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Regularization of hypersingular boundary integral equations: a new approach for axisymmetric elasticity

Subrata Mukherjee*

Department of Theoretical and Applied Mechanics, Kimball Hall, Cornell University, Ithaca, NY 14853, USA

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Abstract

A hypersingular boundary integral equation (HBIE) formulation, for axisymmetric linear elasticity, has been recently presented by de Lacerda and Wrobel [Int. J. Numer. Meth. Engng 52 (2001) 1337]. The strongly singular and hypersingular equations in this formulation are regularized by de Lacerda and Wrobel by employing the singularity subtraction technique. The present paper revisits the same problem. The axisymmetric HBIE formulation for linear elasticity is interpreted here in a 'finite part' sense and is then regularized by employing a 'complete exclusion zone'. The resulting regularized equations are shown to be simpler than those by de Lacerda and Wrobel. © 2002 Elsevier Science Ltd. All rights reserved.

Keywords: Boundary element methods; Hypersingular integrals; Regularization; Linear elasticity

1. Introduction

1.1. Displacement boundary integral equation formulations for axisymmetric problems

The earliest boundary integral equation (BIE) formulations for axisymmetric linear elasticity are due to Kermanidis [2], Mayr [3] and Cruse et al. [4]; with extensions to axisymmetric linear elastic solids with general boundary conditions provided by Rizzo and Shippy [5] and Mayr et al. [6]. Elasto-plastic and elasto-viscoplastic axisymmetric problems, with small strains and rotations, have been considered by Cathie and Banerjee [7] and Sarihan and Mukherjee [8], respectively, while elasto-viscoplastic problems with large strains and rotations have been solved by Rajiyah and Mukherjee [9,10]. Axisymmetric contact problems have been solved by Abdul-Mihsein et al. [11], Graciani et al. [12] and de Lacerda and Wrobel [13]. Another important application area, axisymmetric problems in fracture mechanics, have been considered by, among others, Becker [14], Miyazaki et al. [15], Chen and Farris [16], Selvadurai [17] and Bush [18].

1.2. The hypersingular boundary element method

Hypersingular boundary integral equations (HBIEs) are derived from a differentiated version of the usual BIEs. HBIEs have diverse important applications and are the subject of considerable current research (see, for example, Refs. [19–22] for recent surveys of the field). HBIEs, for example, have been employed for the evaluation of boundary stresses [23–26], in wave scattering [27], in fracture mechanics [21,28–31], to obtain symmetric Galerkin boundary element formulations [32–35], to evaluate nearly singular integrals [36], to obtain the hypersingular boundary contour method [37,38], to obtain the hypersingular boundary node method [39], and for error analysis [39–43] and adaptivity [39].

Of particular interest to the present paper is the elegant approach of regularizing singular and hypersingular integrals using simple solutions, as first proposed by Rudolphi [44]. Several researchers have used this idea to regularize hypersingular integrals before collocating an HBIE at a regular boundary point [[30,45–47]. Finite parts (FPs)

* Fax: +1-607-255-2011.
E-mail address: sm85@cornell.edu (S. Mukherjee).

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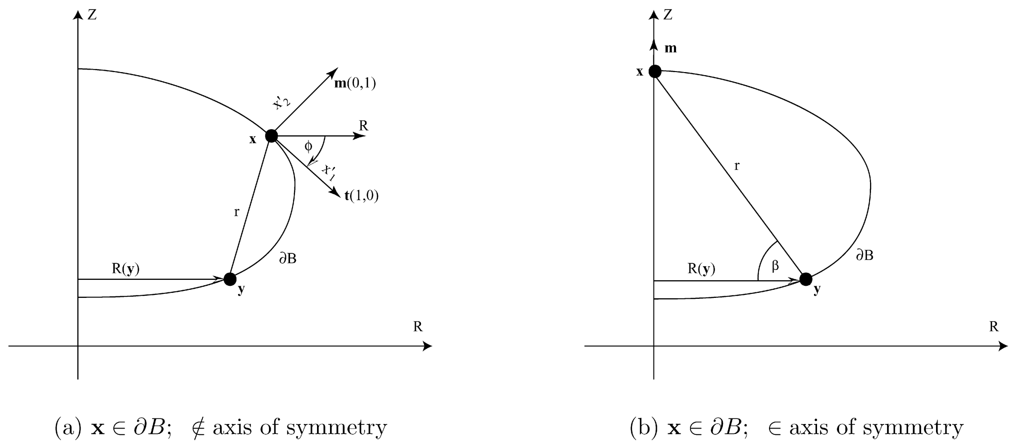

Fig. 1. Geometry of the axisymmetric problem. (a) $\mathbf{x} \in \partial B$; $\notin$ axis of symmetry. (b) $\mathbf{x} \in \partial B$; $\in$ axis of symmetry.

of hypersingular integrals in HBIEs is the subject of Mukherjee [48].

1.3. Hypersingular boundary integral equations for axisymmetric linear elasticity

An elegant treatment of HBIEs for axisymmetric linear elasticity has been recently presented by de Lacerda and Wrobel [1], with a follow-up paper on axisymmetric crack analysis by the same authors [49]. It is demonstrated in Ref. [1] that the cases of a boundary source point being located on and off the axis of symmetry, respectively, must be considered separately. When the source point is not on the axis of symmetry, the axisymmetric fundamental solutions are composed of a series of terms of which the most strongly singular ones are identical to those in the two-dimensional (2D) formulation; while if the source point is located on the axis of symmetry, the ring load becomes a point load, and similarities with three-dimensional (3D) linear elasticity solutions can be observed. The authors of Ref. [1] regularize all strong and hypersingular integrals in their HBIEs using the singularity subtraction technique, so that their final equations are, at most, weakly singular. Some numerical tests are performed to validate their approach.

It is observed that the regularization technique employed in Ref. [1] is analogous to the 'vanishing exclusion zone' approach in the context of Mukherjee [48]. The main contribution of the present paper is as follows. The above problem, i.e. the axisymmetric HBIE formulation for linear elasticity, is first interpreted in the 'finite part' sense of Ref. [48]. This is followed by a regularization technique employing the 'complete exclusion zone' [44,45,48]. It is shown that the final regularized equations obtained in the present paper are simpler than those obtained previously in Ref. [1]. Of course, the regularized equations in Ref. [1], and in the present paper, have to be equivalent.

1.4. Outline of the present paper

The present paper begins with a discussion of the problem when a source point does not lie on the axis of symmetry. This is followed by consideration of the case when a source point does lie on the axis of symmetry. A presentation of a solution strategy for the final regularized equations, for the above two cases, completes the paper.

An interesting topic of future research is to carry out a Galerkin formulation for this problem.

2. Boundary source point not on the axis of symmetry

2.1. Regularized hypersingular equation

Consider a linear elastic solid B with boundary ∂B. The axisymmetric elasticity equation for an internal point $\boldsymbol{\xi} \in B$; $\boldsymbol{\xi} \notin \partial B$ is (see Eq. (5) in Ref. [1])

$$\sigma_{ij}(\boldsymbol{\xi}) = \int_{\partial B} \alpha(\mathbf{y}) D_{ijk}(\boldsymbol{\xi}, \mathbf{y}) \tau_k(\mathbf{y}) \mathrm{d}S(\mathbf{y}) - \int_{\partial B} \alpha(\mathbf{y}) S_{ijk}(\boldsymbol{\xi}, \mathbf{y}) u_k(\mathbf{y}) \mathrm{d}S(\mathbf{y}) \qquad (1)$$

where $i, j, k = R, Z$; σ, τ and u are the stress, traction and displacement, respectively, and $\alpha(\mathbf{y}) = 2\pi R(\mathbf{y})$ (Fig. 1(a)). The kernels $\mathbf{D}$ and $\mathbf{S}$ are $O(1/r)$ and $O(1/r^2)$, respectively, as $r = \|\boldsymbol{\xi} - \mathbf{y}\| \rightarrow 0$. They are discussed in Ref. [1] as well as in Eqs. (3) and (4).

For a source point $\mathbf{x} \in \partial B$; $\notin$ axis of symmetry, one can write

$$\sigma_{ij}(\mathbf{x}) = \int\!\!\!\!\!\!= _{\partial B} \alpha(\mathbf{y}) D_{ijk}(\mathbf{x}, \mathbf{y}) \tau_k(\mathbf{y}) \mathrm{d}S(\mathbf{y}) - \int\!\!\!\!\!\!= _{\partial B} \alpha(\mathbf{y}) S_{ijk}(\mathbf{x}, \mathbf{y}) u_k(\mathbf{y}) \mathrm{d}S(\mathbf{y}) \qquad (2)$$

where the symbol $\fint$ denotes the finite part of the integral in the sense of Mukherjee [48]. Also [1]:

$$D_{ijk}(\mathbf{x},\mathbf{y}) = \frac{1}{\alpha(\mathbf{y})} D_{ijk}^{(2D)}(\mathbf{x},\mathbf{y}) + D_{ijk}^{(w)}(\mathbf{x},\mathbf{y}) \tag{3}$$

$$S_{ijk}(\mathbf{x},\mathbf{y}) = \frac{1}{\alpha(\mathbf{y})} S_{ijk}^{(2D)}(\mathbf{x},\mathbf{y}) + S_{ijk}^{(s)}(\mathbf{x},\mathbf{y}) + S_{ijk}^{(w)}(\mathbf{x},\mathbf{y}) \tag{4}$$

In Eqs. (3) and (4), the superscript 2D denotes the corresponding 2D fundamental solutions of linear elasticity ($\mathbf{D}^{(2D)}$ is $O(1/r)$ and $\mathbf{S}^{(2D)}$ is $O(1/r^2)$, respectively); the superscript (s) denotes a strongly singular kernel (i.e. $\mathbf{S}^{(s)}$ is $O(1/r)$); and, finally, the superscript (w) denotes kernels that are regular, or at most, weakly (logarithmic) singular. Explicit expressions for the hyper and strongly singular kernels are available in Ref. [1].

Following Ref. [48], Eq. (2) can be written as

$$\begin{aligned}\sigma_{ij}(\mathbf{x}) = &\int_{\partial B} \alpha(\mathbf{y}) D_{ijk}(\mathbf{x},\mathbf{y})[\sigma_{kp}(\mathbf{y}) - \sigma_{kp}(\mathbf{x})]n_p(\mathbf{y})\mathrm{d}S(\mathbf{y}) \\ &- \int_{\partial B} \alpha(\mathbf{y}) S_{ijk}(\mathbf{x},\mathbf{y})[u_k(\mathbf{y}) - u_k(\mathbf{x}) \\ &- u_{k,p}(\mathbf{x})(y_p - x_p)]\mathrm{d}S(\mathbf{y}) - A_{ijk}(\partial B)u_k(\mathbf{x}) \\ &+ C_{ijkp}(\partial B)u_{k,p}(\mathbf{x})\end{aligned} \tag{5}$$

where

$$\begin{aligned}A_{ijk}(\partial B) &= \fint_{\partial B} \alpha(\mathbf{y}) S_{ijk}(\mathbf{x},\mathbf{y})\mathrm{d}S(\mathbf{y}) \\ &= \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\partial B} \alpha(\mathbf{y}) S_{ijk}(\boldsymbol{\xi},\mathbf{y})\mathrm{d}S(\mathbf{y})\end{aligned} \tag{6}$$

$$\begin{aligned}C_{ijkp}(\partial B) = &\lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\partial B} E_{mlkp} \alpha(\mathbf{y}) D_{ijm}(\boldsymbol{\xi},\mathbf{y}) n_l(\mathbf{y})\mathrm{d}S(\mathbf{y}) \\ &- \lim_{\boldsymbol{\xi}\to\mathbf{x}} \int_{\partial B} \alpha(\mathbf{y}) S_{ijk}(\boldsymbol{\xi},\mathbf{y})(y_p - \xi_p)\mathrm{d}S(\mathbf{y})\end{aligned} \tag{7}$$

with $\mathbf{E}$ the elasticity tensor in Hooke's law:

$$\begin{aligned}&\sigma_{ml} = E_{mlkp} u_{k,p}, \\ &E_{mlkp} = \lambda \delta_{ml}\delta_{kp} + \mu(\delta_{mk}\delta_{lp} + \delta_{lk}\delta_{mp})\end{aligned} \tag{8}$$

Finally, $\mathbf{n}(\mathbf{y})$ is the unit outward normal to ∂B at a point $\mathbf{y}$ on it and λ and n are the usual lamé constants.

Simple (rigid body and linear) solutions in axisymmetric linear elasticity (see, for example, Refs. [30,45]) are now used in order to determine part of the quantity $\mathbf{A}$ and the entire quantity $\mathbf{C}$. Note that a rigid body displacement in the radial direction is not allowed since such a displacement would violate axisymmetry. Using the rigid body mode $u_z = c$ (c is an arbitrary constant) in Eq. (1), one has:

$$0 = \int_{\partial B} \alpha(\mathbf{y}) S_{ijZ}(\boldsymbol{\xi},\mathbf{y})\mathrm{d}S(\mathbf{y}) \tag{9}$$

Also, it is shown in Ref. [48] that:

$$0 = \int_{\partial B} S_{ijk}^{(2D)}(\boldsymbol{\xi},\mathbf{y})\mathrm{d}S(\mathbf{y}) \tag{10}$$

Using Eqs. (9) and (10) in Eq. (6), together with continuity, one has:

$$\begin{aligned}A_{ijk}(\partial B)u_k(\mathbf{x}) = &u_R(\mathbf{x}) \fint_{\partial B} \alpha(\mathbf{y}) S_{ijR}^{(s)}(\mathbf{x},\mathbf{y})\mathrm{d}S(\mathbf{y}) \\ &+ u_R(\mathbf{x}) \int_{\partial B} \alpha(\mathbf{y}) S_{ijR}^{(w)}(\mathbf{x},\mathbf{y})\mathrm{d}S(\mathbf{y})\end{aligned} \tag{11}$$

(R is the radial direction, no sum over R in Eq. (11)).

Now, using the linear solution

$$\begin{aligned}&u_k = (y_p - \xi_p)u_{k,p}(\boldsymbol{\xi}), \qquad u_{k,m}(\mathbf{y}) = u_{k,m}(\boldsymbol{\xi}), \\ &\tau_k(\mathbf{y}) = \sigma_{km}(\mathbf{y})n_m(\mathbf{y}) = E_{kmqs}u_{q,s}(\boldsymbol{\xi})n_m(\mathbf{y})\end{aligned} \tag{12}$$

in Eq. (1) yields

$$\begin{aligned}\sigma_{ij}(\boldsymbol{\xi}) = u_{k,p}(\boldsymbol{\xi}) &\int_{\partial B} \alpha(\mathbf{y})[E_{mlkp} D_{ijm}(\boldsymbol{\xi},\mathbf{y})n_l(\mathbf{y}) \\ &- S_{ijk}(\boldsymbol{\xi},\mathbf{y})(y_p - \xi_p)]\mathrm{d}S(\mathbf{y})\end{aligned} \tag{13}$$

Taking $\lim_{\boldsymbol{\xi}\to\mathbf{x}}$ of Eq. (13) and comparing with Eq. (7), one has:

$$\sigma_{ij}(\mathbf{x}) = C_{ijkp}(\partial B)u_{k,p}(\mathbf{x}) \tag{14}$$

Comparing Eq. (14) with Eq. (8) yields $\mathbf{C}(\partial B) = \mathbf{E}$.

Finally, Eq. (5) reduces to the simple regularized equation

$$\begin{aligned}0 = &\int_{\partial B} \alpha(\mathbf{y}) D_{ijk}(\mathbf{x},\mathbf{y})[\sigma_{kp}(\mathbf{y}) - \sigma_{kp}(\mathbf{x})]n_p(\mathbf{y})\mathrm{d}S(\mathbf{y}) \\ &- \int_{\partial B} \alpha(\mathbf{y}) S_{ijk}(\mathbf{x},\mathbf{y})[u_k(\mathbf{y}) - u_k(\mathbf{x}) \\ &- u_{k,p}(\mathbf{x})(y_p - x_p)]\mathrm{d}S(\mathbf{y}) - u_R(\mathbf{x}) \\ &\fint_{\partial B} \alpha(\mathbf{y}) S_{ijR}^{(s)}(\mathbf{x},\mathbf{y})\mathrm{d}S(\mathbf{y}) - u_R(\mathbf{x}) \int_{\partial B} \alpha(\mathbf{y}) S_{ijR}^{(w)}(\mathbf{x},\mathbf{y})\end{aligned} \tag{15}$$

(no sum over R in Eq. (15)).

Except for the third, all integrals in Eq. (15) are regular or, at most, logarithmic singular.

From Ref. [50], the strongly singular integral in Eq. (15) is defined as

$$\begin{aligned}&\fint_{\partial B} \alpha(\mathbf{y}) S_{ijR}^{(s)}(\mathbf{x},\mathbf{y})\mathrm{d}S(\mathbf{y}) \\ &= \int\!\!\!\!- _{\partial B} \alpha(\mathbf{y}) S_{ijR}^{(s)}(\mathbf{x},\mathbf{y})\mathrm{d}S(\mathbf{y}) \\ &\quad + \lim_{\epsilon\to 0} \int_{\partial B_\epsilon} \alpha(\mathbf{y}) S_{ijR}^{(s)}(\mathbf{x},\mathbf{y})\mathrm{d}S(\mathbf{y})\end{aligned} \tag{16}$$

where ∂B_ϵ is the boundary of an appropriate inclusion zone around the point $\mathbf{x}$ [1,50].

The second integral on the right hand side of Eq. (16) is evaluated in closed form in Eq. (29) in Ref. [1]. The results are

$$\lim_{\epsilon\to 0}\int_{\partial B_\epsilon}\alpha(\mathbf{y})S^{(s)}_{RRR}(\mathbf{x},\mathbf{y})\mathrm{d}S(\mathbf{y})$$

$$=\lim_{\epsilon\to 0}\int_{\partial B_\epsilon}\alpha(\mathbf{y})S^{(s)}_{ZZR}(\mathbf{x},\mathbf{y})\mathrm{d}S(\mathbf{y})=-\frac{G\nu}{2(1-\nu)R(\mathbf{x})} \quad (17)$$

(where G is the shear modulus of the material), while the rest of the integrals vanish.

The first integral on the right hand side of Eq. (16) is a Cauchy principal value (CPV) integral. The CPV integral

$$I(\mathbf{x})=\fint_{\partial B}\alpha(\mathbf{y})S^{(s)}_{ijk}(\mathbf{x},\mathbf{y})u_k(\mathbf{y})\mathrm{d}S(\mathbf{y}) \quad (18)$$

is also evaluated in Ref. [1] (see their Eq. (43), while simpler ones, containing only (some components) of the kernel **S**, need to be evaluated in the present case. It is also noted that, in the present formulation, the first (CPV) integral on the right hand side of Eq. (16) is the only strongly singular or hypersingular integral whose evaluation needs the specific form of the integrand.

2.2. Circumferential stress

Once the displacements and other stress components are known, the circumferential stress component $\sigma_{\theta\theta}$ can be easily obtained, as a post-processing step, from Hooke's law

$$\sigma_{\theta\theta}=(Eu_R)/R+\nu(\sigma_{RR}+\sigma_{ZZ}) \quad (19)$$

where E and ν are the Young's modulus and Poisson's ratio, respectively, of the linear elastic material.

3. Boundary source point on the axis of symmetry (Z-axis)

3.1. Regularized hypersingular equation

The nonzero displacement, strain, stress, source normal and traction components at the source point **x** in this case are:

$$u_Z(Z),\qquad \epsilon_{ZZ}=u_{Z,Z},\qquad \epsilon_{RR}=u_{R,R}$$

$$\sigma_{RR}=\lambda(u_{R,R}+u_{Z,Z})+2Gu_{R,R}$$

$$\sigma_{\theta\theta}=\lambda(u_{R,R}+u_{Z,Z})$$

$$\sigma_{ZZ}=\lambda(u_{R,R}+u_{Z,Z})+2Gu_{Z,Z}$$

$$m_R=0,\qquad m_Z=\pm 1$$

$$\tau_Z=\sigma_{ZZ}m_Z$$

The nonzero kernels are U_{ZZ}, T_{ZZ}, D_{RRR}, D_{RRZ}, D_{ZZR}, D_{ZZZ}, S_{RRR}, S_{RRZ}, S_{ZZR}, S_{ZZZ}.

This time, Eq. (2) has the same form as before, but $\alpha=2\pi r\cos\beta$ (Fig. 1(b)) and one gets nonzero stress components only for $i=j=R$, $i=j=Z$, i.e. for σ_{RR} and σ_{ZZ}. Also

$$D_{ZZk}(\mathbf{x},\mathbf{y})=D^{(3\mathrm{D})}_{ZZk}(\mathbf{x},\mathbf{y}) \quad (20)$$

$$S_{ZZk}(\mathbf{x},\mathbf{y})=S^{(3\mathrm{D})}_{ZZk}(\mathbf{x},\mathbf{y}) \quad (21)$$

where the superscript 3D denotes the corresponding 3D fundamental solutions of linear elasticity, with $\mathbf{D}^{(3\mathrm{D})}\sim \mathrm{O}(1/r^2)$, and $\mathbf{S}^{(3\mathrm{D})}\sim\mathrm{O}(1/r^3)$, as $r\to 0$. Explicit expressions for these kernels are available in Ref. [1].

Eqs. (5)–(9) remain the same as before. Again, only $i=j=R$, $i=j=Z$ need to be considered.

From Eqs. (6) and (9), one gets:

$$A_{ijk}(\partial B)u_k(\mathbf{x})=u_R(\mathbf{x})\fint_{\partial B}\alpha(\mathbf{y})S_{ijR}(\mathbf{x},\mathbf{y})\mathrm{d}S(\mathbf{y})=0, \quad (22)$$

since $u_R(\mathbf{x})=0$

Also, as before, imposition of linear modes in the R and Z directions yields:

$$\mathbf{C}(\partial B)=\mathbf{E} \quad (23)$$

Finally, the new version of Eq. (15), taking the inner product with the source normal $\mathbf{m}(\mathbf{x})$ (Fig. 1(b)) is:

$$0=m_j(\mathbf{x})\int_{\partial B}\alpha(\mathbf{y})D_{ijk}(\mathbf{x},\mathbf{y})[\sigma_{kp}(\mathbf{y})-\sigma_{kp}(\mathbf{x})]n_p(\mathbf{y})\mathrm{d}S(\mathbf{y})$$

$$-m_j(\mathbf{x})\int_{\partial B}\alpha(\mathbf{y})S_{ijk}(\mathbf{x},\mathbf{y})[u_k(\mathbf{y})-u_k(\mathbf{x})$$

$$-u_{k,p}(\mathbf{x})(y_p-x_p)]\mathrm{d}S(\mathbf{y})$$

Noting that $m_R(\mathbf{x})=0$, $D_{ijk}=0$ for $i\neq j$, Eq. (24) simplifies to:

$$0=\int_{\partial B}\alpha(\mathbf{y})D^{(3\mathrm{D})}_{ZZR}(\mathbf{x},\mathbf{y})[\tau_R(\mathbf{y})-\sigma_{RR}(\mathbf{x})n_R(\mathbf{y})]\mathrm{d}S(\mathbf{y})$$

$$+\int_{\partial B}\alpha(\mathbf{y})D^{(3\mathrm{D})}_{ZZZ}(\mathbf{x},\mathbf{y})[\tau_Z(\mathbf{y})-\sigma_{ZZ}(\mathbf{x})n_Z(\mathbf{y})]\mathrm{d}S(\mathbf{y})$$

$$-\int_{\partial B}\alpha(\mathbf{y})S^{(3\mathrm{D})}_{ZZR}(\mathbf{x},\mathbf{y})[u_R(\mathbf{y})-u_{R,R}(\mathbf{x})y_R]\mathrm{d}S(\mathbf{y})$$

$$-\int_{\partial B}\alpha(\mathbf{y})S^{(3\mathrm{D})}_{ZZZ}(\mathbf{x},\mathbf{y})[u_Z(\mathbf{y})-u_Z(\mathbf{x})$$

$$-u_{Z,Z}(\mathbf{x})(y_Z-x_Z)]\mathrm{d}S(\mathbf{y}) \quad (25)$$

Eq. (25) can be written in abbreviated form as follows

$$0=\int_{\partial B}\alpha(\mathbf{y})D^{(3\mathrm{D})}_{ZZp}(\mathbf{x},\mathbf{y})[\tau_p(\mathbf{y})-\hat{\tau}_p(\mathbf{x},\mathbf{y})]\mathrm{d}S(\mathbf{y})$$

$$-\int_{\partial B}\alpha(\mathbf{y})S^{(3\mathrm{D})}_{ZZp}(\mathbf{x},\mathbf{y})[u_p(\mathbf{y})-u_p(\mathbf{x})-u_{k,k}(\mathbf{x})$$

$$\times(y_p-x_p)]\mathrm{d}S(\mathbf{y}) \quad (26)$$

with

$p = R$ and $p = Z$ (sum over p);
$k = p$, no sum over k;
$\hat{\tau}_R = \sigma_{RR}(\mathbf{x})n_R(\mathbf{y})$, $\hat{\tau}_Z = \sigma_{ZZ}(\mathbf{x})n_Z(\mathbf{y})$
$u_R(\mathbf{x}) = 0$, $x_R = 0$.

Note that the integrals in Eq. (26) are regular since:

$$D_{ZZp}^{(3D)} \sim O(1/r^2), \qquad [\tau_p(\mathbf{y}) - \hat{\tau}_p] \sim O(r),$$

$$\alpha(\mathbf{y}) = 2\pi r \cos\beta \sim O(r)$$

$$S_{ZZp}^{(3D)} \sim O(1/r^3),$$

$$[u_p(\mathbf{y}) - u_p(\mathbf{x}) - u_{k,k}(\mathbf{x})(y_p - x_p)] \sim O(r^2), \quad \alpha(\mathbf{y}) \sim O(r)$$

3.2. Circumferential stress

This time, the circumferential stress is given by:

$$\sigma_{\theta\theta} = \nu(\sigma_{RR} + \sigma_{ZZ}) \tag{27}$$

4. Solution strategy for Eqs. (15) and (26)

Eq. (15) (for $\mathbf{x} \notin$ axis of symmetry) and Eq. (26) (for $\mathbf{x} \in$ axis of symmetry) both require $\nabla\mathbf{u}$ at $\mathbf{x} \in \partial B$. A strategy for obtaining this quantity is described below. Please see Refs. [30,51] for a full discussion of this approach for 3D linear elasticity problems.

4.1. Boundary source point not on the axis of symmetry

Local coordinates, with the x'_1 and x'_2 axes along the tangential and normal directions to ∂B at $\mathbf{x}$, are shown in Fig. 1(a). The components of grad $\mathbf{u}$, in these local coordinates, are obtained as follows.

Tangential derivatives. The tangential derivatives, $\partial u'_i/\partial x'_1$, $i = 1, 2$, are obtained by differentiating the interpolation functions for $\mathbf{u}'$ along the x_1 direction.

Normal derivatives. Using Hooke's law, the normal derivatives $\partial u'_i/\partial x'_2$, $i = 1, 2$, are:

$$\frac{\partial u'_1}{\partial x'_2} = \frac{\tau'_1}{G} - \frac{\partial u'_2}{\partial x'_1} \tag{28}$$

$$\frac{\partial u'_2}{\partial x'_2} = \frac{(1-2\nu)\tau'_2}{2G(1-\nu)} - \left(\frac{\nu}{1-\nu}\right)\left(\frac{\partial u'_1}{\partial x'_1} + \frac{u_R}{R}\right) \tag{29}$$

The components of grad $\mathbf{u}$ in global coordinates are obtained from those in local coordinates by employing the usual tensor transformation laws. Stress components are obtained from the displacement gradients from Hooke's law.

4.2. Boundary source point on the axis of symmetry

This time (Fig. 1(b)) $m_R = 0$, $m_Z = \pm 1$. The nonzero components of the displacement gradient at $\mathbf{x}$ are the tangential derivative $\partial u_R/\partial R$ and the normal derivative $\partial u_Z/\partial Z$. The first quantity is obtained as before by differentiating the interpolation function for u_R with respect to R. The other derivative is obtained from the equation:

$$\frac{\partial u_Z}{\partial Z} = \frac{(1-2\nu)\tau_Z}{2G(1-\nu)m_Z} - \left(\frac{\nu}{1-\nu}\right)\frac{\partial u_R}{\partial R} \tag{30}$$

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References

[1] de Lacerda LA, Wrobel LC. Hypersingular boundary integral equation for axisymmetric elasticity. Int J Numer Meth Engng 2001;52:1337–54.
[2] Kermanidis T. A numerical solution for axially symmetrical elasticity problems. Int J Solids Struct 1975;11:493–500.
[3] Mayr M. The numerical solution of axisymmetric elasticity problems using an integral equation approach. Mech Res Commun 1976;3: 393–8.
[4] Cruse T, Snow DW, Wilson RB. Numerical solutions in axisymmetric elasticity. Comp Struct 1977;7:445–51.
[5] Rizzo FJ, Shippy DJ. A boundary-integral approach to potential and elasticity problems for axisymmetric bodies with arbitrary boundary conditions. Mech Res Commun 1979;6:99–103.
[6] Mayr M, Drexler W, Kuhn G. A semianalytical boundary integral approach for axisymmetric elastic bodies with arbitrary boundary conditions. Int J Solids Struct 1980;16:863–71.
[7] Cathie DN, Banerjee PK. Numerical solutions in axisymmetric elastoplasticity by the boundary element method. Proceedings of the Second International Symposium on Innovative Numerical Analysis in Applied Engineering Sciences, University Press of Virginia, Charlottesvillee, VA; 1980. p. 331–9.
[8] Sarihan V, Mukherjee S. Axisymmetric viscoplastic deformation by the boundary element method. Int J Solids Struct 1982;18:1113–28.
[9] Rajiyah R, Mukherjee S. Boundary element analysis of inelastic axisymmetric problems with large strains and rotations. Int J Solids Struct 1987;23:1679–98.
[10] Rajiyah R, Mukherjee S. A note on the efficiency of the boundary element method for inelastic axisymmetric problems with large strains. ASME J Appl Mech 1989;56:721–4.
[11] Abdul-Mihsein MJ, Bakr AA, Parker AP. A boundary integral equation method for axisymmetric elastic contact problems. Comp Struct 1986;23:787–93.
[12] Graciani E, Mantic V, Paris F. BEM solution of axi-symmetric contact problems by weak application of contact conditions with non-conforming discretizations. Proceedings of the Boundary Element Techniques, Queen Mary and Westfield College, London; 1999.
[13] de Lacerda LA, Wrobel LC. Frictional contact analysis of coated axisymmetric bodies using the boundary element method. J Strain Anal Engng Des 2000;35:423–40.

[14] Becker AA. The boundary element method in engineering—a complete course. London: McGraw Hill; 1982.
[15] Miyazaki N, Ikeda T, Munakata T. Analysis of stress intensity factor using the energy method combined with the boundary element method. Comp Struct 1989;33:867–71.
[16] Chen SY, Farris TN. Boundary element crack closure calculation of axisymmetric stress intensity factors. Comp Struct 1994;50:491–7.
[17] Selvadurai APS. The modelling of axisymmetric basal crack evolution in a borehole indentation problem. Engng Anal Bound Elem 1998;21:377–83.
[18] Bush MB. Simulation of contact-induced fracture. Engng Anal Bound Elem 1999;23:59–66.
[19] Krishnasamy G, Rizzo FJ, Rudolphi TJ. In: Banerjee PK, Kobayashi S, editors. Hypersingular boundary integral equations: their occurrence, interpretation, regularization and computation. Developments in boundary element methods, vol. 7. London: Elsevier; 1992. p. 207–52.
[20] Tanaka M, Sladek V, Sladek J. Regularization techniques applied to boundary element methods. ASME Appl Mech Rev 1994;47:457–99.
[21] Paulino GH. Novel formulations of the boundary element method for fracture mechanics and error estimation. PhD Dissertation. Cornell University, Ithaca, NY; 1995.
[22] Chen JT, Hong HK. Review of dual boundary element methods with emphasis on hypersingular integrals and divergent series. ASME Appl Mech Rev 1999;52:17–33.
[23] Guiggiani M. Hypersingular formulation for boundary stress evaluation. Engng Anal Bound Elem 1994;13:169–79.
[24] Wilde AJ, Aliabadi MH. Direct evaluation of boundary stresses in the 3D BEM of elastostatics. Commun Numer Meth Engng 1998;14:505–17.
[25] Zhao ZY, Lan SR. Boundary stress calculation—a comparison study. Comput Struct 1999;71:77–85.
[26] Chati MK, Mukherjee S. Evaluation of gradients on the boundary using fully regularized hypersingular boundary integral equations. Acta Mech 1999;135:41–5.
[27] Krishnasamy G, Schmerr LW, Rudolphi TJ, Rizzo FJ. Hypersingular boundary integral equations: some applications in acoustic and elastic wave scattering. ASME J Appl Mech 1990;57:404–14.
[28] Cruse TA. Boundary element analysis in computational fracture mechanics. Dordrecht: Kluwer; 1988.
[29] Gray LJ, Martha LF, Ingraffea AR. Hypersingular integrals in boundary element fracture analysis. Int J Numer Meth Engng 1990;29:1135–58.
[30] Lutz ED, Ingraffea AR, Gray LJ. Use of simple solutions for boundary integral methods in elasticity and fracture analysis. Int J Numer Meth Engng 1992;35:1737–51.
[31] Gray LJ, Paulino GH. Crack tip interpolation revisited. SIAM J Appl Math 1998;58:428–55.
[32] Bonnet M. Regularized direct and indirect symmetric variational BIE formulations for three-dimensional elasticity. Engng Anal Bound Elem 1995;15:93–102.
[33] Gray LJ, Balakrishna C, Kane JH. Symmetric Galerkin fracture analysis. Engng Anal Bound Elem 1995;15:103–9.
[34] Gray LJ, Paulino GH. Symmetric Galerkin boundary integral formulation for interface and multizone problems. Int J Numer Meth Engng 1997;40:3085–101.
[35] Gray LJ, Paulino GH. Symmetric Galerkin boundary integral fracture analysis for plane orthotropic elasticity. Comput Mech 1997;20:26–33.
[36] Mukherjee S, Chati MK, Shi X. Evaluation of nearly singular integrals in boundary element contour and node methods for three-dimensional linear elasticity. Int J Solids Struct 2000;37:7633–54.
[37] Phan A-V, Mukherjee S, Mayer JRR. The hypersingular boundary contour method for two-dimensional linear elasticity. Acta Mech 1998;130:209–25.
[38] Mukherjee S, Mukherjee YX. The hypersingular boundary contour method for three-dimensional linear elasticity. ASME J Appl Mech 1998;65:300–9.
[39] Chati MK, Paulino GH, Mukherjee S. The meshless standard and hypersingular boundary node methods—applications to error estimation and adaptivity in three-dimensional problems. Int J Numer Meth Engng 2001;50:2233–69.
[40] Paulino GH, Gray LJ, Zarkian V. Hypersingular residuals—a new approach for error estimation in the boundary element method. Int J Numer Meth Engng 1996;39:2005–29.
[41] Menon G. Hypersingular error estimates in boundary element methods. MS Thesis. Cornell University, Ithaca, NY; 1996.
[42] Menon G, Paulino GH, Mukherjee S. Analysis of hypersingular residual error estimates in boundary element methods for potential problems. Comput Meth Appl Mech Engng 1999;173:449–73.
[43] Paulino GH, Gray LJ. Galerkin residuals for adaptive symmetric-Galerkin boundary element methods. ASCE J Engng Mech 1999;125:575–85.
[44] Rudolphi TJ. The use of simple solutions in the regularization of hypersingular boundary integral equations. Math Comput Modelling 1991;15:269–78.
[45] Cruse TA, Richardson JD. Non-singular Somigliana stress identities in elasticity. Int J Numer Meth Engng 1996;39:3273–304.
[46] Poon H, Mukherjee S, Ahmad MF. Use of simple solutions in regularizing hypersingular boundary integral equations in elastoplasticity. ASME J Appl Mech 1998;65:39–45.
[47] Mukherjee YX, Shah K, Mukherjee S. Thermoelastic fracture mechanics with regularized hypersingular boundary integral equations. Engng Anal Bound Elem 1999;23:89–96.
[48] Mukherjee S. Finite parts of singular and hypersingular integrals with irregular boundary source points. Engng Anal Bound Elem 2000;24:767–76.
[49] de Lacerda LA, Wrobel LC. Dual boundary element method for axisymmetric crack analysis. Int. J. Fract 2002;113:267–84.
[50] Mukherjee S. CPV and HFP integrals and their applications in the boundary element method. Int J Solids Struct 2000;37:6623–34.
[51] Chati MK, Paulino GH, Mukherjee S. The meshless hypersingular boundary node method for three-dimensional potential theory and linear elasticity problems. Engng Anal Bound Elem 2001;25:639–53.