$ to recompute Φ and solving the system (3.29) and (3.30) for a new $\mathbf{u}$ that described in previous section. In all the cases considered, the iteration was stopped when the absolute values of all the unknowns from two consecutive iterations differ by less than 10^{-8}. In all cases tested, convergence was achieved after less than 4 iterations at each time level.

Table 1
The values of $\mathscr{M}$ at different time levels

Time t	$t = 1.0$	$t = 2.0$	$t = 3.0$	$t = 4.0$	$t = 5.0$
Mass	1.6777×10^{-6}	1.6777×10^{-6}	1.6777×10^{-6}	1.6777×10^{-6}	1.6777×10^{-6}

4.1. Example 1

In this example, we consider the C–H equation (2.1) in $[0, 2\pi]$ with the initial condition $u_0(x) = 0.5\cos x$, $\gamma = 0.03$ and $\psi(u) = 0.25u^4 - 0.5u^2$. For numerical solution we set $\Delta t = 0.01$ and $N = 64$ where

$$\{x_i\}_{i=1}^{N} = \{0, 0.1, 0.2, \ldots, 6.1, 6.2, 2\pi\}.$$

In Fig. 1 the evolution from $t = 0$ to $t = 5$ is drawn. In numerical experiments, we observed that the solution $u(x, t)$ for long times remains at patterns that are nearly piecewise constant, and the steady-state solution is very close to a piecewise constant function. In [18,39], also the same results were given. In this example, the patterns hardly changed after $t = 3$. Fig. 2 shows the time dependency of energy functional (2.9) of numerical solutions. Due to the energy functional dissipation property, the energy functional of numerical solutions theoretically decreases as time passes. Ye [39] also obtained the same graph for the energy. As described in Section 2 for the C–H equation, the total mass remains constant as time increases. In Table 1 the values of $\mathscr{M}$ are shown for different time levels up to $t = 5$ where the initial mass is zero. Here the evaluation of energy and mass were performed using the composite trapezoidal rule for integration.

4.2. Example 2

Consider the C–H equation

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial x}\left\{(-0.31 + 2.31u^2)\frac{\partial u}{\partial x} - \frac{\partial^3 u}{\partial x^3}\right\} \tag{4.1}$$

in [0,1] with initial condition

$$u(x, 0) = x^4(1 - x)^4. \tag{4.2}$$

For numerical solution $\Delta t = 0.01$ and $N = 51$ are chosen. Fig. 3 shows the numerical solution up to $t = 10$. We observed that the solution $u(x, t)$ for long times remains at patterns that are nearly constant, and the steady-state solution is very close to the constant function $u = 0.00158730158730$ that is the value of initial mass. As before the energy is plotted in Fig. 4 and the values of $\mathscr{M}$ are presented in Table 2 for different time levels to show the

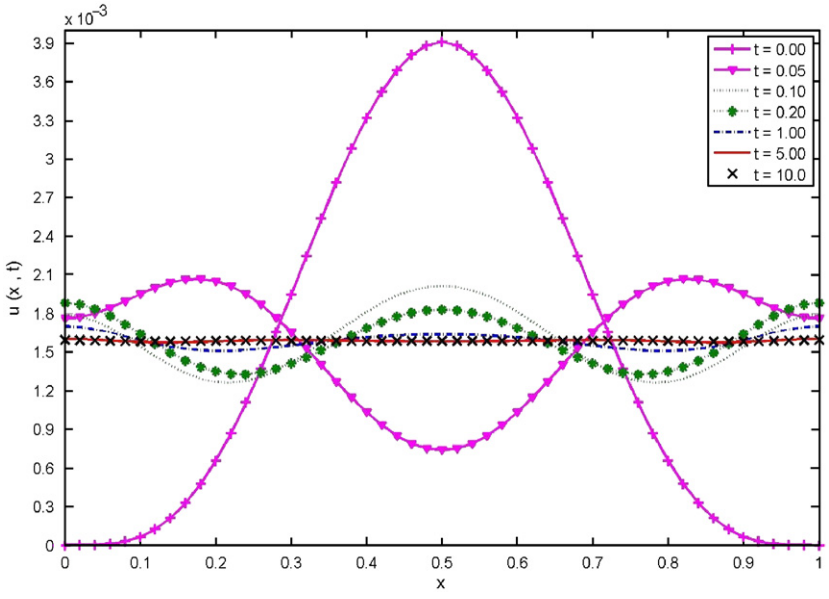

Fig. 3. The evolution from $t = 0$ to 10.

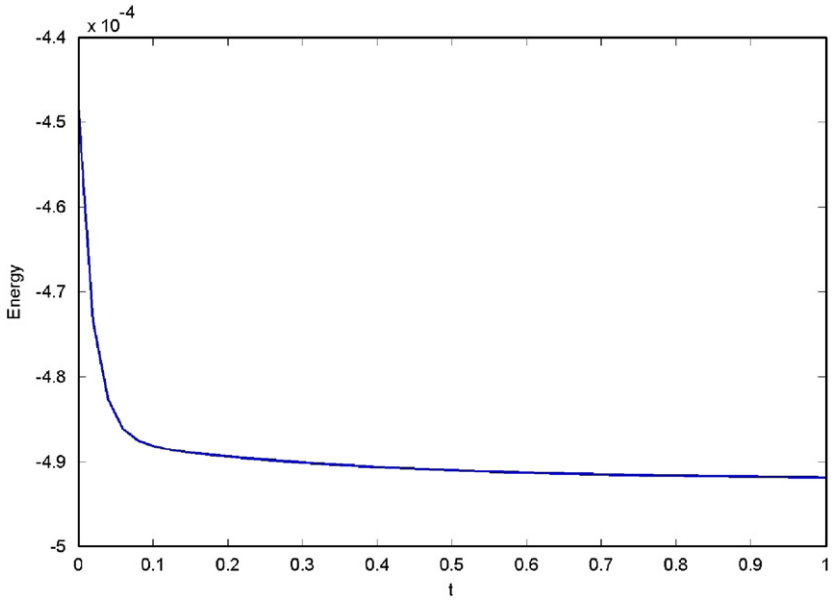

Fig. 4. The energy from $t = 0$ to 1.

Table 2
The values of $\mathscr{M}$ at different time levels

Time t	Mass
$t = 0.00$	0.00158730158730
$t = 0.05$	0.00158730158933
$t = 0.10$	0.00158730158933
$t = 0.20$	0.00158730158933
$t = 1.00$	0.00158730158933
$t = 5.00$	0.00158730158933
$t = 10.00$	0.00158730158933

energy dissipation and mass conservation respectively. Table 2 depicts that the mass remains constant with absolute error 10^{-11} at each time level t.

5. Conclusion

In this paper a numerical method based on the boundary integral equation and dual reciprocity method outlined for the one-dimensional Cahn–Hilliard (C–H) equation. The DRM was applied to eliminate the domain integrals appearing in the boundary integral equation. Linear RBF was used for DRM and an iterative scheme was employed for the time derivative. A predictor–corrector scheme was used to deal with the nonlinearity. Some numerical experiments depicted to demonstrate the efficiency and accuracy of this method. For all examples the energy dissipation and the mass conservation were investigated and the corresponding tables and graphs were presented. These are the known properties of C–H equation. In general, this proposed method could be applied for some one-dimensional partial differential equations (PDEs) such as diffusion and evolution problems. Finally we would like to mention the possibility of employing the new technique developed in the current paper to solve various problems investigated in [56–61].

Acknowledgments

The authors are very grateful to one of the reviewers for his helpful suggestions.

References

[1] Cahn JW. On spinodal decomposition. Acta Metall 1961;9:795–801.
[2] Cahn JW, Hilliard JE. Free energy of a nonuniform system I, interfacial free energy. J Chem Phys 1958;28(2):258–67.
[3] Gurtin M. Generalized Ginzburg–Landau and Cahn–Hilliard equations based on a microforce balance. Physica D 1996;92:178–92.
[4] Miranville A, Piétrus A. A new formulation of the Cahn–Hilliard equation. Nonlinear Anal: Real World Appl 2006;7:285–307.
[5] Wang SQ, Shi Q. Interdiffusion in bilinear polymer mixtures. Macromolecules 1993;26:1091–6.
[6] Jabbari E, Peppas NA. A model for interdiffusion at interfaces of polymers with dissimilar physical properties. Polymer 1995;36:575–86.
[7] Puri S, Binder K. Phenomenological theory for the formation of interfaces via the interdiffusion of layem. Phys Rev B 1991;44:9735–8.
[8] Bai F, Elliott CM, Gardiner A, Spence A, Stuart AM. The viscous Cahn–Hilliard equation: Part I: Computations. Nonlinearity 1995;8:131–60.
[9] Copetti MIM, Elliott CM. Numerical analysis of the Cahn–Hilliard equation with a logarithmic free energy. Numer Math 1992;63:39–65.
[10] Barrett JW, Blowery JF. An error bound for the finite element approximation of the Cahn–Hilliard equation with logarithmic free energy. Numer Math 1995; 72:1–20.
[11] Elliott CM, Zheng SM. On the Cahn–Hilliard equation. Arch Rat Mech Anal 1986;96:339–57.
[12] Yin J. On the existence of nonnegative continuous solutions of the Cahn–Hilliard equation. J Diff Eqs 1992;97:319–27.
[13] Alikakos N, Bates PW, Fusco G. Slow motion for the Cahn–Hilliard equation in one space dimension. J Diff Eqs 1991;90:81–135.
[14] Abels H, Wilke M. Convergence to equilibrium for the Cahn–Hilliard equation with a logarithmic free energy. Nonlinear Anal 2007;67: 3176–93.
[15] Carr J, Gurtin ME, Slemrod M. Structured phase transitions on a finite interval. Arch Rat Mech Anal 1984;86:317–51.
[16] Novick-Cohen A, Segel LA. Nonlinear aspects of the Cahn–Hilliard equation. Physica D 1984;10:277–98.
[17] Novick-Cohen A. Energy method for the Cahn–Hilliard equation. Quart Appl Math 1988;XLVI:681–90.
[18] Elliott CM, French DA. Numerical studies of the Cahn–Hilliard equation for phase separation. IMA J Appl Math 1987;38:97–128.
[19] Elliott CM, French DA. A nonconforming finite element method for the two dimensional Cahn–Hilliard equation. SIAM J Numer Anal 1989;26: 884–903.
[20] Elliott CM, Larsson S. Error estimates with smooth and nonsmooth data for a finite element method for the Cahn–Hilliard equation. Math Comput 1992; 58:603–30.
[21] Elliott CM, French DA, Milner FA. A second order splitting method of the Cahn–Hilliard equation. Numer Math 1989;54:575–90.
[22] French DA, Jensen S. Long-time behaviour of arbitrary order continuous time Galerkin schemes for some one-dimensional phase transition problems. IMA J Numer Anal 1994;14:421–42.
[23] Dean EJ, Glowinski R, Trevas DA. An approximate factorization least squares solution method for a mixed finite element approximation of the Cahn–Hilliard equation. Jpn J Ind Appl Math 1996;13:495–517.

[24] Bañas L, Nürnberg R. Adaptive finite element methods for Cahn–Hilliard equations. J Comput Appl Math, 2008;218(1):2–11.
[25] Kay D, Welford R. A multigrid finite element solver for the Cahn–Hilliard equation. J Comput Phys 2006;212:288–304.
[26] Choo SM, Lee YJ. A discontinuous Galerkin method for the Cahn–Hilliard equation. J Appl Math Comput 2005;18:113–26.
[27] Wells GN, Kuhl E, Garikipati K. A discontinuous Galerkin method for the Cahn–Hilliard equation. J Comput Phys 2006;218:860–77.
[28] Xia Y, Xu Y, Shu CW. Local discontinuous Galerkin methods for the Cahn–Hilliard type equations. J Comput Phys 2007;227(1):472–91.
[29] Furihata D, Onda T, Mori M. A finite difference scheme for the Cahn–Hilliard equation based on a Lyapunov functional. GAKUTO Int Ser Math Sci Appl 1993;2:347–58.
[30] Furihata D. Finite difference schemes for $\partial u/\partial t = (\partial/\partial x)^{\alpha}\delta G/\delta u$ that inherit energy conservation or dissipation property. J Comput Phys 1999;156: 181–205.
[31] Furihata D. A stable and conservative finite difference scheme for the Cahn–Hilliard equation. Numer Math 2001;87:675–99.
[32] Choo SM, Chung SK. Conservative nonlinear difference scheme for the Cahn–Hilliard equation. Comput Math Appl 1998;36(7):31–9.
[33] Choo SM, Chung SK, Kim KI. Conservative nonlinear difference scheme for the Cahn–Hilliard equation—II. Comput Math Appl 2000;39:229–43.
[34] Khiari N, Achouri T, Ben Mohamed ML, Omrani K. Finite difference approximate solutions for the Cahn–Hilliard equation. Numer Methods Partial Differential Equations 2007;23:437–55.
[35] Kim J. A numerical method for the Cahn–Hilliard equation with a variable mobility. Commun Nonlinear Sci Numer Simul 2007;12:1560–71.
[36] de Mello EVL, Filho OT da S. Numerical study of the Cahn–Hilliard equation in one, two and three dimensions. Physica A 2005;347:429–43.
[37] Ye X, Cheng X. The Fourier spectral method for the Cahn–Hilliard equation. Appl Math Comput 2005;171:345–57.
[38] He Y, Liu Y, Tang T. On large time-stepping methods for the Cahn–Hilliard equation. Appl Numer Math 2007;57:616–28.
[39] Ye X. The Fourier collocation method for the Cahn–Hilliard equation. Comput Math Appl 2002;44:213–29.
[40] Ye X. The Legendre collocation method for the Cahn–Hilliard equation. J Comput Appl Math 2003;150:87–108.
[41] Danumjaya P, Nandakumaran AK. Orthogonal cubic spline collocation method for the Cahn–Hilliard equation. Appl Math Comput 2006;182: 1316–29.
[42] Momani Sh, Erturk VS. A numerical scheme for the solution of viscous Cahn–Hilliard equation. Numer Methods Partial Differential Equations 2008;24(2):663–69.
[43] Mitlin V. On the μ-transform: applications to the Cahn–Hilliard equation. Phys Lett A 2006;358:142–8.
[44] Hongjun G, Changchun L. Instability of traveling waves of the convective–diffusive Cahn–Hilliard equation. Chaos, Solitons and Fractals 2004;20: 253–8.
[45] Brebbia CA, Nardini D. Dynamic analysis in solid mechanics by an alternative boundary element procedure. Int J Soil Dyn Earthquake Eng 1983;2:228–33.
[46] Partridge PW, Brebbia CA. The dual reciprocity boundary element method for the Helmholtz equation. In: Brebbia CA, Choudouet-Miranda A, editors. Proceedings of the International Boundary Elements Symposium. Berlin: Computational Mechanics Publications, Springer; 1990. p. 543–55.
[47] Karur SR, Ramachandran PA. Radial basis function approximation in dual reciprocity method. Math Comput Modelling 1994;20(7):59–70.
[48] Ahmad S, Banerjee PK. Free vibration analysis by BEM using particular integrals. J Eng Mech Div ASCE 1986;112:682–95.
[49] Yang MT, Park KH, Banerjee PK. 2D and 3D transient heat conduction analysis by BEM via particular integrals. Comput Methods Appl Mech Eng 2002;191: 1701–22.
[50] Chino E, Tosaka N. Dual reciprocity boundary element analysis of time-independent Burger's equation. Eng Anal Bound Elem 1998;21:261–70.
[51] Ang WT. A numerical method for the wave equation subject to a non-local conservation condition. Appl Numer Math 2006;56:1054–60.
[52] Dehghan M, Mirzaei D. The boundary integral equation approach for numerical solution of the one-dimensional Sine-Gordon equation. Numer Methods Partial Differential Equations, in press.
[53] Chen JT, Wong FC. Analytical derivations for one-dimensional eigenproblems using dual boundary element method and multiple reciprocity method. Eng Anal Bound Elem 1997;20:25–33.
[54] Chen JT, Lee JW. On the spurious eigensolutions for the real-part boundary element method. Eng Anal Bound Elem, accepted for publication.
[55] Dehghan M. Finite difference procedures for solving a problem arising in modeling and design of certain optoelectronic devices. Math Comput Simul 2006;71:16–30.
[56] Dehghan M. Parameter determination in a partial differential equation from the overspecified data. Math Comput Modell 2005;41:196–213.
[57] Dehghan M. Identification of a time-dependent coefficient in a partial differential equation subject to an extra measurement. Numer Methods Partial Differ Equations 2005;21:611–22.
[58] Dehghan M. Implicit collocation technique for heat equation with non-classic initial condition. Int J Non-Linear Sci Numer Simul 2007;7:447–50.
[59] Dehghan M. The one-dimensional heat equation subject to a boundary integral specification. Chaos Solitons Fractals 2007;32:661–757.
[60] Shakeri F, Dehghan M. Numerical solution of the Klein–Gordon equation via He's variational iteration method. Nonlinear Dyn 2008;51:89–97.
[61] Dehghan M. A computational study of the one-dimensional parabolic equation subject to nonclassical boundary specifications. Numer Methods Partial Differ Equations 2006;22:220–57.