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Particular solutions of the multi-Helmholtz-type equation

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Abstract

This paper is devoted to finding analytic particular solutions to a class of fourth-order partial differential equations (PDEs). This is done by using polyharmonic spline approximations to the inhomogeneity. Both the 2-D and the 3-D cases are considered, the 3-D case being simpler than the 2-D one. The solutions to the 3-D case are obtained by using the Neumann expansion of the inverse of the homogeneous operator. For the 2-D case, in addition to the finding of the solutions to the inhomogeneous equation, it is necessary to find an appropriate basis for the radial form of the homogeneous equation. These solutions may have independent interest in obtaining t-Trefftz solutions to the homogeneous multi-Helmholtz-type equation.
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1. Introduction

In recent years, there has been considerable interest in obtaining particular solutions to a variety of partial differential equations (PDEs.) Among these have been the Poisson equation [1], the inhomogeneous Helmholtz and modified Helmholtz equations [1], the inhomogeneous bi-harmonic equation [2], and others. The main reason for doing this has been to eliminate the inhomogeneity of these equations, so that boundary-type methods, such as the boundary element method (BEM), the method of fundamental solutions (MFS), and other Trefftz methods can be employed to solve the resulting homogeneous equations. An important application of these ideas occurs when one wishes to solve time-dependent PDEs, such as the diffusion and wave equations. Various discretization techniques lead one to solve a sequence of inhomogeneous modified Helmholtz equations. As indicated above, to solve the inhomogeneous equation, one needs to find particular solutions for these equations. Over the last 25 years, a variety of techniques have been proposed for doing this. Among these have been integral methods, as proposed by Atkinson [3], domain embedding methods [4], polynomial approximation methods [4], and others. However, probably the most popular method for doing this is the DRM (dual reciprocity method), which uses radial bases functions (RBFs) to approximate the right-hand side. Then, approximate particular solutions are obtained by analytically determining the particular solution corresponding to each basis element. As shown in [5], this required some considerable mathematical skill when thin plate and higher order splines are used as approximating basis functions.

In this paper, we extend this approach to calculate particular solutions for the fourth-order PDE

$$\Delta^2 u(P) + \varepsilon\lambda^4 u(P) = f(P)(\varepsilon = \pm 1), \tag{1}$$

where Δ denotes the Laplacian and $P \in R^d$ ($d = 2$ or 3).

We refer to Eq. (1) as the multi-Helmholtz equation when $\varepsilon = 1$ and the modified multi-Helmholtz equation when $\varepsilon = -1$. Eq. (1) arises in the theory of free flexural vibration of a loaded uniform thin plate [6] and, as a result of applying various discretization techniques or the Laplace transform to the equation [7],

$$D\Delta^2 w(x, y, t) = P(x, y, t) - m\frac{\partial^2}{\partial t^2} w(x, y, t). \tag{2}$$

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D is the rigidity of the plate, m is the mass of the plate, $P(x, y, t)$ is the applied force, and $w(x, y, t)$ is the vertical displacement as described in [7].

One can also consider equations similar to Eqs. (1) and (2) for the 3-D case, describing vibrations of a shell/solid.

The paper is structured in the following way: In Section 2, we give a brief description of the DRM, including approximation methods by splines. In Section 3, we obtain a particular solution for the 3-D case, as the 3-D case happens to be much easier than the 2-D one.

To calculate a particular solution in the 2-D case, we have to first consider obtaining a basis for the radially symmetric version of Eq. (1). This is done in Section 4. In Section 5, we complete the derivation of particular solutions when the right-hand side of Eq. (1) is approximated by radial splines. Finally, in Section 6, we give our conclusions and some directions for future research.

2. The dual reciprocity method (DRM)

The DRM is a technique for obtaining particular solutions to an inhomogeneous PDE. For this purpose, we consider the equation

$$Lu = f, \tag{3}$$

where L is a linear operator. The DRM is based on the classical technique of using particular solutions to solve Eq. (3). In this case, we decompose the solution of Eq. (3) into the sum

$$u = v + u_p, \tag{4}$$

where u_p is a particular solution to Eq. (3), which does not necessarily satisfy any of the boundary conditions and v satisfies the homogeneous equation

$$Lv = 0. \tag{5}$$

When L is a linear partial differential operator, one can often solve Eq. (5) by boundary integral or Trefftz methods. In general, the task of obtaining u_p is a difficult one and usually, only approximate particular solutions can be obtained. As it was pointed out in the Introduction, a number of techniques have been proposed for doing this. Details of many of these can be found in [8]. In this paper, we focus on the DRM as originally given in the paper of Nardini and Brebia [9]. In this method, we first approximate f in Eq. (3) by a function $\hat{f}_N$ where

$$\hat{f}_N = \sum_{j=1}^{N} a_{jN}\phi_{jN}, \tag{6}$$

where $\{\phi_{jN}\}$ is a set of linearly independent basis elements. Then, an approximate particular solution, $\hat{u}_{pN}$, is given by

$$\hat{u}_{pN} = \sum_{j=1}^{N} a_{jN}\psi_{jN}, \tag{7}$$

where ψ_{jN} is an analytical solution to the equation

$$L\psi_{jN} = \phi_{jN}, \quad j = 1, 2, \ldots, N. \tag{8}$$

In this paper, we approximate f by radial spline of the form

$$\hat{f}_N = \sum_{j=1}^{N} a_{jN} r_{jN}^{2n} \ln r_j N + p(x, y) \tag{9}$$

in R^2, where $r_{jN} = \|P - P_{jN}\|$ is the Euclidean distance between P and P_{jN} and $p(x, y)$ is a polynomial of degree n. In R^3, the radial spline is of the form

$$\hat{f}_N = \sum_{j=1}^{N} a_{jN} r_{jN}^{2n-1} + p(x, y, z), \tag{10}$$

where again $r_{jN} = \|P - P_{jN}\|$ and $p(x, y, z)$ is a polynomial of degree n. In Eqs. (9) and (10), and the remaining equations of this section, n is fixed and could take any positive integer value. In these cases, to find approximate particular solutions, we further decompose $\hat{f}_N$ as

$$\hat{f}_N = v_N + p, \tag{11}$$

where

$$v_N = \sum_{j=1}^{N} a_{jN} r_{jN}^{2n-1}, \tag{12}$$

in R^3 and

$$v_N = \sum_{j=1}^{N} a_{jN} r_{jN}^{2n-1} \ln r_{jN} \tag{13}$$

in R^2. Hence, to complete the derivation of the particular solution, we need to solve the equations

$$L\psi_{jN} = r_{jN}^{2n} \ln r_{jN}, \quad j = 1, 2, \ldots, N \tag{14}$$

and

$$L\psi_{N+1} = p(x, y) \tag{15}$$

in R^2. In R^3, we have to solve the equations

$$L\psi_{jN} = r_{jN}^{2n-1}, \quad j = 1, 2, \ldots, N \tag{16}$$

and

$$L\psi_{N+1} = p(x, y, z). \tag{17}$$

In Sections 3 and 5, we will consider the solutions of Eqs. (14) and (16). To obtain the solutions of Eqs. (15) and (17), one can proceed in two ways. In the first, the solutions are obtained by the method of undetermined coefficients. In the second, we specialize to the case when L is the multi-Helmholtz-type operator

$$L = \Delta^2 + \varepsilon\lambda^4 = \varepsilon\lambda^4\left(1 + \frac{\Delta^2}{\varepsilon\lambda^4}\right). \tag{18}$$

In this paper, we use the second way. Then, to solve (15) and (17), we obtain

$$\psi_{N+1} = L^{-1}p, \tag{19}$$

where

$$L^{-1} = \frac{1}{\varepsilon\lambda^4}\left(1 + \frac{\Delta^2}{\varepsilon\lambda^4}\right)^{-1} = -\sum_{k=1}^{\infty} \frac{\Delta^{2k}}{(-\varepsilon\lambda^4)^{k+1}}. \tag{20}$$

Since Δ^2 is a fourth-order operator and p is a polynomial, then $\Delta^{2k}p = 0$ when k is sufficiently big. So, actually, the series in (20) is a finite sum.

3. Particular solutions to the multi-Helmholtz-type equation in R^3

In this section, we will calculate particular solutions to the multi-Helmholtz-type equation, Eq. (1), when the right-hand side is a radial spline. Since, in the previous section, we calculated the particular solution for the polynomial part of the right-hand side, it suffices to obtain the particular solution to the equation

$$\Delta^2 u + \varepsilon\lambda^4 u = r^{2n-1}. \tag{21}$$

Since the right-hand side is radially symmetric, it suffices to find particular solutions to the ODE

$$\Delta_{r,3}^2 u + \varepsilon\lambda^4 u = r^{2n-1}, \tag{22}$$

where $\Delta_{r,3}$ is the operator

$$\Delta_{r,3} = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{d}{dr}\right). \tag{23}$$

To solve Eq. (22), we use a technique analogous to the one used to obtain the particular solution for the polynomial part in the previous section. Denoting the particular solution by u_p, some algebraic manipulation yields

$$u_p = \frac{1}{\varepsilon\lambda^4}\left(1 - \frac{-\varepsilon\Delta_{r,3}^2}{\lambda^4}\right)^{-1} r^{2n-1}. \tag{24}$$

Then, expanding the inverse operator in Eq. (24), we obtain

$$\begin{aligned} u_p &= \frac{1}{\varepsilon\lambda^4}\sum_{k=0}^{\infty} \frac{(-\varepsilon)^k}{\lambda^{4k}} \Delta_{r,3}^{2k}(r^{2n-1}) \\ &= \sum_{k=0}^{[(2n-1)/4]} \frac{(-\varepsilon)^{k+1}}{\lambda^{4k+4}} \Delta_{r,3}^{2k}(r^{2n-1}), \end{aligned} \tag{25}$$

where $[x]$ is the biggest integer that is less than or equal to x.

By successive differentiation, we find $\Delta_{r,3}(r^{2n-1}) = (2n)(2n-1)r^{2n-3}$, $\Delta_{r,3}^2(r^{2n-1}) = (2n)(2n-1)(2n-2)(2n-3) r^{2n-5} = \frac{(2n)!}{(2n-4)!} r^{2n-5}$, $\Delta_{r,3}^4(r^{2n-1}) = (2n)(2n-1)(2n-2)(2n-3)(2n-4)(2n-5)(2n-6)(2n-7)r^{2n-9} = \frac{(2n)!}{(2n-8)!} r^{2n-9}$. By indication, it follows that

$$\Delta_{r,3}^{2k}(r^{2n-1}) = \frac{(2n)!}{(2n-4k)!} r^{2n-4k-1}. \tag{26}$$

Finally, substituting Eq. (26) into Eq. (25) gives the particular solution u_p in the form

$$u_p = -\sum_{k=0}^{[(2n-1)/4]} \frac{(-\varepsilon)^{k+1}}{\lambda^{4k+4}} \frac{(2n)!}{(2n-4k)!} r^{2n-4k-1}. \tag{27}$$

4. Particular solutions to the multi-Helmholtz-type equation in R^2

4.1. Multi-Bessel-type ordinary differential equation

In order to find a particular solution to the 2-D multi-Helmholtz equation with a radial spline as the right-hand side, it is necessary to find an appropriate basis to the homogeneous ODE

$$\Delta_r^2 u + \varepsilon u = 0, \quad (\varepsilon = \pm 1), \tag{28}$$

which we call the multi-Bessel-type ODE. If $\varepsilon = 1$, we refer to Eq. (28) as the multi-Bessel ODE and, if $\varepsilon = -1$, we refer to Eq. (28) as the modified multi- Bessel ODE. Our goal is to find four linearly independent particular solutions, which we denote $BB_k(r)$, $k = 1, 2, 3, 4$. Then, the general solution of Eq. (28) is given by

$$u(r) = \sum_{k=1}^{4} C_k BB_k(r), \tag{29}$$

where C_k $(k = 1, 2, 3, 4)$ are arbitrary real constants. We know that, for the case of $\varepsilon = 1$, Eq. (28) was treated in [10]. However, the basis given in this reference happens to create difficulties while trying to find particular solutions to the inhomogeneous multi-Bessel-type/multi-Helmholtz-type equation in R^2. Using the method of Frobenius, we look for a solution in the form

$$u = \sum_{n=0}^{\infty} a_n r^{n+\alpha}. \tag{30}$$

By straightforward differentiation, we find

$$\Delta_r u = \sum_{n=0}^{\infty} (n+\alpha)^2 a_n r^{n+\alpha-2} \tag{31}$$

and

$$\Delta_r^2 u = \sum_{n=0}^{\infty} (n+\alpha)^2 (n+\alpha-2)^2 a_n r^{n+\alpha-4}. \tag{32}$$

Substituting Eq. (32) into Eq. (28), we get

$$\sum_{n=0}^{\infty} (n+\alpha)^2 (n+\alpha-2)^2 a_n r^{n+\alpha-4} + \varepsilon \sum_{n=4}^{\infty} (n+\alpha)^2 (n+\alpha-2)^2 a_{n-4} r^{n+\alpha-4} = 0. \tag{33}$$

For $n = 0$, we obtain the indicial equation of the multi-Bessel-type ODE

$$\alpha^2(\alpha-2)^2 = 0. \tag{34}$$

For the case $\alpha = 0$, we can start solving Eq. (28) by finding a solution $u_1 = BB_1(r)$ in the form of a power series

$$BB_1(r) = \sum_{n=0}^{\infty} a_n r^n. \tag{35}$$

By repeated differentiation, we find that

$$\Delta_r^2 u_1 = \frac{a_1}{r} + \sum_{n=3}^{\infty} n^2(n-2)^2 a_n r^{n-4}. \tag{36}$$

Substituting Eq. (36) into Eq. (28), the coefficients a_n $(n = 0, 1, 2, \ldots)$ can be determined from

$$\frac{a_1}{r} + \sum_{n=3}^{\infty} n^2(n-2)^2 a_n r^{n-4} + \varepsilon \sum_{n=0}^{\infty} a_n r^n = 0. \tag{37}$$

Hence, choosing $a_0 = 1$ and $a_1 = a_2 = a_3 = 0$, a_n satisfy the equation

$$n^2(n-2)^2 a_n + \varepsilon a_{n-4} = 0, \quad (n = 4, 5, 6, \ldots). \tag{38}$$

Since $a_1 = a_2 = a_3 = 0$, it follows that $a_{4n+1} = a_{4n+2} = a_{4n+3} = 0$ $(n = 0, 1, 2, \ldots)$. For a_{4n}, from Eq. (38), it follows that

$$(4n)^2(4n-2)^2 a_{4n} + \varepsilon a_{4n-4} = 0, \quad (n = 1, 2, 3, \ldots). \tag{39}$$

Eq. (39) can be rewritten as

$$\frac{a_{4k}}{a_{4k-4}} = \frac{-\varepsilon}{2^4(2k)^2(2k-1)^2}. \tag{40}$$

Hence, using Eq. (40), we find

$$a_{4n} = a_0 \prod_{k=1}^{n} \frac{a_{4k}}{a_{4k-4}} = (1) \prod_{k=1}^{n} \frac{(-\varepsilon)}{2^4(2k)^2(2k-1)^2} = \frac{(-\varepsilon)^n}{2^{4n}(2n)!^2}. \tag{41}$$

So, finally, we obtain a particular solution to the multi-Bessel-type ODE as

$$u_1 = BB_1(r) = \sum_{n=0}^{\infty} \frac{(-\varepsilon)^n r^{4n}}{2^{4n}(2n)!^2}. \tag{42}$$

Next, we look for another particular solution in the form

$$u_3 = BB_3(r) = (\ln r) \sum_{n=0}^{\infty} a_{4n} r^{4n} + \sum_{n=0}^{\infty} b_n r^{4n}. \tag{43}$$

By successive differentiation, we obtain

$$\Delta_r^2 u_3 = (\ln r) \sum_{n=1}^{\infty} (4n)^2(4n-2)^2 a_{4n} r^{4n-4} + \sum_{n=1}^{\infty} 4(4n)(4n-1)(4n-2) a_{4n} r^{4n-4} + \sum_{n=1}^{\infty} (4n)^2(4n-2)^2 b_n r^{4n-4}. \tag{44}$$

To find the coefficients b_n, we begin by simplifying the first term in Eq. (44) using the known values of a_{4n} from Eq. (41).

$$\sum_{n=1}^{\infty} (4n)^2(4n-2)^2 a_{4n} r^{4n-4} = \sum_{n=1}^{\infty} \frac{2^4(2n)^2(2n-1)(-\varepsilon)^n}{2^{4n}(2n)!^2} r^{4n-4} = \sum_{n=1}^{\infty} \frac{(-\varepsilon)^n r^{4n-4}}{2^{4n-4}(2n-2)!^2} = \sum_{n=0}^{\infty} \frac{(-\varepsilon)^{n+1} r^{4n}}{2^{4n}(2n)!^2} = -\varepsilon u_1. \tag{45}$$

Using Eqs. (43)–(45), it follows that

$$\Delta_r^2 u_3 + \varepsilon u_3 = \varepsilon \sum_{n=0}^{\infty} b_n r^{4n} + \sum_{n=0}^{\infty} (4n+4)^2(4n+2)^2 b_{n+1} r^{4n} + \sum_{n=0}^{\infty} 4(4n+4)(4n+3)(4n+2) a_{4n+4} r^{4n-4} = 0. \tag{46}$$

Comparing the coefficients of the like terms in Eq. (46) yields the difference equation

$$(4n+4)^2(4n+2)^2 b_{n+1} + \varepsilon b_n + 4(4n+4)(4n+3)(4n+2) a_{4n+4} = 0. \tag{47}$$

Multiplying Eq. (47) by $(4n)!!^2(-\varepsilon)^{n+1}$ and using the values of a_{4n+4} from Eqs. (41), (47) becomes

$$(4n+4)!!^2(-\varepsilon)^{n+1} b_{n+1} - \varepsilon^2(-\varepsilon)^n (4n)!!^2 b_n = \frac{-(-\varepsilon)^{n+1} 4(4n+3)(4n)!!(4n+4)!!(-\varepsilon)^{n+1}}{2^{4n+4}(2n+2)!^2}. \tag{48}$$

Simplification of Eq. (48) and the substitution

$$c_n = 2^{4n}(2n)!^2(-\varepsilon)^n b_n, \tag{49}$$

transforms Eq. (48) into

$$c_{n+1} - c_n = -\frac{1}{2n+1} - \frac{1}{2n+2}. \tag{50}$$

Choosing $b_0 = 0$, i.e. $c_0 = 0$, one can telescope Eq. (50) to get

$$c_n = c_0 + \sum_{k=0}^{n-1} (c_{k+1} - c_k) = 0 + \sum_{k=0}^{n-1} \left(-\frac{1}{2k+1} - \frac{1}{2k+2} \right) = -\sum_{k=1}^{2n} \frac{1}{k}. \tag{51}$$

Hence,

$$b_n = \frac{-(-\varepsilon)^2 \sum_{k=1}^{2n} (1/k)}{2^{4n}(2n)!^2}. \tag{52}$$

Finally, we obtain the formula for u_3 as

$$u_3 = BB_3(r) = BB_1(r) \ln r - \sum_{n=1}^{\infty} \frac{(-\varepsilon)^n \sum_{k=1}^{2n} (1/k)}{2^{4n}(2n)!^2} r^{4n}, \tag{53}$$

where $BB_1(r)$ is given by Eq. (42).

Next, we define u_2 and u_4 as follows:

$$u_2 = BB_2(r) = -4\varepsilon \Delta_r BB_1(r) \tag{54}$$

and

$$u_4 = BB_4(r) = -4\varepsilon\Delta_r BB_3(r). \tag{55}$$

Theorem 1. *$BB_2(r)$ and $BB_4(r)$ are particular solutions of the multi-Bessel-type Equation* (28).

Proof. We already know that

$$\Delta_r^2 BB_k(r) + \varepsilon BB_k(r) = 0, \quad k = 1, 3. \tag{56}$$

Applying the operator $-4\varepsilon\Delta_r$ to both sides of Eq. (56), we obtain

$$\Delta_r^2(-4\varepsilon\Delta_r BB_k(r)) + \varepsilon(-4\varepsilon\Delta_r BB_k(r)) = 0. \tag{57}$$

Recalling the definitions of Eqs. (54) and (55), one can rewrite Eq. (57) as

$$\Delta_r^2 BB_{k+1}(r) + \varepsilon BB_{k+1}(r) = 0, \quad k = 1, 3. \tag{58}$$

This is what we needed to prove. ■

Using the definitions, Eqs. (54) and (55), we find the formulas for u_2 and u_4 as follows:

$$u_2 = BB_2(r) = \sum_{n=0}^{\infty} \frac{(-\varepsilon)^n r^{4n+2}}{2^{4n}(2n+1)!^2} \tag{59}$$

and

$$u_4 = BB_4(r) + BB_2(r)\ln r + \sum_{n=0}^{\infty} \frac{(-\varepsilon)^n r^{4n+2}}{2^{4n}(2n+1)!(2n+2)!} - \sum_{n=1}^{\infty} \frac{(-\varepsilon)^{n+1}\sum_{k=1}^{2n}(1/k)}{2^{4n-4}(2n-1)!^2} r^{4n-3}. \tag{60}$$

Theorem 2. $BB_k(r) = \frac{1}{4}\Delta_r BB_{k+1}(r)$, $k = 1, 3$.

Proof. Applying the operator $\frac{1}{4}\Delta_r$, to Eqs. (54) and (55), we obtain $\frac{1}{4}\Delta_r BB_{k+1}(r) = -\varepsilon\Delta_r^2 BB_k(r) = -\varepsilon(\Delta_r^2 BB_k(r) + \varepsilon BB_k(r)) + BB_k(r) = BB_k(r)$. This concludes the proof of Theorem 2. ■

Theorem 3. *$BB_1(r)$ and $BB_2(r)$ are entire functions* (*analytic for all finite complex values of r*). *$BB_3(r)$ and $BB_4(r)$ have branch points at $r = 0$ and $r =$* infinity *and are analytic elsewhere in the complex r-plane*.

Proof. The statements in Theorem 3 are justified by the facts that all power series in Eqs. (42), (53), (59), and (60) have radii of convergence ∞ and the presence of $\ln(r)$ in $BB_3(r)$ and $BB_4(r)$. ■

Theorem 4. *$BB_k(r)$, ($k = 1, 2, 3, 4$) are linearly independent and hence, the general solution of the multi-Bessel-type ODE, Eq.* (28), *is given by $u(r) = \sum_{k=1}^{4} c_k BB_k(r)$*, i.e. Eq. (29).

Proof. The proof is derived directly from Theorem 3 and the presence of different powers of r in $BB_1(r)$ and $BB_3(r)$ on one side and $BB_2(r)$ and $BB_4(r)$ on the other side. ■

4.2. Particular solution to the multi-Bessel-type inhomogeneous equation with spline right-hand side

We consider the equation

$$\Delta_r^2 u + \varepsilon\lambda^4 u = r^{2m}\ln r. \tag{61}$$

In order to find a solution of maximum differentiability, we look for a solution of the form

$$u = A_3 BB_3(\lambda r) + A_4 BB_4(\lambda r) + w, \tag{62}$$

where

$$w = \sum_{n=0}^{m} c_n r^{2n}\ln r + \sum_{n=0}^{m} d_n r^{2n}. \tag{63}$$

Separating the odd and the even indices, Eq. (63) could be rewritten as

$$w = \sum_{n=0}^{[(m-1)/2]} c_{2n+1} r^{4n+2}\ln r + \sum_{n=0}^{[m/2]} c_{2n} r^{4n}\ln r + \sum_{n=0}^{[(m-1)/2]} d_{2n+1} r^{4n+2} + \sum_{n=0}^{[m/2]} d_{2n} r^{4n}, \tag{64}$$

as w satisfies the equation

$$\Delta_r^2 w + \varepsilon\lambda^4 w = r^{2m}\ln r. \tag{65}$$

Separating the singular terms of the form $Cr^{4n}\ln r$ of u in Eq. (62), we get the function $g(r)$ defined by

$$\begin{aligned} g(r) &= A_3 BB_3(\lambda r) + \sum_{n=0}^{[m/2]-1} c_{2n} r^{4n}\ln r + c_{2[m/2]} r^{4[m/2]}\ln r \\ &= Analyt_1(r) + A_3 \sum_{n=0}^{[m/2]} \frac{(-\varepsilon)^n \lambda^{4n} r^{4n}\ln r}{2^{4n}(2n)!^2} \\ &\quad + \sum_{n=0}^{[m/2]} c_{2n} r^{4n}\ln r + O\left(r^{4[m/2]+4}\ln r\right), \end{aligned} \tag{66}$$

where $Analyt_1(r)$ is an analytic function of r. In order to cancel out the lowest order singular terms in Eq. (66), we choose

$$c_{2n} = \frac{-A_3(-\varepsilon)^n\lambda^{4n}}{2^{4n}(2n)!^2}, \quad \left(n = 0, 1, 2, \ldots, [m/2]-1\right). \tag{67}$$

In a similar fashion, we separate the singular terms of u in Eq. (62) that are of the form $Cr^{4n+2}\ln r$ getting the function $h(r)$ defined by

$$\begin{aligned} h(r) &= A_4 BB_4(\lambda r) + \sum_{n=0}^{[(m-1)/2]} c_{2n+1} r^{4n+2}\ln r \\ &= A_4 \sum_{n=0}^{[(m-1)/2]} \frac{(-\varepsilon)^n \lambda^{4n+2} r^{4n+2}\ln r}{2^{4n}(2n+1)!^2} \\ &\quad + \sum_{n=0}^{[(m-3)/2]} c_{2n+1} r^{4n+2}\ln r + c_{2[(m-1)/2]+1} r^{4[(m-1)/2]+2}\ln r \\ &\quad + O\left(r^{4[(m-1)/2]+6}\ln r\right) + Analyt_2(r), \end{aligned} \tag{68}$$

where $Analyt_2(r)$ is an analytic function of r. In order to cancel out the lowest order singular terms in Eq. (68), we choose

$$c_{2n+1} = \frac{-A_4(-\varepsilon)^n \lambda^{4n+2}}{2^{4n}(2n+1)!^2}, \quad \big(n = 0, 1, 2, \ldots, [(m-3)/2]\big). \tag{69}$$

With the values of c_n found in Eqs. (67) and (69), the solution u of Eq. (62) is guaranteed to be $2m$ times differentiable at point $r = 0$. Next, we need to find the values of A_3, A_4, and d_n ($n = 0, 1, \ldots, m$) by substituting Eq. (64) into Eq. (65). After finding the necessary derivatives of w, substituting into Eq. (65) and cancelation of some of the terms, Eq. (65) becomes

$$\begin{aligned}
&-\sum_{n=0}^{[(m-3)/2]} \frac{A_4(-\varepsilon)^{n+1}(4n+5)\lambda^{4n+6} r^{4n+2}}{2^{4n}(2n+3)!(2n+1)!} \\
&-\sum_{n=0}^{[(m-2)/2]} \frac{A_3(-\varepsilon)^{n+1}(4n+3)\lambda^{4n+4} r^{4n}}{2^{4n}(2n)!(2n+2)!} \\
&+\sum_{n=0}^{[(m-3)/2]} d_{2n+3}(4n+4)^2(4n+6)^2 r^{4n+2} \\
&+\sum_{n=0}^{[(m-2)/2]} d_{2n+2}(4n+4)^2(4n+2)^2 r^{4n} \\
&+\varepsilon\lambda^4 \sum_{n=0}^{[(m-3)/2]} d_{2n+1} r^{4n+2} \\
&+\varepsilon\lambda^4 \sum_{n=0}^{[(m-2)/2]} d_{2n} r^{4n} + \varepsilon\lambda^4 d_{m-1} r^{2m-1} + \varepsilon\lambda^4 d_m r^{2m} \\
&+\frac{A_4(-\varepsilon)^{[(m+1)/2]}\lambda^{4[(m-1)/2]+6} r^{4[(m-1)/2]+2} \ln r}{2^{4[(m-1)/2]}\big(2[(m-1)/2]+1\big)!^2} \\
&+\frac{A_3(-\varepsilon)^{[m/2]+1}\lambda^{4[m/2]+4} r^{4[m/2]} \ln r}{2^{4[m/2]}\big(2[m/2]\big)!^2} = r^{2m} \ln r.
\end{aligned} \tag{70}$$

Equating the coefficients of the like terms in Eq. (70), we obtain the system given by Eqs. (71)–(74):

$$\begin{aligned}
&\varepsilon\lambda^4 d_{2n+1} + 2^4(2n+3)^2(2n+2)^2 d_{2n+3} = -\frac{A_4(-\varepsilon)^{n+1}\lambda^{4n+6}}{2^{4n}(2n+1)!^2} \\
&\quad \times \left(\frac{1}{2n+2} + \frac{1}{2n+3}\right), \quad n = 0, 1, 2, \ldots, [(m-3)/2],
\end{aligned} \tag{71}$$

$$\begin{aligned}
&\varepsilon\lambda^4 d_{2n} + 2^4(2n+1)^2(2n+2)^2 d_{2n+2} = -\frac{A_3(-\varepsilon)^{n+1}\lambda^{4n+4}}{2^{4n}(2n)!^2} \\
&\quad \times \left(\frac{1}{2n+2} + \frac{1}{2n+1}\right), \quad n = 0, 1, 2, \ldots, [(m-2)/2],
\end{aligned} \tag{72}$$

$$d_m = 0, \quad d_{m-1} = 0, \tag{73}$$

$$\begin{aligned}
&\frac{A_4(-\varepsilon)^{[(m+1)/2]}\lambda^{4[(m-1)/2]+6} r^{4[(m-1)/2]+2}}{2^{4[(m-1)/2]}\big(2[(m-1)/2]+1\big)!^2} \\
&\quad + \frac{A_3(-\varepsilon)^{[m/2]+1}\lambda^{4[m/2]+4} r^{4[m/2]}}{2^{4[m/2]}\big(2[m/2]\big)!^2} = r^{2m}.
\end{aligned} \tag{74}$$

In the case when m is even, Eq. (74) becomes

$$\frac{A_4(-\varepsilon)^{m/2}\lambda^{2m+2} r^{2m-2}}{2^{2m-4}(m-1)!^2} + \frac{A_3(-\varepsilon)^{(m/2)+1}\lambda^{2m+4} r^{2m}}{2^{2m} m!^2} = r^{2m}. \tag{75}$$

Equating the coefficients of the like terms in Eq. (75), we obtain

$$A_3 = (-\varepsilon)^{(m/2)+1}\lambda^{-2m-4} 2^{2m} m!^2, \quad A_4 = 0. \tag{76}$$

In the case when m is odd, Eq. (74) becomes

$$\frac{A_4(-\varepsilon)^{(m+1)/2}\lambda^{2m+4} r^{2m}}{2^{2m-2}(m)!^2} + \frac{A_3(-\varepsilon)^{(m+1)/2}\lambda^{2m+2} r^{2m-2}}{2^{2m-2} m!^2} = r^{2m}. \tag{77}$$

Equating the coefficients of the like terms in Eq. (77), we obtain

$$A_3 = 0, \quad A_4 = (-\varepsilon)^{(m+1)/2}\lambda^{-2m-2} 2^{2m-2} m!^2. \tag{78}$$

Comparing Eqs. (76) and (78), we obtain general formulas for A_3 and A_4 as follows:

$$A_3 = \frac{(-1)^m + 1}{2}(-\varepsilon)^{(m+2)/2}\lambda^{-2m-4} 2^{2m} m!^2 \tag{79}$$

and

$$A_4 = \frac{(-1)^{m-1} + 1}{2}(-\varepsilon)^{(m+1)/2}\lambda^{-2m-2} 2^{2m-2} m!^2. \tag{80}$$

Using the values of A_3 and A_4 from Eqs. (79) and (80), we return to solving Eqs. (71)–(73). Defining g_n by

$$g_n = (-\varepsilon)^{[n/2]}\left(\frac{2}{\lambda}\right)^{2n} n!^2 d_n, \tag{81}$$

Eqs. (71)–(73) convert to

$$\begin{aligned}
&g_{2n+1} - g_{2n+3} = 4A_4\left(\frac{1}{2n+2} + \frac{1}{2n+3}\right), \\
&\qquad n = 0, 1, 2, \ldots, [(m-3)/2],
\end{aligned} \tag{82}$$

$$\begin{aligned}
&g_{2n} - g_{2n+2} = A_3\left(\frac{1}{2n+2} + \frac{1}{2n+1}\right), \\
&\qquad n = 0, 1, 2, \ldots, [(m-2)/2],
\end{aligned} \tag{83}$$

$$g_{2[(m-3)/2]+3} = 0, \quad g_{2[(m-2)/2]+2} = 0. \tag{84}$$

Telescoping Eqs. (82) and (83), we get

$$\begin{aligned}
g_{2n+1} &= g_{2[(m-1)/2]+1} + \sum_{k=n}^{[(m-3)/2]} \big(g_{2k+1} - g_{2k+3}\big) \\
&= 4A_4 \sum_{k=n}^{[(m-3)/2]} \left(\frac{1}{2k+2} + \frac{1}{2k+3}\right)
\end{aligned} \tag{85}$$

and

$$g_{2n} = g_{2[m/2]} + \sum_{k=n}^{[m/2]-1} (g_{2k} - g_{2k+2}) = A_3 \sum_{k=n}^{[m/2]-1} \left(\frac{1}{2k+1} + \frac{1}{2k+2} \right). \tag{86}$$

The above telescoping is based also on the following observation:

$$\begin{aligned} 2\left[\tfrac{m-3}{2}\right] + 3 = 2\left[\tfrac{m-1}{2}\right] + 1 = m \quad \text{or} \quad m-1 \\ 2\left[\tfrac{m-2}{2}\right] + 2 = 2\left[\tfrac{m}{2}\right] = m \quad \text{or} \quad m-1. \end{aligned} \tag{87}$$

Since $a_m = a_{m-1} = 0$, then $a_{2[(m-3)/2]+3} = a_{2[(m-2)/2]+2} = a_{2[(m-1)/2]+1} = a_{2[m/2]} = 0$.

Going back to the case when m is even, we obtain the final formulas for the coefficients d_n in that case as follows:

$$d_{2n+1} = 0,$$

$$d_{2n} = (-\varepsilon)^{(m/2)+n+1} 2^{2m-4n} \lambda^{4n-2m-4} \frac{m!^2}{(2n)!^2} \sum_{k=2n+1}^{m} \frac{1}{k}. \tag{88}$$

In the case when m is odd, we obtain the final formulas for the coefficients d_n in that case as follows:

$$d_{2n} = 0,$$

$$d_{2n+1} = (-\varepsilon)^{((m+1)/2)+n} 2^{2m-4n-2} \lambda^{4n-2m} \frac{m!^2}{(2n+1)!^2} \sum_{k=2n+2}^{m} \frac{1}{k}. \tag{89}$$

5. Conclusions

In this paper, we considered finding analytic particular solutions to a class of fourth order PDEs which we refer to as multi-Helmholtz-type PDE. These equations, as we pointed out in the introduction, occur in various problems in elasticity, may be viewed as fourth order generalizations of the classical Helmholtz-type equations. The particular solutions found in this paper are, to the best of our knowledge, new and can be considered as generalizations of the particular solutions obtained in [5]. These particular solutions are expected to be useful in numerically solving the above mentioned equations by the BEM and other boundary methods such as MFS.

Although there are many possible choices of basis functions that can be used to calculate particular solutions, we have chosen a particular class of RBFs, the polyharmonic splines, since such functions seem to be most popular with the engineers for solving inhomogeneous PDEs by DRM or MFS. For completeness, we consider both the 2-D and 3-D cases. Curiously, as with the classical Helmholtz-type equations, the 3-D case is easier to calculate than the 2-D one. Particular solutions for the 3-D case are obtained by the use of the Neumann series expansion of the inverse of the multi-Helmholtz-type operator and similar to the technique used in [8]. The 2-D case turns out to be analytically complex and requires not only a solution to the inhomogeneous equation, but also solutions to the homogeneous case. This is necessary in order to cancel singularities which occur in the solution of the inhomogeneous equation. Both of these calculations use the radial symmetry of the underlying operator. In particular, the homogeneous equation is reduced to a fourth order ODE, which we refer to as a multi-Bessel-type equation and the basis we obtain for this equation appears to be new. This basis may have independent use in constructing t-Trefftz basis for the homogeneous equation in [10]. Future work will be devoted to solving inhomogeneous and time-dependent problems in elasticity [6,7,10].

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