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A generalized Helmholtz equation fundamental solution using a conformal mapping and dependent variable transformation

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Abstract

Fundamental solutions to a generalized Helmholtz equation are determined through dependent variable transforms using the material properties and independent variable transforms based on conformal mapping. This allows variable wave speed media to be examined under some fairly broad material property constraints.

Keywords: Helmholtz equation; Heterogeneous media; Conformal mapping

1. Introduction

The importance of finding fundamental solutions for wave motion problems in heterogeneous media has been stressed elsewhere, e.g. Refs. [1,2]. (Although the descriptors "nonhomogeneous" or "inhomogeneous" are often used in place of "heterogeneous", these also have other meanings in the solution of differential equations and will therefore not be used in this sense here). This is especially true for seismic wave propagation in the earth's upper layers (near-field effects), given the high degree of complexity in the geological structure of the ground where heterogeneity is only one of many complications, e.g. Refs. [3,4]. Of major importance in this context is the scalar wave equation with a depth-dependent wavenumber, e.g. Ref. [5], because it corresponds to (i) sound waves where the acoustic medium density variation over the wavelength is important, (ii) electromagnetic waves where the electric field is polarized and (iii) horizontally (SH) and, under certain restrictions, vertically (SV) polarized elastic shear waves in a continuously heterogeneous medium. Although many wave propagation phenomena can be explained through recourse to SH wave models, it is still necessary to study the vector wave equation and, in practical terms, wave motions under two-dimensional conditions as a first step given the difficulties associated with modeling fully three-dimensional configurations. This is especially true for media where dependence of their material parameters on position is arbitrary; although the wave field can be represented in terms of a position-dependent amplitude and phase angle, it is no longer possible to uniquely divide it into the sum of incident plus reflected waves due to continuous scattering of the signal by the inhomogeneities. Thus, many of the solution techniques developed for homogeneous materials, e.g. Ref. [6], such as vector wave decomposition and the use of potentials, are no longer directly applicable. The relatively few methods applicable to heterogeneous media can be broadly classified as follows [4]: (i) asymptotic methods, which cover a wide range of inhomogeneity and anisotropy but are restricted to high frequencies and to isolated wavefronts; (ii) mode expansions, which are effective when the wavetrain is attributed to a small number of interfering modes; and (iii) generalized ray expansions (e.g. double transform methods, Haskel–Thomson matrix formalism, Cagniard–de Hoop inversion) which are effective for wavetrains described in terms of interference patterns generated by large numbers of rays. The latter two groups of methods are usually restricted to horizontal layering and heterogeneity in the vertical direction.

Of major interest to seismology and earthquake engineering is wave amplification in multi-layered geological media. Solutions for the one-dimensional material representation of this problem were provided by Thomson [7] and Haskel [8,9] using transfer matrices that relate forces and displacements between upper and lower interfaces of an individual layer. The complete solution from bedrock to surface is therefore obtained in a step-by-step approach by considering

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all intermediate layers. In more detail, Ref. [10], the dynamic equilibrium equations of an individual homogeneous layer are solved through decoupling of the wavefield into distortional and dilatational components, followed by an additional decomposition into symmetric and antisymmetric deformation patterns. The final solution is achieved in the form of 2×2 transfer matrices by establishing equilibrium of forces and compatibility of displacements across both upper and lower boundaries of a layer. Furthermore, viscoelastic material behavior can be captured by using modified, frequency dependent material parameters [11]. More recently, Kausel and Roesset [12] introduced a finite element type approach by synthesizing the individual transfer matrices so as to obtain a dynamic stiffness matrix representing the entire layered structure. The practical advantage of this route has provided an impetus for further extension to wave amplification in two-dimensional deposits [13]. Various aspects of wave motion through unidimensional layered media are being continuously explored, e.g. by Chen et al. [14] on regularization of the divergent series which occur in ground motion deconvolution analyses, by Bonnet and Heitz [15] on nonlinear seismic response of soft layers filled with nonlinear materials using the perturbation method, by Pires [16] on the nonlinear stress–strain behavior of layered soil deposits subjected to random seismic loads using equivalent linearization and by Safak [17] on discrete time analysis of seismic site amplification using upgoing and downgoing waves as auxiliary variables and representing each layer by the three basic parameters. Finally, a comparison study on the various transfer matrix methods for wave amplification through elastic layers can be found in Ref. [18].

Also of interest to seismology is the generation of synthetic signals in layered media due to various point sources. For instance, a computationally stable solution for determining surface displacements due to buried dislocation sources in a multi-layered elastic medium was developed by Wang and Herrmann [19] based on Haskell's [8,9] work, who employed the Fourier transformation and evaluated signal time histories for the elastic medium by performing contour integration of Bessel functions in the complex wavenumber plane. In general, solution procedures for wave motions due to buried sources follow along two lines, namely Laplace transform (or Cagniard–de Hoop technique, e.g. Ref. [20]) and the Fourier transform, e.g. Ref. [21]. The former is also known as the generalized ray method because the solution is constructed by tracking the individual seismic signal arrivals, ray by ray from source to receiver. It is valid at high frequencies but not well suited for cases with many layers and large source to receiver distances. In the latter technique, the complete wave solution is expressed in terms of double integral transformations over wavenumber and frequency. The method can handle a large number of plane layers, but requires considerable computational effort at high frequencies. It is also possible to introduce numerical techniques for carrying out the contour integrations, e.g. Ref. [22] or in evaluating the resulting analytical expressions, e.g. Ref. [23].

With the exception of scalar waves, relatively little work is available for wave motions in continuously heterogeneous solid media. Of course, approximate solutions can always be generated, e.g. by decomposing the heterogeneous medium into a stack of laterally varying layers and representing the solution within a layer as a sum of decoupled plane waves, e.g. Ref. [24]. A very general numerical technique for investigating seismic wave propagation in anisotropic and heterogeneous materials is presented in Refs. [25,26]. In particular, different families of algorithms are presented based on a combination of finite integral transforms with finite difference techniques for the computation of complete seismograms in complex, three-dimensional subsurface geometries. Of interest here are (i) the heterogeneous isotropic three-dimensional medium in which the elastic parameters and the density are functions of depth [25], and (ii) SH wave propagation in a heterogeneous medium where the wavespeed is a function of two spatial variables. In the former case, the equations of motion are described in cylindrical coordinates and the methodology used is a double Hankel integral transformation with respect to the two spatial variables, followed by a finite difference solution and inverse transformations. The latter case combines a finite Fourier integral transformation in one spatial coordinate with a finite difference scheme in the other coordinate. All spatial derivatives encountered in the finite difference method are approximated by Fourier series, e.g. Ref. [27] for better accuracy.

As far as analytical techniques are concerned, we mention Acharya's [28] method for determining the wavefield due to a point source in a heterogeneous medium satisfying certain conditions (such as constant density or Poisson's ratio of 0.25 or a linear wavespeed gradient) which allow for independent P and S equations of motion. By assuming cylindrical symmetry and representing the pulse from the point source as a superposition of harmonic waves, the total field is obtained by summing over all plane waves with a given set of direction cosines and then integrating the sum over all values of the direction cosines. Thus, integral expressions are obtained for the compressional and shear potentials that are convergent. Other analytical work addressing wave motions in heterogeneous media is by Hook [29] on the method of separation of variables in order to recast the vector wave equation with position-dependent material parameters into a system of three linearly independent solutions for the corresponding number of scalar potentials, which in turn satisfy second order wave equations. This can always be achieved for SH waves, while formulations for SV and P wave are possible only for certain functional forms of the material properties, such as power laws for the shear modulus and the density and a fixed value of Poisson's ratio. Furthermore, the wave equations for the latter two potentials are coupled, implying that P and SV waves are no longer purely dilatational and rotational. It therefore becomes necessary to impose further constraints on the material parameters in order to achieve two uncoupled P and SV wave equations. This approach was generalized in a later publication [30] through the introduction of a linear transformation for the displacement vector in order to produce a diagonal system matrix for the vector wave equation which was reformulated using matrix notation. The method of separation of variables for heterogeneous media was then implemented for the axisymmetric case (with plane strain being a special case) and the resulting constraints which dictate mathematically acceptable material parameter variations with respect to a single spatial coordinate appear in the form of a nonlinear, ordinary differential equation. Other work along the lines of separation of the displacement vector in a heterogeneous medium into P and S wave potentials is by Gupta [31], who obtained reflection coefficients for a layer (where the elastic parameters are quadratic functions of the depth coordinate and Poisson's ratio is equal to 0.25) sandwiched between two elastic, homogeneous halfspaces. Furthermore, Payton [32] solved the unidimensional wave equation for a pulse traveling in a composite rod exhibiting a constant wavespeed in one part and a quadratically varying one in the other part by using the Laplace transform technique. Finally, a general technique for solving the vector wave equation in an arbitrary heterogeneous medium is by Karal and Keller [33], who introduced an expansion of the solution in terms of asymptotic series. This technique does not require separability in the sense previously discussed, but the calculations required in order to obtain successive terms of the series are very difficult.

2. Formulation

Emphasis here will be placed on analytical fundamental (point source) solutions for scalar time harmonic wave problems to form, among other uses, a "ground truth" for existing numerical schemes. While the usual Helmholtz equation with a position dependent wavenumber suffices for many time harmonic wave problems, some heterogeneous media require a modification to be made, e.g. for underwater acoustics with heterogeneous density and compressibility properties [34]. Consider then the two-dimensional (since conformal mapping will eventually be used) "generalized" heterogeneous Helmholtz equation

$$\nabla\cdot\{K(x,y)\nabla U(x,y)\} + N(x,y)U(x,y) = -Q(x,y) \tag{1}$$

This may be solved by several methods for several types of heterogeneity, e.g. $K = 1/\rho(x,y)$ and $N = k^2(x,y)/\rho(x,y)$ as in Ref. [35]. Note that N is often an inertial term, i.e. a material property multiplied by the frequency squared, $N(x,y) = \tilde{N}(x,y)\omega^2$ for an $\exp(-\mathrm{i}\omega t)$ behavior. If the spatial coordinates are modified to $(\tilde{x},\tilde{y}) = (\omega x, \omega y)$ and the forcing function to $\tilde{Q} = Q/w^2$, the form of Eq. (1) will be unchanged in this modified coordinate system. Since Q will eventually be taken as a delta function in (x,y) which is ω^2 times the delta function in $(\tilde{x},\tilde{y})$, the analogy is complete. Thus the following results may be interpreted directly for a general Helmholtz equation or as modified for a specifically time harmonic form. While this modification of the spatial coordinates is not nondimensional as would be the case if a length scale were used, there are in general several different length scales in these heterogeneous media problems, i.e. the wavelength for time harmonic problems and at least one material length scale which are distinct; at least the frequency is a material independent "fixed" scaling.

The methods to be discussed are primarily "inverse" methods since they require a specific form of the material heterogeneity to "fit" a solution, but there will be a number of degrees of freedom in these material property forms such that this may not be as limiting a constraint as might be thought. This is especially true for those problems where knowledge of the material properties is limited to a few data points, although physical insight may provide some further restrictions on the model used.

3. Transformation of dependent variable

It can be shown, e.g. Ref. [35] for an illustration of this, that a transformation and constraint

$$V(x,y) = K^{1/2}U(x,y) \tag{2a}$$

$$-K^{-1/2}\nabla^2(K^{1/2}) + N/K = \beta^2(x,y) \tag{2b}$$

respectively, lead to a standard form for the Helmholtz equation

$$\nabla^2 V(x,y) + \beta^2(x,y)V(x,y) = -K^{-1/2}(x,y)Q(x,y) \tag{3}$$

For the particular case of $Q(x,y)$ as a delta function, $\delta(x,y;x_0,y_0)$, and β as a constant, β_0, this yields a well-known fundamental solution, i.e. Green's function, for the original problem for a wide variety of material properties where the function V is seen to have a constant wave speed. If a further constraint is introduced, not necessary but convenient

$$N(x,y) = A(x,y)K^{1/2}(x,y) + B(x,y)K(x,y) \tag{4}$$

the equation governing possible forms for K, Eq. (2b), simplifies to

$$\nabla^2(K^{1/2}) + \{\beta^2(x,y) - B(x,y)\}K^{1/2} = A(x,y) \tag{5}$$

Since A and B are arbitrary, K and N may be quite general. When β and B are taken as constants, β_0 and B_0, U (now referred to as the Green's function, G) is given by Eq. (3), with $f(x)\delta(x-x_0) = f(x_0)\delta(x-x_0)$.

$$\begin{aligned}&\nabla^2_{x,y}\{K^{1/2}(x,y)K^{1/2}(x_0,y_0)G(x,y;x_0,y_0)\}\\&\quad + \beta_0^2\{K^{1/2}(x,y)K^{1/2}(x_0,y_0)G(x,y;x_0,y_0)\}\\&\quad = -\delta(x-x_0,y-y_0)\end{aligned} \tag{6}$$

and leading directly to the fundamental solution

$$G = K^{-1/2}(x,y)K^{-1/2}(x_0,y_0)(\mathrm{i}/4)H_0^{(1)}(\beta_0[(x-x_0)^2 + (y-y_0)^2]^{1/2} \tag{7}$$

Ref. [2] gave some one-dimensional material solutions but two-dimensional problems are also relatively straightforward. A simple, but not complete, choice for $A(x,y)$ in two-dimensional could be

$$A(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y + \sum_{n=1}^{n=M_1} A_n \exp(\gamma_n x) + \sum_{n=1}^{n=M_2} B_n \exp(\epsilon_n y) \tag{8}$$

which leads to solutions for the material parameter $K(x,y)$ as

$$K^{1/2} = c_1 \exp(\mu_0 x) + c_2 \exp(-\mu_0 x) + c_3 \exp(\nu_0 y) + c_4 \exp(-\nu_0 y) + (\alpha_0 + \alpha_1 x + \alpha_2 y)/(\beta_0^2 - B_0) + \sum_{n=1}^{n=M_1} A_n \exp(\gamma_n x)/[\gamma_n^2 + \beta_0^2 - B_0] + \sum_{n=1}^{n=M_2} B_n \exp(\epsilon_n y)/[\epsilon_n^2 + \beta_0^2 - B_0] \tag{9}$$

with $\mu_0^2 + \nu_0^2 = (B_0 - \beta_0^2)$. For example, for $A(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y$, i.e. $c_1 = c_2 = c_3 = c_4 = A_n = B_n = 0$, the material behavior K, N and the slowness, $s = (N/K)^{1/2}$ are

$$K(x,y) = (\alpha_0 + \alpha_1 x + \alpha_2 y)^2/(\beta_0^2 - B_0)^2 \tag{10a}$$

$$N(x,y) = \beta_0^2(\alpha_0 + \alpha_1 x + \alpha_2 y)^2/(\beta_0^2 - B_0)^2 \tag{10b}$$

$$s(x,y) = \beta_0 \tag{10c}$$

and the corresponding Green's function would be Eq. (7), i.e.

$$G(x,y;x_0,y_0) = \{(\alpha_0 + \alpha_1 x_0 + \alpha_2 y_0)^{-1}((\alpha_0 + \alpha_1 x + \alpha_2 y)/\nu_0^4)^{-1}\}(\mathrm{i}/4)H_0^{(1)}[\beta_0\{(x-x_0)^2 + (y-y_0)^2\}^{1/2}] \tag{10d}$$

Note that if the coordinates have been stretched by the frequency as mentioned above, the slowness is actually a wavenumber. While this solution has a constant slowness everywhere, the addition of even one more term, e.g. use $A(x,y) = \alpha_0 + \alpha_1 x + \alpha_2 y + B_1 \exp(\epsilon_1 y)$, i.e. with $B_1 \neq 0$, will lead to a variable local slowness, e.g.

$$s = \{[\beta_0^2 + B_0 P]/[1+P]\}^{1/2} \tag{11a}$$

$$P = B_1(\beta_0^2 - B_0)\exp(\epsilon_1 y)/[(\epsilon_1^2 + \beta_0^2 - B_0) \times (\alpha_0 + \alpha_1 x + \alpha_2 y)] \tag{11b}$$

with G given by

$$G(x,y;x_0,y_0) = \{[\alpha_0 + \alpha_1 x_0 + \alpha_2 y_0 + B_1 \exp(\epsilon_1 y_0)^{-1}) \times (\alpha_0 + \alpha_1 x + \alpha_2 y + B_1 \exp(\epsilon_1 y)/\nu_0^4)^{-1}\} \times (\mathrm{i}/4)H_0^{(1)}[\beta_0\{(x-x_0)^2 + (y-y_0)^2\}^{1/2}] \tag{11c}$$

The slowness here varies with position but approaches a constant far field value for $\epsilon < 0$.

It is clear that any number of coefficients may be used to model the material properties; furthermore, this approach may be extended to three-dimensional material variations. For example, a simple three-dimensional form could be $K^{1/2}(x,y,z) = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z$ which clearly satisfies Eq. (5), but since the emphasis here is on two-dimensional problems, this point will be left to a later paper.

4. Conformal mapping (independent variable transformation)

Consider a coordinate transformation based on a conformal mapping

$$Z = X + \mathrm{i}Y = f(z) = f(x + \mathrm{i}y) \tag{12}$$

which allows the Laplacian operator in (x,y) to be rewritten as

$$\nabla^2_{x,y}U(x,y) = J(X,Y;x,y)\nabla^2_{X,Y}U(X,Y) \tag{13}$$

where J is the Jacobian of this transformation or "mapping", $|\mathrm{d}Z/\mathrm{d}z|^2 = \partial(X,Y)/\partial(x,y)$. For the case of constant $K = K_0$ and variable N, this maps Eq. (1), with a point source forcing function, to

$$JK_0\nabla^2 G(X,Y;X_0,Y_0) + N(x,y)G(X,Y;X_0,Y_0) = -J\delta(X,Y;X_0,Y_0) \tag{14}$$

where the mapped delta function is still a point source modified by a Jacobian, as discussed in Appendix A. If $N(x,y)$ is $k_0^2 K_0 J(x,y)$, the problem has been reduced to a standard Helmholtz form for which the fundamental Green's function is known.

$$\nabla^2 G(X,Y;X_0,Y_0) + k_0^2 G(X,Y;X_0,Y_0) = -\delta(X,Y;X_0,Y_0) \tag{15}$$

but in mapped coordinates, (X,Y). While there are an infinite number of possible mappings, many will not turn out to be of practical use in engineering problems. Nevertheless, several "new" solutions have be determined, e.g.

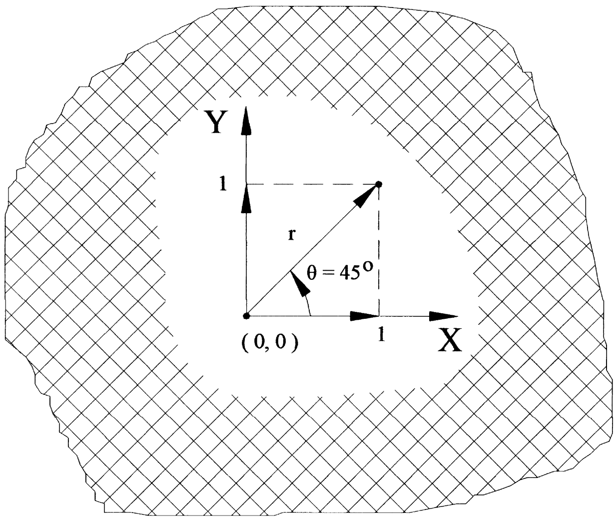


Fig. 1. Geometry of the source versus receiver configuration.

Ref. [36]; a simple illustration would be the mapping

$$Z = \exp(z);\ \ X = \exp(x)\ \text{and}\ \ Y = \mathrm{i}\theta\,(-\pi < \theta < \pi);$$

$$J = \exp(2x) \tag{16}$$

Then, for $N(x) = k_0^2 K_0 \exp(2x)$, the slowness is simply k_0 and Green's function is, upon returning to the (x, y) system

$$G(x, y; x_0, y_0) = (\mathrm{i}/4)H_0^{(1)}[k_0\{\mathrm{e}^{2x} + \mathrm{e}^{2x_0} - 2\,\mathrm{e}^{x+x_0} \times \cos(y - y_0)\}^{1/2}] \tag{17}$$

Another example would be the mapping

$$Z = Z^n;\ \ X = r^n \cos(n\theta)\ \text{and}\ Y = r^n \sin(n\theta);\ \ J = n^2 r^{2(n-1)} \tag{18}$$

which for $k^2 = k_0^2 n^2 (x^2 + y^2)^{(n-1)}$ leads to a fundamental solution

$$G(x, y; x_0, y_0) = (\mathrm{i}/4)H_0^{(1)}[k_0\{r^{2n} + r_0^{2n} - 2r^n r_0^n \cos(\theta - \theta_0)\}^{1/2}] \tag{19}$$

as found by another method by Shaw and Gipson [37].

5. Combined mapping and dependent variable transformation

Here the combination of these two approaches is considered. Eq. (6) for the fundamental solution is, in the new coordinates

$$\nabla^2_{X,Y}\{K^{1/2}(x, y)K^{1/2}(x_0, y_0)G(x, y; x_0, y_0)\} + \{\beta^2(x, y)/J(x, y)\}\{K^{1/2}(x, y)K^{1/2}(x_0, y_0)G(x, y; x_0, y_0)\} = -\delta(x - x_0, y - y_0)/J(x, y) = -\delta(X - X_0, Y - Y_0) \tag{20}$$

The term $\{K^{1/2}(x, y)K^{1/2}(x_0, y_0)G(x, y; x_0, y_0)\} = W$ may be determined in (X, Y) space for various forms of the wavenumbers, $\beta/J^{1/2}$, including the simplest, β^2/J being a constant, β_0^2, which is a simple variation on the solution given above as Eq. (7). Clearly X and Y are functions of x and y through the mapping. Thus the material parameters, K and N, as well as the Green's function, G, may be thought of as functions of either (x, y) or (X, Y). Some care must be taken, however, in considering the regions of applicability, e.g. a whole space in (x, y) may map into a half space, a strip, etc. in (X, Y) space and vice versa. The procedure may then be summarized as a Helmholtz equation, e.g. Eq. (6), and a constraint equation corresponding to Eq. (2b) but now in terms of (X, Y) coordinate

$$-K^{-1/2}\nabla^2_{X,Y}(K^{1/2}) + N/(JK) = \beta^2/J \tag{21}$$

With the useful additional constraint of Eq. (4) relating N to K, this can be used to determine a class of material variations for which this procedure works, i.e. if

$$N(x, y) = k_0^2 J(x, y)\{AK^{1/2} + BK\} \tag{22}$$

and β^2/J is a constant, β_0^2, as above, the equation on $K^{1/2}$ becomes the same as Eq. (5) but in (X, Y) coordinates rather than (x, y) coordinates. As an example of this, consider the mapping

$$Z = \ln(z) = \ln(r) + \mathrm{i}\theta \tag{23}$$

such that, for $-\pi < \theta < \pi$,

$$X = \ln(r) = \ln[(x^2 + y^2)^{1/2}];\qquad Y = \theta = \tan^{-1}(y/x) \tag{24}$$

The Jacobian of this transformation is

$$J = (x^2 + y^2)^{-1} = 1/r^2 = \exp(-2X) \tag{25}$$

If we take

$$\beta^2 = \beta_0^2 J = k_0^2 \exp(-2X) \tag{26a}$$

$$B = B_0 J = B_0 \exp(-2X) \tag{26b}$$

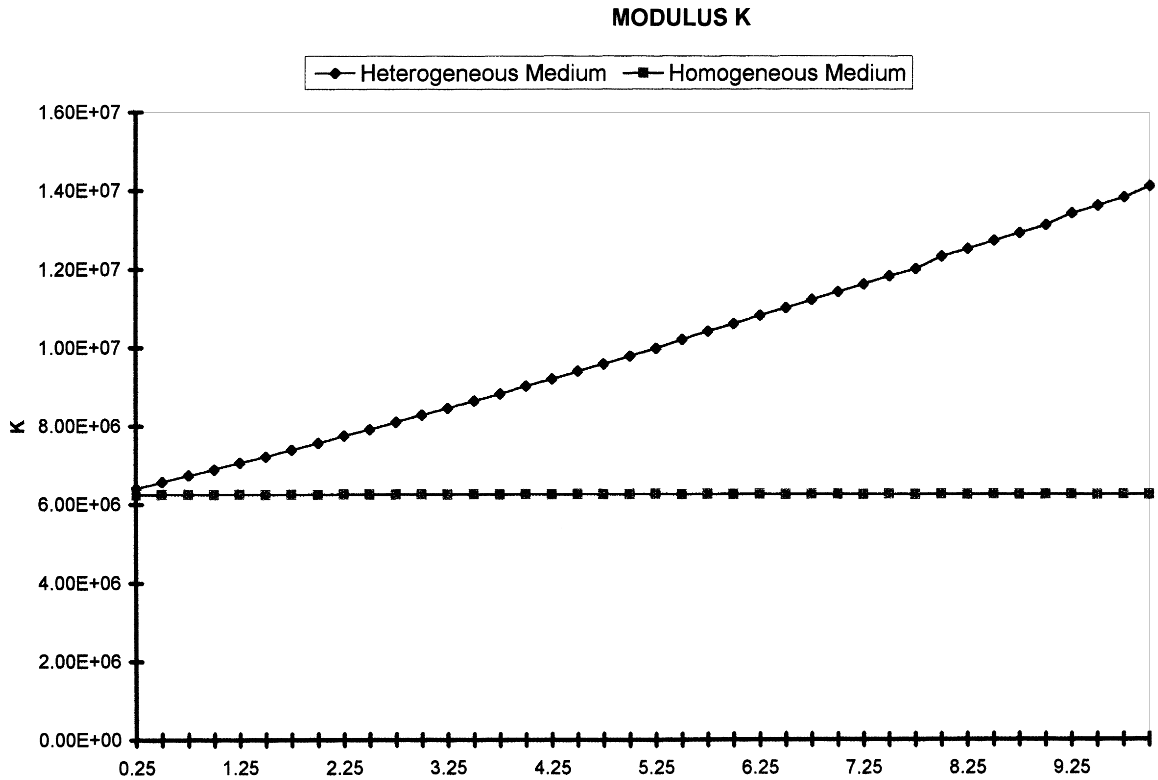
$$A = \left\{\alpha_0 + \alpha_1 X + \sum_{n=1}^{n=M} A_n \exp(\gamma_n X)\right\}J \tag{26c}$$

we have the constraint equation on $K^{1/2}$ in the form

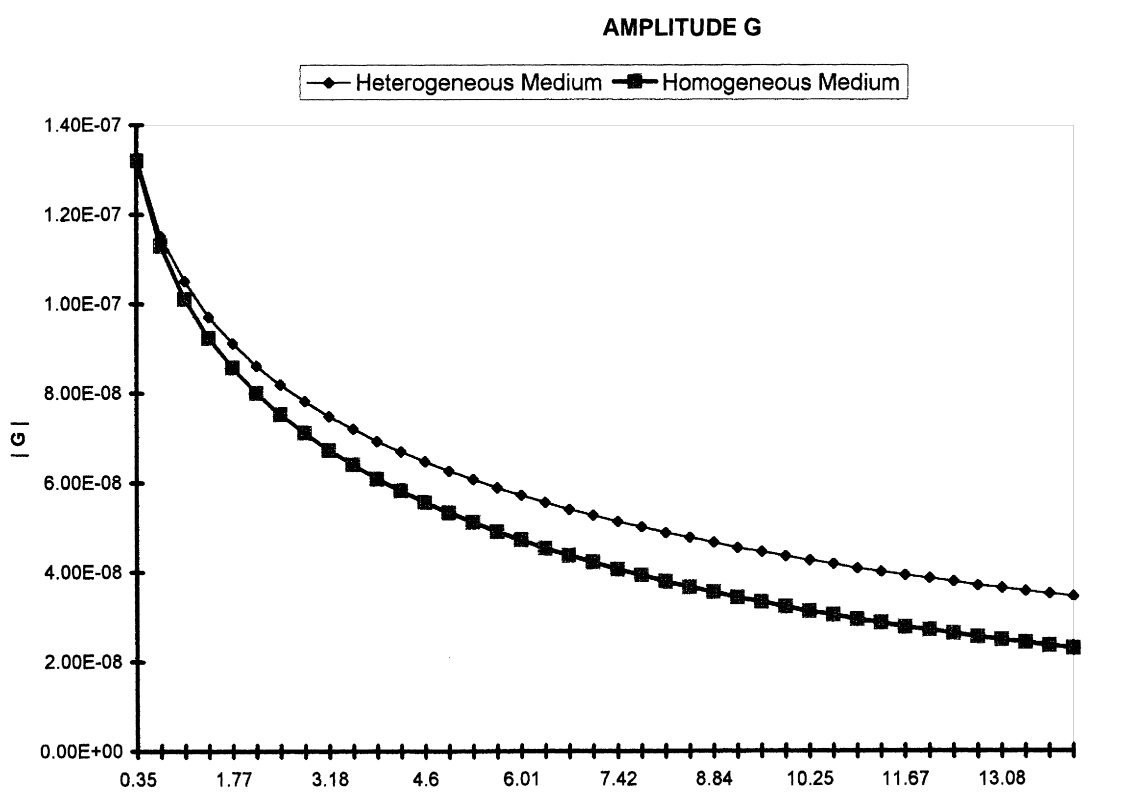
$$\mathrm{d}^2(K^{1/2})/\mathrm{d}X^2 + (\beta_0^2 - B_0)(K^{1/2}) = \alpha_0 + \alpha_1 X + \sum_{n=1}^{n=M} A_n \exp(\gamma_n X) \tag{27}$$

which has a simple solution

$$K^{1/2}(X) = C_1 \exp\{(B_0 - \beta_0^2)^{1/2}X\} + C_2 \exp \times\{-(B_0 - \beta_0^2)^{1/2}X\} + (\alpha_0 + \alpha_1 X)/(\beta_0^2 - B_0) + \sum_{n=1}^{n=M} A_n \exp(\gamma_n X)/(\gamma_n^2 + \beta_0^2 - B_0) \tag{28}$$



(a)



(b)

Fig. 2. (a) Modulus K, (b) amplitude of G and (c) phase angle of G for the DVT method for $\alpha_0 = 1.0$; $\alpha_1 = \alpha_2 = 0.025$; $\beta_0 = 0.02$.

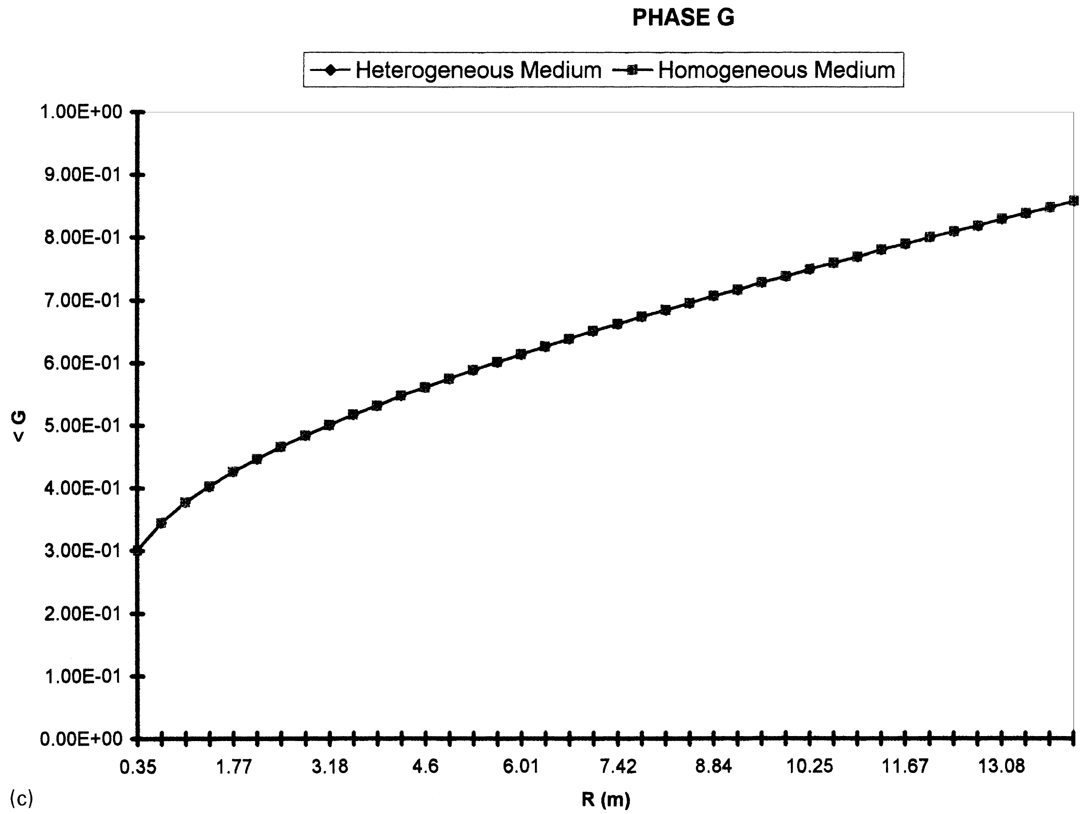


Fig. 2. (*continued*)

with N given by Eq. (4). Since X is $\ln(r)$, K and N will be fairly general functions of r, i.e. axisymmetric, with a large number of degrees of freedom in the coefficients,C_1, C_2 and A_n. The actual Green's function or fundamental solution then is the solution to Eqs. (10a)–(10d) with $\beta^2/J = B_0^2$, i.e.

$$G = K^{-1/2}(r)K^{-1/2}(r_0)\{(\mathrm{i}/4)H_0^{(1)}(\beta_0[\ln(r/r_0)^2 + (\theta - \theta_0)^2]^{1/2})\} \tag{29a}$$

$$K^{1/2}(r) = C_1 r^{(B_0-\beta_0^2)^{1/2}} + C_2 r^{-(B_0-\beta_0^2)^{1/2}} + (\alpha_0 + \alpha_1 \ln(r))/[\beta_0^2 - B_0] + \sum_{n=1}^{n=M} A_n r^{\gamma_n}/(g_n^2 + b_0^2 - B_0) \tag{29b}$$

with $N(r)$ given by Eq. (4). There are enough unknown coefficients to fit almost any actual data. Note that Green's function is not axisymmetric, i.e. the locations (x, y) and (x_0, y_0) are general. This form is a generalization of that given by Shaw and Gipson [37].

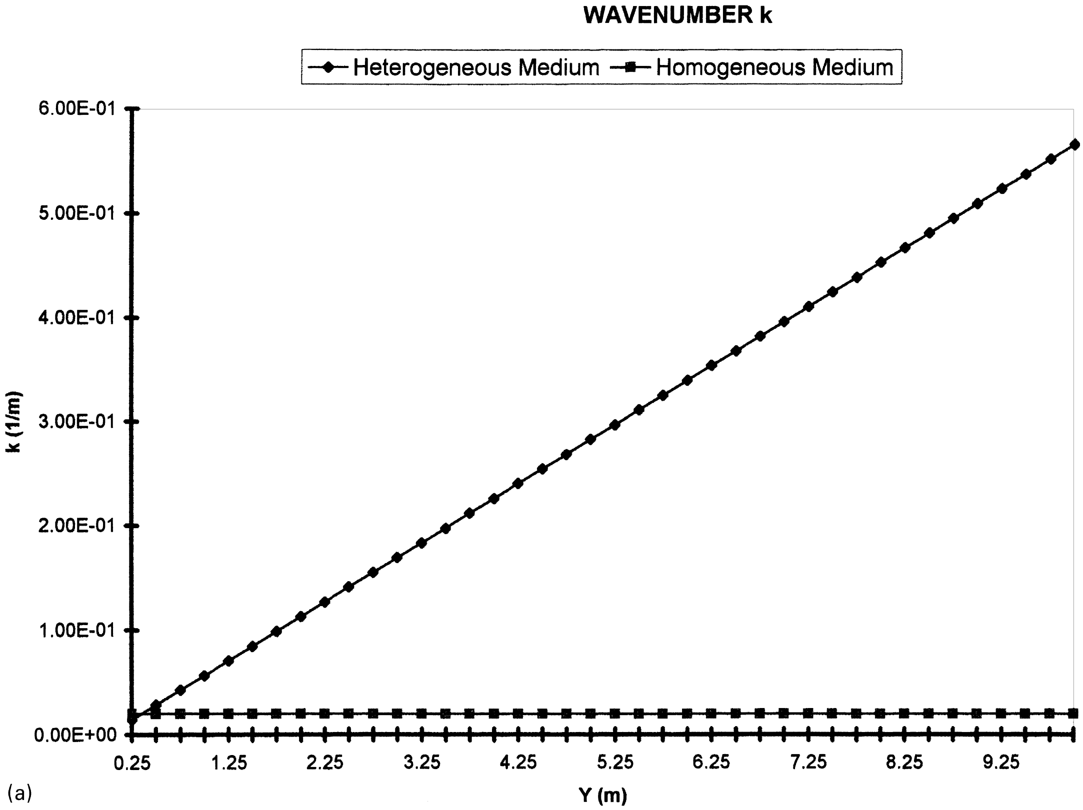
6. Numerical examples

Three numerical examples will be presented here so as to illustrate the previously developed methods. All are referred to the source/receiver configuration depicted in Fig. 1, whereby receivers are placed along the line r making a 45° angle with the horizontal, while the source is at the origin of the coordinate system. We note that the distance to the last receiver is $r = (100 + 100)^{1/2} = 14.1$ m.

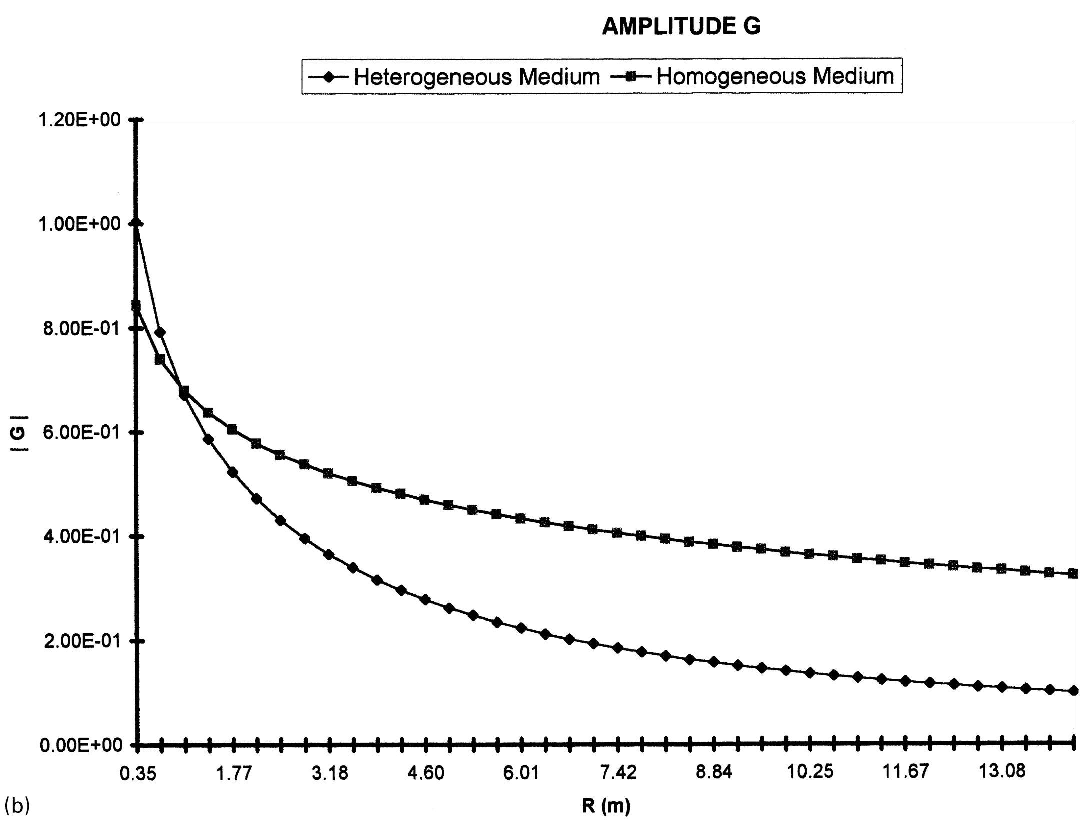
Example 1. This example pertains to the dependent variable transformation (DVT) method. The only non-zero constants are

$$\alpha_0 = 1.0; \quad \alpha_1 = \alpha_2 = 0.025; \quad \beta_0 = 0.02 \tag{30}$$

which give a reference modulus value of $K = K_0 = \alpha_0^2/\beta_0^4 = 6.25 \times 10^6$, defined at the source $(x_0, y_0) = (0, 0)$. We will assume a unit frequency $\omega = 1$ so that K does indeed coincide with a material property, e.g. a shear modulus if Eq. (1) pertains to horizontally polarized, elastic shear waves. Thus, in this case K and N physically correspond to a typical geological material such as sandstone. Fig. 2(a) depicts the variation of $K(x, y)$ along the vertical (e.g. depth) coordinate, while Fig. 2(b) and (c), respectively, plot the amplitude and phase angle of the fundamental solution G along the radial distance r. In all cases, the analogous results for the reference homogeneous medium with $\alpha_0 = \alpha_1 = \alpha_2 = 0$ are concurrently plotted for comparison purposes. We observe a decrease in the response of the heterogeneous medium, as compared to the homogeneous one, which is reasonable to expect since the former becomes stiffer in the direction of propagation. Finally, this type of transformation leaves the phase angle of G unchanged.

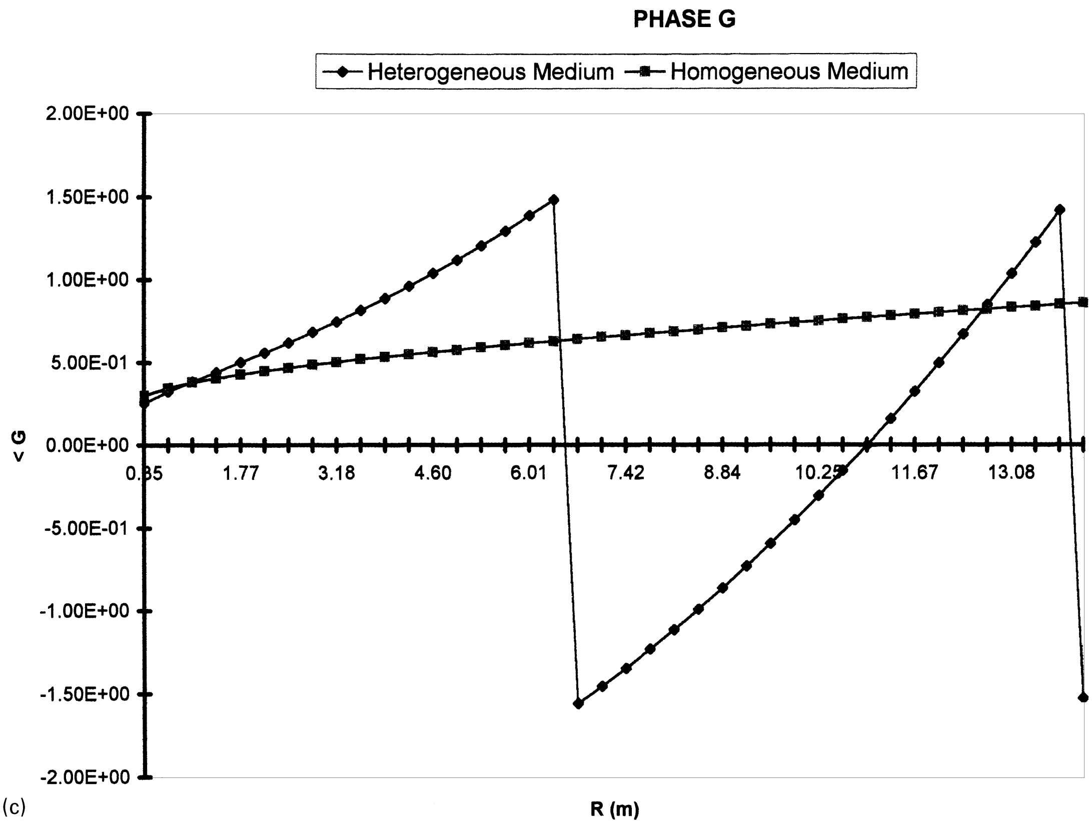


(a)



(b)

Fig. 3. (a) Wavenumber k, (b) amplitude of G and (c) phase angle of G for the CMT method for $n = 2$.



(c)

Fig. 3. (*continued*)

Example 2. This example serves to illustrate the conformal mapping transformation (CMT) method. We focus on the mapping given by Eq. (18) with $n = 2$. Thus, the resulting variable wavenumber is where the reference wavenumber is $k^2(\vec{r}) = k_0^2 n^2 r^{2(n-1)}$, where the reference wavenumber is $k_0 = (N_0/K_0)^{1/2}$ as used in Eq. (19). Fig. 3(a) plots the variable wavenumber k, which increases rapidly with radial distance from the source. Given the fact that this situation corresponds to a heterogeneous medium which becomes softer in the direction of propagation, it is not surprising to see that the amplitude of the fundamental solution shown in Fig. 3(b) decreases with r when compared to the results for the reference homogeneous medium. Finally, as shown in Fig. 3(c), the phase angle changes sign twice in the interval $(0 < r < 14.14)$ for the heterogeneous medium.

Example 3. In this last example, we examine the combined dependent variable—conformal mapping transformation (DVT–CMT) method. Specifically, we consider the logarithmic mapping of Eq. (21) and following transformation parameters are assumed to be non-zero:

$$k_0 = \beta_0 = 0.02; \quad \alpha_0 = 1.0; \quad \alpha_1 = 0.025 \tag{31}$$

We observe a rather complicated behavior for modulus K in Fig. 4(a), since it is less than the reference homogeneous material value in the near field around the source and increases with increasing distance from it. Furthermore, both amplitude and phase angle of the fundamental solution G given in Fig. 4(b) and (c), respectively, mirror this situation in that the near field solution is larger in the near field and smaller in the far field when compared with the reference homogeneous material results.

Appendix A

A conformal mapping will transform a delta function to another delta function with a multiplicative factor of the Jacobian just as it does the Laplace operator. This is seen through the limiting process$x \longrightarrow x_0$ and $y \longrightarrow y_0$. The argument of the mapped delta function is $\{(X - X_0)^2 + (Y - Y_0)^2\}^{1/2}$ which, in the neighborhood of the source point (x_0, y_0) is simply $\{J[(x - x_0)^2 + (y - y_0)^2]\}^{1/2}$. Thus the rate of the approach to zero from any direction is the same in either form apart from a multiplicative constant, i.e. the symmetry properties are maintained. The actual delta function is defined through its integral

$$\oint_c \delta(x, y)\, \mathrm{d}x\, \mathrm{d}y = -1 = \oint_C \delta(X, Y)\, \mathrm{d}X\, \mathrm{d}Y \tag{A1}$$

The transformation $X(x, y)$ and $Y(x, y)$ has $\mathrm{d}X\, \mathrm{d}Y = J\, \mathrm{d}x\, \mathrm{d}y$ and thus

$$\delta(x, y) = J(x, y)\delta(X, Y) \tag{A2}$$

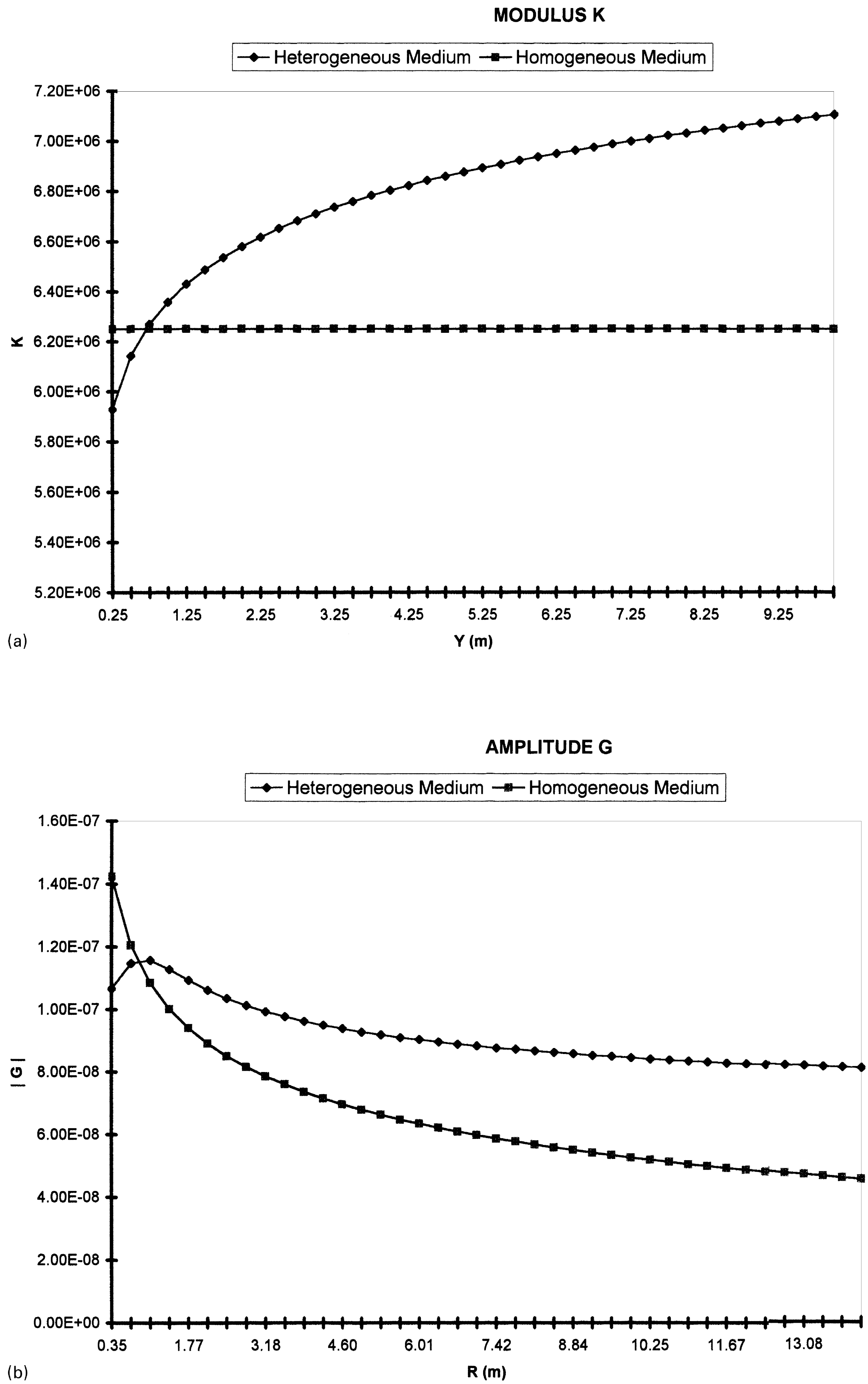


Fig. 4. (a) Modulus K; (b) amplitude of G and (c) phase angle of G for the combined DVT–CMT method for $\alpha_0 = 1.0$; $\alpha_1 = 0.025$; $k_0 = \beta_0 = 0.02$.

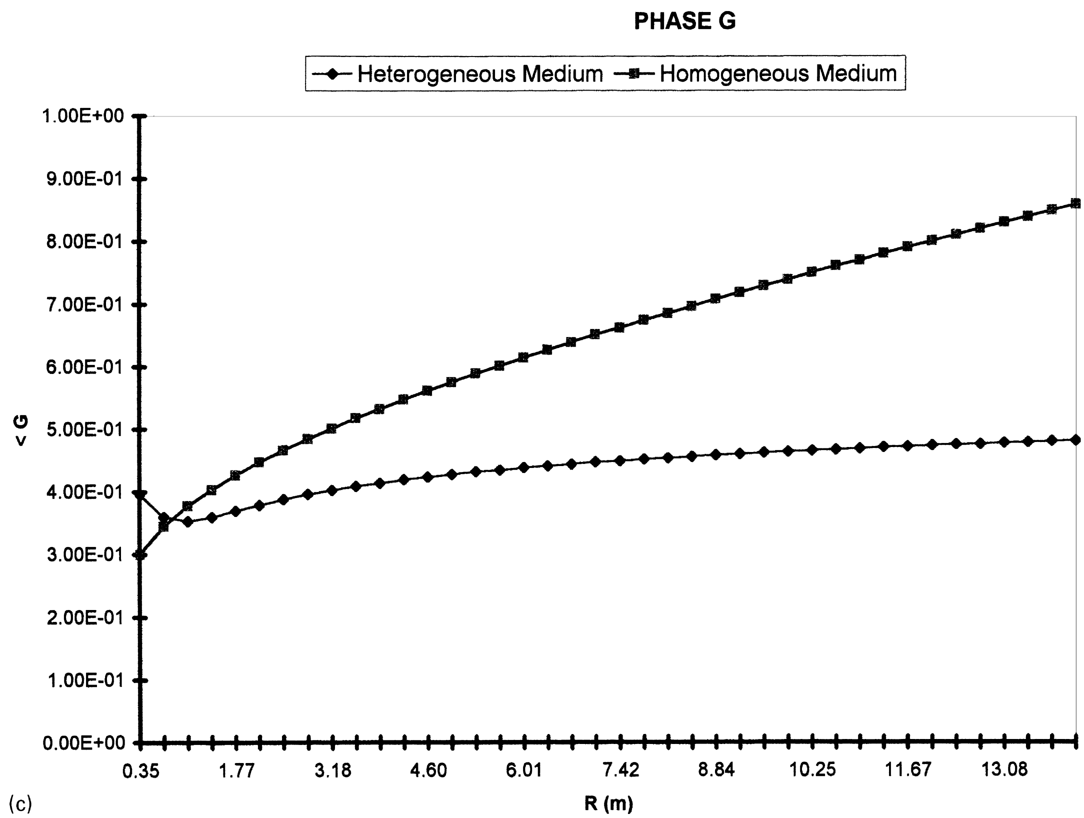


Fig. 4. (*continued*)

Finally as a conformal mapping preserves angles and, in some cases circles as well, the use of Green's theorem and the exclusion of the source point from either the volume or the boundary by circular or spherical segments as is the custom in BEM is maintained.

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