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Crack analysis using an enriched MFS domain decomposition technique

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Abstract

In this paper we consider the application of the method of fundamental solutions to solve crack problems. These problems present difficulties, which are not only related to the intrinsic singular nature of the problem, instead they are mainly related to the impossibility in choosing appropriate point sources to write the solution as a whole. In this paper we present: (1) a domain decomposition technique that allows to express a piecewise approximation of the solution using a method of fundamental solutions applied to each subdomain; (2) an enriched approximation whereby singular functions (fully representing the singular behaviour around the cracks or other sources of boundary singularities) are used. An application of the proposed techniques to the torsion of cracked components is carried out.

Keywords: Method of fundamental solutions; Crack problems; Domain decomposition; Enriched approximations

1. Introduction

The Method of Fundamental Solutions (MFS) is a simple but powerful technique that has been used to obtain highly accurate numerical approximations of PDE solutions with simple codes and small computational effort. It has been introduced in the 1960 s (cf. [4,15]) as an alternative numerical method to the boundary integral methods in several elliptic homogeneous problems (e.g. [6,8,14,19], see also [2,9] for extensions to nonhomogeneous PDEs).

The application of the MFS can be justified by density results that have been established for simply connected domains with regular boundary (e.g. [2,6]); its generalization to multiply connected domains is straightforward. However, the application of the MFS, with external sources, cannot be justified if the domain includes cracks, that is, if boundary irregularities occur.

The study of the solution behavior in cracked domains is important in problems such as cracked bars under torsion [7], or to identify the presence of cracks inside a domain (e.g. [1,3]). The solution of crack problems presents difficulties due to its singular behavior at the crack. The MFS relies on the use of point-sources outside the domain (which are analytic functions inside the domain) whose linear combination cannot adequately model the singular behavior at the cracks.

- Here we present a model problem in potential theory where cracks are considered. One aim of this work is to investigate a possible application of this technique to the Cauchy–Navier equations of elasticity. In the current framework these cracks may also be seen as mode III cracks.
- Some previous works, e.g. [12,13] or [20], consider the application of the of method of fundamental solutions to singular problems. For instance, in [20] a mixed Dirichlet–Neumann boundary condition was considered, and in symmetric situations these mixed boundary conditions can be a simplified setting to solve crack problems.

On the other hand, the MFS can be seen as a Trefftz type method (cf. [22]), based on the use of a superposition of solutions of the PDE. The pure Trefftz method is based on an expansion of the solution as a series and it is also a simple computational method that has been applied to several PDE problems (e.g. [10]). More recently Trefftz methods have been applied to crack analysis (e.g. [17]), using a domain decomposition technique (c.f. [16]). The application of domain decomposition techniques together with meshless methods has been a subject of recent research in other contexts (e.g. [5,11,18]).

Here we consider a general domain decomposition technique applied to the method of fundamental solutions and

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a procedure to enrich the approximations. To the standard basis of functions used in the MFS (translations of the fundamental solution itself) a new set of singular functions is added so that problems exhibiting boundary singularities (such as crack problems) may be solved.

In particular, here we consider the problem of cracked bars under torsion as in [7]. We give several numerical tests that show high quality approximations of the singular solution, when using the enriched MFS, or even better when using an enriched MFS with a domain decomposition technique.

2. Statement of a general crack problem

Consider a crack (or several cracks) γ embedded in a bounded domain $\Omega \subset \mathbb{R}^d$ (we will be interested in dimensions $d=2$ or 3), with regular boundary $\Gamma = (\Omega$ In a general situation, we look for the solution u on Ω/γ of the boundary value problem

$$\begin{cases} \mathcal{D}u = 0 & \text{in } \Omega\backslash\bar{\gamma}, \\ \mathcal{B}_\gamma u = g & \text{on } \gamma, \\ \mathcal{B}_\Gamma u = f & \text{on } \Gamma = \partial\Omega. \end{cases} \tag{1}$$

Here $\mathcal{D}$ stands for an elliptic differential operator, with fundamental solution Φ such that $\mathcal{D}\Phi = \delta$. The boundary condition on $(\Omega$ is given by the operator $\mathcal{B}_\Gamma$ and on the crack we consider a boundary condition given by the operator $\mathcal{B}_\gamma$. $\bar{\gamma}$ stands for the closure of γ, which includes its tips.

3. An enriched MFS for crack problems

The basic features of all Trefftz-based techniques [22] is the use, as approximating functions, of actual solutions of the homogeneous differential equation. In this context, the Method of Fundamental Solutions (MFS) may be seen as Trefftz technique. In both cases boundary-only collocation may be used, the difference being the location of the sources and the approximating functions used. In the MFS an arbitrary number of sources are located at an artificial boundary and the same function is used at all sources - *the fundamental solution*, whereas in the classical Trefftz collocation method an arbitrary number of functions, located at a single source, are used - *the so-called T-functions* (e.g. [10]).

The MFS is a simple technique that has been used to obtain highly accurate numerical approximations when the domain presents no cracks. The presence of cracks produces singular solutions near the crack tips that lead to difficulties in the approximation. The MFS relies on the use of point-sources outside the domain and the singular behavior near the crack tips cannot be reproduced by a combination of these analytic functions.

A technique to provide an approximation of the solution using the MFS is to add, to the regular approximation, singular particular solutions (e.g. [20]). This approach has also been used in the context of Trefftz methods (e.g. [17]) for crack problems.

Consider that the solution may be written as the sum of a regular and a singular solution as follows,

$$u(x) = u_{\mathrm{R}}(x) + u_{\mathrm{S}}(x), \quad \forall x \in \Omega.$$

This technique is based on an *a priori* knowledge of the behaviour of the singular solution itself; to appropriately model the singular behaviour, actual singular particular solutions are used.

–The approximation of u_{R} is made using regular solutions, i.e. the set of fundamental solutions $\{\phi_1, \ldots, \phi_{m_R}\}$ with $\phi_k(x) = \Phi(x - y_k)$ where the set of point-sources $Y = \{y_1, \ldots, y_{m_R}\} \subset \mathbb{R}^d\backslash\bar{\Omega}$ is usually considered at an artificial boundary $\hat{\Gamma} \subset \mathbb{R}^d\backslash\bar{\Omega}$ that encloses $\Gamma = (\Omega$

$$u_{\mathrm{R}}(x) \approx \sum_{k=1}^{m_{\mathrm{R}}} \alpha_k \phi_k(x).$$

These regular solutions verify $\mathcal{D}u_{\mathrm{R}} = 0$ in the whole Ω

–On the other hand, the approximation of u_{s} is made using singular solutions, i.e. a set of solutions $\{\psi_1, \ldots, \psi_{m_S}\}$ that may be used to characterize the singular behaviour of the solution-near the crack,

$$u_{\mathrm{S}}(x) \approx \sum_{k=1}^{m_{\mathrm{S}}} \beta_k \psi_k(x). \tag{2}$$

These functions verify the domain equation, i.e. $\mathcal{D}u_{\mathrm{S}} = 0$ in $\Omega\backslash\bar{\gamma}$.

3.1. Least-squares fitting

We consider two sets of collocation points $X_\Gamma = \{x_1, \ldots, x_{n_\Gamma}\} \subset \partial\Omega$ and $X_\gamma = \{\xi_1, \ldots, \xi_{n_\gamma}\} \subset \gamma$. The collocation system is

$$\begin{bmatrix} \mathcal{B}_\Gamma\Phi(x_i - y_j) & \mathcal{B}_\Gamma\Phi_j(x_i) \\ \mathcal{B}_\gamma\Phi(\xi_i - y_j) & \mathcal{B}_\gamma\Phi_j(\xi_i) \end{bmatrix} \begin{bmatrix} \alpha_j \\ \beta_j \end{bmatrix} = \begin{bmatrix} f(x_i) \\ g(\xi_i) \end{bmatrix}, \tag{3}$$

Imposing $n_\Gamma + n_\gamma > m_{\mathrm{R}} + m_{\mathrm{S}}$ we solve the system using discrete least squares, to obtain the approximation

$$\tilde{u}(x) = \sum_{k=1}^{m_{\mathrm{R}}} \alpha_k \phi_k(x) + \sum_{k=1}^{m_{\mathrm{S}}} \beta_k \psi_k(x)$$

that minimizes the error at the collocation points, i.e.:

$$E(\tilde{u}) = ||\mathcal{B}_\Gamma(\tilde{u}) - f||_{l^2(X_\Gamma)} + ||\mathcal{B}_\gamma(\tilde{u}) - g||_{l^2(X_\gamma)}.$$

4. Domain decomposition technique

To perform a domain decomposition technique that allows to express the solution u on γ with two layers, we divide the domain into two subdomains Ω_1 and Ω_2. These subdomains are such that $\bar{\Omega}_1 \cup \bar{\Omega}_2 = \bar{\Omega}$ and the interface $\Gamma_\gamma = \bar{\Omega}_1 \cap \bar{\Omega}_2$ contains the crack γ. This interface Γ_γ is a $d-1$ dimensional manifold that separates the domain Ω in the two simply connected subdomains Ω_1,Ω_2. We will call *artificial interface*

to the part of the interface $\gamma^* = \Gamma_\gamma \backslash \gamma$ that is defined by the arbitrary domain decomposition.

Note that the boundary of Ω_j is now given by $(\Omega_j = \Gamma_j \cup \Gamma_\gamma$ with $\Gamma_j = (\partial\Omega \cap \bar{\Omega}_j)$, and therefore $\Gamma = \Gamma_1 \cup \Gamma_2$

Having decomposed the domain into two parts, we separate the solution u,

$$u_1 = u|_{\Omega_1} \quad \text{and } u_2 = u|_{\Omega_2}. \tag{4}$$

The solution is analytic through the artificial interface γ^* but, on the other hand, there is a jump of the solution $[u] = u_2 - u_1$ through γ, which is called the crack opening.

In terms of the partial solutions u_1 and u_2 this can be ensured by imposing Cauchy transmission conditions on γ^*

$$u_1 = u_2 \quad \text{and } \partial_n u_1 = \partial_n u_2 \quad (\text{on } \gamma^*), \tag{5}$$

and imposing on γ the boundary condition (for each partial solution),

$$\mathcal{B}_\gamma u_1 = \mathcal{B}_\gamma u_2 = g \quad (\text{on } \gamma). \tag{6}$$

4.1. The MFS applied to domain decomposition

Using the MFS we propose to construct each partial solution u_j with point-sources y_k located at the artificial boundaries $\hat{\Gamma}_j \subset \mathbb{R}^N \backslash \bar{\Omega}_j$. These closed artificial boundaries $\hat{\Gamma}_j$ surround the subdomains Ω_j. More precisely, we consider $\hat{\Omega}_j \supset \Omega_j$ and $\hat{\Gamma}_j = \partial\hat{\Omega}_j$. Note that we could also consider a single artificial boundary $\hat{\Gamma} = \partial\hat{\Omega}$ with $\hat{\Omega} \supset \Omega$, that would be used for both subdomains, but in practical terms it is better to separate the solution by means of the two different artificial boundaries.

For each part of the solution u_α (with $\alpha = 1,2$) we consider an approximation

$$u_\alpha(x) \approx \sum_{j=1}^{m_\alpha} a_j^{[\alpha]} \Phi(x - \hat{y}_j^{[\alpha]}), \quad \text{with } \hat{y}_j^{[\alpha]} \in \hat{\Gamma}_\alpha, \tag{7}$$

and being m_α the number of source points used in each subdomain $\hat{\Gamma}_\alpha$. Recall that the solution will be given by

$$u(x) = \begin{cases} u_1(x) & \text{if } x \in \Omega_1, \\ u_2(x) & \text{if } x \in \Omega_2. \end{cases} \tag{8}$$

We have to solve a system such that the following conditions are fulfilled:

$$\begin{aligned} &\mathcal{B}_\Gamma u_1 = f \text{ on } \Gamma_1, \quad \mathcal{B}_\Gamma u_2 = f \text{ on } \Gamma_2, \\ &\mathcal{B}_\gamma u_1 = g \text{ on } \gamma, \quad \mathcal{B}_\gamma u_2 = g \text{ on } \gamma, \\ &u_2 = u_1 \text{ on } \gamma^*, \quad \partial_n u_2 = \partial_n u_1 \text{ on } \gamma^*. \end{aligned} \tag{9}$$

First, we need to consider collocation points on the domain boundary $\Gamma = \Gamma_1 \cup \Gamma_2$ that may be grouped into two sets:

$$\begin{aligned} X_{\Gamma_1} &= \{x_1^{[1]}, \ldots, x_{n_1}^{[1]}\} \subset \Gamma_1, \\ X_{\Gamma_2} &= \{x_1^{[2]}, \ldots, x_{n_2}^{[2]}\} \subset \Gamma_2. \end{aligned} \tag{10}$$

Second, we consider collocation points at the interface boundary $\Gamma_\gamma = \gamma \cup \gamma^*$ also divided into two sets

$$X_\gamma = \{x_1^{[3]}, \ldots, x_{n_3}^{[3]}\} \subset \gamma, \qquad X_{\gamma^*} = \{x_1^{[4]}, \ldots, x_{n_4}^{[4]}\} \subset \gamma^*. \tag{11}$$

Defining the matrices $\mathbb{G}_{ij}^{[\alpha,\beta]} = \Phi(x_i - \hat{y}_j^{[\alpha]})$ with $x_i \in \{x_1^{[\beta]}, \ldots, x_{n_\beta}^{[\beta]}\}$, we get (using Einstein summation convention on the repeated index $\alpha = 1,2$)

$$u_\alpha(x_i^{[\beta]}) = \mathbb{G}_{ij}^{[\alpha,\beta]} a_j^{[\alpha]}, \qquad (\beta = 1,2,3,4). \tag{12}$$

This allows to fulfil the conditions in the collocation points by imposing

$$\begin{aligned} &\mathcal{B}_\Gamma \mathbb{G}_{ij}^{[1,1]} a_j^{[1]} = f(x_i^{[1]}), \quad \mathcal{B}_\Gamma \mathbb{G}_{ij}^{[2,2]} a_j^{[2]} = f(x_i^{[2]}), \\ &\mathcal{B}_\gamma \mathbb{G}_{ij}^{[1,3]} a_j^{[1]} = g(x_i^{[3]}), \quad \mathcal{B}_\gamma \mathbb{G}_{ij}^{[2,3]} a_j^{[2]} = g(x_i^{[3]}), \\ &\mathbb{G}_{ij}^{[1,4]} a_j^{[1]} = \mathbb{G}_{ij}^{[2,4]} a_j^{[2]}, \quad \partial_n \mathbb{G}_{ij}^{[1,4]} a_j^{[1]} = \partial_n \mathbb{G}_{ij}^{[2,4]} a_j^{[2]}, \end{aligned} \tag{13}$$

where $\partial_n \mathbb{G}_{ij}^{[\alpha,\beta]} = \mathbf{n}(x_i^{[\beta]}) \cdot \nabla\Phi(x_i^{[4]} - \hat{y}_j^{[\alpha]})$. Note that for $x \in \Gamma_\gamma$ we have two normal directions, $\mathbf{n}_1(x)$ that points outwards Ω_1 and the symmetric $\mathbf{n}_2$(x) that points outwards Ω_2 Without any loss of generality, we choose $\mathbf{n} = \mathbf{n}_1 - \mathrm{n}_2$ and therefore $({}_n u_2 = g$ on γ leads to the condition $-\partial_n \mathbb{G}_{ij}^{[2,3]} a_j^{[2]} = g(x_i^{[3]})$.

From (13) we build the whole linear system by taking $\mathbb{G}^{[\alpha,\beta]}$ and $\partial_n \mathbb{G}^{[\alpha,\beta]}$ as submatrices,

$$\begin{bmatrix} \mathcal{B}_\Gamma \mathbb{G}^{[1,1]} & \mathbf{0} \\ \mathbf{0} & \mathcal{B}_\Gamma \mathbb{G}^{[2,2]} \\ \mathcal{B}_\gamma \mathbb{G}^{[1,3]} & \mathbf{0} \\ \mathbf{0} & \mathcal{B}_\gamma \mathbb{G}^{[2,3]} \\ \mathbb{G}^{[1,4]} & -\mathbb{G}^{[2,4]} \\ \partial_n \mathbb{G}^{[1,4]} & -\partial_n \mathbb{G}^{[2,4]} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{[1]} \\ \mathbf{a}^{[2]} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{[1]} \\ \mathbf{f}^{[2]} \\ \mathbf{g} \\ \mathbf{g} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{14}$$

where $\mathbf{a}^{[\alpha]} = (a_1^{[\alpha]}, \ldots, a_{m_\alpha}^{[\alpha]})$, $\mathbf{f}^{[\beta]} = (f(x_1^{[\beta]}), \ldots, f(x_{n_\beta}^{[\beta]}))$, and $\mathbf{g} = (g(x_1^{[3]}), \ldots, g(x_{n_3}^{[3]}))$.

Briefly, we write this system as $\mathbf{Ga} = \mathbf{b}$ where $\mathbf{b}$ is the right hand side N dimensional vector with $N = n_1 + n_2 + 2n_3 + 2n_4$ the vector of the unknowns a is a M dimensional vector with $M = m_1 + m_2$ and finally G denotes the whole $N \times M$ matrix. We can solve the whole system in the interpolation sense, by choosing $N = M$ but we will prefer to consider a solution in the least squares sense, i.e. we will consider $\mathbf{G}^{\mathrm{T}}\mathbf{Ga} = \mathbf{G}^{\mathrm{T}}\mathbf{b}$.

Here we considered single domain decomposition into two subdomains, however it is clear that this procedure can be extended to an arbitrary number of subdomains, keeping a similar structure as in (14).

4.2. An enriched MFS domain decomposition technique

Using the previous notations and defining the matrices $\mathbb{G}_{ij}^{[S,\beta]} = \Psi_j(x_i^{[\beta]})$ for the particular solutions with singular behavior Ψ_k, as in (2), we obtain the system:

$$\begin{bmatrix} \mathcal{B}_\Gamma\mathbb{G}^{[1,1]} & \mathbf{0} & \mathcal{B}_\Gamma\mathbb{G}^{[S,1]} \\ \mathbf{0} & \mathcal{B}_\Gamma\mathbb{G}^{[2,2]} & \mathcal{B}_\Gamma\mathbb{G}^{[S,2]} \\ \mathcal{B}_\gamma\mathbb{G}^{[1,3]} & \mathbf{0} & \mathcal{B}_\gamma\mathbb{G}^{[S,3]} \\ \mathbf{0} & \mathcal{B}_\gamma\mathbb{G}^{[2,3]} & \mathcal{B}_\gamma\mathbb{G}^{[S,3]} \\ \mathbb{G}^{[1,4]} & -\mathbb{G}^{[2,4]} & \mathbf{0} \\ \partial_n\mathbb{G}^{[1,4]} & -\partial_n\mathbb{G}^{[2,4]} & \mathbf{0} \end{bmatrix} \begin{bmatrix} \mathbf{a}^{[1]} \\ \mathbf{a}^{[2]} \\ \mathbf{a}^{[S]} \end{bmatrix} = \begin{bmatrix} \mathbf{f}^{[1]} \\ \mathbf{f}^{[2]} \\ \mathbf{g} \\ \mathbf{g} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \tag{15}$$

where $\mathbf{a}^{[S]} = (a_1^{[S]}, \ldots, a_{m_S}^{[S]})$. The approximation is now given by the superposition

$$u_\alpha(x_i^{[\beta]}) = \mathbb{G}_{ij}^{[\alpha,\beta]} a_j^{[\alpha]} + \mathbb{G}_{ij}^{[S,\beta]} a_j^{[S]} \tag{16}$$

Remark: Quality of the MFS domain decomposition approximation.

The domain decomposition technique applied to the MFS, as presented, produces two approximations:

$\tilde{u}_1 \approx u|_{\Omega_1}$ and $\tilde{u}_2 \approx u|_{\Omega_2}$.

The quality of these approximations can be directly checked in the true boundaries Γ and γ and in the artificial interface γ^*, by the following errors

$$\begin{aligned} &E_1 = ||\tilde{u}_1 - f||_{\Omega_1}, \qquad E_2 = ||\tilde{u}_2 - f||_{\Omega_2} \\ &E_3 = ||\partial\tilde{u}_1 - g||_\gamma + ||\partial\tilde{u}_2 - g||_\gamma, \\ &E_4 = ||\tilde{u}_2 - \tilde{u}_1||_{\gamma^*} + ||\partial_n\tilde{u}_2 - \partial_n\tilde{u}_1||_{\gamma^*}. \end{aligned} \tag{17}$$

5. A problem in potential theory

This general setting will be considered here only for the Laplace differential operator $\mathcal{D} = \nabla^2 = \nabla\cdot\nabla$, in 2D, where the fundamental solution is given by

$$\Phi(r) = \frac{-1}{2\pi}\log(r).$$

We will be mainly interested in the Dirichlet problem

$$\begin{cases} \nabla^2 u = 0 & \text{in } \Omega\backslash\bar{\gamma}, \\ u = g & \text{on } \gamma, \\ u = f & \text{on } \Gamma, \end{cases} \tag{18}$$

where the boundary operators $\mathcal{B}_\Gamma$ and $\mathcal{B}_\gamma$ are assumed to be the identity operator.

In the crack analysis with the Trefftz method (cf. [17]) it is appropriated to use solutions of the form

$$\psi_k(\rho,\theta) = \rho^{p_k}\cos(p_k\theta),$$

in polar coordinates, centred at the crack tip (i.e. $\rho = |x - c_{\text{tip}}|$ and θ is the angle defined by the point $x - c_{\text{tip}}/\rho$ in the unit circle). In the pure Trefftz method, for the Dirichlet problem, we may consider $p_k = k - 0.5$ and use exclusively the expansion

$$u_S(x) \approx \sum_{k=1}^{\infty} \beta_k \rho^{k-0.5}\cos((k-0.5)\theta)$$

to fit the boundary conditions on (Ω with the unknown coefficients β_k. It should be mentioned that in the pure Trefftz method the complete series also includes terms of the form $\rho^{p_k}\sin(p_k\theta)$ with integer $p_k \geq 1$. However, in the MFS framework those terms are already taken into account by the fundamental solutions.

Here we propose to fit the boundary conditions by augmenting the regular approximation with a singular expansion of the type shown above. The choice of powers p_k is, naturally, geometry dependent: the p_k sequence starts with 1/2 in the case of a crack (the faces of the crack are parallel to each other); different p_k sequences are required in the case of notches (the faces of the notch concur at the tip but are not parallel). It may also be seen that smaller values of p_k will produce larger crack opening displacements, and only the first terms in the sequence 1/2,3/2,5/2,… will be used to fit this opening. In fact, only the term $p_k = 1/2$ or $p_k < 1$ will produce a jump with singular derivative in the crack tip, for $p_k > q$, with $q \in \mathbb{N}$, the functions ψ_k produce jumps with null q-order derivative across the crack tip.

6. Application to elastic torsion problems

Consider a prismatic bar (along coordinate x_3) subjected to torsional end moments. If the cross section (in the x_1,x_2 plane) is radially symmetric then the cross sections will remain plane and rotate (with a constant rate) without distortion during twist (Coulomb hypotheses).

For the non-radially symmetric case the cross section no longer remains plane, i.e. warping (extension of the fibers along the direction where the moment is applied) occurs. Saint-Venant's hypothesis consists in assuming warping to be constant.

By using the above simplifying hypotheses the displacement components for the torsion problem may be expressed as [21]:

$$\begin{aligned} &v_1(\mathbf{x}) = -\theta x_2 x_3, \quad v_2(\mathbf{x}) = \theta x_1 x_3 \\ &\text{and } v_3(\mathbf{x}) = \theta \mathrm{w}(x_1, x_2) \end{aligned} \tag{19}$$

where θ is the twist angle per unit length and $\mathrm{w}(x_1,x_2)$ is the warping function.

The strain components are:

$$\begin{aligned} &\gamma_{13} = \frac{\partial v_3}{\partial x_1} + \frac{\partial v_1}{\partial x_3} = -x_2\theta + \theta\frac{\partial \mathrm{w}}{\partial x_1} \\ &\text{and } \gamma_{23} = \frac{\partial v_3}{\partial x_2} + \frac{\partial v_2}{\partial x_3} = x_1\theta + \theta\frac{\partial \mathrm{w}}{\partial x_2} \end{aligned} \tag{20}$$

and the corresponding stresses are:

$$\sigma_{13} = -Gx_2\theta + G\theta\frac{\partial \mathrm{w}}{\partial x_1} \qquad \text{and } \sigma_{23} = Gx_1\theta + G\theta\frac{\partial \mathrm{w}}{\partial x_2} \tag{21}$$

where G is the shear modulus.

At this stage, and to simplify the boundary conditions, the stress function $U(x_1,x_2)$ may be introduced thus leading to the following definitions for the stress components:

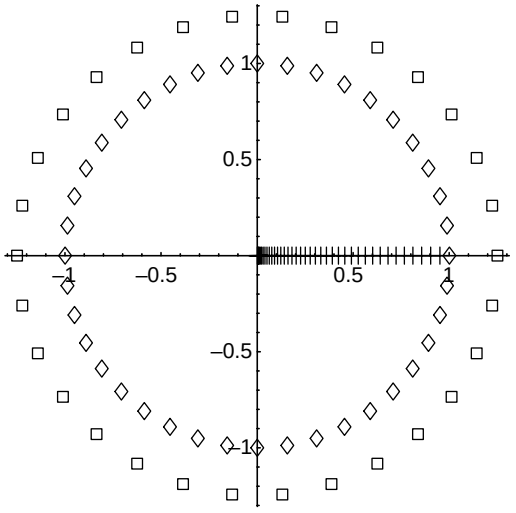

Fig. 1. Standard MFS: distribution of sources and collocation points. Point sources (□) and collocation points on the boundary Γ (◇) and on the crack γ (symbol: +).

$$\sigma_{13} = \frac{\partial U}{\partial x_2} \qquad \text{and } \sigma_{23} = -\frac{\partial U}{\partial x_1}. \tag{22}$$

Using the equilibrium equation

$$\frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} = 0 \tag{23}$$

and after eliminating w the following Poisson problem is obtained,

$$\nabla^2 U = -2G\theta \quad \text{in } \Omega, \tag{24}$$

subjected to $U=0$ on the whole boundary Γ

Now, it may be convenient to reformulate the problem such that a global solution is obtained from the superposition of the solution of the homogeneous problem u_h and a particular solution u_p

So the actual problem that is going to be solved in this work is the following:

$$\nabla^2 u_h = 0 \quad \text{in } \Omega, \tag{25}$$

subjected to $u_h = -u_p$ on the whole boundary Γ

After finding $U=u_h+u_p$ the twisting moment T may be obtained from:

$$T = \int_D (x_1\sigma_{23} - x_2\sigma_{13}) = 2\int_D U \tag{26}$$

which straightforwardly leads to the torsion rigidity $R=T\theta$

6.1. The case of a cracked circular cross section

Consider now the case of torsion of a cracked circular cross section, as presented in [7]. The crack tip is located at point (0,0) and the crack length is equal to the radius of the circular cross section. The sides of the crack are denoted by γ^+ and γ^- and Γ is the remaining boundary.

The torsion problem can now be stated as follows:

$$\nabla^2 u_h = 0, \quad \text{in } \Omega \setminus \bar{\gamma}, \tag{27}$$

subjected to $u_h = -u_p$ on the whole boundary $\partial(\Omega \setminus \bar{\gamma}) = \Gamma \cup \bar{\gamma}$. where $u_p(x_1,x_2) = (x_1^2 + x_2^2)/2$ is a particular solution.

A result of interest (for comparison purposes) is the torsion rigidity R (the twisting moment for unit θ) which may be obtained in the form of a domain integral (as presented above in (26)) or in the form of a boundary integral only after integrating (26) by parts and using the Green's theorem:

$$T = -G\theta \int_{\partial\Omega} (x_1^2 + x_2^2)^2 \frac{\partial U}{\partial n} \mathrm{d}s - \frac{G\theta}{16} \int_{\partial\Omega} \frac{\partial}{\partial n}(x_1^2 + x_2^2)^2 \mathrm{d}s \tag{28}$$

The analytic solution (the potential U in polar coordinates r,ϕ centered at the crack tip for the circular cross section with a crack length a to diameter d ratio of $a/d=0.5$ is (e.g. Chen et al. [7]):

$$U(r,\varphi) = 32\frac{a^2}{\pi} \times \sum_{n=0}^{\infty} \frac{\left(\frac{r}{a}\right)^{(2n+1)/2} - \left(\frac{r}{a}\right)^2}{(2n+1)[16-(2n+1)^2]} \sin\frac{(2n+1)\varphi}{2} \tag{29}$$

$$R = Ga^4 \sum_{n=0}^{\infty} \frac{1}{(2n+1)^2(2n+5)[16-(2n+1)^2]} - \frac{\pi}{2} = 0.878Ga^4 \tag{30}$$

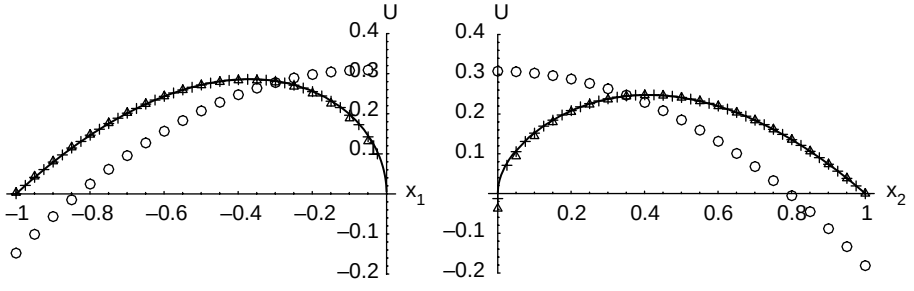

Fig. 2. Single Domain: (a) variation of U at $x_2=0$; (b) variation of U at $x_1=0$; analytical solution (solid line), and approximations by standard MFS (○), one term enriched MFS (△), two terms enriched MFS (+).

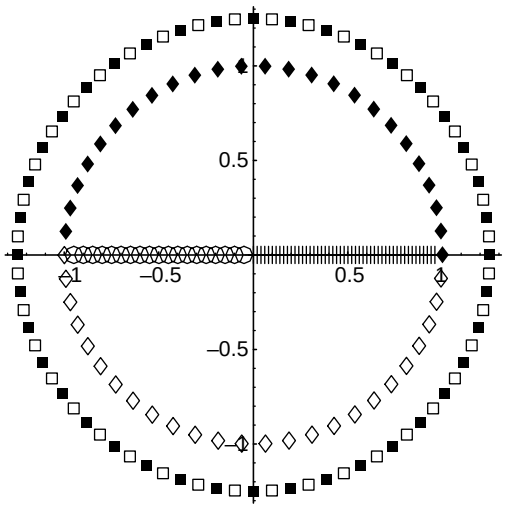

Fig. 3. Domain Decomposition: distribution of sources and collocation points. Filled symbols refer to Ω_1 and empty symbols to Ω_2. Point sources ($\square$) and collocation points on the boundary Γ ($\diamond$) and on the crack γ (symbol: $+$). Extra collocation points ($\bigcirc$) refer to the interface boundary.

6.2. Enriched MFS for a single domain

We consider the previous setting with $\Omega = B(0,1)$ the unit disk, and a single crack $\gamma =]0,1[\times \{0\}$ with a crack tip at the origin (0.0) In Fig. 1 we plot the collocation and source points needed to apply the standard MFS. The square symbols ($\square$) denote the position of 30 source points, equally distributed in a circle of radius 1.25. The diamond symbols ($\diamond$) represent the 40 collocation points equally spaced on Γ (boundary of Ω) Finally, the cross symbols ($+$) represent the 39 collocation points on the crack γ, which are not equally distributed, they acumulate at the crack tip (0,0). This distribution (using a square function) was found to be better than equally spaced collocation points.

In Fig. 2 we consider two plots of the exact solution (solid line) for the variation of U at $x_2 = 0$ (left picture) and for the variation of U at $x_1 = 0$ (right picture). These results are compared with the approximations given by the standard MFS ($\bigcirc$), the one term enriched MFS ($\triangle$), the two terms enriched MFS ($+$).

It is clear that the standard MFS ($\bigcirc$) does not produce a good approximation since it is unable to follow the variation of U specially near the crack tip. Using more source points or more collocation points does not produce better results. The standard MFS approximation uses analytic functions that are unable to fit the descontinuities at the crack.

On the other hand, just by enriching the MFS with one singular term $p_1 = 1/2$ the quality of the approximation changes dramatically. The symbols ($\triangle$) follow perfectly the solid line, and only a small deviation is seen near the crack tip. No major change is seen by enriching the approximation with another singular term (with $p_2 = 3/2$), but we may refer that the symbols ($+$) follow better the solid line near the crack tip.

6.3. Enriched MFS for multiple domains

We consider the same setting to apply a domain decomposition technique. The domain Ω is decomposed into $\Omega_1 = B^+(0,1)$ and $\Omega_2 = B^-(0,1)$, the interface is given by the crack γ and by $\gamma^* =]-1,0[\times \{0\}$ (where Cauchy transmission conditions are imposed).

In Fig. 3 we plotted the collocation and source points needed to apply the Domain Decomposition MFS. The square symbols ($\square$) denote the positions of 80 source points (the filled squares represent 40 source points used in $\hat{\Gamma}_1$ and the empty squares represent 40 source points used in $\hat{\Gamma}_2$). The empty 25 diamond symbols ($\diamond$) represent the collocation points on Γ_2 and the 25 filled symbols are collocation points on Γ_1 The cross symbols ($+$) represent the 50 collocation points on the crack γ, which are also equally distributed and the circle symbols ($\bigcirc$) represent the 20 equally spaced collocation points used on the interface γ^*.

In Fig. 4 we consider the same plots as in Fig. 2. Now, the approximations to the analytical solution (solid line) are given by the standard Domain Decomposition MFS (DD-MFS) and by the enriched DD-MFS.

We observe that the standard DD-MFS ($\bigcirc$) produces an approximation that it is better than the MFS without domain decomposition, but it still fails to give a good approximation to the exact solution. Like in the MFS, just by enriching the DD-MFS with one singular term $p_1 = 1/2$ the quality of the approximation changes dramatically. The symbols ($\triangle$) follow perfectly the solid line, and no significant deviation is seen near the crack tip.

We must refer that although the results given by the enriched DD-MFS are better than the ones obtained for the enriched MFS,

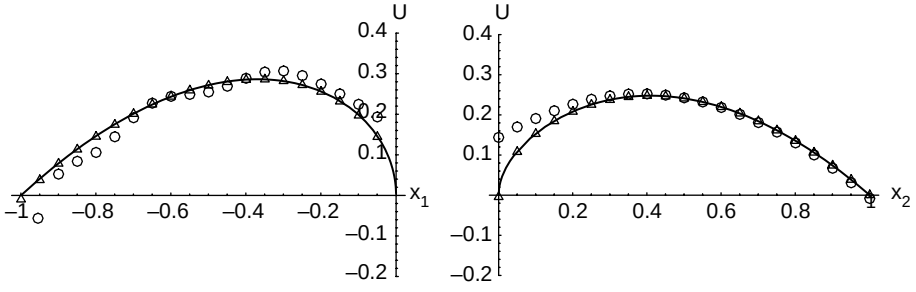

Fig. 4. Domain Decomposition: (a) variation of U at $x_2 = 0$; (b) variation of U at $x_1 = 0$: analytical solution (solid line), and approximations by standard DD-MFS ($\bigcirc$), one term enriched DD-MFS ($\triangle$).

we also used almost double the number of collocation and source points. A more careful analysis relating the choice of point sources and collocation points is being carried out.

7. Conclusions

The main purpose of this work was to contribute to the development of the Method of Fundamental Solutions by presenting a domain decomposition technique applied to the analysis of singular problems, namely the torsion of cracked components.

Standard MFS approximations do not seem to appropriately describe the singular behaviour around cracks. The enrichment of the approximations with extra (singular) terms is, as may be concluded from the above results, one way of addressing problems exhibiting singularities. Further work on the subject is now being carried out.

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