Finite parts of singular and hypersingular integrals with irregular boundary source points

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Abstract

The subject of this paper is the evaluation of finite parts (FPs) of certain singular and hypersingular integrals, that appear in boundary integral equations (BIEs), when the source point is an irregular boundary point (situated at a corner on a one-dimensional plane curve or at a corner or edge on a two-dimensional surface). Two issues addressed in this paper are: an unified, consistent and practical definition of a FP with an irregular boundary source point, and numerical evaluation of such integrals together with solution strategies for hypersingular BIEs (HBIEs). The proposed formulation is compared with others that are available in the literature and interesting connections are made between this formulation and those of other researchers. © 2000 Elsevier Science Ltd. All rights reserved.

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1. Introduction

1.1. The hypersingular boundary element method

Hypersingular boundary integral equations (HBIEs) are derived from a differentiated version of the usual boundary integral equations (BIEs). HBIEs have diverse important applications and are a subject of considerable current research (see, for example, Krishnasamy et al. [1], Tanaka et al. [2], Paulino [3] and Chen and Hong [4] for recent surveys of the field). HBIEs, for example, have been employed for the evaluation of boundary stresses (e.g. Guiggiani et al. [5], Wilde and Aliabadi [6], Zhao and Lan [7], Chati and Mukherjee [8]), in wave scattering (e.g. Krishnasamy et al. [9]), in fracture mechanics (e.g. Cruse [10], Gray et al. [11], Lutz et al. [12], Paulino [3], Gray and Paulino [13]), to obtain symmetric Galerkin boundary element formulations (e.g. Bonnet [14], Gray et al. [15], Gray and Paulino ([16,17]), to evaluate nearly singular integrals (Mukherjee et al. [18]), to obtain the hypersingular boundary contour method (HBCM) (Phan et al. [19], Mukherjee and Mukherjee [20]), to obtain the hypersingular boundary node method (Chati et al. [21]), and for error analysis (Paulino et al. [22], Menon [23], Menon et al. [24], Chati et al. [21], Paulino and Gray [25])) and adaptivity [21].

Of particular interest to the present paper is the elegant approach of regularizing singular and hypersingular integrals using simple solutions, as first proposed by Rudolphi [26]. Several researchers have used this idea to regularize hypersingular integrals before collocating an HBIE at a regular boundary point. Examples are Cruse and Richardson [27], Lutz et al. [12], Poon et al. [28] and Mukherjee et al. [29].

1.2. Hypersingular integrals with irregular boundary source points

A regular boundary point is one at which the boundary of the body is locally smooth. It is sometimes useful to collocate an HBIE at an irregular boundary point (a point located at a corner on a one-dimensional (1-D) boundary of a 2-D body, or at a corner or an edge on a 2-D boundary of a 3-D body.) Gray and Manne [30] have carried this out successfully for the 2-D Laplace equation and Gray and Lutz [31] for 3-D problems, using a limit to the boundary (LTB) approach. This approach is a unified one that works for both singular and hypersingular integrals. Typically, symbolic programming is used to evaluate certain integrals. It is interesting to note that Gray and Manne [30], referring to one of the prevailing definitions of the finite part (FP) at that time, state that “the FP method fails for a boundary corner”. In fact, the FP approach discussed in the present paper is closely related to the LTB approach of Gray and his co-authors, and it is shown here that this approach does, indeed, work at a boundary corner.

Guiggiani et al. [5], Telles and Prado [32] and Chen and Hong [33] have (independently) considered the same
problem in 2-D for a corner with locally straight segments. Free terms have been derived in these papers when an HBIE for the 2-D Laplace equation is collocated at such a corner. The 3-D Laplace equation is also considered in detail in Ref. [5] but closed-form explicit expressions for the free terms are only derived for the 2-D case.

Guiggiani [34] and Mantič and Paris [35] have also independently considered this same problem and have demonstrated (by asymptotic analysis with a vanishing exclusion zone) that the hypersingular integral equation for Laplace’s equation, collocated at a corner, produces, in the most general case (i.e. at a corner with locally curved segments), two (bounded) free terms. One of these multiplies the potential function and the other the potential gradient at the source point. Each of these researchers present the closed-form expression for the term that multiplies the potential for the case of 2-D potential theory and Mantič and Paris [35] also present explicit expressions for the 3-D case. The expressions for the free terms multiplying the potential gradient appear in earlier papers by Guiggiani et al. [5], Telles and Prado [32] and Chen and Hong [33]. Very recently, Paulino and Gray [25] have considered corners in the context of the Galerkin BIE, and have proposed an elegant approach for dealing with this matter. Also, Cruse and Richardson [27] have proved that their HBIE for linear elasticity, regularized by using “simple solutions” (see their equation (28)), is also valid at an irregular boundary point, provided that the primary variables satisfy certain smoothness requirements.

1.3. Finite parts of integrals

There exists an intimate relationship between HBIEs and FP integrals in the sense of Hadamard [36]. Krishnasamy et al. [9], for example, have proved that, provided that the primary variables satisfy certain smoothness requirements, a HBIE integral, obtained in the limit as an internal or external source point approaches a regular point on the boundary, can be interpreted as a FP integral. This author feels, however, that a conventional FP definition, such as in Martin [37] (see, also, Toh and Mukherjee [38] equation (1)) leaves something to be desired if one is interested in the actual evaluation of FPs of integrals. Also, various existing interpretations of FPs are not consistent — compare, for example, equation (46) in Mantič and Paris [35] and equation (29) in Cruse and Richardson [27], with Eqs. (15) and (52), respectively, in the present paper.

Toh and Mukherjee [38] proposed a consistent and practical definition of the FP of a singular integral. Their definition is general in that it applies to integrals with any order of singularity (provided that the primary variables are sufficiently smooth) in any number of physical dimensions. This paper also presents a regularization scheme for the evaluation of such integrals. This interpretation is based on the LTB concept. Results obtained from this interpretation agree with the results from Krishnasamy et al. [9] on scattering of acoustic waves and are completely consistent with complex variable formulations for 2-D BIEs (Hui and Mukherjee [39]). Finally, in a recent paper (Mukherjee and Mukherjee [20]), this FP interpretation has been shown to be consistent with a HBCM formulation for 3-D linear elasticity. The relationship between the Cauchy principal value (CPV) and FP of an integral, when its CPV exists, is explored in Mukherjee [40]. It is important to state, however, that the work of Toh and Mukherjee [38] and Mukherjee [40] is limited to the case where a boundary source (collocation) point is regular. Irregular source points are of concern in the present paper.

1.4. Outline of the present paper

The primary contributions of the present paper are:

- Present a unified, consistent and practical definition of the FP of a singular or hypersingular integral that is valid for both a regular as well as an irregular boundary collocation point.
- Present a strategy for numerical evaluation of a regularized version of the FP above, and then for solving the associated HBIE using suitable boundary elements.

The next section of this paper presents the definition of the FP together with a proof of its relationship with the LTB approach of Gray and his co-authors. Similar to the LTB approach (see Gray and Manne [30]), the present formulation is also able to handle both strongly singular and hypersingular integrals in a unified manner. The next section considers an example problem — the gradient BIE for the 2-D Laplace equation collocated at a corner. It is shown that the results from the present formulation are consistent with those of Guiggiani [34] and Mantič and Paris [35] obtained by considering a vanishing exclusion zone. It is also demonstrated next that this same FP formulation, this time with a “complete exclusion zone”, leads to the regularized equations (obtained from use of simple solutions) of Rudolphi [26], for potential theory, and of Cruse and Richardson [27] for linear elasticity.

The last topic discussed in this paper is smoothness requirements for interpolation functions for collocation of an HBIE at an irregular point (Martin et al. [41]). Mukherjee and Mukherjee [42] have recently proved that, for a certain admissible class of problems, interpolation functions employed in the boundary contour method (BCM) satisfy all such smoothness requirements. Therefore, such functions are possible candidates for the hypersingular boundary element method (HBEM), with regular as well as irregular collocation points.

2. The FP of a hypersingular integral collocated at an irregular boundary point

2.1. Definition

Consider, for specificity, the space $\mathbb{R}^3$, and let $S$ be a
There are several equivalent ways for evaluating \( A \) and \( B \).

**Method one.** Replace \( S \) by \( \hat{S} \) and \( \hat{S} \) by \( \hat{\hat{S}} \) in Eq. (2). Now, setting \( \phi(y) = 1 \) in Eq. (2) and using Eq. (3), one gets:

\[
A(\hat{S}) - A(\hat{\hat{S}}) = \int_{\hat{\hat{S}}} K(x, y) \, dS(y)
\]  

(5)

Next, setting \( \phi(y) = (y_p - x_p) \) (note that, in this case, \( \phi(x) = 0 \) and \( \phi_p(x) = 1 \) in Eq. (2), and using Eq. (4), one gets:

\[
B_p(\hat{S}) - B_p(\hat{\hat{S}}) = \int_{\hat{\hat{S}}} K(x, y)(y_p - x_p) \, dS(y)
\]  

(6)

The formulae (5) and (6) are most useful for obtaining \( A \) and \( B \) when \( \hat{S} \) is an open surface and Stoke regularization is employed. An example is the application of the FP definition (2) (for a regular collocation point) in Toh and Mukherjee [38], to regularize a hypersingular integral that appears in the HBIE formulation for the scattering of acoustic waves by a thin scatterer. The resulting regularized equation is shown in Ref. [38] to be equivalent to the result of Krishnasamy et al. [9]. Eqs. (5) and (6) are also used in Mukherjee and Mukherjee [20] and in Section 3.2 of this paper.

**Method two.** From Eq. (5):

\[
A(\hat{S}) - A(\hat{\hat{S}}) = \int_{\hat{\hat{S}}} K(x, y) \, dS(y) = \lim_{\xi \to x} \int_{\hat{\hat{S}}} K(\xi, y) \, dS(y)
\]  

(7)

The second equality above holds since \( K(x, y) \) is regular for \( x \in \hat{S} \) and \( y \in \hat{\hat{S}} \). Assuming that the \( \lim_{\xi \to x} \int_{\hat{\hat{S}}} K(\xi, y) \, dS(y) \) and \( \lim_{\xi \to x} \int_{\hat{\hat{S}}} K(\xi, y) \, dS(y) \) exist, then:

\[
A(\hat{S}) = \lim_{\xi \to x} \int_{\hat{\hat{S}}} K(\xi, y) \, dS(y)
\]  

(8)

Similarly:

\[
B_p(\hat{S}) = \lim_{\xi \to x} \int_{\hat{\hat{S}}} K(\xi, y)(y_p - x_p) \, dS(y)
\]  

(9)

Eqs. (8) and (9) are most useful for evaluating \( A \) and \( B \) when \( \hat{S} = \partial B \), a closed surface that is the entire boundary of a body \( B \). An example appears in Section 3.3 of this paper.

**Method three.** A third way for evaluation of \( A \) and \( B \) is to use an auxiliary surface (or “tent”) as first proposed for fracture mechanics analysis by Lutz et al. [12]. (see, also, Mukherjee et al. [29], Mukherjee [43] and Section 3.2.1 of this paper). This method is useful if \( S \) is an open surface.

### 2.2. The FP and the LTB

There is a very simple connection between the FP, defined above, and the LTB approach employed by Gray and his co-authors. With, as before, \( \xi \not\in S \), \( x \in S \) (\( x \) can be an irregular point on \( S \)), \( K(x, y) = C(|x - y|^{-3}) \) as \( y \to x \).
and \( \phi(y) \in C^{1,\alpha} \) at \( y = x \), this can be stated as:

\[
\lim_{\xi \to x} \int_{S} K(\xi, y) \phi(y) \, dS(y) = \int_{S} K(x, y) \phi(y) \, dS(y) \tag{10}
\]

Of course, \( \xi \) can approach \( x \) from either side of \( S \).

2.2.1. Proof of Eq. (10)

Consider the first and second terms on the right hand side of Eq. (2). Since these integrands are regular in their respective domains of integration, one has:

\[
\int_{S} K(x, y) \phi(y) \, dS(y) = \lim_{\xi \to x} \int_{S} K(\xi, y) \phi(y) \, dS(y) \tag{11}
\]

and

\[
\int_{S} K(x, y)[\phi(y) - \phi(x) - \phi,p(x)(y_p - x_p)] \, dS(y) \tag{12}
\]

Use of Eqs. (8), (9), (11) and (12) in Eq. (2) proves Eq. (10).

3. The gradient BIE for the 2-D Laplace equation collocated at a corner

Collocation of the gradient BIE for the 2-D Laplace equation, at an irregular point, is considered as an example problem in this section. It is proved that with a vanishing exclusion zone, one recovers, from the proposed FP definition (Eq. (2)), the results in Guiggianni [34], Guiggianni et al. [5] and Mantić and Paris [35], while, with a complete exclusion zone, one recovers the results from Rudolphi [26]. (It should be noted that while Rudolphi [26] only considers collocation at a regular boundary point, his equation (20) is also valid at an irregular boundary point.)

3.1. FP Formulation

Consider a body \( B \) (an open set) with boundary \( \partial B \). With \( \xi \not\in \partial B \), the well-known BIE for Laplace’s equation \( \nabla^2 u = 0 \) in 2-D can be written as:

\[
\gamma(\xi)u(\xi) = \int_{\partial B} [U(\xi, y)q(y) - T(\xi, y)u(y)] \, dS(y) \tag{13}
\]

and its differentiated form, the HBIE:

\[
\gamma(\xi) \frac{\partial u(\xi)}{\partial n} = \int_{\partial B} [-D(\xi, y)q(y) + S(\xi, y)u(y)] \, dS(y) \tag{14}
\]

In the above, \( q = (\partial u/\partial n)(y), \quad \gamma(\xi) = 1 \) for \( \xi \in B \), \( \gamma(\xi) = 0 \) for \( \xi \not\in B \), and the kernels are given in Appendix A.

Let \( \xi \to x \) where \( x \in \partial B \) can be an irregular point, and \( u(y) \in C^{1,\alpha} \) at \( y = x \). With an exclusion zone \( \hat{S} \) (Fig. 2), and \( \xi \) an exterior point, \( \gamma(\xi) = 0 \). Now, using the LTB property of the FP from Eq. (10), (14) becomes:

\[
0 = \int_{\partial B} [S(\xi, y)u(y) - D(\xi, y)q(y)] \, dS(y) \tag{15}
\]

With \( S = \partial B \), application of the definition of the FP (Eq. (2)) allows Eq. (15) to be written as:

\[
0 = \int_{\partial B, \hat{S}} [S(\xi, y)u(y) - D(\xi, x, y)q(y)] \, dS(y)
\]

\[
+ \int_{\hat{S}} S(\xi, y)[u(y) - u(x) - u_p(x)(y_p - x_p)] \, dS(y)
\]

\[
- \int_{\hat{S}} D(\xi, x, y)[u_p(x) - u_p(x)\eta_p(y)] \, dS(y) + u(x)A_1(\hat{S})
\]

\[
+ u_p(x)[B_p(\hat{S}) - E_p(\hat{S})] \tag{16}
\]

where, using Eqs. (5) and (6), one gets:

\[
A_1(\hat{S}) - A_1(\hat{S}) = \int_{\hat{S}} S(\xi, x, y) \, dS(y) \tag{17}
\]

\[
B_p(\hat{S}) - B_p(\hat{S}) = \int_{\hat{S}} S(\xi, x, y)(y_p - x_p) \, dS(y) \tag{18}
\]

and

\[
E_p(\hat{S}) - E_p(\hat{S}) = \int_{\hat{S}} D(\xi, x, y)\eta_p(y) \, dS(y) \tag{19}
\]

Finally, define \( C = B - E \)

3.2. Vanishing exclusion zone

A vanishing exclusion zone, \( S_\epsilon \to 0 \) (Fig. 2) is considered for this problem by Guiggianni et al. [5], Guiggianni [34] and Mantić and Paris [35]. \( S_\epsilon \) is an arc of a circle in these papers, whereas it is arbitrary in the present work. Clearly, \( S_\epsilon \to 0 \) implies that \( \hat{S} \to 0 \), so that the second and third terms (with regular integrands) on the right hand side of Eq. (16) vanish in this limit. The first term and the quantity \( A \) are of \( \mathcal{O}(1/\epsilon) \) as \( S_\epsilon \to 0 \) (here \( \epsilon \) is a length scale for \( S_\epsilon \), its radius if \( S_\epsilon \) is a circular arc centered at \( P \) in Fig. 2); while \( C \) has a finite limit as \( \epsilon \to 0 \). This situation is discussed next.

3.2.1. The vector \( A \)

Applying Stokes’ theorem in the form:

\[
\int_{\partial a} (g \cdot n_j - g \cdot n_i) \, dS = \epsilon_{ij} a \int_{\partial a} b_d \tag{20}
\]
Using asymptotic analysis (see Fig. 3), Guiggiani [34] and Mantić and Paris [35] have independently proved that:

$$A_1(S_e) = \left[ \frac{b_1}{e} + a_i + C(e) \right]$$

(28)

so that, in the limit as $S_e \rightarrow 0$ (i.e. $e \rightarrow 0$), one has $\mathcal{F}(A_1) = a_i$. (Here $\mathcal{F}$ denotes the FP of the appropriate quantity as in Mantić and Paris [35] — their equations (7) and (8)). These researchers have also obtained the explicit values of $a_i$. The final result, from Refs. [34,35], in the present notation, is:

$$a_1 = \mathcal{F}(A_1) = 1/4\pi[\cos \phi_2(0)k_2(0) + \cos \phi_1(0)k_1(0)]$$

$$a_2 = \mathcal{F}(A_2) = 1/4\pi[\sin \phi_2(0)k_2(0) + \sin \phi_1(0)k_1(0)]$$

(29)

where $\phi_1(0)$ and $\phi_2(0)$ are the angles made by the tangents to the curves $C_1(\phi_1(e))$ and $C_2(\phi_2(e))$ at $x$ (in Fig. 3), with the positive $y_1$ axis; and $k_1(0)$ and $k_2(0)$ are the signed curvatures of $C_1$ and $C_2$ at $x$. In general, for a plane curve parameterized as $y_1(\xi)$, $y_2(\xi)$, the signed curvature is defined here as in [34]:

$$\kappa(\xi) = \frac{\mathbf{p} \times \mathbf{d} \cdot \mathbf{k}}{\|\mathbf{p}\|} = \frac{-y_1y_2 + y_2y_1}{[y_1'^2 + (y_2')^2]^{3/2}}$$

(30)

where $\mathbf{p} = i y_1 + j y_2$, $\mathbf{k} = i \times j$ and a superscribed dot denotes a derivative with respect to the parameter $\xi$.

Guiggiani [34] combines the term $(\mathbf{b}_i(\xi)/e)u(\xi)$ (see Eq. (28)) with the first integral in the right hand side of Eq. (16) in order to get a finite result as $S_e \rightarrow 0$. He has shown that (in the present notation), $b_1 = -(1/(2\pi))[\sin \phi_2(0) - \sin \phi_1(0)]$, $b_2 = (1/(2\pi))[\cos \phi_2(0) - \cos \phi_1(0)]$. Details are available in Guiggiani et al. [5].

3.2.2. The tensor $C$

From Eqs. (18) and (19):

$$C_{ij}(\tilde{S}) = C_{ij}(\check{S}) = \int_{\tilde{S}} [S_j(x,y)(y_p - y_p) - D_i(x,y)n_p(y)] dS(y)$$

(31)

Using the expressions in Appendix A, one has:

$$C_{ij}(\tilde{S}) - C_{ij}(\check{S}) = \int_{\tilde{S}} [W_j(x,y)(y_p - y_p) - U_j(x,y)\delta_{ij}]n_p(y) dS(y)$$

$$= \int_{\tilde{S}} [W_j(x,y)(y_p - y_p) - U_j(x,y)\delta_{ij}]n_p(y) dS(y)$$

(32)

The procedure followed now is quite analogous to that used in the previous section for determining $A$. Using:

$$g = W_j(y_p - y_p) - U\delta_{ij}$$

(33)

in Stokes’ theorem (20), and noting that $g_{j} = 0$ for $x \in \tilde{S}$

...
and \( y \in \hat{S} \setminus \tilde{S} \), one gets:

\[
C_{ip}(\hat{S}) - C_{ip}(\tilde{S}) = e_{ij}\left[ W_j(y_p - x_p) - U \delta_{ij} \right] \text{[35]}
\]

where \( s \) is a length coordinate that is measured around \( \partial B \) (Fig. 2) in a clockwise direction, the second term in Eq. (34) becomes:

\[
I_2 = -\frac{1}{2\pi} \left[ (\phi_2(0) - \phi_1(0)) \right]
\]

Using Eqs. (34) and (36), one has:

\[
C_{ip}(\hat{S}) = C_{ip}(\tilde{S})
\]

\[
e_{ij}\left[ W_j(y_p - x_p) - U \delta_{ij} \right] \text{[35]}
\]

In the limit \( S_v \to 0 \), one gets (as in Guiggiani et al. [5] but with a sign change due to the sign differences in the definitions of the kernels \( S \) and \( D \) in [34] and in the present paper):

\[
C_{11} = -\frac{1}{2\pi} \left[ (\phi_2(0) - \phi_1(0)) \right] + \frac{\sin(2\phi_2(0)) - \sin(2\phi_1(0))}{2}
\]

\[
C_{22} = -\frac{1}{2\pi} \left[ (\phi_2(0) - \phi_1(0)) - \frac{\sin(2\phi_2(0)) - \sin(2\phi_1(0))}{2} \right]
\]

\[
C_{12} = C_{21} = -\frac{1}{2\pi} \left[ \sin^2(\phi_2(0)) - \sin^2(\phi_1(0)) \right]
\]

Note that the term \( [U^b]_0 \) does not contribute to \( C \).

### 3.2.3. Final equation

The final form of Eq. (16), with a vanishing exclusion zone, has two free terms. It is (see Guiggiani [34]):

\[
0 = a_i(x)u(x) + C_{ip}(x)u_p(x) + \lim_{\hat{S} \to 0} \int_{\partial B \setminus S} [S_i(x,y)u(y) - D_i(x,y)q(y)] \text{d}S(y) + \frac{b_i(x)}{\varepsilon} u(x)
\]

An alternative equivalent form (see Mantić and Paris [35]) is:

\[
0 = \mathcal{F}(A_i(x))u(x) + C_{ip}(x)u_p(x) + \mathcal{F}\left\{ \int_{\partial B \setminus \tilde{S}} [S_i(x,y)u(y) - D_i(x,y)q(y)] \text{d}S(y) \right\}
\]

with the FPs \( \mathcal{F} \) evaluated for the limit \( \hat{S} \to 0 \).

It is interesting to compare (the equivalent) Eqs. (15) and (42). One contains two free terms, the other does not. Both contain FPs, defined in different ways. The LTB definition of Gray and Manne [30] is equivalent to the FP Eq. (15) although they do not use the designation "finite part".

### 3.3. Complete exclusion zone

The situation in which \( \hat{S} = \partial B \) is referred to as a "complete" exclusion zone. Now, the FP Eq. (15) (see Eq. (16)) has the form:

\[
0 = \int_{\partial B} S_i(x,y)[u(y) - u(x) - u_p(x)(y_p - x_p)] \text{d}S(y)
\]

\[
- \int_{\partial B} D_i(x,y)[u_p(x) - u_p(x)] \text{d}S(y) + u(x)A_i(\partial B)
\]

\[
+ u_p(x)C_{ip}(\partial B)
\]

with (see Eqs. (8) and (9)):

\[
A_i(\partial B) = \lim_{\xi \to x} \int_{\partial B} S_i(\xi,y) \text{d}S(y)
\]

\[
C_{ip}(\partial B) = \lim_{\xi \to x} \int_{\partial B} [S_i(\xi,y)(y_p - \xi_p) - D_i(\xi,y)n_p(\xi)] \text{d}S(y)
\]

\[
S_i(\xi,y) \text{d}S(y) = 0
\]

while use of the linear solution:

\[
u(y) = c(y_p - \xi_p), \quad q(y) = cn_p(y) \quad (\text{with } p = 1, 2)
\]

in Eq. (14) (with \( \gamma = 0 \)) gives:

\[
\int_{\partial B} [S_i(\xi,y)(y_p - \xi_p) - D_i(\xi,y)n_p(\xi)] \text{d}S(y) = 0
\]
Therefore, (assuming continuity) \( A = C = 0 \) and one obtains a simple, fully regularized form of Eq. (43) as:

\[
0 = \int_{\partial B} S(x, y)[u(y) - u(x) - u_p(x)(yp - xp)] \, dS(y) - \int_{\partial B} D(x, y)[u_p(y) - u_p(x)]n_p(y) \, dS(y) \tag{49}
\]

A few comments are in order. First, Eq. (49) is the same as Rudolphi’s [26] equation (20) with (his) \( \kappa = 1 \) and (his) \( S_0 \) set equal to \( S \) and renamed \( \partial B \). (See, also, Kane [45], equation (17.34)). Second, this equation can also be shown to be valid for the case \( \xi \in \partial B \), i.e. for an inside approach to the boundary point \( x \). Third, as noted before, \( x \) can be a corner point on \( \partial B \) (provided, of course, that \( u(y) \in C^{1,\alpha} \) at \( y = x \) — Rudolphi had only considered a regular boundary collocation point in his excellent paper ten years ago). Finally, as discussed in the next section, Eq. (49) is analogous to the regularized stress BIE in linear elasticity — equation (28) in Cruse and Richardson [27].

4. The stress BIE for 3-D linear elasticity collocated at an irregular boundary point

The case of the hypersingular stress BIE for 3-D linear elasticity, collocated at an irregular boundary point, is briefly discussed below. Only the case of the complete exclusion zone is considered here.

The well-known BIE for 3-D linear elasticity, with \( \xi \in \partial B \) (Rizzo [46], Cruse [47]) is:

\[
\gamma(\xi)u_\lambda(\xi) = \int_{\partial B} [U_{\lambda}(\xi, y)\tau_\kappa(y) - T_{\lambda}(\xi, y)u_k(y)] \, dS(y) \tag{50}
\]

The hypersingular stress BIE in 3-D linear elasticity, with \( \xi \in \partial B \) (Cruse [47]) is:

\[
\gamma(\xi)\sigma_{\lambda\kappa}(\xi) = \int_{\partial B} [-D_{\lambda\kappa}(\xi, y)\tau_\kappa(y) + S_{\lambda\kappa}(\xi, y)u_k(y)] \, dS(y) \tag{51}
\]

In the above, \( u \) is the displacement field, \( \tau \) is the traction field, \( \sigma \) is the stress field, \( \gamma(\xi) = 1 \) for \( \xi \in B \), \( \gamma(\xi) = 0 \) for \( \xi \in \partial B \), and the kernels are given in Appendix A.

Let \( \xi \to x \), where \( x \in \partial B \) can be an irregular point, and \( u(y) \in C^{1,\alpha} \) at \( y = x \). With an exclusion zone \( \hat{S} \), and \( \xi \) an exterior point, \( \gamma(\xi) = 0 \). Now, using the LTB property of the FP from Eq. (10), Eq. (51) becomes:

\[
0 = \int_{\partial B} [S_{\lambda\kappa}(x, y)u_k(y) - D_{\lambda\kappa}(x, y)\tau_\kappa(y)] \, dS(y) \tag{52}
\]

With \( S = \partial B \), application of the definition of the FP (Eq. (2)) allows Eq. (52) to be written as:

\[
0 = \int_{\partial B} S_{\lambda\kappa}(x, y)[u_k(y) - u_k(x) - u_{kp}(x)(yp - xp)] \, dS(y)
- \int_{\partial B} D_{\lambda\kappa}(x, y)[\sigma_{\lambda\kappa}(y) - \sigma_{\lambda\kappa}(x)]n_p(y) \, dS(y)
+ A_{\lambda\kappa}(\partial B)u_k(x) + C_{\lambda\kappa}(\partial B)u_{kp}(x) \tag{53}
\]

with (see Eqs. (8) and (9)):

\[
A_{\lambda\kappa}(\partial B) = \lim_{\xi \to x} \int_{\partial B} S_{\lambda\kappa}(\xi, y) \, dS(y) \tag{54}
\]

\[
C_{\lambda\kappa}(\partial B) = \lim_{\xi \to x} \int_{\partial B} [S_{\lambda\kappa}(\xi, y)(yp - \xi_p)
- E_{kp\lambda\epsilon}D_{jm}(\xi, y)n_{\epsilon}(y)] \, dS(y) \tag{55}
\]

Simple (rigid body and linear) solutions in linear elasticity (see, for example, Lutz et al., [12], Cruse and Richardson [27]) are now used in order to determine the quantities \( A \) and \( C \). With \( \xi \) an exterior point, \( \gamma(\xi) = 0 \) Eq. (51). Using the rigid body mode \( u_k = c_k \) \( (c_k \) are arbitrary constants) in Eq. (51), one has:

\[
0 = \int_{\partial B} S_{\lambda\kappa}(\xi, y) \, dS(y) \tag{56}
\]

while, using the linear mode:

\[
u_k = c_k(y_p - \xi_p),
\tag{57}
\]

\[
\tau_m = \sigma_{m\lambda\epsilon}n_\epsilon = E_{m\lambda\kappa}c_k n_\epsilon = E_{kp\lambda\epsilon}c_k n_\epsilon
\tag{57}
\]

in Eq. (51) gives:

\[
0 = \int_{\partial B} [S_{\lambda\kappa}(x, y)(yp - \xi_p) - E_{kp\lambda\epsilon}D_{jm}(\xi, y)n_{\epsilon}(y)] \, dS(y) \tag{58}
\]

Invoking continuity, \( A = C = 0 \), so that Eq. (53) reduces to the simple regularized form:

\[
0 = \int_{\partial B} S_{\lambda\kappa}(x, y)[u_k(y) - u_k(x) - u_{kp}(x)(yp - xp)] \, dS(y)
- \int_{\partial B} D_{\lambda\kappa}(x, y)[\sigma_{\lambda\kappa}(y) - \sigma_{\lambda\kappa}(x)]n_p(y) \, dS(y) \tag{59}
\]

which is equation (28) of Cruse and Richardson [27] in the present notation. As is the case in the present work, Cruse and Richardson [27] have also proved that their equation (28) is valid at a corner point, provided that the stress is continuous there.

It has been proved in this section that the regularized stress BIE (28) of Cruse and Richardson [27] can also be obtained from the FP definition (2) with a complete exclusion zone.
5. Solution strategy for a HBIE collocated at an irregular boundary point

Hypersingular BIEs for a body \( B \) with boundary \( \partial B \) are considered here. Regularized HBIEs, obtained by using complete exclusion zones, e.g. Eq. (49) for potential theory or Eq. (59) for linear elasticity, are recommended as starting points.

An irregular collocation point \( x \) for 3-D problems is considered next. Let \( \partial B_n, (n = 1, 2, 3, \ldots, N) \) be smooth pieces of \( \partial B \) that meet at an irregular point \( x \in \partial B \). Also, let a source point, with coordinates \( y_s \), be denoted by \( P \), and a field point, with coordinates \( y_t \), be denoted by \( Q \).

Martin et al. [41] state the following requirements for collocating a regularized HBIE, such as Eq. (59) at an irregular point \( P \in \partial B \). These are:

\[ u_t^i(Q_n; P) = u_t(P) + u_{ij}(P)[y_t(Q_n) - y_j(P)] \]  

(60)

In the above, \( r_n = |y(Q_n) - x(P)|, Q_n \in \partial B_n \), and \( \alpha > 0 \). Also

\[ u_t^i(Q_n; P) = u_t(P) + u_{ij}(P)[y_t(Q_n) - y_j(P)] \]

(60)

There are two important issues to consider here.

The first is that, if there is to be any hope for collocating Eq. (59) at an irregular point \( P \), the exact solution of a boundary value problem must satisfy conditions (i)–(iv) in Box 1. Clearly, one should not attempt this collocation if, for example, the stress is unbounded at \( P \) (this can easily happen — see an exhaustive study on the subject in Glushkov et al. [48]), or is bounded but discontinuous at \( P \) (e.g. at the tip of a wedge — see, for example, Zhang and Mukherjee [49]). The discussion in the rest of this paper is limited to the class of problems, referred to as the admissible class, whose exact solutions satisfy conditions (i)–(iv).

The second issue refers to smoothness requirements on the interpolation functions for \( u, \sigma \) and the traction \( \tau = n\sigma \) in Eq. (59). It has proved very difficult, in practice, to find BEM interpolation functions that satisfy, a priori, (ii) (b)–(iv)) in Box 1, for collocation at an irregular surface point on a 3-D body [41]. It has recently been proved in Mukherjee and Mukherjee [42], however, that interpolation functions used in the BCM (see, for example, Mukherjee et al. [50], Mukherjee and Mukherjee [20]) satisfy these conditions a priori. Another important advantage of using these interpolation functions is that \( \nabla u \) can be directly computed from them at an irregular boundary point [20], without the need to use the (undefined) normal and tangent vectors at this point.

Numerical results from the hypersingular BCM, collocated on edges and at corners, are available in Mukherjee and Mukherjee [42]. In principle, these BCM interpolation functions can also be used in the BEM. This is a recommended topic for future research.

6. Concluding remarks

The following are the highlights of this paper:

- Presents a consistent unified definition of the FP of a hypersingular integral collocated at an irregular boundary point. This definition is valid for an arbitrary exclusion zone. There is no need to consider a direction of a limit process as for the LTB approach.
- Unifies, in some sense, the work of researchers such as Cruse, Rizzo, Gray, Mantić, Guiggiani, Rudolph, Lutz, and their co-authors, in this subject area.
- Recommends the use of the complete exclusion zone for regularization of HBIEs defined on closed surfaces. This procedure has the following advantages, that become most obvious for 3-D problems:
  - no need to evaluate free terms;
  - no need to compute limits, requiring, perhaps, symbolic programming.
- Recommends the use of Stoke regularization or auxiliary surfaces for HBIEs on open surfaces such as in fracture mechanics problems.
- Recommends research towards the use of HBCM interpolation functions, that satisfy smoothness requirements a priori, for obtaining numerical solutions of HBIEs that involve collocation at irregular boundary points.
- Recommends consideration of adoption of the FP definition, proposed in this paper, as the standard one for future research in this subject area.

Appendix A. BIE and HBIE kernels

The kernels for the BIE (13) for Laplace’s equation in 2-D are:

\[ U(\xi, \eta) = \frac{1}{2\pi} \ln \left( \frac{1}{r} \right) \]

\[ W_r(\xi, \eta) = \frac{\partial U}{\partial \eta} = -\frac{r_i}{2\pi r} = -\frac{(y_i - \xi_i)}{2\pi r^2} \]

\[ T(\xi, \eta) = \frac{\partial U}{\partial n(\eta)} = W_r(\eta) = -\frac{1}{2\pi r} \frac{\partial r}{\partial n}(\eta) \]
The corresponding kernels for the gradient BIE (14) are:

\[
D_{ik}(\xi, y) = -\frac{\partial U}{\partial \xi_k} = W_{ik}
\]

\[
S_{ik}(\xi, y) = -\frac{\partial T}{\partial \xi_k} = -\frac{\partial W}{\partial \xi_k} n_j(y) = \frac{\partial^2 U}{\partial y_i \partial y_j} n_j(y) = \frac{1}{2\pi r^2} \left( 2r \frac{\partial r}{\partial n}(y) - n_j(y) \right)
\]

The Kelvin kernels for the displacement BIE (50) for 3-D linear elasticity are:

\[
U_{ik}(\xi, y) = \frac{1}{16\pi G(1-\nu)} [(3-4\nu)\delta_{ik} + r_j r_k]
\]

\[
T_{ik}(\xi, y) = -\frac{1}{8\pi(1-\nu) r^2} \times \left[ (1-2\nu)\delta_{ik} + 3r_j r_k \right] \frac{\partial r}{\partial n} + (1-2\nu)\left( r_j n_i - r_i n_j \right)
\]

The corresponding kernels for the stress BIE (51) are:

\[
D_{ik}(\xi, y) = -E_{i j m n} \frac{\partial U_{mk}}{\partial \xi_n} = \frac{1}{8\pi(1-\nu) r^2} \left[ (1-2\nu)(\delta_{ik}r_j + \delta_{ij}r_k - \delta_{jk}r_i) + 3r_j r_k \delta_{ik} \right]
\]

\[
S_{ik}(\xi, y) = -E_{i j m n} \frac{\partial T_{mk}}{\partial \xi_n} = -\frac{4G}{4\pi(1-\nu) r^2} \left[ (1-2\nu)\delta_{ik}r_j + \nu(\delta_{ik}r_j + \delta_{jk}r_i) \right.
\]

\[
\left. - 5r_j r_k \delta_{ik} \right] \frac{\partial r}{\partial n} + 3 \nu(n_i r_j r_k + n_j r_i r_k) \right) - (1-4\nu)n_i \delta_{ik} + (1-2\nu)(3n_j r_k r_i + n_i \delta_{jk} + n_j \delta_{ik})
\]

In the above, \( r = y - \xi \) with \( r = \|y - \xi\| \) the distance between a source point \( \xi \) and a field point \( y \). \( G \) and \( \nu \) are the shear modulus and Poisson’s ratio, respectively, \( \delta_{ik} \) are the components of the Kronecker delta and, \( k = (\partial/\partial y_j) \). Also, the components of the (unit outward) normal \( n \), as well as the normal derivative \( \partial r/\partial n \), are evaluated at the field point \( y \) and \( E_{i j m n} \) are components of the elasticity tensor for a homogeneous, isotropic elastic solid.

References


