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Error estimation and adaptive mesh refinement in boundary element method, an overview

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Abstract

Further to the previous review article (Adv Engng Software 19(1) (1994) 21–32), this paper reviews more recent studies on the same subject by citing more than one hundred papers.

The adaptive mesh refinement process is composed of three processes; the error estimation, the adaptive tactics and the mesh refinement processes. Therefore, in this paper, the existing studies are classified and discussed according to the processes.

The error estimation schemes are classified into the residual-type, the interpolation-type, the integral equation-type, the node sensitivity-type and the solution difference type. The mesh refinement schemes are classified into h-, p-, r-schemes and the others. The adaptive tactics are closely related to the mesh refinement schemes. Therefore, they are discussed individually.

The discussion presented herein is an extension of the previous article and focuses our principal attention on the following points. Some interesting studies for the error estimation scheme are added; e.g. new schemes named as 'nodal design sensitivity', 'hyper-singular residual type' and 'solution difference type'. Some studies for the adaptive tactics are added; e.g. the tactics based on the convergence property of the error, the extension of the extended error indicator to r- and hr-adaptive schemes and so on.

Keywords: Error estimation; Adaptive mesh refinement; Boundary element method; Adaptive tactics

1. Introduction

Boundary element method as well as finite element method are recognized as very powerful tools for the numerical analysis and thus, are applied to several numerical simulations. The boundary element method can solve the problem by its boundary discretization alone when the object under consideration is governed by linear and homogeneous differential equations. Input data generation for the boundary element method is easier and more efficient than the domain type solution procedures such as the finite element and the finite difference methods. Moreover, this feature is very important for the computer simulation of some kinds of problems. The free and the moving boundary problems are considered as the typical ones. In these problems, the profile of the object under consideration is modified interactively. When the domain type solutions are applied to these problems, the mesh and the grid are distorted by the iterative modification of the profile, worsening the computational accuracy. If the boundary element method is applied to these problems instead of the domain type solutions, the distortion of the mesh is not so terrible and the re-generation of the mesh is easier than the domain type solutions.

For increasing its validity still more, the accuracy insurance for the solution is necessary. Many researchers have been studying the error estimation and the adaptive mesh refinement schemes for the boundary element method. This paper reviews more than one hundred papers on the error estimation and the adaptive mesh refinement schemes for the boundary element method. Some review papers have already been published for the same purpose [47,79,87]. This paper is a further review to the previous article [79]. In this paper, the adaptive mesh refinement process is composed of three processes; the error estimation, the adaptive tactics and the mesh refinement processes. Therefore, in the following sections, the existing studies are individually classified and discussed according to the processes.

This paper is organized as follows. In Section 2, the whole process of the adaptive scheme is described. In Section 3, the boundary element formulation is described briefly for the potential problem. In Section 4, the error estimation schemes are classified into the residual-type,

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the interpolation-type, the integral equation-type, the node sensitivity-type and the solution difference type. In Section 5, the mesh refinement schemes are classified into h-, p-, r-schemes and the others. In Section 6, the adaptive tactics is described. Finally, the discussions are summarized in Section 7.

2. Adaptive process

The adaptive mesh refinement process can be summarized as follows.

Input initial data: the mathematical models related to the objects under consideration are derived and then, several data on their geometric shapes and boundary conditions are defined.

Initial mesh discretization: boundary element discretization for the initial analysis is constructed.

Boundary element analysis: boundary element analysis for the object is performed.

Convergence satisfaction: the satisfaction of the convergence criterion is judged. If the criterion is satisfied, the process is terminated. If not so, the process goes to the next step.

Error estimation: discretization errors of the boundary element solutions are estimated.

Adaptive tactics: the elements to be refined and the relevant mesh refinement scheme are selected.

Mesh refinement: the mesh is refined actually.

We can say that the main processes in the adaptive mesh refinement process are the error estimation, the adaptive tactics and the mesh refinement processes. Therefore, the discussions on the following sections are performed individually for three processes.

3. Boundary element analysis

In this section, we shall explain briefly boundary element analysis in the two-dimensional potential problem in order to facilitate the discussions in this paper [9,12].

3.1. Boundary integral equation

The governing equation and the boundary conditions in the two-dimensional potential problem are described as:

$$\nabla^2 u = 0 \ \text{ in } \Omega \tag{1}$$

$$u = \bar{u} \ \text{ on } \Gamma_u, \qquad q \equiv \frac{\partial u}{\partial n} = \bar{q} \ \text{ on } \Gamma_q \tag{2}$$

where u and q are the potential and its normal derivative (flux), respectively. Ω, Γ_u and Γ_q denote the region occupied by the object under consideration, its potential-specified and flux-specified boundaries, respectively. n and (–) denote the unit vector in the normal direction on the boundary and the specified value on the boundary, respectively.

Taking the fundamental solution u^* which satisfies

$$\nabla^2 u^* + \delta = 0 \tag{3}$$

where δ denotes Dirac's delta function, we have

$$c(p)u(p) - \int_\Gamma [qu^* - uq^*]\, \mathrm{d}\Gamma = 0 \tag{4}$$

where $c(p)$ is the parameter dependent on where the source point p is placed; $c(p) = 1$ when p is inside Ω, $c(p) = 0$ when outside Ω, $c(p) = 1/2$ when on the smooth boundary.

3.2. Discretization, error and residual

Discretizing the boundary Γ for Eq. (4), we have

$$L(u(p_i)) \equiv c(p_i)u(p_i) - \sum_{m=1}^{M} \int_{\Gamma_m} [qu^* - uq^*]\, \mathrm{d}\Gamma = 0 \tag{5}$$

where Γ_m denotes boundary element and M is total number of the boundary elements. The potential and flux values on each element are approximated by the linear combination of their nodal values with the interpolation functions. Denoting the approximate solutions of u and q by $\hat{u}$ and $\hat{q}$, respectively, the potential and flux errors, e_u and e_q, are

$$e_u = u - \hat{u}, \qquad e_q = q - \hat{q} \tag{6}$$

If we substitute $\hat{u}$ and $\hat{q}$ into Eq. (5), instead of u and q, Eq. (5) is not satisfied perfectly and therefore, the residual R arises

$$R(p_i) = L(\hat{u}(p_i)) = c(p_i)\hat{u}(p_i) - \sum_{m=1}^{M} \int_{\Gamma_m} [\hat{q}u^* - \hat{u}q^*]\, \mathrm{d}\Gamma \tag{7}$$

In the collocation formulation, which is the most popular in the boundary element formulations, R is forced to be zero at the boundary points taken as the collocation points. Denoting the collocation points by p_i, we have

$$R(p_i) = 0 \ \ (i = 1, \ldots, N) \tag{8}$$

where N is total number of the collocation points. Collecting these equations, we have

$$\mathbf{Hu} = \mathbf{Gq} \tag{9}$$

where $\mathbf{u}$ and $\mathbf{q}$ denote the vectors of the nodal potential and flux values, respectively, and $\mathbf{H}$ and $\mathbf{G}$ are their coefficient matrices. Introducing the specified values on the boundary to Eq. (9), we have

$$\mathbf{Ax} = \mathbf{b} \tag{10}$$

where $\mathbf{x}$, $\mathbf{A}$ and $\mathbf{b}$ are the vector of the boundary unknowns, its coefficient matrix and the vector related to the specified values on the boundary.

Although the Galerkin formulation is not so popular, it is often applied for the boundary element formulation. In this formulation, the product of the residual R and (global) interpolation function Φ is integrated on the boundary, which is forced to be zero

$$\int_\Gamma R(p_i)\Phi\,d\Gamma = \int_\Gamma L(\hat{u}(p_i))\Phi\,d\Gamma$$
$$= \int_\Gamma \left\{ c(p_i)\hat{u}(p_i) - \sum_{m=1}^{M} \int_{\Gamma_m} [\hat{q}u^* - \hat{u}q^*]\,d\Gamma\Phi\,d\Gamma \right\} = 0 \tag{11}$$

Collecting them as a system of equations, we have

$$\mathbf{Hu} = \mathbf{Gq} \tag{12}$$

and finally,

$$\mathbf{Ax} = \mathbf{b}. \tag{13}$$

Although the above equations are written by the same notations as the collocation formulation, their definitions are different. Especially, we must notice that the boundary integrals appear twice in the above formulation although only once in the collocation formulation, which is the main reason why the Galerkin formulation is not so popular.

In the Galerkin BEM, it is proved that $\|e_u\|$ can be bounded by $\|R\|$ [41,43,110];

$$c_1\|R\| \le \|e_u\| \le c_2\|R\| \tag{14}$$

where c_1 and c_2 are the constants independent of $e(p_i)$ and $R(p_i)$ and $\|\cdot\|$ denotes the norm dependent on the subspace of the approximate solution. Finally, we should notice that the similar relationship is not proved mathematically for the collocation BEM.

4. Error estimation schemes

Error estimation process is the most important part of the adaptive process. We shall classify the existing studies on error estimation into residual type, interpolation error type, boundary integral equation error type, nodal design sensitivity type and solution difference type. The terminology is employed only in this paper for their classification and thus, the readers should recognize that it is not general.

4.1. Residual type

In the Galerkin-type boundary element method, the solution errors can be bounded by the residual of the boundary integral equation and therefore, the error is often estimated by the residual of boundary integral equation.

The error and the residual are estimated by error norms; L_1 norm [129–133,151], $H^0(L_2)$ norm [24,26–28,39,66,102–105,126–128], $H^{1/2}$ norm [41–43] and H^2 norm [109–112], which are dependent on the subspace spanned by the solution. They have individual advantages as well as disadvantages. Hsiao et al. [41,43] compared H^0 and $H^{1/2}$ norms and obtained the following conclusions:

1. $H^0(L_2)$ norm is relatively easy to calculate and may give appropriate error estimation when the boundary and the data are smooth enough.
2. When the boundary and the data are non-smooth, H^0 norm underestimates the actual solution error and in this case, $H^{1/2}$ norm may give better error estimation.

The residual type error estimation schemes are very popular in the finite element method. Since the finite element method is formulated by means of the Galerkin method or the variational approach and thus, it can be proved mathematically that the error is bounded with the residual, as shown in Eq. (14). The boundary element method is very often formulated by the help of the collocation method and therefore, there are some uncertainties when the residual type schemes are applied to the collocation-type boundary element method. Firstly, Eq. (14) is not proved mathematically for the collocation BEM and only confirmed by numerical examples. Secondly, as shown in Eq. (8), the residual disappears at the initial collocation points and therefore, a special technique is necessary for computation of the residuals of the equations. Parreira et al. [26,27,101–105] and Rank [109–112] estimated the residual at the collocation points which were different from those for the initial analysis. In these cases, the additional collocation points are often placed halfway between those for the initial analysis. Sun et al. [129–133,151] and Abe et al. [1–4] derived the relationship between the residual and the error from the boundary integral equation. The error is estimated in some ways and then, the residuals is calculated from the relationship. The way to derive the relationship is very similar to that described in the Section 4.3, which is explained in the following section.

Hyper-singular residual type. Recently, Paulino et al. [90,106,107] have presented the new residual type scheme named as 'hyper-singular residual type'. The similar scheme was also presented by Liang et al. [86].

According to the boundary element formulation, the boundary integral equation is given as Eq. (4). Taking the direct differentiation of Eq. (4) in the normal direction at the point p, we have the hyper-singular integral equation:

$$c(p)q(p) - \int_\Gamma \left[q\frac{\partial u^*}{\partial n} - u\frac{\partial q^*}{\partial n} \right] d\Gamma = 0 \tag{15}$$

Since the numerical solutions $\hat{u}$ and $\hat{q}$ do not satisfy Eq. (15), the residual yields as follows

$$R(p) = c(p)\hat{q}(p) - \int_\Gamma \left[\hat{q}\frac{\partial u^*}{\partial n} - \hat{u}\frac{q^*}{\partial n} \right] d\Gamma \ne 0 \tag{16}$$

On the other hand, the exact solutions u and q satisfy Eq. (15) and therefore,

$$0 = c(p)\hat{q}(p) - \int_\Gamma \left[\hat{q}\frac{\partial u^*}{\partial n} - \hat{u}\frac{q^*}{\partial n} \right] d\Gamma \ne 0 \tag{17}$$

Subtracting Eqs. (16) and (17), we have the error indicator:

$$\mathrm{EI}(p) = c(p)\hat{q}(p) - \int_{\Gamma}\left[\hat{q}\frac{\partial u*}{\partial n} - \hat{u}\frac{q^*}{\partial n}\right]\mathrm{d}\Gamma \neq 0 \tag{18}$$

4.2. Interpolation error type

Interpolation error estimation schemes were presented by Rencis et al. [113–118,134,135] and Kita et al. [76–78]. Exact solution is predicted by approximating numerical solution by higher order interpolation function than the initial analysis and then, the difference between the numerical and predicted exact solutions is estimated as the error (Fig. 1). Denoting the predicted exact solutions of potential and flux by u^0 and q^0 and the numerical solutions by $\hat{u}$ and $\hat{q}$, respectively, the potential and flux error $\|e_u\|$ and $\|e_q\|$ at each element are

$$\|e_u\|^2 = \int_{\Gamma_i}(u^0 - \hat{u})^2\,\mathrm{d}\Gamma \tag{19}$$

$$\|e_q\|^2 = \int_{\Gamma_i}(q^0 - \hat{q})^2\,\mathrm{d}\Gamma \tag{20}$$

where $\|\cdot\|$ denotes the norm.

Although these schemes have the great uncertainty that the computational accuracy of the predicted solutions is not guaranteed, the similar schemes are employed widely in the finite element method as well as the boundary element method [6,158,159]. This is because these schemes have some attractive features; their algorithms are very simple and their computational cost is cheaper than the others.

4.3. Boundary integral equation error type

The relationship between the solution error and the residual is derived from the boundary integral equation. The residual is estimated in some ways and then, the relationship is solved for the error. As mentioned above, Sun et al. [129–133,151] and Abe et al. [1–4] employed the same relationship for computation of the residual, while Alarcon et al. [7,15–17,84,119], Cerrolaza [18], Kamiya et al. [48–50,53,54,56–60,62–64,67–75,80,124,125] and Ye et al. [139,140] directly estimated the error from it. Besides,

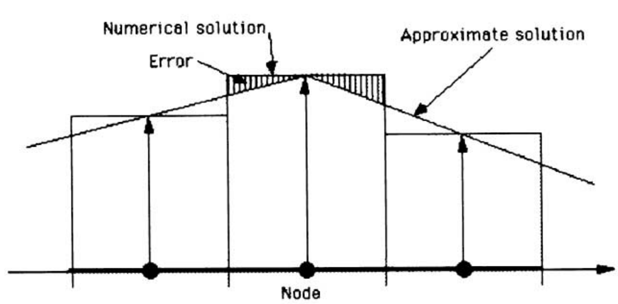

Fig. 1. Relationship between error and solution in case of interpolation type error estimation scheme.

Kamiya et al. presented first, the formulation on the two-dimensional potential and elastic problems [57–61,67–72,74,75] and then, extended to the subdomain-partitioned problem [62,63], the thermo-elastostatic problem [49], the two-dimensional elastic body under body force [52], the problems with mixed boundary conditions [50,51], the object governed by the quasi-harmonic differential equations [80,73] and the plate bending problems [124,125].

Alarcon et al. [15–17,84], Cerrolaza [18] and Kamiya et al. [48–50,53,54,56–60,62–64,67–75,80,124,125] derived the relationship as follows. Applying the collocation formulation to the boundary integral equation at a source point p_i, we have

$$c(p_i)u(p_i) = \int_{\Gamma}[qu^* - uq^*]\,\mathrm{d}\Gamma \tag{21}$$

Substituting the approximate solutions $\hat{u}$ and $\hat{q}$ into this equation, instead of u and q, we have

$$c(p_i)\hat{u}(p_i) = \int_{\Gamma}[\hat{q}u^* - \hat{u}q^*]\,\mathrm{d}\Gamma \tag{22}$$

Subtracting Eq. (22) from Eq. (21), we have

$$r(p_i) = \int_{\Gamma}[e_q u^* - e_u q^*]\,\mathrm{d}\Gamma \tag{23}$$

where e_u and e_q denote the errors of the potential and the flux, respectively. $r(p_i)$is the residual of the integral equation defined as

$$r(p_i) \equiv cu(p_i) - c\hat{u}(p_i) \tag{24}$$

The residual is estimated in some ways and then, Eq. (23) is discretized and solved for the solution error. In this case, there is great difficulty. Since, in the collocation-type boundary element method, the residual always disappears on the initial collocation points, special techniques are necessary for this purpose.

The popular way to estimate the residual is to take the other collocation points for computation of the residual than those for the initial analysis. Since the conforming linear boundary elements are employed for the initial analysis, Kamiya et al. [48–50,53,54,56–60,62–64,67–71] placed the additional collocation points on the middle of each element and then, it is assumed that the error is maximum at the center of each element and zero at both ends. Since, in the studies of Sawaki et al. [124,125], the constant elements are employed for the initial analysis, the additional collocation points are taken at the other place rather than the middle of each element. In this case, the accuracy of the error estimation is dependent on the placement of the additional points. For overcoming this difficulty, Kita et al. [73,74,80] estimated the mean value of the residual distribution on the element, instead of taking the additional collocation points. According to their formulation, the residual r on the element

Γ_m is estimated from Eq. (24) as follows:

$$r = \frac{1}{L_m} \int_{\Gamma_m} [cu(p_i) - c\hat{u}(p_i)]\, \mathrm{d}\Gamma \tag{25}$$

where L_m denotes the length of the element Γ_m.

4.4. Nodal design sensitivity type

Paulino et al. [108] have presented a new methodology for the error estimation. A similar scheme is also presented by Guiggiani [35]. In their studies, the solution error is estimated as the derivative of the solution with respect to the movement in the tangential direction of the boundary collocation points.

It is considered that their studies are motivated by the work of Guiggiani et al. [31,32,37,38]. In the scheme, the initial analysis is done by the conforming quadratic elements and then, two analyses are performed for the error estimation. At the first analysis, the center node is placed on the middle between both end points and at the second, it is a little far from the middle. The solution error is defined as the difference between the solutions at the first and the second analyses (Fig. 2). Guiggiani et al. defined as the solution error the variation of the solution with respect to the variation of the position of the center node. On the other hand, Paulino et al. [90,106–108] presented the more sophisticated scheme. In the scheme, the variation of the solution is estimated directly from the integral equation derived from direct differentiation of the original integral equation in the tangential direction. They named the variation of the solution as 'nodal sensitivity'. This scheme, however, includes a great difficulty. Since the original integral equation has the singular property due to the fundamental solution, the equation with respect to the nodal sensitivity has hyper-singularity and therefore, special techniques are necessary for estimation of the hyper-singular equation. During the 1990s, many researchers have studied the formulations for estimating the hyper-singular integral equation related to the design sensitivity analysis in order to solve the structural optimization problems; e.g. Barone et al. [10,11], Kwak et al. [82], Matsumoto et al. [88,89], Guiggiani et al. [33,34,36], Zhao et al. [153–155] and others. We may say that in the nodal sensitivity type scheme, a similar reasoning is applied to the error estimation.

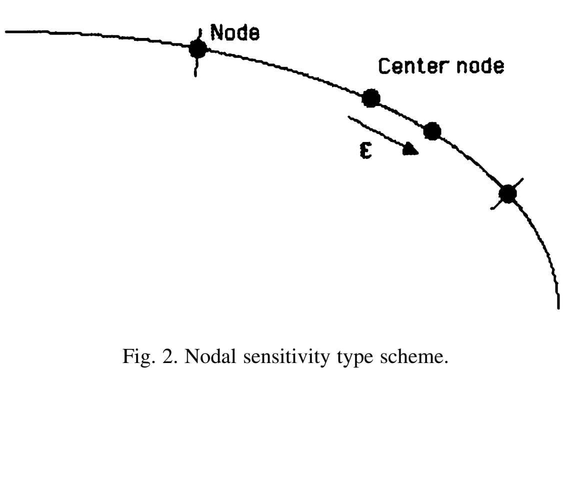

Fig. 2. Nodal sensitivity type scheme.

4.5. Solution difference type

Some researchers presented the schemes in which the solution error is defined as the difference between the solutions of the first and the second analyses. We shall call them as 'solution difference type'. While the ordinary boundary element method is applied for the first analysis, several schemes are employed as the second one. According to the schemes for the second analysis, they can be classified as follows.

(1) Mullen and Rencis: in their scheme [99], after the first analysis is done by the boundary element mesh with N equal elements, the second one is by the finer mesh with $2N$ equal elements. The solution in the second analysis is expected to be more accurate than the initial and then, the error is defined by the difference between both solutions.

(2) Yuuki et al.: in their scheme [146–150], the problem is solved by the ordinary and non-singular boundary element methods. In the first one, which is called 'direct singular method (DSM)', source points are on the boundary. In the second one, which is so-called 'direct regular method (DRM)', source points are placed outside the boundary of the object under consideration. The error is estimated by the difference of both solutions.

(3) Charafi et al.: in this scheme [19–21], after boundary elements analysis for the initial mesh, an arbitrary element is divided into equal length sub-elements and 'local re-analysis' is carried out for predicting more accurate solution at the element. The error is defined by the difference between both solutions. The local re-analysis is carried out over each element and the elements with bigger error are refined. Kamiya et al. [65] applied this scheme to the error estimation of the eigenvalue analysis problem of the Helmholtz equation.

We shall compare here the scheme of Charafi et al. with the nodal sensitivity type. While, in the nodal sensitivity type, the error is defined from the variation (sensitivity) of the solution with respect to the position of the central node, in the scheme of Charafi et al., the variation of the solution with respect to the addition of the node leads to the error. The scheme of Charafi et al. may be classified into the nodal sensitivity type.

(4) Muci-Küchler et al.: recently, Muci-Küchler et al. [94–98] presented the new scheme for the error estimation of the boundary element solutions. In their scheme, the error indicator is defined as the difference between a solution obtained with Hermite elements and a second 'reduced' solution. The second one is obtained by approximating the first one obtained with Hermite elements using Lagrangian shape functions inside each element.

4.6. Summary

In this section, the error estimation schemes are classified

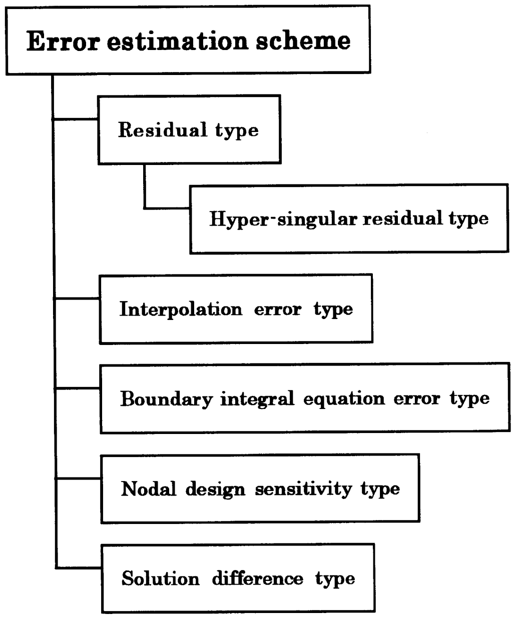

Fig. 3. Error estimation schemes.

into the residual type, the interpolation type, the boundary integral equation error type, the nodal sensitivity type and the solution difference type schemes (Fig. 3). In the residual type, the solution error is estimated by the residual of the integral equation. In the interpolation error type, the exact solution is predicted from the numerical solution by using the higher order interpolation functions and then, the error is defined by the difference between the numerical and predicted solutions. In the boundary integral equation error type, the relationship between the error and the residual, which is derived from the boundary integral equation, is discretized and solved for the error. In the nodal sensitivity type, the error is defined from the sensitivity of the solution with respect to the tangential movement of the node. Finally, in the solution difference type, the error is defined as the difference between the solutions of the first and the second analyses.

From the viewpoint of the computational accuracy, the residual type may have a difficulty. In the residual type, it is assumed that the solution error can be bounded by the residual of the integral equations. We, however, should notice that this relationship is not proved for the collocation BEM, which is the most usual formulation of the boundary element method.

From the viewpoint of the computational cost, the interpolation type is less expensive than the other schemes. This is because the computational cost of the interpolation type depends only on the approximation of the BEM solutions using the higher-order interpolation functions. In the schemes other than the interpolation type and the scheme of Muci-Küchler et al., the cost of the error estimation is almost equal to that of the initial BEM analysis. Therefore, some researchers have presented the schemes for reducing the cost of the error estimation; local reanalysis [19–21], parallel processing [53–56] and so on.

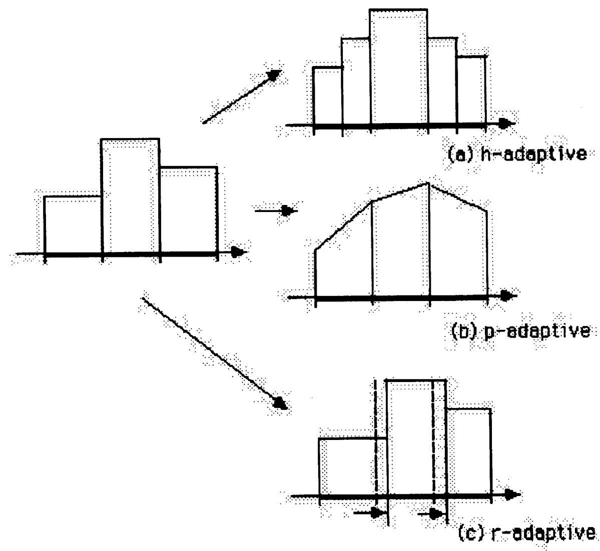

Fig. 4. h-, p- and r-Refinement scheme.

5. Mesh refinement

Mesh refinement schemes for the boundary element method, according to the terminology for the finite element method, are basically classified into the h-, p-, r-refinement and their combinations schemes (Fig. 4). In this paper, the singular element method is also included into the mesh refinement schemes.

5.1. h-Refinement scheme

In the h-refinement scheme, total number of elements is increased but the order of interpolation function remains invariant [26,27,37,38,76–78,100,105]. The algorithm is simple and easy to implement on boundary element solver. Since, however, the global matrix must be formulated after each mesh refinement, the computational cost is expensive. For overcoming this difficulty, h-hierarchical refinement scheme was presented [26,27,37,38,81,105,156], which employs h-hierarchical interpolation function and therefore, mesh refinement can be implemented in the sense of h-refinement although the elements are not divided into sub-elements actually.

Besides Kita and Kamiya presented an alternative h-refinement scheme in which not only elements with bigger error are divided but also the elements with relatively smaller error are combined. They named it as 'reverse h-adaptive' [76–78].

h-Hierarchical refinement scheme: Parreira [101] firstly presented the h-hierarchical refinement scheme using the constant h-hierarchical interpolation function and then, the schemes using the linear and quadratic h- hierarchical interpolation functions were presented [26,27,37,38,105]. Linear

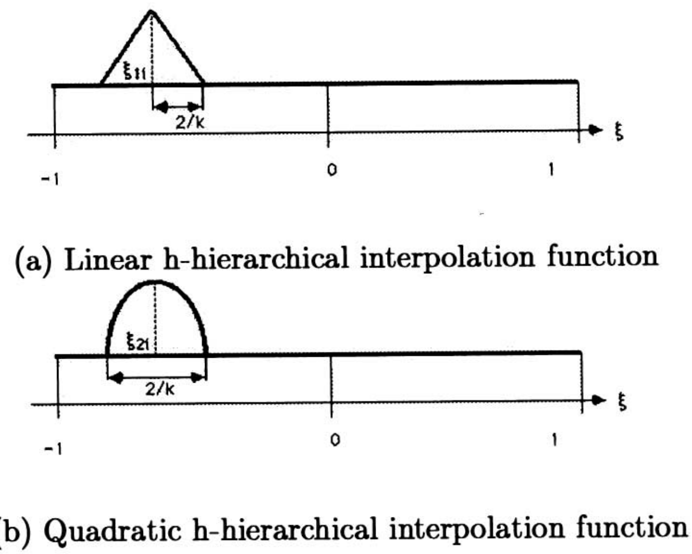

Fig. 5. h-Hierarchical interpolation functions.

and quadratic h-hierarchical interpolation functions are defined as follows:

$$\text{Linear type}: \ \Phi_{1,kl} = \begin{cases} 0 & \text{if } |\xi - \xi_{1l}| \geq 2/k \\ 1 - \dfrac{|\xi - \xi_{1l}|}{2/k} & \text{if } |\xi - \xi_{1l}| \leq 2/k \end{cases} \tag{26}$$

quadratic type : $\Phi_{2,kl}$

$$= \begin{cases} 0 & \text{if} |\xi - \xi_{2l}| \geq 1/k \\ 1 - \dfrac{(\xi - \xi_{2k})^2}{(1/k)^2} & \text{if } |\xi - \xi_{2l}| \leq 1/k \end{cases} \tag{27}$$

where k is the number of division of element. ξ is the local coordinate on each element and ξ_{1i} and ξ_{2i} the center of each sub-element shown in Fig. 5.

Standard and h-hierarchical linear interpolation functions are shown in Fig. 6 [27,105]. Fig. 6(a) indicates the interpolation functions on the element before mesh refinement and Fig. 6(b) and (c) indicate them after the elements are divided into two or three equal length sub-elements, respectively. Additional h-hierarchical functions are indicated by broken lines.

In the hierarchical h-refinement scheme, the initial analysis is carried out by using standard interpolation functions and then, the solution accuracy is improved by adding the h-hierarchical interpolation functions to the solution for the mesh before refinement. Denoting the initial and improved solutions by $\hat{u}$ and $\hat{\hat{u}}$, we have

$$\hat{\hat{u}} = \hat{u} + \sum \Phi_{i,k} \tilde{u}_{i,k} \tag{28}$$

where $\tilde{u}_{i,k}$ is the parameter related to the newly generated elements.

The matrix equation for the initial assumed mesh

$$\mathbf{A}_{11}\mathbf{x}_1 = \mathbf{b}_1 \tag{29}$$

is refined by the h-hierarchical scheme to give

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{Bmatrix} \tag{30}$$

where $\mathbf{A}_{11}$, $\mathbf{b}_1$ and $\mathbf{x}_1$ are the matrix and the vectors related to the initial assumed mesh and $\mathbf{A}_{12}$, $\mathbf{A}_{21}$, $\mathbf{A}_{22}$, $\mathbf{b}_2$ and $\mathbf{x}_2$ are the matrices and the vectors related to the refined mesh. In this case, since the coefficient matrix $\mathbf{A}_{11}$ of Eq. (30) is the same as that of Eq. (29), the coefficient matrix for the refined mesh can be constructed by only calculating $\mathbf{A}_{12}$, $\mathbf{A}_{21}$ and

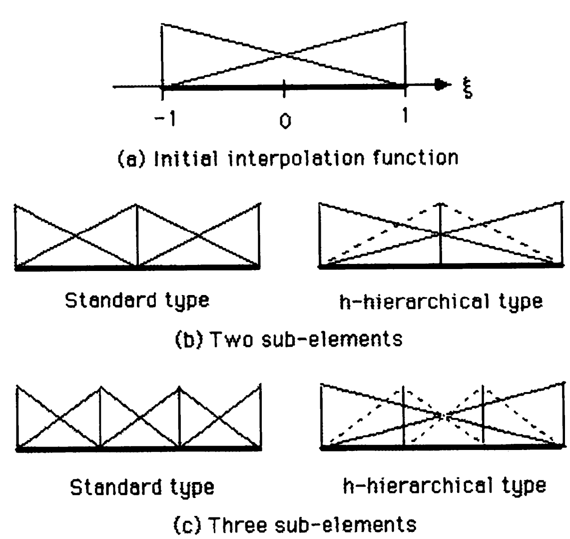

Fig. 6. Interpolation functions of ordinary and h-hierarchical linear elements.

$\mathbf{A}_{22}$. On the other hands, in the h-refinement scheme using the ordinary interpolation function, the matrix equation on the refined mesh is as follows:

$$\begin{bmatrix} \mathbf{A}'_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{Bmatrix} = \begin{Bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \end{Bmatrix} \tag{31}$$

In this case, the coefficient matrix for the initial mesh $\mathbf{A}_{11}$ is also changed, which may increase the computational cost to generate the coefficient matrix on the refined mesh.

Reverse h-adaptive. In the h-adaptive scheme, the rough mesh is specified as the initial and then, successive improvement of meshes is performed by iterations. The mesh refinement is carried out so as to increase monotonously the number of elements. This process has the following disadvantages.

- The computational cost is expensive.
- Fine final mesh may have more degrees of freedom exceeding the computer storage.

The reverse h-adaptive is designed in order to overcome the above-mentioned disadvantages [76–78]. In this scheme, a relatively fine initial mesh is employed, which is usually constructed by equal length elements and whose number of elements is determined by user. During successive mesh refinement, not only the elements with bigger error are divided but also the elements with relatively smaller error are combined. While the computational accuracy is improved by dividing the elements with bigger error, the combination of the elements with relatively small error prevents the increase of the total number of elements and therefore, the computational cost can be reduced. As a result, this scheme has the following advantages.

- The degrees of freedom of the refined mesh do not exceed the computer storage.
- In the ordinary scheme, iterative refinement of rough initial mesh leads to fairly fine final mesh, which is one of the reason why the computational cost is expensive. Since this scheme employs relatively fine initial mesh, the computational cost may by saved fairly.

5.2. p-Refinement scheme

In p-refinement scheme, the initially assumed mesh is not changed during the entire iterative process but the order of interpolation function is increased uniformly or selectively. The p-refinement schemes are classified into 'ordinary type' scheme using the ordinary interpolation function [134,135] and 'hierarchical type' scheme using the p-hierarchical interpolation function [7,15–18,23,30,84,102,103,119]. Since the most attention focuses on the latter, we will discuss it still more in the following.

p-Hierarchical refinement scheme. In this scheme, the conforming linear interpolation function is often taken as the initial one, which is defined as

$$\phi_0 = \frac{1}{2}(1 - \xi) \qquad \phi_1 = \frac{1}{2}(1 + \xi) \tag{32}$$

where ξ is the local coordinate on the element. The accuracy is improved by adding the p-hierarchical interpolation functions to the numerical solution. As the p-hierarchical interpolation functions, the Legendre polynomials [15,18,84,103,102] and the Peano's family [7,16,17,119] are employed:

$$\text{Legendre polynomials}: \ \phi_k = \frac{1}{2^{k-2}(k-1)!}\frac{d^{k-2}}{d\xi^{k-2}} \times [(1-\xi^2)^{k-1}]$$

$$\text{Peano's family}: \ \phi_k = \frac{1}{k!}(\xi^k - b)$$

where $b = 1$ if k is odd number and $b = \xi$ if k is even number. Parreira [103] compared the Legendre polynomials and the Peano's family and concluded as follows.

- Using the Legendre polynomials, the condition number of the coefficient matrices is smaller than that in the case of Peano's family. The error of the numerical integration, however, becomes larger due to their oscillatory character.
- Using the Peano's family, the interelement continuity on a continuous model is easier to be implemented and more accurate numerical integration can be achieved. So, the Peano's family seems more convenient to use.

Placement scheme of new collocation point. When the order of the interpolation function is increased, new collocation points are located on the element to be refined. The ways of adding new collocation points are classified into 'symmetric type' and 'true hierarchical type' (Fig. 7). In

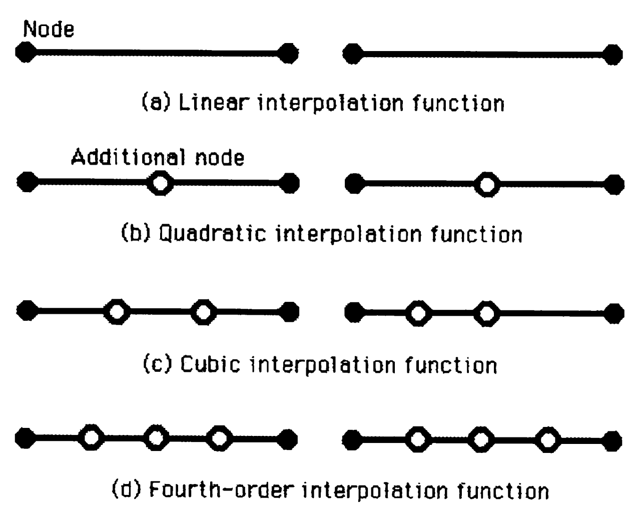

Fig. 7. Placement of new node for p-hierarchical refinement scheme.

the 'symmetric type', the collocation points are located symmetrically and uniformly on the element and therefore, the coefficient matrices related to not only new collocation points but also all replaced points should be re-calculated. Since, in the 'true hierarchical type', the nodes are located so as not to change the distribution of the collocation points at the present step, only coefficient matrices related to the newly added collocation points are re-calculated after each mesh refinement. Therefore, as pointed out by Crim and Basu [22,23], the 'true hierarchical type' scheme is more efficient than the 'symmetric type' scheme. We should notice, however, that the 'true hierarchical type' scheme has lower computational accuracy due to the non-symmetric distribution of collocation points.

5.3. r-Refinement scheme

In the r-refinement scheme, both the total number of elements and the order of the interpolation function are kept invariant but the mesh points are relocated within fixed topology in order minimize the global error of the mesh.

The r-refinement schemes have been studied by Ingber et al. [44–46], Sun et al. [129–132,151], Hall et al. [40], Kita et al. [72–75] and Abe et al. [5].

Since this scheme does not allow the increase of degrees of freedom, the desired accuracy may not be obtained if the initial mesh does not have sufficient degrees of freedom. Therefore, the combination of the r-adaptive scheme and the other schemes such as the h-adaptive scheme should be studied still more.

5.4. Combination schemes

Since the above-mentioned schemes have individual advantages as well as disadvantages, the combination schemes are devised in order to overcome their disadvantages.

hp-Refinement scheme. Rank [110–112,109], Stephan et al. [128,127], Demkowicz [25] and Bachtold et al. [8] presented hp-refinement scheme, in which h-refinement is applied for the elements adjacent to a singular point, while the p-refinement is applied for the other.

Rank [109–112] compared h-, p- and hp-refinement schemes by numerical experiments and indicated the following features:

- The convergence rate of the h-refinement scheme is less dependent on the singularity of the exact solution than the p-refinement scheme.
- If the exact solution is smooth, the p-refinement scheme has faster convergence rate than the h-refinement scheme (the former is almost twice as fast as the latter).
- Finally, the presented hp-refinement scheme has the exponential convergence rate even when the exact solution has singularity.

hr-Refinement scheme. hr-Adaptive schemes are presented by Sun and Zamani [130,132,133,151], Yuuki et al. [149] and Kita et al. [72,75]. In this case, the initial mesh is firstly refined by the h-refinement scheme and then, more in detail by the r-refinement scheme.

5.5. Singular element method

Although, usually, the singular element method is not included into the adaptive scheme, it is worth considering it here.

The boundary element method is usually formulated by the conforming elements. When, however, the exact solution has singularity, the conforming elements cannot approximate the singularity accurately. Hsiao et al. [43] and Li [85], therefore, pointed out that the singular elements should be employed near singular points. If the singular points could be predicted in advance, it is easy to place the singular elements near the singular points. If not so, the analysis starts with the conforming elements and then, the conforming elements should be replaced with the singular elements.

5.6. Summary

The mesh refinement schemes are basically classified into h-, p-, r-, hp- and hr-schemes. In this paper, the singular element method is also included in the mesh refinement scheme (Fig. 8).

In the h-scheme, the elements with large error are subdivided but the order of the interpolation functions is invariant. In the p-scheme, the order of the interpolation functions is changed but the number of the elements and the boundary element mesh are invariant. In the r-scheme, the number of the elements and the order of the functions is invariant but the boundary element mesh is modified.

The h-and p-schemes are mainly classified into the

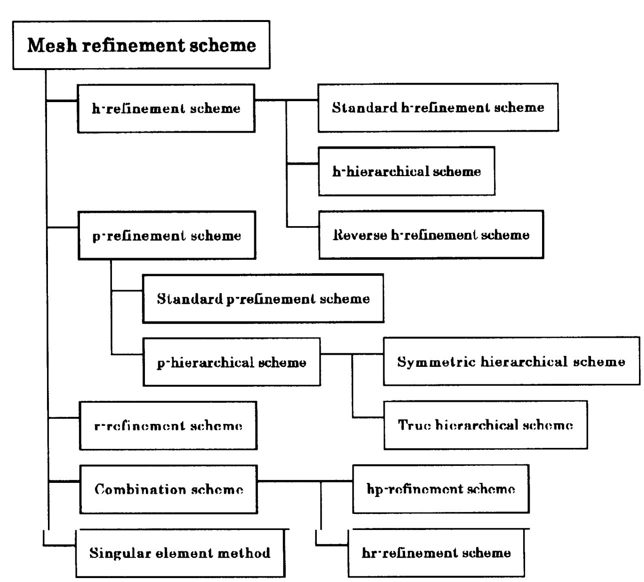

Fig. 8. Mesh refinement schemes.

standard and the hierarchical schemes. In the standard scheme, the elements are actually divided or the interpolation functions are changed. On the other hand, the hierarchical schemes perform them by adding the hierarchical interpolation functions. From the view point of the computational efficiency, the hierarchical schemes are better than the standard ones. This is because the computational cost of the coefficient matrices can be reduced after the mesh refinement.

From the viewpoint of the convergence speed of the error, the p-scheme is the most attractive. However, in the problem of which the exact solution has the singular property, the h-scheme has good error convergence property around the singular point. Besides, the h-scheme tends to generate the fine mesh divided with many elements and therefore, the r-scheme should be combined with the h-scheme in order to restrict the exponential increase of the elements. Finally, the hpr-scheme may be the best.

6. Adaptive tactics

The role of the adaptive tactics is to determine the elements to be refined and the related mesh refinement scheme according to the result of the error estimation. Since its algorithm is strongly dependent on the error estimation and mesh refinement schemes, it is often difficult to separate this process from the others.

6.1. Adaptive tactics for h-refinement

Reference value method: the elements are refined when their errors are bigger than the prescribed reference value. Denoting the error at the element i by λ_i, the reference value $\bar{\lambda}$ may be defined say, by

$$\bar{\lambda} = \text{Average of } \lambda_i \tag{33}$$

or

$$\bar{\lambda} = \eta \times \max(\lambda_i) \ \ (0 < \eta < 1). \tag{34}$$

When using the former reference value, the error at each element is minimized on the average and we often encounter very fine final mesh. Such a difficulty may not happen when employing the latter. But it is, in general, difficult to specify the value of η adequately. On the contrary, it may be also effective that the value of η is changed according to the convergence property of the error.

Multi-division method of the elements is also considered. In this case, individual reference values are specified for individual division numbers of elements and then, the elements are divided into two or more sub-elements according to the magnitude of their error. The multi-division method seems to have higher convergence rate than simple two-division method. In general, it is very difficult to specify the adequate reference values.

Error convergence method: in the Galerkin BEM, it was proved that the solution error converges in proportion to the square of the length of element [137]:

$$\int_\Gamma (u^{k+1} - u^k)^2 \, \mathrm{d}T = K(h^k)^2 + o(h^k)^3 \tag{35}$$

where u^k and $u^{k+1}$are the solutions at the kth and $(k+1)$th iterations, respectively. h^k is the standard length of the element at the kth iteration and K is a parameter. From these equations, Rencis et al. [113,114,116,117], Wang et al. [136] and Rodriguez et al. [120] derived a relationship to determine the division number of elements for obtaining the desired accuracy. In their descending studies [115,118,134,135], they derived the similar relationship from the assumption that the solution error converged in proportion to h^k but its theoretical background is not evident. Yokoyama et al. [121–123,142,143] also presented the similar method. They determined the division number of elements from the assumption that the error converged in proportion to $(h^k)^\beta$, where the parameter β was determined by numerical experiments. For the error convergence method, we should note the following points.

- The relationship between the error and the length of the element has been proved for the Galerkin BEM but its theoretical background is not evident for the collocation BEM.
- The division number of the elements is determined and then, mesh refinement is carried out by dividing the elements into equal length sub-elements. So, when the exact solution has singularity, it may be difficult to generate the graded mesh to a singular point.

Equilibrium criterion method: Leal and Mota Soares [83] presented the method based on the equilibrium of the external force and the tangential stresses on the boundary.

The global error of the mesh is estimated by the equilibrium of external forces. The global equilibrium criterion without body forces is expressed as

$$I_i = \int_\Gamma t_i \, \mathrm{d}\Gamma \ \ i = 1, 2, 3 \tag{36}$$

where t_i is the traction component in the x_i direction. If I_i is greater than the prescribed value, each element is divided into equal length sub-elements.

After the global equilibrium is achieved, the local equilibrium criterion is applied in order to generate the graded meshes. The local equilibrium criterion is defined by the difference of the tangential stress components on adjacent elements. If the difference is greater than the prescribed value, the adjacent elements are divided into equal length sub-elements.

6.2. Adaptive tactics for hierarchical p-refinement

Alarcon et al.: in the scheme presented by Alarcon et al. [7,15–17,84,119], and Cerrolaza [18], initial analysis is carried out by conforming linear elements and then, the solution error is reduced by adding p-hierarchical functions to the initial interpolation of the solution. Denoting the initial solutions by $\hat{u}$ and $\hat{q}$, the improved solutions $\hat{\hat{u}}$ and $\hat{\hat{q}}$ are expressed as

$$\hat{\hat{u}} = \hat{u} + a_{m+1}N_{m+1}, \qquad \hat{\hat{q}} = \hat{q} + b_{n+1}N_{n+1} \tag{37}$$

where N_{m+1} and N_{n+1} are p-hierarchical interpolation functions. a_{m+1} and b_{n+1} are unkown coefficients, which are calculated by applying the algorithm employed in FEM [157] with some modification.

Parreira et al.: in the above-mentioned scheme, the solution error is reduced by the mesh refinement. On the other hand, in the scheme presented by Parreira [102–104], the residual of the integral equation is reduced by the mesh refinement. In this case, the residual of the integral equation, Eq. (7), is expressed by the product of p-hierarchical interpolation function N_{m+1} and unknown coefficient a_{m+1}:

$$R(p_i) = L(\hat{u}(p_i)) \equiv a_{m+1}N_{m+1} \tag{38}$$

$a_{m+1}$is determined by the collocation method or the Galerkin method.

Error convergence method: Yokoyama et al. [141,144,145,152] extend to the p-refinement the error convergence method which they have already presented for the h-refinement [121–123,142,143]. It is assumed that the solution error converges in proportion to the inverse number of the order of the interpolation function p;

$$\int_\Gamma (u^{k+1} - u^k)^2 \, d\Gamma = K\left(\frac{1}{p^k}\right)^\beta \tag{39}$$

The parameters K and β are determined from the numerical expermients.

6.3. Adaptive tactics for r-refinement

The tactics for r-refinement scheme often have the algorithm defined as an optimization problem. The object function expresses the total error of the employed mesh, which is minimized by appropriate mesh redistribution. For example, Ingber and Mitra [44] defined the following object function and constraint condition.

$$\sum_{i=1}^{k} h_i \|e_u\|_\infty \tag{40}$$

$$\sum_{i=1}^{k} h_i = 1, \;\; h_i > 0 \tag{41}$$

where $\|e_u\|\infty$ is the maximum error norm at each element and h_i is the length of the element normalized as

$$\sum_{i=1}^{k} h_i = 1 \tag{42}$$

and besides, k is the total number of elements. This problem is solved by interative algorithm.

Ingber and Mitra [44] defined the object function by using the maximum error norm, while Abe [1–3] and Sun and Zamani [129–133,151] employed the residual of the integral equation.

Carey et al. [13,14], Sun and Zamani [129–133,151] and Abe [1,2] employed the grading function in order to relate the error and the mesh distribution. Yuuki et al. [149] also employed mesh density function related to the error of each element. Kita et al. introduced the relationship between the error and the length of the element. The similar relationship is employed for the h-adaptive scheme by Rencis et al. [113,114,116,117].

6.4. Adaptive tactics for hp-refinement

The adaptive tactics for hp-refinement was presented by Rank [109–112], which is considered to be one of the above-mentioned 'reference value method' for h-refinement. The solution error at each element is expressed by 'error indicator' which is defined by the residual of the equation. Initial analysis is done by conforming linear elements and then, if the error indicator is greater than the reference value, the elements may be refined as follows:

- if the element is not adjacent to a singular point, the polynomial degree is increased by one (p-refinement).
- if the element is adjacent to a singular point, it is divided into two sub-elements (h-refinement). The polynomial degree of the newly generated element adjacent to a singular point is 1 and the other gets polynomial degree 2.

Note that, on the elements adjacent to singular points, the polynomial degree is always 1.

As indicated by Rank [109–112] and Stephan et al. [28,39], in this scheme, the solution error converges exponentially even when the exact solution has singularity. This is the reason why hp-refinement is thought to be the most promising.

6.5. Extended error indicator

In the above-mentioned various schemes, the bigger error elements are refined because the error is thought to have the local property; i.e. the error at each element is thought to have negligible effect to the solution at the other elements. This assumption, in the finite element method, is guaranteed by the local property of certain kinds of governing differential equations and the interpolation function [111]. In the boundary element method, however, it is not guaranteed because the local error has some influence on the global solutions due to the fundamental solution of influence function type. Kamiya et al. [48–50,53,54,56–60,62–64,67–75,80,124,125] paid attention to this point and presented 'extended error indicator'. The extended error indicator is defined by integrating the product of the solution error and the related fundamental solution at each element. Therefore, not only the magnitude of the error but also the global property of the error can be reflected to the mesh refinement.

They employed it to the h-refinement scheme, in which the elements to be refined are determined by the magnitude of the extended error indicator, instead of the local error level. Recently, they applied the extended error indicator to the r- and hr-adaptive schemes [72,74,75].

6.6. Summary

The adaptive tactics are classified into them for the h-, p-, r- and hp-schemes and the extended error indicator (Fig. 9).

The reference value method is considered to be the most popular among the tactics for the h-scheme. However, there does not exist the theoretical way to specify the reference value in advance, which is the great difficulty. Besides, it is usual that the element with larger error than the reference value is divided into two equal-length elements. However, the element with very large error should be divided into more than two elements in order to improve the convergence speed still more. For overcoming these difficulties, the relationship between the computational error and the number of the elements should be derived theoretically.

Regarding the p- and r-schemes, the theoretical background of the tactics is established better than that for the h-scheme. In the p-scheme, the computational error or the residual of the integral equation is related to the order of the interpolating functions, which leads to the tactics. In the r-scheme, the mesh refinement process is formulated as the optimization problem in which the total error is minimized by changing the distribution of the elements. The tactics for the r-scheme, however, is very difficult to formulate.

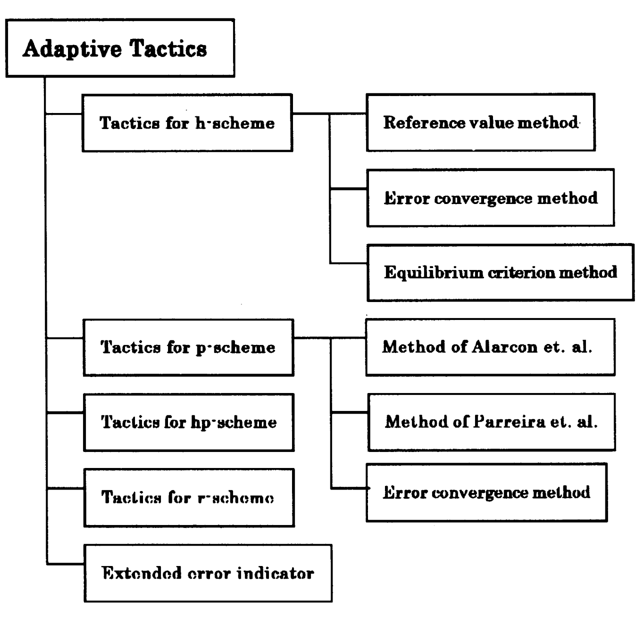

Fig. 9. Adaptive tactics.

7. Conclusions

This paper discussed the error estimation and the adaptive mesh refinement schemes on boundary element method. We considered here that the adaptive process was composed of three main processes such as the error estimation, the adaptive tactics and the mesh refinement processes. Therefore, the existing schemes were classified and discussed from these view-points.

In Section 4, we discussed the error estimation schemes. The schemes are classified into the residual type, the interpolation error type, the boundary integral equation error type, the nodal sensitivity type and the solution difference type. While, in the residual type, the solution error is estimated by the residual of boundary integral equation, the other schemes directly estimate the solution error. In the interpolation error type, the exact solution is predicted from the numerical solution by using the higher order interpolation functions and then, the error is defined by the difference between the numerical and predicted solutions. In the residual type and the boundary integral equation error type, the relationship between the error and the residual, which is derived from the boundary integral equation, is discretized and solved for the error. In the nodal sensitivity type, the error is defined from the sensitivity of the solution with respect to the tangential movement of the node. Finally, in the solution difference type, the error is defined as the difference between the solutions of the first and the second analyses. The computational cost of the boundary integral equation type and the nodal sensitivity type is more expensive than the others. The high computational accuracy is the most important feature for the error estimation scheme. However, in order to select the most adequate error estimation scheme for the problem to be solved, we should take into consideration the other features such as the computational cost, the level of the required accuracy, the compatibility with the mesh refinement schemes and the adaptive tactics.

In Section 5, we discussed the mesh refinement schemes. The schemes are classified mainly into h-, p-, r- and their combination schemes. Since they have the individual advantages and disadvantages, it is difficult to select the best one among them. If forced to determine, the hp-refinement scheme seems to be the most promising among them because of its higher error convergence. In the h-refinement, the mesh refinement is done by dividing the elements with the bigger error into equal length sub-elements. Since, however, the non-uniform division of the elements and the so-called 'graded mesh' are thought to be effective for some problems [91–93], the combination of the h- and r-adaptive schemes should be studied still more. Finally, the attention will be focused on the development of the hpr-scheme. The singular element method is thought to be effective for the problems whose solutions have the singularity and therefore, the scheme to automatically exchange the conforming and the singular elements according to the property of the solution is necessary.

In Section 6, we discussed the adaptive tactics. They are strongly dependent on the error estimation, the mesh refinement and the programmer's policy and favorite. It is very difficult to determine the best one among them and therefore, we should study it still more.

Finally, we would like to mention new trends in the adaptive boundary element method. The first is to decrease the computational cost of the error estimation. The computational accuracy and the efficiency of the error estimation schemes conflict with each other. The accurate error estimation scheme is usually time-consuming and on the contrary, the simple scheme has relatively lower accuracy. Kamiya et al. [53–56], therefore, employed the workstation cluster system in order to improve the computational efficiency of the error estimation scheme. The second is to apply the adaptive boundary element method to the three-dimensional and the actual engineering problems. The attention mainly focuses on the basic studies of the adaptive boundary element method except for some studies [8,28,29,65,80,138] and therefore, we should study still more.

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