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Analysis of anisotropic plate using multiquadric radial basis function

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Abstract

In this paper, multiquadric radial basis function is used for dynamic and static analysis of anisotropic plates. Multiquadric radial basis function is applied for spatial discretization and Newmark implicit scheme is used for temporal discretization. The spatial discretization of the differential equations generates greater number of algebraic equations than the unknown coefficients. The multiple linear regression analysis, which is based on the least squares error norm, is employed to obtain the coefficients. Considering simple supported boundary conditions, an analogy between isotropic skew plates and rectangular anisotropic plates is used for solutions. The effect of fiber orientation is observed in clamped square and rectangular anisotropic plates. The results obtained by this method are compared with those obtained by other analytical methods.

Keywords: Isotropic skew plates; Anisotropic plates; Multiquadric radial basis function; Analogy; Fiber orientation

1. Introduction

There is a need to have an efficient analytical method for the study of lightweight structure made of anisotropic materials. Various numerical and analytical methods are available for static and dynamic analyses of anisotropic plates. But analytical methods are restricted to simple geometries and boundary conditions [1,2]. Finite-element methods [3] are used for the analysis of complex anisotropic plates but generation of mesh is a time-consuming task. To avoid the mesh-generation process for the analysis of complex anisotropic plates, recently classes of new methods known as meshless methods are developed. Some of these methods are: meshless local Petrov–Galerkin (MLPG) method [4], the method of finite sphere [5], element-free Galerkin method (EFG) [6], local boundary integral equation method [7], natural neighbor Galerkin method [8], and partition of unity method [9]. In this paper, multiquadric radial basis function (MQRBF) method has been used, which was developed by Kansa [10]. Franke [11] studied the evaluation of radial basis function (RBF) for scattered data interpolation in terms of timing, accuracy and ease of implementation. Hardy [12] used the MQRBF for the interpolation of geographically scattered data. Coleman [13] employed RBFs in the analysis of elliptic boundary value problems. Ferreira applied RBF for the analysis of laminated composite beams [14] and plates [15]. Wendland [16] used it as interpolation function and estimated the error. Hon and Wu [17] solved the stiff ordinary differential equations. Chen et al. [18] studied the free-vibration analysis of circular and rectangular plate. In this paper, anisotropic rectangular plates with simply supported and clamped boundary conditions subjected to uniformly distributed transverse load have been studied.

2. Definitions and basic equations of anisotropic plate

Consider a small deflection in a thin anisotropic elastic plate. Using Kirchhoff's plate theory, strain–displacement relation is defined [19] as

$$\begin{bmatrix} \varepsilon_{x^*} \\ \varepsilon_{y^*} \\ \gamma_{x^*y^*} \end{bmatrix} = -z^*\kappa w^*. \qquad (1)$$

The strain at a distance z^* from the middle plane can be defined in terms of curvature of the surface developed due

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Nomenclature

m	mass of the plate
a,b	dimension of plates
C_v^*, C_v	viscous damping, dimensionless viscous damping
D_{11}, D_{22}, D_{12}, D_{66}	flexural rigidity of plates
D_{16}, D_{26}	bending–twisting coupling stiffness terms
E_x, E_y	Young's modulus in x^*, y^* directions, respectively
E_L	Young's modulus parallel to the fibers
E_T	Young's modulus transverse to the fibers
v_{LT}	major Poisson's ratio
G_{LT}	shear modulus relative to the $x_L^* - x_T^*$ plane
θ	orientation of the fiber
h	thickness of plates
q, Q	transverse load, dimensionless transverse load
t^*, t	time, dimensionless time
λ	aspect ratio (a/b)
ρ	mass density of plates
v	Poisson's ratio
w^*	displacement in z^* direction
w	dimensionless displacement in z direction
ε_{x^*}, ε_{y^*}	normal strains in x^*, y^* directions, respectively
$\gamma_{x^*y^*}$	shear strains in x^*, y^* directions, respectively
σ_{x^*}, σ_{y^*}	normal stress in x^*, y^* directions, respectively
$\tau_{x^*y^*}$	shear stress in x^*, y^* directions, respectively

to bending load. The curvature κ is defined as

$$\kappa = \begin{bmatrix} \frac{\partial}{\partial x^*} & 0 \\ 0 & \frac{\partial}{\partial y^*} \\ \frac{\partial}{\partial y^*} & \frac{\partial}{\partial x^*} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x^*} \\ \frac{\partial}{\partial y^*} \end{bmatrix}. \quad (2)$$

2.1. Moment–curvature relationship

Stresses (normal and shear) are distributed over the thickness of the plate and cause bending and twisting moments as well as vertical shear forces. As a result of stresses developed in the plate, moments are given by [20]

$$\begin{bmatrix} M_{x^*} \\ M_{y^*} \\ M_{x^*y^*} \end{bmatrix} = -D\kappa w^*, \quad (3)$$

where D is the flexural rigidity matrix. It is given as

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{21} & D_{22} & D_{26} \\ D_{61} & D_{62} & D_{66} \end{bmatrix}. \quad (4)$$

The governing equation of the plate under vibration is written as

$$\kappa^{\mathrm{T}} D\kappa w + \rho h\ddot{w} = q. \quad (5)$$

2.2. Governing equations

Fig. 1 shows the geometry, coordinate system and loading. Neglecting the in-plane and rotary inertia, equation of anisotropic plate is expressed in non-dimensional form as

$$(w_{xxxx} + 2\lambda^2\eta w_{xxyy} + \lambda^4\psi w_{yyyy} + 4\lambda\phi w_{xxxy} + 4\lambda^3\mu w_{xyyy}) + w_{tt} + c_v w_t - Q(x,y,t) = 0, \quad (6)$$

where the subscript denotes the partial derivative with respect to the following suffix.

The non-dimensional quantities are defined by

$$w = w*/h, \quad x = x^*/a, \quad y = y^*/b, \quad \lambda = a/b,$$
$$t = t*\sqrt{D_{11}/(\rho a^4 h)},$$
$$Q = qa^4/(D_{11}h), \quad C_v = (C_v^*/m)\sqrt{(\rho a^4 h)/D_{11}}, \quad m = \rho abh,$$
$$\eta = (D_{12} + 2*D_{66})/D_{11}, \quad \psi = D_{22}/D_{11},$$
$$\phi = D_{16}/D_{11}, \quad \mu = D_{26}/D_{11}. \quad (7)$$

The boundary conditions considered are as follows:

1. For all four edges simply supported

$$x = 0,1, \quad w = 0, \quad w_{xx} = 0, \quad (8)$$

$$y = 0,1, \quad w = 0, \quad w_{yy} = 0. \quad (9)$$

2. For all four edges clamped

$$x = 0,1, \quad w = 0, \quad w_x = 0, \quad (10)$$

$$y = 0,1, \quad w = 0, \quad w_y = 0. \quad (11)$$

3. For initial conditions, it is assumed that at $t = 0$

$$w = 0, \quad w_t = 0, \quad w_{tt} = 0. \quad (12)$$

2.3. Radial basis function method

Consider a general differential equation

$$Lw = f(x,y) \quad \text{in} \quad \Omega, \quad (13)$$

$$L_B w = g(x,y) \quad \text{on} \quad \partial\Omega, \quad (14)$$

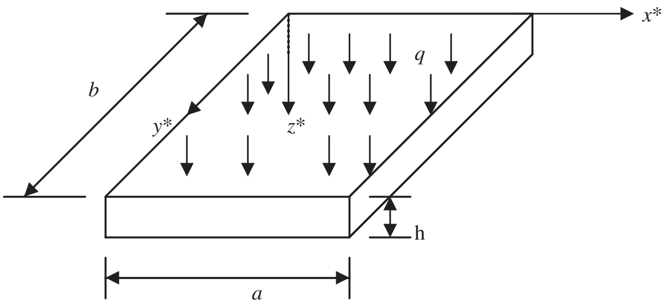

Fig. 1. Geometry of the rectangular plate.

where Ω is domain and $\partial\Omega$ is boundary of the domain. L is a linear differential operator and L_B is a linear boundary operator imposed as boundary conditions, such as Dirichlet, Neumann, and Robin.

Let us denote $\{P_i = (x_i, y_i)\}_{i=1}^{N}$ to be N collocation points in Ω of which $\{(x_i, y_i)\}_{i=1}^{N_I}$ are interior points; $\{(x_i, y_i)\}_{i=N_{I+1}}^{N}$ are boundary points. In Kansa's method, it is assumed that the approximate solution for problem (8) can be expressed as

$$w(x,y) = \sum_{j=1}^{N} w_j \varphi_j(x,y), \tag{15}$$

where $\{w_j\}_{j=1}^{N}$ are the unknown coefficients to be determined, and $\varphi_j(x_j, y_j)$ is a basis function. Substitution of Eq. (15) into Eqs. (13) and (14) yields

$$\sum_{j=1}^{N} (L\varphi_j)(x_i, y_i) w_j = f(x_i, y_i), \quad i = 1, 2 \ldots\ldots N_I, \tag{16}$$

$$\sum_{j=1}^{N} (L_B\varphi_j)(x_i, y_i) w_j = g(x_i, y_i) i = N_{I+1}, \quad N_{I+2}, \ldots\ldots N, \tag{17}$$

which suggests the solution of $N \times N$ linear algebraic systems.

Now the most widely used RBFs are

$\varphi(r) = r^3$	cubic
$\varphi(r) = r^2 \log(r)$	thin plate splines
$\varphi(r) = (1-r)^m + p(r)$	Wendland functions
$\varphi(r) = e^{-(cr)^2}$	Gaussian
$\varphi(r) = \sqrt{c^2 + r^2}$	multiquadrics
$\varphi(r) = (c^2 + r^2)^{-1/2}$	inverse multiquadrics

Here $r = \|P - P_j\|$ is the Euclidean norm between points $P = (x,\ y)$ and $P_j = (x_j, y_j)$. The Euclidian distance r is real and non-negative and c is a shape parameter, a positive constant. Ling and Kansa [21] discussed in details about the shape parameter.

2.4. Multiquadric method for governing differential equation

Substitution of RBF in Eq. (1) gives

$$\left(\sum_{j=1}^{N} w_j \frac{\partial^4}{\partial x^4} \varphi_j + 2\lambda^2 \eta \sum_{j=1}^{N} w_j \frac{\partial^4}{\partial x^2 \partial y^2} \varphi_j + \lambda^4 \psi \sum_{j=1}^{N} w_j \frac{\partial^4}{\partial y^4} \varphi_j \right.$$
$$\left. + 4\lambda \varphi \sum_{j=1}^{N} w_j \frac{\partial^4}{\partial x^3 \partial y} \varphi_j 4\lambda^3 \mu \sum_{j=1}^{N} w_j \frac{\partial^4}{\partial x \partial y^3} \varphi_j \right)$$
$$+ \sum_{j=1}^{N} w_j \frac{\partial^2}{\partial t^2} \varphi_j + C_v w_j \frac{\partial}{\partial t} \varphi_j - Q = 0. \tag{18}$$

2.5. Boundary conditions

(a) For simple supported edge

$$x = 0, 1 \quad \sum_{j=1}^{N} w_j \varphi_j = 0, \tag{19}$$

$$y = 0, 1 \quad \sum_{j=1}^{N} w_j \varphi_j = 0, \tag{20}$$

$$x = 0, 1 \quad \sum_{j=1}^{N} w_j \frac{\partial^2}{\partial x^2} \varphi_j = 0, \tag{21}$$

$$y = 0, 1 \quad \sum_{j=1}^{N} w_j \frac{\partial^2}{\partial y^2} \varphi_j = 0. \tag{22}$$

(b) For all four edges clamped

$$x = 0, 1 \quad \sum_{j=1}^{N} w_j \varphi_j = 0, \tag{23}$$

$$y = 0, 1 \quad \sum_{j=1}^{N} w_j \varphi_j = 0, \tag{24}$$

$$x = 0, 1 \quad \sum_{j=1}^{N} w_j \frac{\partial}{\partial x} \varphi_j = 0, \tag{25}$$

$$y = 0, 1 \quad \sum_{j=1}^{N} w_j \frac{\partial}{\partial y} \varphi_j = 0. \tag{26}$$

(c) For initial conditions at $t = 0$

$$w = 0 \quad \text{and} \quad w_t = 0, w_{tt} = 0 \tag{27}$$

The generating Eq. (18) gives rise to N_I algebraic equations. The four clamped or simply supported boundary conditions through Eqs. (19)–(26) give $2 \times (N - N_I) + 8$ algebraic equations. The total number of equations obtained are $2N - N_I + 8$ and total number of unknowns w_j are N. It can be noted that the total number of equations are more than the total number of unknowns. In order to have a compatible solution, the multiple regression analysis based on least squares error norms is used which leads to compatibility. Dynamic terms are transformed to the right side so the left side matrix is invariant with respect to marching variable time t or loading Q. The set of Eqs. (18)–(27) can be expressed in the matrix form as

$$A\,a = p, \tag{28}$$

where A is $(l^{*}k)$ coefficient matrix, a is $(k^{*}1)$ vector, p is $(l^{*}1)$ load vector. Multiple regression analysis gives

$$a = (A^T A)^{-1} A^T p \tag{29}$$

or

$$a = B\,P. \tag{30}$$

Details are given in Appendix I. The matrix A is evaluated once and stored for subsequent usages. Same procedure is adopted for clamped plate.

3. Numerical results

In this paper, the governing equations of equilibrium of anisotropic plate subjected to uniformly distributed transverse static and dynamic loadings are considered. The equations of motion are solved in space domain using multiquadric radial basis function and Newmark time-marching technique is used for the time domain discretization. Results are presented based on a viscous damping factor $C_v = 1.25, 6$ and 10.

3.1. Case studies

3.1.1. Case (1): simply supported anisotropic plates

Whitney [22] has presented the exact and energy solutions of static simply supported anisotropic plates with an analogy to skew isotropic plates. Present results of the simply supported anisotropic plates obtained by MQRBF method are in very close agreement to those of Whitney as seen in Table 1.

Further dynamic analyses of the simply supported anisotropic plates have also been performed and the behaviors of their equivalent skew isotropic plates have been analyzed. Figs. 2 and 3 compare the damped response results by finite difference and present methods for the simply supported anisotropic plates equivalent to 60° and 63° skew isotropic plates. It is observed that the results are in good agreement. Figs. 4 and 5 show the damped response of the simply supported anisotropic plate equivalent to 54° and 90° skew isotropic plates, respectively. It is observed that for the same damping factor, 54°-skew isotropic plates will stabilize faster than the 90°-skew isotropic plates.

Table 1
Comparison of simply supported rectangular anisotopic plate solutions with skew-isotropic plate

Equivalent skew angle Φ [23]	Exact solutions k [23]	Energy solutions k [23]	Present Solutions of anisotropic plate k
	$w_{\max} = (ka^4/D_{11})q_o$		
90°	0.00406	0.00406	0.0041
80°	0.00411	0.00408	0.0041
63°	0.00444	0.00422	0.0042
60°	0.00452	0.00425	0.0042
54°	0.00476	0.00430	0.0044

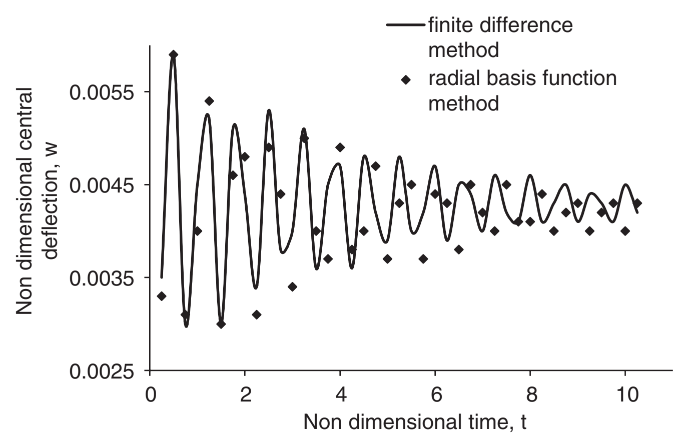

Fig. 2. Damped response of simple supported anisotropic plate equivalent to 60° skew isotropic plate.

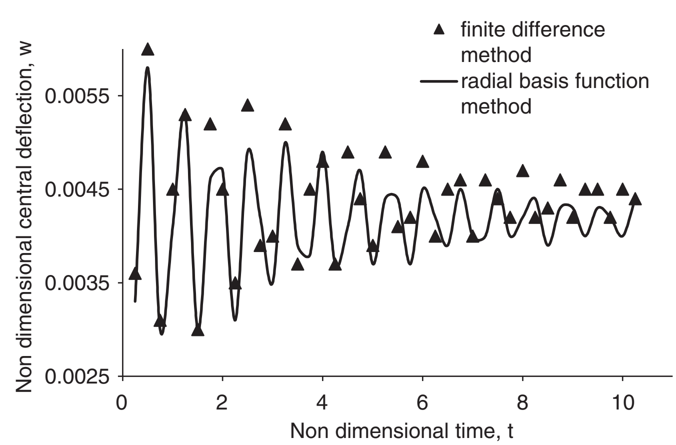

Fig. 3. Damped response of simple supported anisotropic plate equivalent to 63° skew isotropic plate.

3.1.2. Case (2): clamped anisotropic plates

At different fiber orientations behavior of clamped anisotropic plates made of fiber and matrix, with $E_L/E_T = 10, G_{LT}/E_T = 0.25$ and $v_{LT} = 0.3$ have been analyzed. Central deflections of static clamped edges square and rectangular anisotropic plates under uniform load have been obtained by MQRBF method and the results comparing those by Ritz method [23] are presented in Table 2. Figs. 6 and 7 show the variation of non-dimensional central deflection, k, with fiber orientation, θ, for clamped edges square and rectangular plates, respectively. It is seen that the results obtained by MQRBF method are in good agreement with those of Ritz method.

Dynamic analyses of various clamped plates with different fiber orientations have also been performed. Figs. 8–10 show the damped responses of clamped anisotropic plates at 0°, 15° and 75° fiber orientations respectively. Table 3 presents the dynamic response data from Figs. 8–10 for $C_v = 1.25, 6$ and 10. The non-dimensional amplitude in plates with 75° fiber orientation is much less as compared to 0° and 15° fiber orientations since the rigidity of the plate with 75° fiber orientation is much higher. It is further observed that the clamped plates with $C_v = 1.25$ take much longer time to stabilize. However, for both $C_v = 6$ and 10, with increase in fiber orientation, stability time increases in spite of decrease in amplitude.

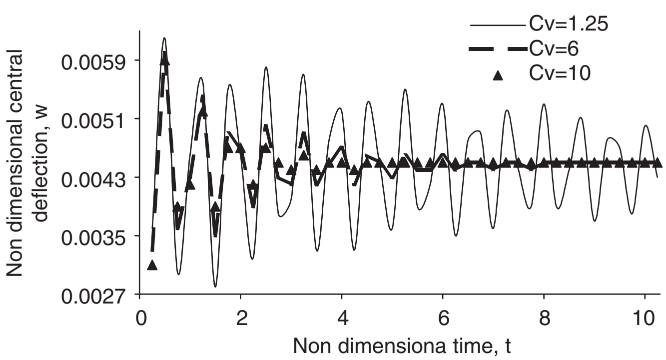

Fig. 4. Damped response of the simple supported anisotropic plate equivalent to 54° skew isotropic plate.

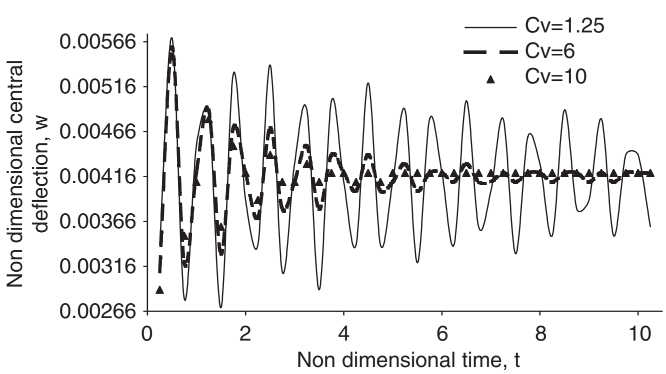

Fig. 5. Damped response of the simple supported anisotropic plate equivalent to 90° skew isotropic plate.

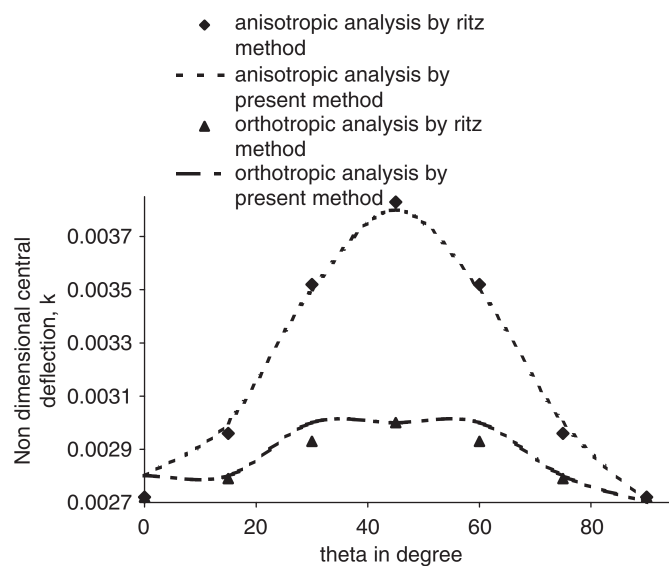

Fig. 6. Non-dimensional central deflection of square plate.

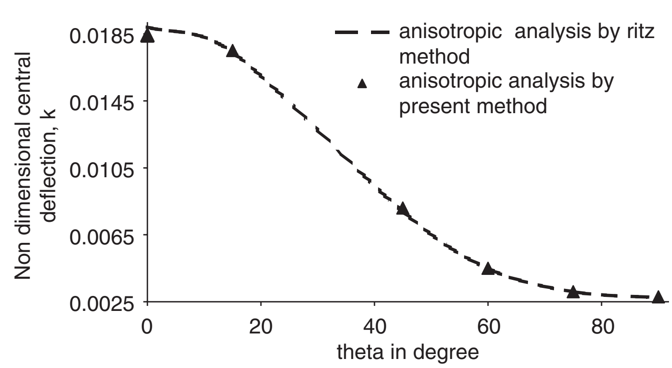

Fig. 7. Non-dimensional central deflection of rectangular clamped plate.

Table 2
Maximum deflection of clamped edges square anisotropic plate under uniform load

$w_{\max} = kq_o(b^4/D_{11}) \times 10^{-3} \quad D_{11} = E_L h^3/\{12(1 - v_{LT}^2 E_T/E_L)\}$

	Orientation		0°	15°	30°	45°	60°	75°	90°
	Anisotropic $a = b$	Ritz method [22]	2.72	2.96	3.52	3.83	3.52	2.96	2.72
		Present method	2.80	3.00	3.50	3.80	3.50	3.00	2.70
k	Orthotropic $a = b$	Ritz method [22]	2.72	2.79	2.93	3.00	2.93	2.79	2.72
		Present method	2.80	2.80	3.00	3.00	3.00	2.80	2.70
	Anisotropic $a = 2b$	Ritz method [22]	18.9	17.4	—	7.89	4.49	3.10	2.75
		Present method	18.4	17.5	—	8.10	4.50	3.10	2.80

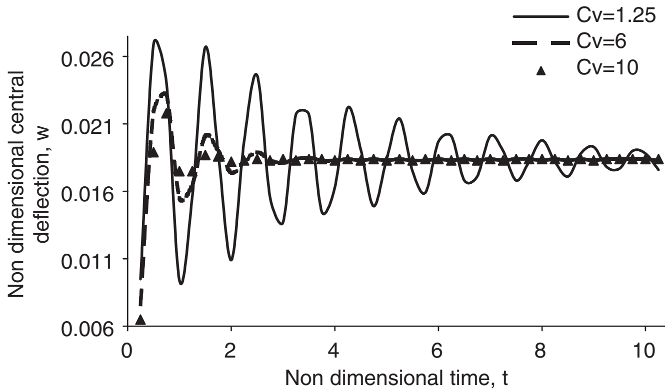

Fig. 8. Damped response of clamped supported anisotropic plate 0° orientation of the fiber.

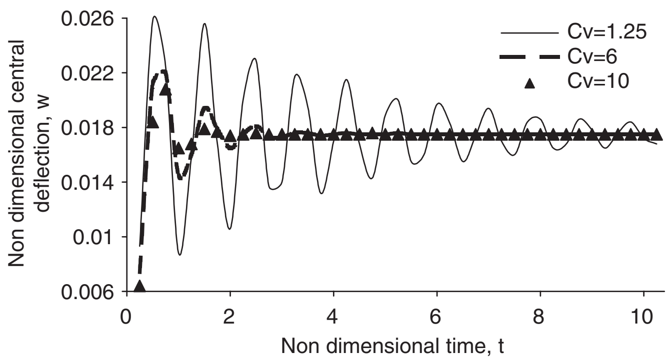

Fig. 9. Damped response of clamped supported anisotropic plate 15° orientation of the fiber.

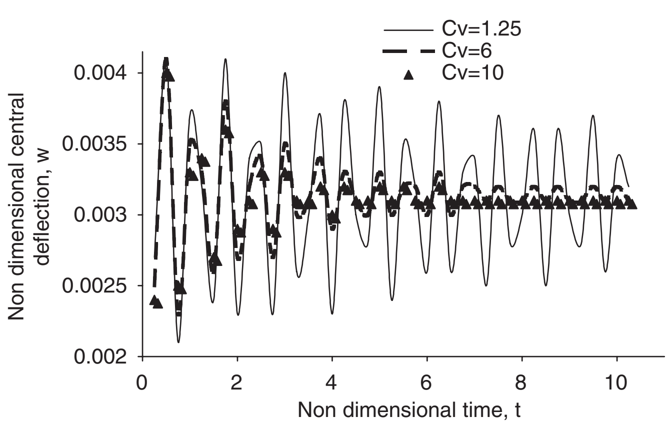

Fig. 10. Damped response of clamped supported anisotropic plate 75° orientation of the fiber.

4. Conclusions

This method has many advantages as compared to the commonly used finite elements and finite difference methods. These advantages can be summarized as follows:

(1) No connectivity is needed because the discretization process is based on the employment of a set of randomly distributed boundary and internal nodes.

Table 3
Dynamic behavior of clamped edges anisotropic plates at various orientation of the fiber

Fiber orientation	Non-dimensional load	Non-dimensional maximum amplitude	Non-dimensional time to stabilize
$C_v = 1.25$			
0°	1	0.0173	Did not
15°	1	0.0170	stabilize even
75°	1	0.0032	after 10.4
$C_v = 6$			
0°	1	0.0188	4
15°	1	0.0175	5.4
75°	1	0.00316	10.0
$C_v = 10$			
0°	1	0.0188	2.2
15°	1	0.0175	2.6
75°	1	0.00310	6.5

(2) After performing the analysis on classical laminated plates, it is observed that this method is very suitable for computer implementation.
(3) Numerical examples have shown the accuracy and good convergence of the results that prove the effectiveness of the approach.

Appendix I

Multiple regression analysis gives

$$Aa = p,$$

where A is ($l^* k$) coefficient matrix, a is ($k^* 1$) vector, p is ($l^* 1$) load vector.

Approximating the solution by introducing the error vector e, one gets

$$p = Aa + e,$$

where e is ($l 1$) vector.

To minimize the error norm, a function S is defined as

$$S(a) = e^{\mathrm{T}} e = (p - Aa)^{\mathrm{T}} (p - Aa).$$

The least squares norm must satisfy the condition

$$(\partial S/\partial a)_a = -2A^{\mathrm{T}} p + 2A^{\mathrm{T}} Aa = 0.$$

This can be expressed as

$$a = (A^T A)^{-1} A^T p$$

or $a = Bp$

The matrix B is evaluated once and stored for subsequent usages.

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