ELSEVIER

Engineering Analysis with Boundary Elements 24 (2000) 767–776

ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

www.elsevier.com/locate/enganabound

Finite parts of singular and hypersingular integrals with irregular boundary source points

S. Mukherjee*

Department of Theoretical and Applied Mechanics, 212 Kimball Hall, Cornell University, Ithaca, NY 14853-1503, USA

Abstract

The subject of this paper is the evaluation of finite parts (FPs) of certain singular and hypersingular integrals, that appear in boundary integral equations (BIEs), when the source point is an irregular boundary point (situated at a corner on a one-dimensional plane curve or at a corner or edge on a two-dimensional surface). Two issues addressed in this paper are: an unified, consistent and practical definition of a FP with an irregular boundary source point, and numerical evaluation of such integrals together with solution strategies for hypersingular BIEs (HBIEs). The proposed formulation is compared with others that are available in the literature and interesting connections are made between this formulation and those of other researchers.

Keywords: Boundary integral equations; Boundary element method; Singular and hypersingular integrals; Cauchy principal value; Finite part

1. Introduction

1.1. The hypersingular boundary element method

Hypersingular boundary integral equations (HBIEs) are derived from a differentiated version of the usual boundary integral equations (BIEs). HBIEs have diverse important applications and are the subject of considerable current research (see, for example, Krishnasamy et al. [1], Tanaka et al. [2], Paulino [3] and Chen and Hong [4] for recent surveys of the field). HBIEs, for example, have been employed for the evaluation of boundary stresses (e.g. Guiggiani et al. [5], Wilde and Aliabadi [6], Zhao and Lan [7], Chati and Mukherjee [8]), in wave scattering (e.g. Krishnasamy et al. [9]), in fracture mechanics (e.g. Cruse [10], Gray et al. [11], Lutz et al. [12], Paulino [3], Gray and Paulino [13]), to obtain symmetric Galerkin boundary element formulations (e.g. Bonnet [14], Gray et al. [15], Gray and Paulino ([16,17]), to evaluate nearly singular integrals (Mukherjee et al. [18]), to obtain the hypersingular boundary contour method (HBCM) (Phan et al. [19], Mukherjee and Mukherjee [20]), to obtain the hypersingular boundary node method (Chati et al. [21]), and for error analysis (Paulino et al. [22], Menon [23], Menon et al. [24], Chati et al. [21], Paulino and Gray [25])) and adaptivity [21].

Of particular interest to the present paper is the elegant approach of regularizing singular and hypersingular integrals using simple solutions, as first proposed by Rudolphi [26]. Several researchers have used this idea to regularize hypersingular integrals before collocating an HBIE at a regular boundary point. Examples are Cruse and Richardson [27], Lutz et al. [12], Poon et al. [28] and Mukherjee et al. [29].

1.2. Hypersingular integrals with irregular boundary source points

A regular boundary point is one at which the boundary of the body is locally smooth. It is sometimes useful to collocate an HBIE at an irregular boundary point (a point located at a corner on a one-dimensional (1-D) boundary of a 2-D body, or at a corner or an edge on a 2-D boundary of a 3-D body.) Gray and Manne [30] have carried this out successfully for the 2-D Laplace equation and Gray and Lutz [31] for 3-D problems, using a limit to the boundary (LTB) approach. This approach is a unified one that works for both singular and hypersingular integrals. Typically, symbolic programming is used to evaluate certain integrals. It is interesting to note that Gray and Manne [30], referring to one of the prevailing definitions of the finite part (FP) at that time, state that “the FP method fails for a boundary corner”. In fact, the FP approach discussed in the present paper is closely related to the LTB approach of Gray and his co-authors, and it is shown here that this approach does, indeed, work at a boundary corner.

Guiggiani et al. [5], Telles and Prado [32] and Chen and Hong [33] have (independently) considered the same

* Tel.: +1-607-255-7143; fax: +1-607-255-2011.
E-mail address: sm85@cornell.edu (S. Mukherjee).

0955-7997/00/$ - see front matter
PII: S0955-7997(00)00059-X

problem in 2-D for a corner with locally straight segments. Free terms have been derived in these papers when an HBIE for the 2-D Laplace equation is collocated at such a corner. The 3-D Laplace equation is also considered in detail in Ref. [5] but closed-form explicit expressions for the free terms are only derived for the 2-D case.

Guiggiani [34] and Mantič and Paris [35] have also independently considered this same problem and have demonstrated (by asymptotic analysis with a vanishing exclusion zone) that the hypersingular integral equation for Laplace's equation, collocated at a corner, produces, in the most general case (i.e. at a corner with locally curved segments), two (bounded) free terms. One of these multiplies the potential function and the other the potential gradient at the source point. Each of these researchers present the closed-form expression for the term that multiplies the potential for the case of 2-D potential theory and Mantič and Paris [35] also present explicit expressions for the 3-D case. The expressions for the free terms multiplying the potential gradient appear in earlier papers by Guiggiani et al. [5], Telles and Prado [32] and Chen and Hong [33]. Very recently, Paulino and Gray [25] have considered corners in the context of the Galerkin BIE, and have proposed an elegant approach for dealing with this matter. Also, Cruse and Richardson [27] have proved that their HBIE for linear elasticity, regularized by using "simple solutions" (see their equation (28)), is also valid at an irregular boundary point, provided that the primary variables satisfy certain smoothness requirements.

1.3. Finite parts of integrals

There exists an intimate relationship between HBIEs and FP integrals in the sense of Hadamard [36]. Krishnasamy et al. [9], for example, have proved that, provided that the primary variables satisfy certain smoothness requirements, a HBIE integral, obtained in the limit as an internal or external source point approaches a regular point on the boundary, can be interpreted as a FP integral. This author feels, however, that a conventional FP definition, such as in Martin [37] (see, also, Toh and Mukherjee [38] equation (1)) leaves something to be desired if one is interested in the actual evaluation of FPs of integrals. Also, various existing interpretations of FPs are not consistent — compare, for example, equation (46) in Mantič and Paris [35] and equation (29) in Cruse and Richardson [27], with Eqs. (15) and (52), respectively, in the present paper.

Toh and Mukherjee [38] proposed a consistent and practical definition of the FP of a singular integral. Their definition is general in that it applies to integrals with any order of singularity (provided that the primary variables are sufficiently smooth) in any number of physical dimensions. This paper also presents a regularization scheme for the evaluation of such integrals. This interpretation is based on the LTB concept. Results obtained from this interpretation agree with the results from Krishnasamy et al. [9] on scattering of acoustic waves and are completely consistent with complex variable formulations for 2-D BIEs (Hui and Mukherjee [39]). Finally, in a recent paper (Mukherjee and Mukherjee [20]), this FP interpretation has been shown to be consistent with a HBCM formulation for 3-D linear elasticity. The relationship between the Cauchy principal value (CPV) and FP of an integral, when its CPV exists, is explored in Mukherjee [40]. *It is important to state, however, that the work of Toh and Mukherjee [38] and Mukherjee [40] is limited to the case where a boundary source (collocation) point is regular. Irregular source points are of concern in the present paper.*

1.4. Outline of the present paper

The primary contributions of the present paper are:

- Present a unified, consistent and practical definition of the FP of a singular or hypersingular integral that is valid for both a regular as well as an irregular boundary collocation point.
- Present a strategy for numerical evaluation of a regularized version of the FP above, and then for solving the associated HBIE using suitable boundary elements.

The next section of this paper presents the definition of the FP together with a proof of its relationship with the LTB approach of Gray and his co-authors. Similar to the LTB approach (see Gray and Manne [30]), the present formulation is also able to handle both strongly singular and hypersingular integrals in a unified manner. The next section considers an example problem — the gradient BIE for the 2-D Laplace equation collocated at a corner. It is shown that the results from the present formulation are consistent with those of Guiggiani [34] and Mantič and Paris [35] obtained by considering a vanishing exclusion zone. It is also demonstrated next that this same FP formulation, this time with a "complete exclusion zone", leads to the regularized equations (obtained from use of simple solutions) of Rudolphi [26], for potential theory, and of Cruse and Richardson [27] for linear elasticity.

The last topic discussed in this paper is smoothness requirements for interpolation functions for collocation of an HBIE at an irregular point (Martin et al. [41]). Mukherjee and Mukherjee [42] have recently proved that, for a certain admissible class of problems, interpolation functions employed in the boundary contour method (BCM) satisfy all such smoothness requirements. Therefore, such functions are possible candidates for the hypersingular boundary element method (HBEM), with regular as well as irregular collocation points.

2. The FP of a hypersingular integral collocated at an irregular boundary point

2.1. Definition

Consider, for specificity, the space $\mathbf{R}^3$, and let S be a

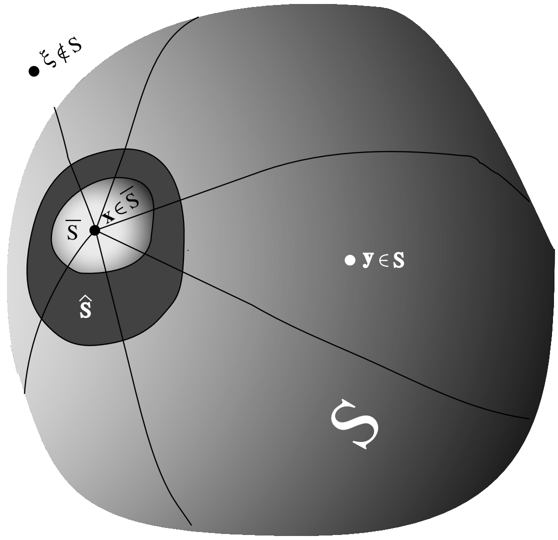

Fig. 1. A surface S with regions $\hat{S}$ and $\bar{S}$ and points $\boldsymbol{\xi}$, $\mathbf{x}$ and $\mathbf{y}$.

surface in $\mathbf{R}^3$. Let the points $\mathbf{x} \in S$ and $\boldsymbol{\xi} \notin S$. Also, let $\hat{S}$ and $\bar{S} \subset \hat{S}$ be two neighborhoods (in S) of $\mathbf{x}$ such that $\mathbf{x} \in \bar{S}$ (Fig. 1). The point $\mathbf{x}$ can be an irregular point on S.

Let the function $K(\mathbf{x},\mathbf{y})$, $\mathbf{y} \in S$, have its only singularity at $\mathbf{x} = \mathbf{y}$ of the form $1/r^3$ where $r = |\mathbf{x} - \mathbf{y}|$, and let $\phi(\mathbf{y})$ be a function that has no singularity in S and is of class $C^{1,\alpha}$ at $\mathbf{y} = \mathbf{x}$ for some $\alpha > 0$.

The FP of the integral

$$I(\mathbf{x}) = \int_S K(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \tag{1}$$

is defined as:

$$\begin{aligned} &⨍_S K(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \\ &\quad = \int_{S\setminus\hat{S}} K(\mathbf{x},\mathbf{y})\phi(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) + \int_{\hat{S}} K(\mathbf{x},\mathbf{y})[\phi(\mathbf{y}) - \phi(\mathbf{x}) \\ &\qquad - \phi_{,p}(\mathbf{x})(y_p - x_p)]\, \mathrm{d}S(\mathbf{y}) + \phi(\mathbf{x})A(\hat{S}) + \phi_{,p}(\mathbf{x})B_p(\hat{S}) \end{aligned} \tag{2}$$

where $\hat{S}$ *is any arbitrary neighborhood (in S) of* $\mathbf{x}$ and:

$$A(\hat{S}) = ⨍_{\hat{S}} K(\mathbf{x},\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \tag{3}$$

$$B_p(\hat{S}) = ⨍_{\hat{S}} K(\mathbf{x},\mathbf{y})(y_p - x_p)\, \mathrm{d}S(\mathbf{y}) \tag{4}$$

The above FP definition can be easily extended to any number of physical dimensions and any order of singularity of the kernel function $K(\mathbf{x},\mathbf{y})$. Please refer to Toh and Mukherjee [38] for further discussion of a previous closely related FP definition for the case when $\mathbf{x}$ is a regular point on S, and to Mukherjee [40] for a discussion of the relationship of this FP to the CPV of an integral when the CPV exists.

2.1.1. Evaluation of A and $\mathbf{B}$

There are several equivalent ways for evaluating A and $\mathbf{B}$.

Method one. Replace S by $\hat{S}$ and $\hat{S}$ by $\bar{S}$ in Eq. (2). Now, setting $\phi(\mathbf{y}) = 1$ in Eq. (2) and using Eq. (3), one gets:

$$A(\hat{S}) - A(\bar{S}) = \int_{\hat{S}\setminus\bar{S}} K(\mathbf{x},\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \tag{5}$$

Next, setting $\phi(\mathbf{y}) = (y_p - x_p)$ (note that, in this case, $\phi(\mathbf{x}) = 0$ and $\phi_{,p}(\mathbf{x}) = 1$) in Eq. (2), and using Eq. (4), one gets:

$$B_p(\hat{S}) - B_p(\bar{S}) = \int_{\hat{S}\setminus\bar{S}} K(\mathbf{x},\mathbf{y})(y_p - x_p)\, \mathrm{d}S(\mathbf{y}) \tag{6}$$

The formulae (5) and (6) are most useful for obtaining A and $\mathbf{B}$ when $\hat{S}$ is an open surface and Stoke regularization is employed. An example is the application of the FP definition (2) (for a regular collocation point) in Toh and Mukherjee [38], to regularize a hypersingular integral that appears in the HBIE formulation for the scattering of acoustic waves by a thin scatterer. The resulting regularized equation is shown in Ref. [38] to be equivalent to the result of Krishnasamy et al. [9]. Eqs. (5) and (6) are also used in Mukherjee and Mukherjee [20] and in Section 3.2 of this paper.

Method two. From Eq. (5):

$$A(\hat{S}) - A(\bar{S}) = \int_{\hat{S}\setminus\bar{S}} K(\mathbf{x},\mathbf{y})\, \mathrm{d}S(\mathbf{y}) = \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\hat{S}\setminus\bar{S}} K(\boldsymbol{\xi},\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \tag{7}$$

The second equality above holds since $K(\mathbf{x},\mathbf{y})$ is regular for $\mathbf{x} \in \bar{S}$ and $\mathbf{y} \in \hat{S}\setminus\bar{S}$. Assuming that the $\lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\hat{S}} K(\boldsymbol{\xi},\mathbf{y})\, \mathrm{d}S(\mathbf{y})$ and $\lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\bar{S}} K(\boldsymbol{\xi},\mathbf{y})\, \mathrm{d}S(\mathbf{y})$ exist, then:

$$A(\hat{S}) = \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\hat{S}} K(\boldsymbol{\xi},\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \tag{8}$$

Similarly:

$$B_p(\hat{S}) = \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\hat{S}} K(\boldsymbol{\xi},\mathbf{y})(y_p - x_p)\, \mathrm{d}S(\mathbf{y}) \tag{9}$$

Eqs. (8) and (9) are most useful for evaluating A and $\mathbf{B}$ when $\hat{S} = \partial B$, a closed surface that is the entire boundary of a body B. An example appears in Section 3.3 of this paper.

Method three. A third way for evaluation of A and $\mathbf{B}$ is to use an auxiliary surface (or "tent") as first proposed for fracture mechanics analysis by Lutz et al. [12]. (see, also, Mukherjee et al. [29], Mukherjee [43] and Section 3.2.1 of this paper). This method is useful if S is an open surface.

2.2. The FP and the LTB

There is a very simple connection between the FP, defined above, and the LTB approach employed by Gray and his co-authors. With, as before, $\boldsymbol{\xi} \notin S$, $\mathbf{x} \in S$ ($\mathbf{x}$ can be an irregular point on S), $K(\mathbf{x},\mathbf{y}) = \mathcal{O}(|\mathbf{x} - \mathbf{y}|^{-3})$ as $\mathbf{y} \to \mathbf{x}$

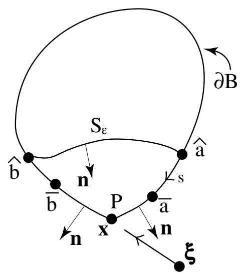

Fig. 2. Exclusion zone for the Laplace problem. $\hat{S} = \hat{a}P\hat{b}$, $\bar{S} = \bar{a}P\bar{b}$.

and $\phi(\mathbf{y}) \in C^{1,\alpha}$ at $\mathbf{y} = \mathbf{x}$, this can be stated as:

$$\lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_S K(\boldsymbol{\xi}, \mathbf{y})\phi(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) = \oint_S K(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \tag{10}$$

Of course, $\boldsymbol{\xi}$ can approach $\mathbf{x}$ from either side of S.

2.2.1. Proof of Eq. (10)

Consider the first and second terms on the right hand side of Eq. (2). Since these integrands are regular in their respective domains of integration, one has:

$$\int_{S\setminus\hat{S}} K(\mathbf{x}, \mathbf{y})\phi(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) = \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{S\setminus\hat{S}} K(\boldsymbol{\xi}, \mathbf{y})\phi(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \tag{11}$$

and

$$\int_{\hat{S}} K(\mathbf{x}, \mathbf{y})[\phi(\mathbf{y}) - \phi(\mathbf{x}) - \phi_{,p}(\mathbf{x})(y_p - x_p)]\, \mathrm{d}S(\mathbf{y})$$
$$= \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\hat{S}} K(\boldsymbol{\xi}, \mathbf{y})[\phi(\mathbf{y}) - \phi(\boldsymbol{\xi}) - \phi_{,p}(\boldsymbol{\xi})(y_p - \xi_p)]\, \mathrm{d}S(\mathbf{y}) \tag{12}$$

Use of Eqs. (8), (9), (11) and (12) in Eq. (2) proves Eq. (10).

3. The gradient BIE for the 2-D Laplace equation collocated at a corner

Collocation of the gradient BIE for the 2-D Laplace equation, at an irregular point, is considered as an example problem in this section. It is proved that with a vanishing exclusion zone, one recovers, from the proposed FP definition (Eq. (2)), the results in Guiggiani [34], Guiggiani et al. [5] and Mantič and Paris [35], while, with a complete exclusion zone, one recovers the results from Rudolphi [26]. (It should be noted that while Rudolphi [26] only considers collocation at a regular boundary point, his equation (20) is also valid at an irregular boundary point.)

3.1. FP Formulation

Consider a body B (an open set) with boundary ∂B. With $\boldsymbol{\xi} \notin \partial B$, the well-known BIE for Laplace's equation $\nabla^2 u = 0$ in 2-D can be written as:

$$\gamma(\boldsymbol{\xi})u(\boldsymbol{\xi}) = \int_{\partial B} [U(\boldsymbol{\xi}, \mathbf{y})q(\mathbf{y}) - T(\boldsymbol{\xi}, \mathbf{y})u(\mathbf{y})]\, \mathrm{d}S(\mathbf{y}) \tag{13}$$

and its differentiated form, the HBIE:

$$\gamma(\boldsymbol{\xi})\frac{\partial u(\boldsymbol{\xi})}{\partial \xi_i} = \int_{\partial B} [-D_i(\boldsymbol{\xi}, \mathbf{y})q(\mathbf{y}) + S_i(\boldsymbol{\xi}, \mathbf{y})u(\mathbf{y})]\, \mathrm{d}S(\mathbf{y}) \tag{14}$$

In the above, $q = (\partial u/\partial n)(\mathbf{y})$, $\gamma(\boldsymbol{\xi}) = 1$ for $\boldsymbol{\xi} \in B$, $\gamma(\boldsymbol{\xi}) = 0$ for $\boldsymbol{\xi} \notin B$, and the kernels are given in Appendix A.

Let $\boldsymbol{\xi} \to \mathbf{x}$ where $\mathbf{x} \in \partial B$ can be an irregular point, and $u(\mathbf{y}) \in C^{1,\alpha}$ at $\mathbf{y} = \mathbf{x}$. With an exclusion zone $\hat{S}$ (Fig. 2), and $\boldsymbol{\xi}$ an exterior point, $\gamma(\boldsymbol{\xi}) = 0$. Now, using the LTB property of the FP from Eq. (10), (14) becomes:

$$0 = \oint_{\partial B} [S_i(\mathbf{x}, \mathbf{y})u(\mathbf{y}) - D_i(\mathbf{x}, \mathbf{y})q(\mathbf{y})]\, \mathrm{d}S(\mathbf{y}) \tag{15}$$

With $S = \partial B$, application of the definition of the FP (Eq. (2)) allows Eq. (15) to be written as:

$$0 = \int_{\partial B\setminus\hat{S}} [S_i(\mathbf{x}, \mathbf{y})u(\mathbf{y}) - D_i(\mathbf{x}, \mathbf{y})q(\mathbf{y})]\, \mathrm{d}S(\mathbf{y})$$
$$+ \int_{\hat{S}} S_i(\mathbf{x}, \mathbf{y})[u(\mathbf{y}) - u(\mathbf{x}) - u_{,p}(\mathbf{x})(y_p - x_p)]\, \mathrm{d}S(\mathbf{y})$$
$$- \int_{\hat{S}} D_i(\mathbf{x}, \mathbf{y})[u_{,p}(\mathbf{y}) - u_{,p}(\mathbf{x})]n_p(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) + u(\mathbf{x})A_i(\hat{S})$$
$$+ u_{,p}(\mathbf{x})[B_{ip}(\hat{S}) - E_{ip}(\hat{S})] \tag{16}$$

where, using Eqs. (5) and (6), one gets:

$$A_i(\hat{S}) - A_i(\bar{S}) = \int_{\hat{S}\setminus\bar{S}} S_i(\mathbf{x}, \mathbf{y})\, \mathrm{d}S(\mathbf{y}) \tag{17}$$

$$B_{ip}(\hat{S}) - B_{ip}(\bar{S}) = \int_{\hat{S}\setminus\bar{S}} S_i(\mathbf{x}, \mathbf{y})(y_p - x_p)\, \mathrm{d}S(\mathbf{y}) \tag{18}$$

and

$$E_{ip}(\hat{S}) - E_{ip}(\bar{S}) = \int_{\hat{S}\setminus\bar{S}} D_i(\mathbf{x}, \mathbf{y})n_p(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \tag{19}$$

Finally, define $\mathbf{C} = \mathbf{B} - \mathbf{E}$

3.2. Vanishing exclusion zone

A vanishing exclusion zone, $S_\epsilon \to 0$ (Fig. 2) is considered for this problem by Guiggiani et al. [5], Guiggiani [34] and Mantič and Paris [35]. *S_ϵ is an arc of a circle in these papers, whereas it is arbitrary in the present work*. Clearly, $S_\epsilon \to 0$ implies that $\hat{S} \to 0$, so that the second and third terms (with regular integrands) on the right hand side of Eq. (16) vanish in this limit. The first term and the quantity $\mathbf{A}$ are of $\mathcal{O}(1/\epsilon)$ as $S_\epsilon \to 0$ (here ϵ is a length scale for S_ϵ, its radius if S_ϵ is a circular arc centered at P in Fig. 2); while $\mathbf{C}$ has a finite limit as $\epsilon \to 0$. This situation is discussed next.

*3.2.1. The vector **A***

Applying Stokes' theorem in the form:

$$\int_a^b (g_{,i}n_j - g_{,j}n_i)\, \mathrm{d}S = \epsilon_{ij3}[g]_a^b \tag{20}$$

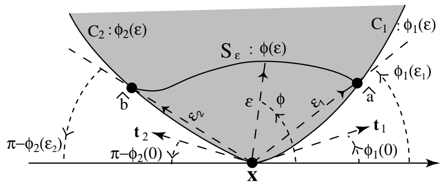

Fig. 3. Close-up of exclusion zone.

(here $\boldsymbol{\epsilon}$ is the alternating symbol), with $g = (\partial U)/(\partial y_j) = W_j$; and using the facts that (see Appendix A):

$$S_i(\mathbf{x}, \mathbf{y}) = \frac{\partial^2 U(\mathbf{x}, \mathbf{y})}{\partial y_i \partial y_j} n_j(\mathbf{y}) = g_{,i} n_j \tag{21}$$

and that:

$$\frac{\partial^2 U(\mathbf{x}, \mathbf{y})}{\partial y_j \partial y_j} = 0 \qquad \text{for } \mathbf{y} \in \hat{S} \backslash \bar{S} \tag{22}$$

one gets:

$$A_i(\hat{S}) - A_i(\bar{S}) = \epsilon_{ij3}[[W_j]_{\hat{a}}^{\bar{a}} + [W_j]_{\bar{b}}^{\hat{b}}] = \epsilon_{ij3}[[W_j]_{\hat{a}}^{\hat{b}} - [W_j]_{\bar{a}}^{\bar{b}}] \tag{23}$$

so that:

$$A_i(\hat{S}) = \epsilon_{ij3}[W_j]_{\hat{a}}^{\hat{b}} \tag{24}$$

In the present work, S_ϵ is an arbitrary curve joining the points $\hat{a}$ and $\hat{b}$. This curve can be considered to be an example of an auxiliary surface to $\hat{S}$ as first proposed by Lutz et al. [12] (see, also, Mukherjee [43]) for evaluating hypersingular integrals (method 3 in the present paper). The identity (see equation (20) in Mukherjee [43]):

$$\oint_{\tilde{S}} S_i(\mathbf{x}, \mathbf{y}) \, \mathrm{d}S(\mathbf{y}) = 0 \tag{25}$$

can be obtained by using the uniform solution $u(\mathbf{y}) = c$ (c is a constant) in Eq. (15) *for any closed surface* $\tilde{S}$. Choosing $\tilde{S} = \hat{S} \cup S_\epsilon$, one gets (see Eqs. (17) and (3)):

$$\oint_{\hat{S}} S_i(\mathbf{x}, \mathbf{y}) \, \mathrm{d}S(\mathbf{y}) = -\int_{S_\epsilon} S_i(\mathbf{x}, \mathbf{y}) \, \mathrm{d}S(\mathbf{y}) \Rightarrow A_i(\hat{S}) = A_i(S_\epsilon) \tag{26}$$

in view of the directions of the normals on $\hat{S}$ and on S_ϵ in Fig. 2.

With $r_{,1} = \cos\phi$ and $r_{,2} = \sin\phi$, one has:

$$A_1(S_\epsilon) = -\left[\frac{\sin\phi(\epsilon)}{2\pi\epsilon}\right]_{\hat{a}}^{\hat{b}}, \qquad A_2(S_\epsilon) = \left[\frac{\cos\phi(\epsilon)}{2\pi\epsilon}\right]_{\hat{a}}^{\hat{b}}, \tag{27}$$

Here, the angle ϕ is a function of the distance ϵ (see Fig. 3) because the corner is a meeting of curved elements. Please note that below his equation (14), Guiggiani [34], with S_ϵ a circular arc of radius ϵ centered at P, has $F_1 = \sin\phi/2\pi$, $F_2 = -\cos\phi/2\pi$; the sign differences being due to the difference in the definition of the kernel $\mathbf{S}$ in Ref. [34] and in the present paper.

Using asymptotic analysis (see Fig. 3), Guiggiani [34] and Mantič and Paris [35] have independently proved that:

$$A_i(S_\epsilon) = \left[\frac{b_i}{\epsilon} + a_i + \mathscr{O}(\epsilon)\right]_{\hat{a}}^{\hat{b}} \tag{28}$$

so that, in the limit as $S_\epsilon \to 0$ (i.e. $\epsilon \to 0$), one has $\mathscr{F}(A_i) = a_i$. (Here $\mathscr{F}$ denotes the FP of the appropriate quantity as in Mantič and Paris [35] — their equations (7) and (8)). These researchers have also obtained the explicit values of a_i. The final result, from Refs. [34,35], in the present notation, is:

$$a_1 = \mathscr{F}(A_1) = 1/4\pi[\cos\phi_2(0)\kappa_2(0) + \cos\phi_1(0)\kappa_1(0)]$$

$$a_2 = \mathscr{F}(A_2) = 1/4\pi[\sin\phi_2(0)\kappa_2(0) + \sin\phi_1(0)\kappa_1(0)] \tag{29}$$

where $\phi_1(0)$ and $\phi_2(0)$ are the angles made by the tangents to the curves $C_1(\phi_1(\epsilon))$ and $C_2(\phi_2(\epsilon))$ at $\mathbf{x}$ (in Fig. 3), with the positive y_1 axis; and $\kappa_1(0)$ and $\kappa_2(0)$ are the signed curvatures of C_1 and C_2 at $\mathbf{x}$. In general, for a plane curve parameterized as $y_1(\boldsymbol{\xi})$, $y_2(\boldsymbol{\xi})$, the signed curvature is defined here as in [34]:

$$\kappa(\boldsymbol{\xi}) = \frac{(\dot{\boldsymbol{\rho}} \times \ddot{\boldsymbol{\rho}}) \cdot \mathbf{k}}{\|\dot{\boldsymbol{\rho}}\|^3} = \frac{-\ddot{y}_1\dot{y}_2 + \ddot{y}_2\dot{y}_1}{[(\dot{y}_1)^2 + (\dot{y}_2)^2]^{3/2}} \tag{30}$$

where $\boldsymbol{\rho} = \mathbf{i}y_1 + \mathbf{j}y_2$, $\mathbf{k} = \mathbf{i} \times \mathbf{j}$ and a superscribed dot denotes a derivative with respect to the parameter ξ.

Guiggiani [34] combines the term $(b_i(\mathbf{x})/\epsilon)u(\mathbf{x})$ (see Eq. (28)) with the first integral in the right hand side of Eq. (16) in order to get a finite result as $S_\epsilon \to 0$. He has shown that (in the present notation), $b_1 = -(1/(2\pi))[\sin(\phi_2(0) - \sin(\phi_1(0))]$, $b_2 = (1/(2\pi))[\cos(\phi_2(0) - \cos(\phi_1(0))]$. Details are available in Guiggiani et al. [5].

3.2.2. The tensor C

From Eqs. (18) and (19):

$$C_{ip}(\hat{S}) - C_{ip}(\bar{S}) = \int_{\hat{S}\backslash\bar{S}} [S_i(\mathbf{x}, \mathbf{y})(y_p - x_p) - D_i(\mathbf{x}, \mathbf{y})n_p(\mathbf{y})] \, \mathrm{d}S(\mathbf{y}) \tag{31}$$

Using the expressions in Appendix A, one has:

$$\begin{aligned} C_{ip}(\hat{S}) - C_{ip}(\bar{S}) &= \int_{\hat{S}\backslash\bar{S}} [W_{j,i}(\mathbf{x}, \mathbf{y})(y_p - x_p) - U_{,i}(\mathbf{x}, \mathbf{y})\delta_{pj}]n_j(\mathbf{y}) \, \mathrm{d}S(\mathbf{y}) \\ &= \int_{\hat{S}\backslash\bar{S}} [W_j(\mathbf{x}, \mathbf{y})(y_p - x_p) - U(\mathbf{x}, \mathbf{y})\delta_{pj}]_{,i} n_j(\mathbf{y}) \, \mathrm{d}S(\mathbf{y}) \\ &\quad - \int_{\hat{S}\backslash\bar{S}} W_j(\mathbf{x}, \mathbf{y})\delta_{pi}n_j(\mathbf{y}) \, \mathrm{d}S(\mathbf{y}) \end{aligned} \tag{32}$$

The procedure followed now is quite analogous to that used in the previous section for determining $\mathbf{A}$. Using:

$$g = W_j(y_p - x_p) - U\delta_{pj} \tag{33}$$

in Stokes' theorem (20), and noting that $g_{,j} = 0$ for $\mathbf{x} \in \bar{S}$

and $\mathbf{y} \in \hat{S} \setminus \bar{S}$, one gets:

$$\begin{aligned} & C_{ip}(\hat{S}) - C_{ip}(\bar{S}) \\ & \quad = \epsilon_{ij3}[[W_j(y_p - x_p) - U\delta_{pj}]_{\hat{a}}^{\hat{b}} - [W_j(y_p - x_p) - U\delta_{pj}]_{\bar{a}}^{\bar{b}}] \\ & \qquad - \int_{\hat{S}\setminus\bar{S}} T\delta_{pi}\, \mathrm{d}S(\mathbf{y}) \\ & \quad \equiv I_1 + I_2 \end{aligned} \tag{34}$$

Using the fact that (see, for example, Ghosh et al. [44]):

$$T(\mathbf{x},\mathbf{y}) = -\frac{1}{2\pi r}\frac{\partial r}{\partial n}(\mathbf{y}) = \frac{1}{2\pi}\frac{\partial \phi}{\partial s} \tag{35}$$

where s is a length coordinate that is measured around ∂B (Fig. 2) in a clockwise direction, the second term in Eq. (34) becomes:

$$I_2 = -\frac{1}{2\pi}[[\phi]_{\hat{a}}^{\bar{a}} + [\phi]_{\bar{b}}^{\hat{b}}] = -\frac{1}{2\pi}[[\phi]_{\hat{a}}^{\hat{b}} - [\phi]_{\bar{a}}^{\bar{b}}] \tag{36}$$

Using Eqs. (34) and (36), one has:

$$\begin{aligned} & C_{ip}(\hat{S}) = C_{ip}(S_\epsilon) \\ & \qquad = \epsilon_{ij3}[W_j(y_p - x_p) - U\delta_{pj}]_{\hat{a}}^{\hat{b}} - (1/(2\pi))[\phi]_{\hat{a}}^{\hat{b}}\delta_{pi} \end{aligned} \tag{37}$$

In the limit $S_\epsilon \to 0$, one gets (as in Guiggiani et al. [5] but with a sign change due to the sign differences in the definitions of the kernels **S** and **D** in [34] and in the present paper):

$$C_{11} = -\frac{1}{2\pi}\left[(\phi_2(0) - \phi_1(0)) + \frac{\sin(2\phi_2(0)) - \sin(2\phi_1(0))}{2}\right] \tag{38}$$

$$C_{22} = -\frac{1}{2\pi}\left[(\phi_2(0) - \phi_1(0)) - \frac{\sin(2\phi_2(0)) - \sin(2\phi_1(0))}{2}\right] \tag{39}$$

$$C_{12} = C_{21} = -\frac{1}{2\pi}[\sin^2(\phi_2(0)) - \sin^2(\phi_1(0))] \tag{40}$$

Note that the term $[U]_{\hat{a}}^{\hat{b}}$ does not contribute to **C**.

3.2.3. Final equation

The final form of Eq. (16), with a vanishing exclusion zone, has two free terms. It is (see Guiggiani [34]):

$$\begin{aligned} 0 = a_i(\mathbf{x})u(\mathbf{x}) + C_{ip}(\mathbf{x})u_{,p}(\mathbf{x}) + \lim_{\hat{S}\to 0}\Bigg\{ & \int_{\partial B\setminus\hat{S}} [S_i(\mathbf{x},\mathbf{y})u(\mathbf{y}) \\ & - D_i(\mathbf{x},\mathbf{y})q(\mathbf{y})]\, \mathrm{d}S(\mathbf{y}) + \frac{b_i(\mathbf{x})}{\epsilon}u(\mathbf{x})\Bigg\} \end{aligned} \tag{41}$$

An alternative equivalent form (see Mantič and Paris [35]) is:

$$\begin{aligned} 0 = {} & \mathscr{F}(A_i(\mathbf{x}))u(\mathbf{x}) + C_{ip}(\mathbf{x})u_{,p}(\mathbf{x}) \\ & + \mathscr{F}\left\{\int_{\partial B\setminus\hat{S}} [S_i(\mathbf{x},\mathbf{y})u(\mathbf{y}) - D_i(\mathbf{x},\mathbf{y})q(\mathbf{y})]\, \mathrm{d}S(\mathbf{y})\right\} \end{aligned} \tag{42}$$

with the FPs $\mathscr{F}$ evaluated for the limit $\hat{S} \to 0$.

It is interesting to compare (the equivalent) Eqs. (15) and (42). One contains two free terms, the other does not. *Both contain FPs, defined in different ways.* The LTB definition of Gray and Manne [30] is equivalent to the FP Eq. (15) although they do not use the designation "finite part".

3.3. Complete exclusion zone

The situation in which $\hat{S} = \partial B$ is referred to as a "complete" exclusion zone. Now, the FP Eq. (15) (see Eq. (16)) has the form:

$$\begin{aligned} 0 = {} & \int_{\partial B} S_i(\mathbf{x},\mathbf{y})[u(\mathbf{y}) - u(\mathbf{x}) - u_{,p}(\mathbf{x})(y_p - x_p)]\, \mathrm{d}S(\mathbf{y}) \\ & - \int_{\partial B} D_i(\mathbf{x},\mathbf{y})[u_{,p}(\mathbf{y}) - u_{,p}(\mathbf{x})]n_p(\mathbf{y})\, \mathrm{d}S(\mathbf{y}) + u(\mathbf{x})A_i(\partial B) \\ & + u_{,p}(\mathbf{x})C_{ip}(\partial B) \end{aligned} \tag{43}$$

with (see Eqs. (8) and (9)):

$$A_i(\partial B) = \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\partial B} S_i(\boldsymbol{\xi},\mathbf{y})\, \mathrm{d}S(\mathbf{y}) \tag{44}$$

$$C_{ip}(\partial B) = \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\partial B} [S_i(\boldsymbol{\xi},\mathbf{y})(y_p - \xi_p) - D_i(\boldsymbol{\xi},\mathbf{y})n_p(\mathbf{y})]\, \mathrm{d}S(\mathbf{y}) \tag{45}$$

Since an exclusion zone is implied here, one has $\boldsymbol{\xi} \notin B$, $\mathbf{x} \in \partial B$, i.e. the (possibly irregular) boundary point $\mathbf{x}$ is approached from outside the body B.

The quantities **A** and **C** can be easily evaluated using the imposition of simple solutions. Following Rudolphi [26], use of the uniform solution $u(\mathbf{y}) = c$ (c is a constant) in Eq. (14) gives:

$$\int_{\partial B} S_i(\boldsymbol{\xi},\mathbf{y})\, \mathrm{d}S(\mathbf{y}) = 0 \tag{46}$$

while use of the linear solution:

$$u(\mathbf{y}) = c(y_p - \xi_p), \quad q(\mathbf{y}) = cn_p(\mathbf{y}) \ \ (\text{with } p = 1, 2) \tag{47}$$

in Eq. (14) (with $\gamma = 0$) gives:

$$\int_{\partial B} [S_i(\boldsymbol{\xi},\mathbf{y})(y_p - \xi_p) - D_i(\boldsymbol{\xi},\mathbf{y})n_p(\mathbf{y})]\, \mathrm{d}S(\mathbf{y}) = 0 \tag{48}$$

Therefore, (assuming continuity) $\mathbf{A} = \mathbf{C} = 0$ and one obtains a simple, fully regularized form of Eq. (43) as:

$$0 = \int_{\partial B} S_i(\mathbf{x},\mathbf{y})[u(\mathbf{y}) - u(\mathbf{x}) - u_{,p}(\mathbf{x})(y_p - x_p)]\,\mathrm{d}S(\mathbf{y}) - \int_{\partial B} D_i(\mathbf{x},\mathbf{y})[u_{,p}(\mathbf{y}) - u_{,p}(\mathbf{x})]n_p(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) \tag{49}$$

A few comments are in order. First, Eq. (49) is the same as Rudolphi's [26] equation (20) with (his) $\kappa = 1$ and (his) S_0 set equal to S and renamed ∂B. (See, also, Kane [45], equation (17.34)). Second, this equation can also be shown to be valid for the case $\boldsymbol{\xi} \in B$, i.e. for an inside approach to the boundary point $\mathbf{x}$. Third, as noted before, $\mathbf{x}$ can be a corner point on ∂B (provided, of course, that $u(\mathbf{y}) \in C^{1,\alpha}$ at $\mathbf{y} = \mathbf{x}$ — Rudolphi had only considered a regular boundary collocation point in his excellent paper ten years ago). Finally, as discussed in the next section, Eq. (49) is analogous to the regularized stress BIE in linear elasticity — equation (28) in Cruse and Richardson [27].

4. The stress BIE for 3-D linear elasticity collocated at an irregular boundary point

The case of the hypersingular stress BIE for 3-D linear elasticity, collocated at an irregular boundary point, is briefly discussed below. Only the case of the complete exclusion zone is considered here.

The well-known BIE for 3-D linear elasticity, with $\boldsymbol{\xi} \notin \partial B$ (Rizzo [46], Cruse [47]) is:

$$\gamma(\boldsymbol{\xi})u_i(\boldsymbol{\xi}) = \int_{\partial B} [U_{ik}(\boldsymbol{\xi},\mathbf{y})\tau_k(\mathbf{y}) - T_{ik}(\boldsymbol{\xi},\mathbf{y})u_k(\mathbf{y})]\,\mathrm{d}S(\mathbf{y}) \tag{50}$$

The hypersingular stress BIE in 3-D linear elasticity, with $\boldsymbol{\xi} \notin \partial B$ (Cruse [47]) is:

$$\gamma(\boldsymbol{\xi})\sigma_{ij}(\boldsymbol{\xi}) = \int_{\partial B} [-D_{ijk}(\boldsymbol{\xi},\mathbf{y})\tau_k(\mathbf{y}) + S_{ijk}(\boldsymbol{\xi},\mathbf{y})u_k(\mathbf{y})]\,\mathrm{d}S(\mathbf{y}) \tag{51}$$

In the above, $\mathbf{u}$ is the displacement field, $\boldsymbol{\tau}$ is the traction field, $\boldsymbol{\sigma}$ is the stress field, $\gamma(\boldsymbol{\xi}) = 1$ for $\boldsymbol{\xi} \in B$, $\gamma(\boldsymbol{\xi}) = 0$ for $\boldsymbol{\xi} \notin B$, and the kernels are given in Appendix A.

Let $\boldsymbol{\xi} \to \mathbf{x}$, where $\mathbf{x} \in \partial B$ can be an irregular point, and $\mathbf{u}(\mathbf{y}) \in C^{1,\alpha}$ at $\mathbf{y} = \mathbf{x}$. With an exclusion zone $\hat{S}$, and $\boldsymbol{\xi}$ an exterior point, $\gamma(\boldsymbol{\xi}) = 0$. Now, using the LTB property of the FP from Eq. (10), Eq. (51) becomes:

$$0 = \oint\!\!\!\!\!\!-_{\partial B} [S_{ijk}(\mathbf{x},\mathbf{y})u_k(\mathbf{y}) - D_{ijk}(\mathbf{x},\mathbf{y})\tau_k(\mathbf{y})]\,\mathrm{d}S(\mathbf{y}) \tag{52}$$

With $S = \partial B$, application of the definition of the FP (Eq. (2)) allows Eq. (52) to be written as:

$$0 = \int_{\partial B} S_{ijk}(\mathbf{x},\mathbf{y})[u_k(\mathbf{y}) - u_k(\mathbf{x}) - u_{k,p}(\mathbf{x})(y_p - x_p)]\,\mathrm{d}S(\mathbf{y}) - \int_{\partial B} D_{ijk}(\mathbf{x},\mathbf{y})[\sigma_{kp}(\mathbf{y}) - \sigma_{kp}(\mathbf{x})]n_p(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) + A_{ijk}(\partial B)u_k(\mathbf{x}) + C_{ijkp}(\partial B)u_{k,p}(\mathbf{x}) \tag{53}$$

with (see Eqs. (8) and (9)):

$$A_{ijk}(\partial B) = \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\partial B} S_{ijk}(\boldsymbol{\xi},\mathbf{y})\,\mathrm{d}S(\mathbf{y}) \tag{54}$$

$$C_{ijkp}(\partial B) = \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\partial B} [S_{ijk}(\boldsymbol{\xi},\mathbf{y})(y_p - \xi_p) - E_{kpm\ell}D_{ijm}(\boldsymbol{\xi},\mathbf{y})n_\ell(\mathbf{y})]\,\mathrm{d}S(\mathbf{y}) \tag{55}$$

Simple (rigid body and linear) solutions in linear elasticity (see, for example, Lutz et al., [12], Cruse and Richardson [27]) are now used in order to determine the quantities $\mathbf{A}$ and $\mathbf{C}$. With $\boldsymbol{\xi}$ an exterior point, $\gamma(\boldsymbol{\xi}) = 0$ Eq. (51). Using the rigid body mode $u_k = c_k$ (c_k are arbitrary constants) in Eq. (51), one has:

$$0 = \int_{\partial B} S_{ijk}(\boldsymbol{\xi},\mathbf{y})\,\mathrm{d}S(\mathbf{y}) \tag{56}$$

while, using the linear mode:

$$u_k = c_k(y_p - \xi_p), \qquad \tau_m = \sigma_{m\ell}n_\ell = E_{mlkp}c_kn_l = E_{kpm\ell}c_kn_\ell \tag{57}$$

in Eq. (51) gives:

$$0 = \int_{\partial B} [S_{ijk}(\boldsymbol{\xi},\mathbf{y})(y_p - \xi_p) - E_{kpm\ell}D_{ijm}(\boldsymbol{\xi},\mathbf{y})n_\ell(\mathbf{y})]\,\mathrm{d}S(\mathbf{y}) \tag{58}$$

Invoking continuity, $\mathbf{A} = \mathbf{C} = 0$, so that Eq. (53) reduces to the simple regularized form:

$$0 = \int_{\partial B} S_{ijk}(\mathbf{x},\mathbf{y})[u_k(\mathbf{y}) - u_k(\mathbf{x}) - u_{k,p}(\mathbf{x})(y_p - x_p)]\,\mathrm{d}S(\mathbf{y}) - \int_{\partial B} D_{ijk}(\mathbf{x},\mathbf{y})[\sigma_{kp}(\mathbf{y}) - \sigma_{kp}(\mathbf{x})]n_p(\mathbf{y})\,\mathrm{d}S(\mathbf{y}) \tag{59}$$

which is equation (28) of Cruse and Richardson [27] in the present notation. As is the case in the present work, Cruse and Richardson [27] have also proved that their equation (28) is valid at a corner point, provided that the stress is continuous there.

It has been proved in this section that the regularized stress BIE (28) of Cruse and Richardson [27] can also be obtained from the FP definition (2) with a complete exclusion zone.

5. Solution strategy for a HBIE collocated at an irregular boundary point

Hypersingular BIEs for a body B with boundary ∂B are considered here. Regularized HBIEs, obtained by using complete exclusion zones, e.g. Eq. (49) for potential theory or Eq. (59) for linear elasticity, are recommended as starting points.

An irregular collocation point $\mathbf{x}$ for 3-D problems is considered next. Let ∂B_n, $(n = 1, 2, 3, \ldots, N)$ be smooth pieces of ∂B that meet at an irregular point $\mathbf{x} \in \partial B$. Also, let a source point, with coordinates x_k, be denoted by P, and a field point, with coordinates y_k, be denoted by Q.

Martin et al. [41] state the following requirements for collocating a regularized HBIE, such as Eq. (59) at an irregular point $P \in \partial B$. These are:

Box 1: Requirements for collocation of a HBIE at an irregular point (from Ref. [41]).

(i) The displacement $\mathbf{u}$ must satisfy the equilibrium equations in B.
(ii) (a) The stress $\boldsymbol{\sigma}$ must be continuous in B; (b) The stress $\boldsymbol{\sigma}$ must be continuous on ∂B.
(iii) $|u_i(Q_n) - u_i^L(Q_n; P)| = \mathcal{O}(r_n^{(1+\alpha)})$ as $r_n \to 0$, for each n.
(iv) $[\sigma_{ij}(Q_n) - \sigma_{ij}(P)]n_j(Q_n) = \mathcal{O}(r_n^{\alpha})$ as $r_n \to 0$, for each n.

In the above, $r_n = |\mathbf{y}(Q_n) - \mathbf{x}(P)|$, $Q_n \in \partial B_n$, and $\alpha > 0$. Also

$$u_i^L(Q_n; P) = u_i(P) + u_{i,j}(P)[y_j(Q_n) - x_j(P)] \tag{60}$$

There are two important issues to consider here.

The first is that, if there is to be any hope for collocating Eq. (59) at an irregular point P, the exact solution of a boundary value problem must satisfy conditions (i)–(iv) in Box 1. Clearly, one should not attempt this collocation if, for example, the stress is unbounded at P (this can easily happen — see an exhaustive study on the subject in Glushkov et al. [48]), or is bounded but discontinuous at P (e.g. at the tip of a wedge — see, for example, Zhang and Mukherjee [49]). The discussion in the rest of this paper is limited to the class of problems, *referred to as the admissible class*, whose exact solutions satisfy conditions (i)–(iv).

The second issue refers to smoothness requirements on the interpolation functions for $\mathbf{u}$, $\boldsymbol{\sigma}$ and the traction $\boldsymbol{\tau} = \mathbf{n}\cdot\boldsymbol{\sigma}$ in Eq. (59). It has proved very difficult, in practice, to find BEM interpolation functions that satisfy, a priori, (ii (b)–(iv)) in Box 1, for collocation at an irregular surface point on a 3-D body [41]. It has recently been proved in Mukherjee and Mukherjee [42], however, that interpolation functions used in the BCM (see, for example, Mukherjee et al. [50], Mukherjee and Mukherjee [20]) satisfy these conditions a priori. Another important advantage of using these interpolation functions is that $\nabla \mathbf{u}$ can be directly computed from them at an irregular boundary point [20], without the need to use the (undefined) normal and tangent vectors at this point. Numerical results from the hypersingular BCM, collocated on edges and at corners, are available in Mukherjee and Mukherjee [42]. In principle, these BCM interpolation functions can also be used in the BEM. This is a recommended topic for future research.

6. Concluding remarks

The following are the highlights of this paper:

- Presents a consistent unified definition of the FP of a hypersingular integral collocated at an irregular boundary point. This definition is valid for an arbitrary exclusion zone. There is no need to consider a direction of a limit process as for the LTB approach.
- Unifies, in some sense, the work of researchers such as Cruse, Rizzo, Gray, Mantič, Guiggiani, Rudolphi, Lutz, and their co-authors, in this subject area.
- Recommends the use of the complete exclusion zone for regularization of HBIEs defined on closed surfaces. This procedure has the following advantages, that become most obvious for 3-D problems:
  - no need to evaluate free terms;
  - no need to compute limits, requiring, perhaps, symbolic programming.
- Recommends the use of Stoke regularization or auxiliary surfaces for HBIEs on open surfaces such as in fracture mechanics problems.
- Recommends research towards the use of HBCM interpolation functions, that satisfy smoothness requirements a priori, for obtaining numerical solutions of HBIEs that involve collocation at irregular boundary points.
- Recommends consideration of adoption of the FP definition, proposed in this paper, as the standard one for future research in this subject area.

Appendix A. BIE and HBIE kernels

The kernels for the BIE (13) for Laplace's equation in 2-D are:

$$U(\boldsymbol{\xi}, \mathbf{y}) = \frac{1}{2\pi} \ln\left(\frac{1}{r}\right)$$

$$W_i(\boldsymbol{\xi}, \mathbf{y}) = \frac{\partial U}{\partial y_i} = -\frac{r_{,i}}{2\pi r} = -\frac{(y_i - \xi_i)}{2\pi r^2}$$

$$T(\boldsymbol{\xi}, \mathbf{y}) = \frac{\partial U}{\partial n(\mathbf{y})} = W_j n_j(\mathbf{y}) = -\frac{1}{2\pi r}\frac{\partial r}{\partial n}(\mathbf{y})$$

The corresponding kernels for the gradient BIE (14) are:

$$D_i(\boldsymbol{\xi}, \mathbf{y}) = -\frac{\partial U}{\partial \xi_i} = W_i$$

$$S_i(\boldsymbol{\xi}, \mathbf{y}) = -\frac{\partial T}{\partial \xi_i} = -\frac{\partial W_j}{\partial \xi_i} n_j(\mathbf{y}) = \frac{\partial^2 U}{\partial y_i \partial y_j} n_j(\mathbf{y})$$

$$= \frac{1}{2\pi r^2}\left(2r_{,i}\frac{\partial r}{\partial n}(\mathbf{y}) - n_i(\mathbf{y})\right)$$

The Kelvin kernels for the displacement BIE (50) for 3-D linear elasticity are:

$$U_{ik}(\boldsymbol{\xi}, \mathbf{y}) = \frac{1}{16\pi G(1-\nu)r}[(3-4\nu)\delta_{ik} + r_{,i}r_{,k}]$$

$$T_{ik}(\boldsymbol{\xi}, \mathbf{y}) = -\frac{1}{8\pi(1-\nu)r^2}$$

$$\times\left[\{(1-2\nu)\delta_{ik} + 3r_{,i}r_{,k}\}\frac{\partial r}{\partial n} + (1-2\nu)(r_{,k}n_i - r_{,i}n_k)\right]$$

The corresponding kernels for the stress BIE (51) are:

$$D_{ijk}(\boldsymbol{\xi}, \mathbf{y}) = -E_{ijmn}\frac{\partial U_{mk}}{\partial \xi_n}$$

$$= -\frac{1}{8\pi(1-\nu)r^2}[(1-2\nu)(\delta_{ik}r_{,j} + \delta_{kj}r_{,i} - \delta_{ji}r_{,k}) + 3r_{,i}r_{,j}r_{,k}]$$

$$S_{ijk}(\boldsymbol{\xi}, \mathbf{y}) = -E_{ijmn}\frac{\partial T_{mk}}{\partial \xi_n}$$

$$= -\frac{G}{4\pi(1-\nu)r^3}\Bigg[\{(1-2\nu)\delta_{ij}r_{,k} + \nu(\delta_{ik}r_{,j} + \delta_{jk}r_{,i})$$

$$- 5r_{,i}r_{,j}r_{,k}\}3\frac{\partial r}{\partial n} + 3\nu(n_i r_{,j}r_{,k} + n_j r_{,k}r_{,i})$$

$$- (1-4\nu)n_k\delta_{ij} + (1-2\nu)(3n_k r_{,i}r_{,j} + n_j\delta_{ki} + n_i\delta_{jk})\Bigg]$$

In the above, $\mathbf{r} = \mathbf{y} - \boldsymbol{\xi}$ with $r = \|\mathbf{y} - \boldsymbol{\xi}\|$ the distance between a source point $\boldsymbol{\xi}$ and a field point $\mathbf{y}$, G and ν are the shear modulus and Poisson's ratio, respectively, δ_{ik} are the components of the Kronecker delta and, $k \equiv (\partial/\partial y_k)$. Also, the components of the (unit outward) normal $\mathbf{n}$, as well as the normal derivative $\partial r/\partial n$, are evaluated at the field point $\mathbf{y}$ and E_{ijmn} are components of the elasticity tensor for a homogeneous, isotropic elastic solid.

References

[1] Krishnasamy G, Rizzo FJ, Rudolphi TJ. Hypersingular boundary integral equations: their occurrence, interpretation, regularization and computation. In: Banerjee PK, Kobayashi S, editors. Developments in boundary element methods — 7. London: Elsevier Applied Science, 1992. p. 207–52.

[2] Tanaka M, Sladek V, Sladek J. Regularization techniques applied to boundary element methods. ASME Appl Mech Rev 1994;47:457–99.

[3] Paulino GH.,Novel formulations of the boundary element method for fracture mechanics and error estimation. PhD dissertation. Cornell University, Ithaca, NY, 1995.

[4] Chen JT, Hong HK. Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series. ASME Appl Mech Rev 1999;52:17–33.

[5] Guiggiani M, Krishnasamy G, Rudolphi TJ, Rizzo FJ. A general algorithm for the numerical solution of hypersingular boundary integral equations. ASME J Appl Mech 1992;59:604–14.

[6] Wilde AJ, Aliabadi MH. Direct evaluation of boundary stresses in the 3D BEM of elastostatics. Commun Numer Meth Engng 1998;14:505–17.

[7] Zhao ZY, Lan SR. Boundary stress calculation — a comparison study. Comput Struct 1999;71:77–85.

[8] Chati MK, Mukherjee S. Evaluation of gradients on the boundary using fully regularized hypersingular boundary integral equations. Acta Mech 1999;135:41–5.

[9] Krishnasamy G, Schmerr LW, Rudolphi TJ, Rizzo FJ. Hypersingular boundary integral equations: some applications in acoustic and elastic wave scattering. ASME J Appl Mech 1990;57:404–14.

[10] Cruse TA. Boundary element analysis in computational fracture mechanics. Dordrecht: Kluwer, 1988.

[11] Gray LJ, Martha LF, Ingraffea AR. Hypersingular integrals in boundary element fracture analysis. Int J Numer Meth Engng 1990;29:1135–58.

[12] Lutz ED, Ingraffea AR, Gray LJ. Use of simple solutions for boundary integral methods in elasticity and fracture analysis. Int J Numer Meth Engng 1992;35:1737–51.

[13] Gray LJ, Paulino GH. Crack tip interpolation revisited. SIAM J Appl Math 1998;58:428–55.

[14] Bonnet M. Regularized direct and indirect symmetric variational BIE formulations for three-dimensional elasticity. Engng Anal Bound Elem 1995;15:93–102.

[15] Gray LJ, Balakrishna C, Kane JH. Symmetric Galerkin fracture analysis. Engng Anal Bound Elem 1995;15:103–9.

[16] Gray LJ, Paulino GH. Symmetric Galerkin boundary integral formulation for interface and multizone problems. Int J Numer Meth Engng 1997;40:3085–101.

[17] Gray LJ, Paulino GH. Symmetric Galerkin boundary integral fracture analysis for plane orthotropic elasticity. Comput Mech 1997;20:26–33.

[18] Mukherjee S, Chati MK, Shi X. Evaluation of nearly singular integrals in boundary element contour and node methods for three-dimensional linear elasticity. Int J Solids Struct 2000;37:7633–54.

[19] Phan A-V, Mukherjee S, Mayer JRR. The hypersingular boundary contour method for two-dimensional linear elasticity. Acta Mech 1998;130:209–25.

[20] Mukherjee S, Mukherjee YX. The hypersingular boundary contour method for three-dimensional linear elasticity. ASME J Appl Mech 1998;65:300–9.

[21] Chati MK, Paulino GH, Mukherjee S. The meshless standard and hypersingular boundary node methods — applications to error estimation and adaptivity in three-dimensional problems. Int J Numer Meth Engng 2000 (in press).

[22] Paulino GH, Gray LJ, Zarkian V. Hypersingular residuals-a new approach for error estimation in the boundary element method. Int J Numer Meth Engng 1996;39:2005–29.

[23] Menon G. Hypersingular error estimates in boundary element methods. MS thesis. Cornell University, Ithaca, NY, 1996.

[24] Menon G, Paulino GH, Mukherjee S. Analysis of hypersingular residual error estimates in boundary element methods for potential problems. Comput Meth Appl Mech Engng 1999;173:449–73.

[25] Paulino GH, Gray LJ. Galerkin residuals for adaptive symmetric-Galerkin boundary element methods. ASCE J Engng Mech 1999;125:575–85.

[26] Rudolphi TJ. The use of simple solutions in the regularization of hypersingular boundary integral equations. Math Comput Modelling 1991;15:269–78.
[27] Cruse TA, Richardson JD. Non-singular Somigliana stress identities in elasticity. Int J Numer Meth Engng 1996;39:3273–304.
[28] Poon H, Mukherjee S, Ahmad MF. Use of simple solutions in regularizing hypersingular boundary integral equations in elastoplasticity. ASME J Appl Mech 1998;65:39–45.
[29] Mukherjee YX, Shah K, Mukherjee S. Thermoelastic fracture mechanics with regularized hypersingular boundary integral equations. Engng Anal Bound Elem 1999;23:89–96.
[30] Gray LJ, Manne LL. Hypersingular integrals at a corner. Engng Anal Bound Elem 1993;11:327–34.
[31] Gray LJ, Lutz ED. On the treatment of corners in the boundary element method. J Comp Appl Math 1990;32:369–86.
[32] Telles JCF, Prado AA. Hypersingular boundary formulation for 2-D potential problems. In: Aliabadi MH, Brebbia CA, editors. Advanced formulations in boundary element methods. Southampton, UK: Elsevier Applied Science/Computational Mechanics Publications, 1993. p. 191–213 (chap. 6).
[33] Chen JT, Hong HK. Dual boundary integral equations at a corner using contour approach around singularity. Adv Engng Software 1994;21:169–78.
[34] Guiggiani M. Hypersingular boundary integral equations have an additional free term. Comput Mech 1995;16:245–8.
[35] Mantič V, Paris F. Existence and evaluation of two free terms in the hypersingular boundary integral equation of potential theory. Engng Anal Bound Elem 1995;16:253–60.
[36] Hadamard J. Lectures on Cauchy's problem in linear partial differential equations. New Haven, CT: Yale University Press, 1923.
[37] Martin PA. End-point behavior of solutions to hypersingular integral equations. Proc R Soc London, A 1991;432:301–20.
[38] Toh K-C, Mukherjee S. Hypersingular and finite part integrals in the boundary element method. Int J Solids Struct 1994;31:2299–312.
[39] Hui CY, Mukherjee S. Evaluation of hypersingular integrals in the boundary element method by complex variable techniques. Int J Solids Struct 1997;34:203–21.
[40] Mukherjee S. CPV and HFP integrals and their applications in the boundary element method. Int J Solids Struct 2000;37:6623–34.
[41] Martin PA, Rizzo FJ, Cruse TA. Smoothness-relaxation strategies for singular and hypersingular integral equations. Int J Numer Meth Engng 1998;42:885–906.
[42] Mukherjee S, Mukherjee YX. Error analysis and adaptivity in three-dimensional linear elasticity by the usual and hypersingular boundary contour method. Int J Solids Struct 2000;38:161–78.
[43] Mukherjee S. On boundary integral equations for cracked and for thin bodies. Math Mech Solids 2000 (in press).
[44] Ghosh N, Rajiyah H, Ghosh S, Mukherjee S. A new boundary element method formulation for linear elasticity. ASME J Appl Mech 1986;53:69–76.
[45] Kane JH. Boundary element analysis in engineering continuum mechanics. Englewood Cliffs, NJ: Prentice Hall, 1994.
[46] Rizzo FJ. An integral equation approach to boundary value problems of classical elastostatics. Qly Appl Math 1967;25:83–95.
[47] Cruse TA. Numerical solutions in three-dimensional elastostatics. Int J Solids Struct 1969;5:1259–74.
[48] Glushkov E, Glushkova N, Lapina O. 3-D elastic stress analysis at polyhedral corner points. Int J Solids Struct 1999;36:1105–28.
[49] Zhang Q, Mukherjee S. Design sensitivity coefficients for linear elastic bodies with zones and corners by the derivative boundary element method. Int J Solids Struct 1991;27:983–98.
[50] Mukherjee YX, Mukherjee S, Shi X, Nagarajan A. The boundary contour method for three-dimensional linear elasticity with a new quadratic boundary element. Engng Anal Bound Elem 1997;20:35–44.