Engineering Analysis with Boundary Elements 31 (2007) 10–21

ENGINEERING ANALYSIS with BOUNDARY ELEMENTS

www.elsevier.com/locate/enganabound

The method of fundamental solutions for problems of free vibrations of plates

S.Yu. Reutskiy

Magnetism Division, Institute of Electrodynamics, National Academy of Science of Ukraine, Industrialnaya St., 19, 61106 Kharkov, Ukraine

Received 13 March 2006; accepted 29 June 2006
Available online 11 September 2006

Abstract

In this paper a new boundary method for problems of free vibrations of plates is presented. The method is based on mathematically modelling of the physical response of a system to external excitation over a range of frequencies. The response amplitudes are then used to determine the resonant frequencies. So, contrary to the traditional scheme, the method described does not involve evaluation of determinants of linear systems. The method shows a high precision in simply and doubly connected domains. The results of the numerical experiments justifying the method are presented.

Keywords: Free vibrations; Plates; Method of fundamental solutions; Multipole expansion

1. Introduction

The free vibrations of an isotropic thin elastic plate are described by the following equation:

$$\rho h \frac{\partial^2 u}{\partial t^2} + D\nabla^4 u, \quad u = u(\mathbf{x}, t), \quad \mathbf{x} \in \Omega \subset \mathscr{R}^2. \tag{1}$$

Here u is the normal displacement of the middle surface of the plate, ρ, h and D are the volume density, the thickness and the rigidity of the plate.

Considering harmonic vibrations

$$u(\mathbf{x}, t) = w(\mathbf{x}) \exp(i\omega t)$$

the governing equation can be written in the following dimensionless form

$$\nabla^4 w - k^4 w = 0, \quad k^4 = \frac{\rho h a^4 \omega^2}{D}, \tag{2}$$

where a is a typical linear size of the plate. The problem of free vibration is to find the real k for which there exist non-null functions w verifying (2) and some homogeneous boundary conditions:

$$\mathbf{B}[w] = 0, \quad \mathbf{x} \in \partial\Omega. \tag{3}$$

The operator of the boundary conditions $\mathbf{B}[\ldots]$ will be specified below.

The problems (2), (3) is a classical problem of mathematical physics. Apart from a few analytically solvable cases [1–3], there is no general solution of this problem. Therefore, a large number of numerical methods have been developed for many practical problems. The usual approach for eigenvalue problems with a positive defined operator is to use the Rayleigh minimal principle. See [4–6] for more details and references. Then, using an approximation for w with a finite number of free parameters, one gets the same problem in a finite-dimensional subspace which can be solved by a standard procedure of linear algebra, e.g., see [7,8]. The global basis functions [9–11] as well as finite elements [12,13] are used for this approximation.

Recently, some new powerful numerical techniques have been developed in this field. These are the differential quadrature methods proposed by Bellman and coworkers in 1972 [14], its recent version—the generalized differential quadrature (GDQ) approach [15,16] and the discrete singular convolution (DSC) algorithm which can be regarded as a local spectral method [17,18].

The boundary methods [19], in particular, the method of fundamental solutions (MFS) [20,21] are convenient in application to the problems (2), (3).

E-mail address: reutskiy@bars.net.ua.

0955-7997/$ - see front matter © 2006 Elsevier Ltd. All rights reserved.
doi:10.1016/j.enganabound.2006.06.004

In the framework of the boundary methods a general approach to solving these problems is as follows. First, using an integral representation of w in the BEM, or an approximation over fundamental solutions in MFS, one gets a homogeneous linear system $\mathscr{A}(k)\mathbf{q} = \mathbf{0}$ with matrix elements depending on the wave number k. To obtain the non-trivial solution the determinant of this matrix must be zero:

$$\det[\mathscr{A}(k)] = 0. \quad (4)$$

To get the eigenvalues this equation must be investigated analytically or numerically. This technique is described in [22–26] with more details. In the two latest papers there is a complete bibliography on the subject considered.

Another technique is proposed in [27–29]. This is a mathematical model of physical measurements when the resonance frequencies of a system are determined by the amplitude of response to some external excitation.

Let us consider the eigenvalue problem:

$$L[w] + \lambda w = 0, \quad \mathbf{x} \in \Omega \subset \mathscr{R}^2, \quad B[w] = 0, \quad \mathbf{x} \in \partial\Omega. \quad (5)$$

The method presented is as follows. Let us extend the operator of the problem from the initial domain Ω into a more wider Ω_0. In particular case $\Omega_0 = \mathscr{R}^2$. Let $w_p(\mathbf{x})$ be a particular solution of the PDE

$$L[w] + \lambda w = f(\mathbf{x}), \quad \mathbf{x} \in \Omega_0,$$

where $f(\mathbf{x}) = 0$ for $\mathbf{x} \in \Omega \subset \Omega_0$. If w_h is the solution of the boundary value problem

$$L[w_h] + \lambda w_h = 0, \quad \mathbf{x} \in \Omega,$$
$$B[w_h(\mathbf{x})] = -B[w_p(\mathbf{x})], \quad \mathbf{x} \in \partial\Omega,$$

then, the sum $w(\mathbf{x}, \lambda) = w_h + w_p$ satisfies (5). Let $F(\lambda)$ be some norm of the solution w. This function of λ has extremums at the eigenvalues and, under some conditions described below, can be used for their determining.

The outline of this paper is as follows. The main algorithm is described in Section 2. In Section 3, we give numerical examples to illustrate the method presented for simply and multiple connected domains. In particular, the case of doubly connected region with the inner region of vanishing maximal dimension which is important for technical applications is considered here.

2. The main algorithm

2.1. 1D case

For the sake of simplicity, let us consider 1D problem of free vibrations of a homogeneous beam with simply supported endpoints (SS conditions).

$$\frac{\partial^2 u}{\partial t^2} + \frac{EI}{\rho S}\frac{\partial^4 u}{\partial x^4} = 0, \quad 0 \leqslant x \leqslant l, \quad (6)$$

$$u(0,t) = \frac{\partial^2 u}{\partial x^2}(0,t) = 0, \quad u(l,t) = \frac{\partial^2 u}{\partial x^2}(l,t) = 0,$$

where E is Young's modulus, ρ is density, S and I are the area and moment of inertia of the cross section. Let us consider the harmonic vibration

$$u(x,t) = w(x)e^{i\omega t}.$$

The eigenvalue problem, can be written in the dimensionless form as follows:

$$\frac{d^4 w}{dx^4} - k^4 w = 0, \quad (7)$$

$$w(0) = w^{(2)}(0) = w(1) = w^{(2)}(1) = 0, \quad (8)$$

where

$$k^4 = \frac{\rho S l^4 \omega^2}{EI}. \quad (9)$$

It can be proved that k is a dimensionless value. The problem (7), (8) has a well-known solution:

$$k_n = n\pi, \quad \varphi_n(x) = \sin(n\pi x).$$

On the other hand, the differential operator of the problem can written as a product

$$\frac{d^4}{dx^4} - k^4 = \left(\frac{d^2}{dx^2} - k^2\right)\left(\frac{d^2}{dx^2} + k^2\right) \equiv \mathscr{L}_2(k)\mathscr{L}_1(k).$$

Let us assume that $k \neq 0$, then, the two singular solutions corresponding these two operators are

$$\Phi_1(x,\xi) = \exp(ik|x-\xi|), \quad \Phi_2(x,\xi) = \exp(k|x-\xi|). \quad (10)$$

The MFS solution of (7), (8) can be written in the following way:

$$w = q_1\Phi_1(x,\xi_1) + q_2\Phi_2(x,\xi_1) + q_3\Phi_1(x,\xi_2) + q\Phi_2(x,\xi_2)$$
$$= q_1 e^{ik(x-\xi_1)} + q_2 e^{k(x-\xi_1)} + q_3 e^{-ik(x-\xi_2)} + q_4 e^{-k(x-\xi_2)},$$

where $\xi_1 < 0$ and $\xi_2 > 1$ are the positions of the MFS source points.

Using the boundary conditions (8) and setting equal to zero the determinant of resulting linear system we get

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ e^{ik} & e^{k} & e^{-ik} & e^{-k} \\ -e^{ik} & e^{k} & -e^{-ik} & e^{-k} \end{vmatrix} = 0,$$

or after simple transforms:

$$(e^{ik} - e^{-ik})(e^{k} - e^{-k}) = 0.$$

We get the wave numbers k_n as solutions: $\sin(k) = 0$, or $k = n\pi$. Thus, MFS gives the exact solution. Note that in multidimensional cases such computations are not so simple and are time consuming.

According to the technique presented we solve the inhomogeneous problem:

$$\frac{d^4 w}{dx^4} - k^4 w = f(x), \quad w(0) = w^{(2)}(0) = w(1) = w^{(2)}(1) = 0. \quad (11)$$

Here $x \in [A, B]$, where $[A, B] \supset [0, 1]$ is a large enough interval. The function $f(x)$ is defined in $[A, B]$ and $f(x) = 0$ for $x \in [0, 1]$. Under this condition when $x \in [0, 1]$, any solution of (11) satisfies (7). The general solution of (11) now can be written in the form

$$w = q_1 e^{ik(x-\xi_1)} + q_2 e^{k(x-\xi_1)} + q_3 e^{-ik(x-\xi_2)} + q_4 e^{-k(x-\xi_2)} + w_p, \tag{12}$$

where w_p is the particular solution corresponding to the right-hand side f. We can take

$$w_p = e^{ik|x-\xi_{ext}|} \quad \text{or} \quad w_p = e^{k|x-\xi_{ext}|}, \quad \xi_{ext} \notin [0, 1],$$

i.e., in the same form as the MFS basis functions (10). The particular solution can be also taken in the form of a travelling wave $w_p = e^{\pm ikx}$ or as a function with the singular point in infinity $w_p = e^{\pm kx}$. All these particular solutions correspond to f equal to zero inside the solution domain. Note that we do not need the explicit form of f because we deal with the particular solution only.

Substituting (12) in the same homogeneous boundary conditions (8) now we get an *inhomogeneous* 4×4 linear system for each k:

$$\begin{cases} q_1 e^{-ik\xi_1} + q_2 e^{-k\xi_1} + q_3 e^{ik\xi_2} + q_4 e^{k\xi_2} = -w_p(0), \\ -k^2 q_1 e^{-ik\xi_1} + k^2 q_2 e^{-k\xi_1} - k^2 q_3 e^{ik\xi_2} + k^2 q_4 e^{k\xi_2} = -w_p^{(2)}(0), \\ q_1 e^{ik(1-\xi_1)} + q_2 e^{k(1-\xi_1)} + q_3 e^{-ik(1-\xi_2)} + q_4 e^{-k(1-\xi_2)} = -w_p(1), \\ -k^2 q_1 e^{ik(1-\xi_1)} + k^2 q_2 e^{k(1-\xi_1)} - k^2 q_3 e^{-ik(1-\xi_2)} \\ \quad +k^2 q_4 e^{-k(1-\xi_2)} = -w_p^{(2)}(1). \end{cases} \tag{13}$$

Let us introduce the norm of the solution as

$$F(k) = \sqrt{\frac{1}{N_t} \sum_{l=1}^{N_t} |w(x_{t,l})|^2}, \quad F_d(k) = F(k)/F(k_0), \tag{14}$$

where $F(k_0)$ is the scaling value, k_0 is a reference wave number and the points $x_{t,l}$ are randomly distributed in [0, 1]. In all the calculations presented in this section we use $N_t = 7$. Note that the method is not very sensitive to the number of points $x_{t,l}$. But it is more important, they should be placed in an irregular way. For instance, the mode $w_n = \sin(n\pi x)$ is equal to zero when $x = 1/n$. The function $F_d(k)$ characterizes the relative value of the response of the system to the outer excitation.

In Fig. 1 the value F_d as a function of the wave number k is shown. The graph contains large sharp peaks at the positions of eigenvalues. Generally speaking, this resonance curve can be used to determine the eigenvalues in the same way as $\det[\mathscr{A}(k)]$ in the technique described above. However, the graph $F_d(k)$ is a non-smooth one, as it is shown in the right-hand part of the figure with more details. This can be explained by the following reasons. For any choice of the particular solution w_p listed above, the system (13) has an *exact solution*, e.g., $q_1 = 0$, $q_2 = -e^{k\xi_1}$, $q_3 = 0$, $q_4 = 0$ for $w_p = e^{kx}$. As a result the total solution $w(x) = 0$, for $x \in [0, 1]$. So, here we have $F(k)$ which is equal to zero with machine precision accuracy when k is far from eigenvalues; $F(k)$ grows considerably in the neighbourhood of the eigenvalues when the linear system becomes almost degenerated. And a smoothing procedure is needed to get an appropriate curve which is convenient for applying an optimization procedure.

The smoothing procedure consists of introducing a small parameter in the governing equation. Let us consider the problem (cf. (11)):

$$\frac{d^4 w}{dx^4} - (k^4 - i\varepsilon k^2)w = f, \quad w(0) = w^{(2)}(0) = w(1) = w^{(2)}(1) = 0. \tag{15}$$

Here ε is a small parameter. From the mathematical point of view it means that we shift the spectra of differential operator from the real axis. On the other hand, from the physical point of view, it means that the initial equation (6) is changed by the equation

$$\frac{\partial^2 u}{\partial t^2} + c\frac{\partial u}{\partial t} + \frac{EI}{\rho S}\frac{\partial^4 u}{\partial x^4} = 0, \quad 0 \leqslant x \leqslant l, \tag{16}$$

which describes free vibrations of a homogeneous beam with a friction term $c\partial u/\partial t$. The singular solutions of (15) are

$$\Psi_1(x, \xi) = \exp(i\chi|x - \xi|), \quad \Psi_2(x, \xi) = \exp(\chi|x - \xi|),$$
$$\chi(k, \varepsilon) = (k^4 - i\varepsilon k^2)^{1/4}. \tag{17}$$

Note that the boundary value problem (15) has a unique non-zero solution for all real k. The resonance curve corresponding to $\varepsilon = 10^{-6}$ is shown in Fig. 2.

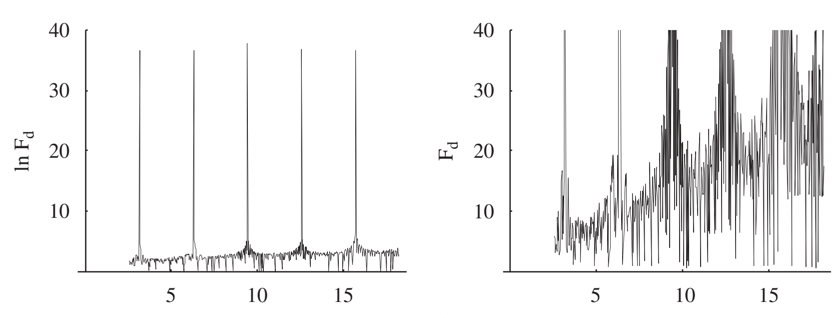

Fig. 1. The resonance curve without regularization.

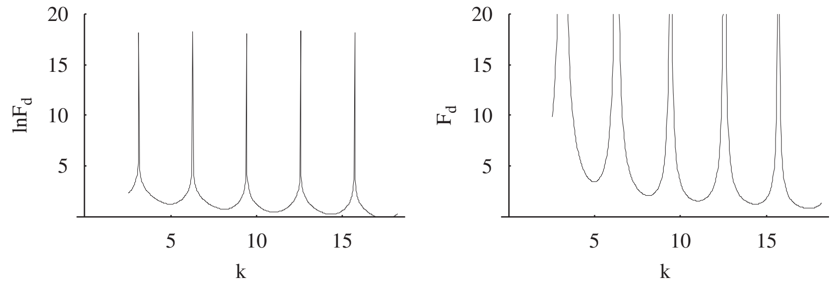

Fig. 2. The resonance curve. Regularization by an additional friction term: $\varepsilon = 10^{-6}$.

Now this is a smooth curve with separated maximums at the positions of eigenvalues. To find the eigenvalues we use the following algorithm throughout the paper. Let us look for the eigenvalues on the interval $k \in [a, b]$. Then

(A)

Step 0:	*Choose* $\Delta k > 0$;
	if $F(a) > F(a + \Delta k)$ *goto step* 5;
Step 1:	$k_1 = a$; $F1 = F(k_1)$;
Step 2:	$k_2 = k_1 + \Delta k$; $F2 = F(k_2)$;
	if $k_2 > b$ *stop*;
Step 3:	*if* $F2 > F1$ *then* $[F1 = F2; k_1 = k_2;$ *goto step* 2];
Step 4:	*find the maximum point* x_m *of* $F(k)$ *on* $[k_2 - 2\Delta k, k_2]$;
Step 5:	$k_1 = a$; $F1 = F(k_1)$;
Step 6:	$k_2 = k_1 + \Delta k$; $F2 = F(k_2)$;
	if $k_2 > b$ *stop*;
Step 7:	*if* $F2 < F1$ *then* $[F1 = F2; k_1 = k_2;$ *goto step* 6];
	else goto step 2.

In other words, using the step Δk we find the intervals $[k_2 - 2\Delta k, k_2]$ which contain only maximum of $F_d(k)$. Then, we make the position of the eigenvalue more precise using an optimization procedure. Note that any univariate optimization procedure can be used at Step 4. In particular, we applied Brent's method based on a combination of parabolic interpolation and bisection of the function near the extremum (see [30, Chapter 10], [31, Chapter 5]). The step is taken $\Delta k = 0.01$ throughout the paper if this is not specified.

Example 1. The data placed in Table 1 are obtained by applying this technique with $\varepsilon = 0.1,\ 10^{-3}$. The other parameters are: $\xi_1 = -1.0$, $\xi_2 = 2.0$, $\xi_{ext} = 2.2$. We place the relative errors

$$e_r = |k_i - k_i^{(ex)}| / k_i^{(ex)} \tag{18}$$

in the calculation of the first ten eigenvalues of (7), (8) in the left part of the table. The data in the right part of the table correspond to the SC (simply supported—clamped) boundary conditions:

$$w(0) = w^{(2)}(0) = w(1) = w^{(1)}(1) = 0.$$

Table 1

1D eigenproblem $w^{(4)} - k^2 w = 0$, SS conditions: $w(0) = w^{(2)}(0) = w(1) = w^{(2)}(1) = 0$, SC conditions: $w(0) = w^{(2)}(0) = w(1) = w^{(1)}(1) = 0$

SS conditions			SC conditions		
$k_i^{(ex)}$	$\varepsilon = 0.1$	$\varepsilon = 10^{-3}$	$k_i^{(ex)}$	$\varepsilon = 0.1$	$\varepsilon = 10^{-3}$
π	3×10^{-6}	3×10^{-10}	3.926602312	2×10^{-6}	1×10^{-10}
2π	2×10^{-8}	2×10^{-11}	7.068582745	2×10^{-7}	2×10^{-11}
3π	4×10^{-8}	2×10^{-12}	10.21017612	5×10^{-8}	4×10^{-12}
4π	1×10^{-7}	4×10^{-13}	13.35176877	1×10^{-8}	5×10^{-12}
5π	5×10^{-9}	3×10^{-12}	16.49336143	7×10^{-9}	5×10^{-13}
6π	2×10^{-9}	2×10^{-12}	19.63495408	2×10^{-9}	6×10^{-13}
7π	1×10^{-9}	2×10^{-12}	22.77654674	8×10^{-10}	4×10^{-13}
8π	8×10^{-10}	3×10^{-12}	25.91813939	8×10^{-10}	9×10^{-13}
9π	5×10^{-10}	1×10^{-12}	29.05973204	6×10^{-10}	8×10^{-14}
10π	3×10^{-10}	5×10^{-13}	32.20132470	4×10^{-10}	1×10^{-13}

The relative errors in calculations of the eigenvalues.

Here the exact eigenvalues are the roots of the equation $\tanh(k) = \tan(k)$.

2.2. 2D case

Let us return to the eigenproblems (2), (3). According to the technique proposed, we consider the following BVP:

$$\nabla^4 w - k^4 w = f, \quad \mathbf{x} \in \Omega \subset \mathscr{R}^2,$$
$$\mathbf{B}[w] = 0, \quad \mathbf{x} \in \partial\Omega, \tag{19}$$

where f describes an external exciting source, i.e., $f = 0$ for $\mathbf{x} \in \Omega$.

In application to this problem, the MFS technique is similar to the one considered in the previous section. Because of the splitting

$$(\nabla^4 - k^4) = (\nabla^2 + k^2) \times (\nabla^2 - k^2) \tag{20}$$

the singular solutions are of two types: the fundamental solutions of the Helmholtz operator $\nabla^2 + k^2$:

$$\Phi_n^{(1)}(\mathbf{x}) = H_0^{(1)}(k|\mathbf{x} - \zeta_n|) \tag{21}$$

and the fundamental solutions of the modified Helmholtz operator $\nabla^2 - k^2$:

$$\Phi_n^{(2)}(\mathbf{x}) = K_0(k|\mathbf{x} - \zeta_n|), \tag{22}$$

where $H_0^{(1)}$ is the Hankel function and K_0 is the modified Bessel function of the second kind and of order zero. This is the so-called Kupradze basis [32]. So, an approximate solution is sought in the form of the linear combination:

$$w(\mathbf{x}|\mathbf{q}_1,\mathbf{q}_2) = w_p(\mathbf{x}) + \sum_{n=1}^{N} q_{1,n}\Phi_n^{(1)}(\mathbf{x}) + \sum_{n=1}^{N} q_{2,n}\Phi_n^{(2)}(\mathbf{x}), \tag{23}$$

where $w_p(\mathbf{x})$ is a particular solution corresponding to the external source f. It can be taken as a singular solution with the singular point placed outside the solution domain:

$$w_p(\mathbf{x}) = H_0^{(1)}(k|\mathbf{x}-\zeta_{ext}|), \quad w_p(\mathbf{x}) = K_0(k|\mathbf{x}-\zeta_{ext}|), \tag{24}$$

where $\zeta_{ext} \subset \mathscr{R}^2\backslash\Omega$, or in the form of the travelling wave:

$$w_p(\mathbf{x}) = \exp\, \mathrm{i}k(x\cos\theta + y\sin\theta). \tag{25}$$

Here $0\leqslant\theta\leqslant 2\pi$ is the angle of incidence, i.e., the one between direction of the wave-propagation and the x-axis.

The free parameters $q_{1,n}, q_{2,n}$ are determined from the boundary conditions:

$$\mathbf{B}[w] = 0 \quad \text{or} \quad B_1[w] = 0, \quad B_2[w] = 0, \quad \mathbf{x} \in \partial\Omega,$$

where the scalar operators B_1, B_2 are described in the Appendix.

The free parameters $q_{1,n}, q_{2,n}$ are looked for as a solution of the minimization problem:

$$\min_{\mathbf{q}_1,\mathbf{q}_2} \sum_{i=1}^{N_c} \left\{ B_1[w_p(\mathbf{x}_i)] + \sum_{n=1}^{N} q_{1,n} B_1[\Phi_n^{(1)}(\mathbf{x}_i)] + \sum_{n=1}^{N} q_{2,n} B_1[\Phi_n^{(2)}(\mathbf{x}_i)] \right\}^2, \tag{26}$$

$$\min_{\mathbf{q}_1,\mathbf{q}_2} \sum_{i=1}^{N_c} \left\{ B_2[w_p(\mathbf{x}_i)] + \sum_{n=1}^{N} q_{1,n} B_2[\Phi_n^{(1)}(\mathbf{x}_i)] + \sum_{n=1}^{N} q_{2,n} B_2[\Phi_n^{(2)}(\mathbf{x}_i)] \right\}^2. \tag{27}$$

Here the collocation points $\mathbf{x}_i$, $i = 1,\ldots,N_c$ are distributed uniformly on the boundary $\partial\Omega$. We take N_c approximately twice as large as the number of free parameters $2N$. As a result we get an overdetermined $2N_c \times 2N$ linear system which is solved by the standard least squares procedure. Note that (26), (27) can be regarded as a result of discretization of the integral condition:

$$\min_{\mathbf{q}_1,\mathbf{q}_2} \int_{\partial\Omega} B_1[w(\mathbf{x}|\mathbf{q}_1,\mathbf{q}_2)]^2\,\mathrm{d}s, \quad \min_{\mathbf{q}_1,\mathbf{q}_2} \int_{\partial\Omega} B_2[w(\mathbf{x}|\mathbf{q}_1,\mathbf{q}_2)]^2\,\mathrm{d}s.$$

The same trial functions can also be used when dealing with problems in multiply connected domains. The source points should be placed inside each hole as it is depicted in Fig. 3a.

The special basis functions associated with each hole can be used as an alternative approach. Let us consider again the splitting of the operator (20). We have two sets of the singular solutions corresponding to each operator $\nabla^2 + k^2$ and $\nabla^2 - k^2$:

$$\Psi_{s,1}^{(1)}(\mathbf{x}) = H_0^{(1)}(kr_s), \quad \Psi_{s,2n+1}^{(1)}(\mathbf{x}) = H_n^{(1)}(kr_s)\cos n\theta_s,$$
$$\Psi_{s,2n}^{(1)}(\mathbf{x}) = H_n^{(1)}(kr_s)\sin n\theta_s, \tag{28}$$

$$\Psi_{s,1}^{(2)}(\mathbf{x}) = K_0(kr_s), \quad \Psi_{s,2n+1}^{(2)}(\mathbf{x}) = K_n(kr_s)\cos n\theta_s,$$
$$\Psi_{s,2n}^{(2)}(\mathbf{x}) = K_n(kr_s)\sin n\theta_s. \tag{29}$$

Here $r_s = |\mathbf{x} - \mathbf{x}_s|, \theta_s$ is the local polar coordinate system with the origin at the point $\mathbf{x}_s$ of multipoles location (see Fig. 3b). This is the so-called Vekua basis [33,34] or multipole expansion. In this case instead of (23) we use

$$w(\mathbf{x}|\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_{1,s},\mathbf{p}_{2,s}) = w_p(\mathbf{x}) + \sum_{n=1}^{N} q_{1,n}\Phi_n^{(1)}(\mathbf{x}) + \sum_{n=1}^{N} q_{2,n}\Phi_n^{(2)}(\mathbf{x}) + \sum_{s=1}^{S}\sum_{m=1}^{M} p_{1,s,m}\Psi_{s,m}^{(1)}(\mathbf{x}) + \sum_{s=1}^{S}\sum_{m=1}^{M} p_{2,s,m}\Psi_{s,m}^{(2)}(\mathbf{x}), \tag{30}$$

where S is the number of holes and M is the number of terms in each multipole expansion.

We apply the same smoothing procedure to get a smooth resonance curve $F_d(k)$. The governing equation should be replaced by the following one:

$$\nabla^4 w - (k^4 - \mathrm{i}\varepsilon k^2)w = f$$

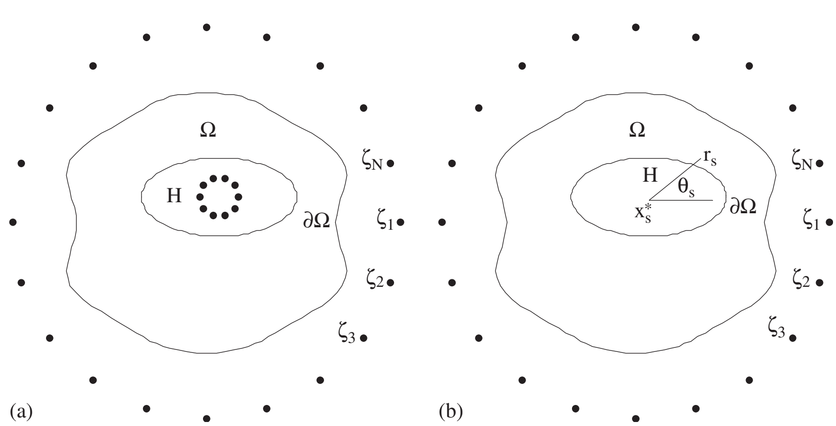

Fig. 3. Geometry configuration of a doubly connected domain.

and so the arguments of the basis functions $\Phi_n^{(1)}(\mathbf{x})$, $\Phi_n^{(2)}(\mathbf{x})$ should be modified in the following way:

$$\Phi_n^{(1)}(\mathbf{x}) = H_0^{(1)}(\chi|\mathbf{x}-\zeta_n|), \quad \Phi_n^{(2)}(\mathbf{x}) = K_0(\chi|\mathbf{x}-\zeta_n|),$$
$$\chi = (k^4 - \mathrm{i}\varepsilon k^2)^{1/4}.$$

In the similar way, the arguments of the basis functions $\Psi_{s,m}^{(1)}(\mathbf{x})$, $\Psi_{s,m}^{(2)}(\mathbf{x})$ should be modified too, i.e., instead of kr_s one should use χr_s.

3. Numerical examples

In this section, numerical examples are given to examine the method presented. The resonance curve $F(k)$ is always computed using $N_t = 15$ testing points $\mathbf{x}_{t,l} \in \Omega$ (see (14)). They are distributed inside Ω with the help of RNUF generator of pseudorandom numbers from the Microsoft IMSL Library. The particular solution corresponding to the exciting source is taken as $w_p(\mathbf{x}) = H_0^{(1)}(k|\mathbf{x}-\zeta_{ex}|)$ (see (24)).

3.1. Eigenvalues

Example 2. A circular plate with the radius $r = 1$ subjected to the boundary conditions of three types (A.12)–(A.14) is considered. The exciting source is placed at the position $\zeta_{ext} = (5, 5)$; the singular points ζ_n of the fundamental solutions (21), (22) are located on the circle with the radius $R = 3$. The number of the MFS sources $N = 30$. Note that the number of free parameters is $2N$ because at each source point we have two basis functions. The data presented in Table 2 are obtained using the smoothing procedure with $\varepsilon = 10^{-6}$. Here we place the relative errors (18). The exact eigenvalues $k_i^{(ex)}$ are the roots of the equation: $J_n'(k)I_n(k) - J_n(k)I_n'(k) = 0$—conditions C;

$$J_n(k)I_{n+1}(k) + I_n(k)J_{n+1}(k) - \frac{2k}{1-\nu}J_n(k)I_n(k) = 0 \tag{31}$$

—conditions S; and we get the equation for free edge (condition F):

$$\begin{aligned}&\{((1-\nu)(n^2-n)-k^2)J_n(k)+k(1-\nu)J_{n+1}(k)\}\\&\quad\times\{(k^2-(1-\nu)n^2)(nI_n(k)+kI_{n+1}(k))+(1-\nu)n^2I_n(k)\}\\&\quad-\{((1-\nu)(n^2-n)-k^2)I_n(k)-k(1-\nu))_{n+1}(k)\}\\&\quad\times\{-(k^2-(1-\nu)n^2)(nJ_n(k)-kJ_{n+1}(k))\\&\quad+(1-\nu)n^2J_n(k)\}=0.\end{aligned} \tag{32}$$

Example 3. In the next example, the square plate $[-0.5, +0.5] \times [-0.5, +0.5]$ with different boundary conditions is considered. In all the calculations presented the MFS sources are located on the circle with the radius $R = 2$. First, we consider the plate with simply supported edge. The boundary conditions are

$$w|_{x=\pm a/2} = 0, \quad \frac{\partial^2 w}{\partial x^2}|_{x=\pm a/2} = w|_{y=\pm b/2} = \frac{\partial^2 w}{\partial y^2}|_{y=\pm b/2} = 0, \tag{33}$$

i.e., SSSS boundary conditions. This problem has an analytical solution: $k^{(ex)} = \pi\sqrt{i^2+j^2}$, $i, j = 1, 2, \ldots$. Table 3 shows the relative error (18) for different numbers of the MFS sources N. The calculations were performed with $\varepsilon = 10^{-6}$.

Example 4. The same square plate with clamped edges (CCCC) is taken. The boundary conditions are

$$w|_{x=\pm a/2} = 0, \quad \left.\frac{\partial w}{\partial x}\right|_{x=\pm a/2} = w|_{y=\pm a/2} = \left.\frac{\partial w}{\partial y}\right|_{y=\pm a/2} = 0. \tag{34}$$

The MFS source points are located on the circle with the radius $R = 2$. The results shown in Table 4 correspond to $\varepsilon = 10^{-6}$. In Tables 5–10 we present the results of the calculations for the square plate with different boundary conditions. The exact solutions are not available here. Thus, we compare our results with the ones obtained in [2,3], GDQ [15,16], DSC [17,18]. Note that to compare the results we place the dimensionless frequencies $\widehat{\omega}_i = k_i^2$.

Example 5. Let us consider an annular case. The solution domain is the region between two circles. The inner and outer radii of an annular domain are $r_1 = 0.5$ and $r_2 = 2$ correspondingly. The singular points are distributed on the circles with the radii $a = 3$ (outside the domain) and $b = 0.2$ (inside the hole). The number of the singular points on each auxiliary contour is equal to N. The exciting source is placed at the position $\zeta_{ex} = (5, 5)$. The general expression

Table 2
A circular plate with different boundary conditions

Clamped edge		Simply supported edge		Free edge	
$k_i^{(ex)}$	e_r	$k_i^{(ex)}$	e_r	$k_i^{(ex)}$	e_r
3.1962206069	5×10^{-8}	2.2045701221	4×10^{-11}	2.3475907027	8×10^{-11}
4.6108998881	4×10^{-8}	3.7195305760	2×10^{-13}	2.9815924124	2×10^{-10}
5.9056782349	2×10^{-8}	5.0550750234	5×10^{-11}	3.5699204653	6×10^{-12}
6.3064370439	7×10^{-9}	5.4462973275	2×10^{-11}	4.5176539670	6×10^{-11}
7.1435310492	2×10^{-8}	6.3166259662	2×10^{-9}	4.7235536456	2×10^{-8}

The relative errors in calculation of the first five eigenvalues. The number of the MFS sources $N = 30$; ε procedure with $\varepsilon = 10^{-6}$.

Table 3
A square plate with SSSS boundary conditions

i	$N=20$	$N=30$	$N=40$
1	8.0×10^{-7}	4.7×10^{-10}	6.5×10^{-12}
2	1.1×10^{-5}	6.0×10^{-9}	8.0×10^{-12}
3	3.9×10^{-5}	7.5×10^{-8}	6.7×10^{-12}
4	2.7×10^{-4}	2.3×10^{-9}	2.9×10^{-10}
5	5.3×10^{-4}	1.8×10^{-6}	1.1×10^{-9}

The relative errors in calculation of the first five eigenvalues.

Table 4
A square plate with clamped edges

i	$N=34$	$N=38$	$N=42$	$N=46$	Liessa	DSC	GDQ
1	35.978	35.978	35.982	35.983	35.992	35.989	35.985
2	73.442	73.379	73.385	73.387	73.413	73.407	73.394
3	108.189	108.169	108.181	108.202	108.27	108.249	108.210
4	131.580	131.581	131.581	131.580	131.64	131.622	131.580
5	132.200	132.200	132.202	132.228	132.24	132.244	132.200
6	164.995	164.997	164.997	165.030	—	165.074	165.000
7	210.522	210.521	210.524	210.519	—	—	—
8	220.034	220.028	220.036	219.932	—	—	—
9	242.317	242.140	242.155	242.155	—	—	—
10	296.712	243.132	243.143	243.177	—	—	—

The first 10 eigenfrequencies.

Table 5
A square plate with CSSS boundary conditions

i	$N=34$	$N=38$	$N=42$	$N=46$	Liessa	DSQ	GDQ
1	31.818	31.822	31.823	31.824	31.829	31.828	31.826
2	63.351	63.343	63.338	63.335	63.347	63.338	63.331
3	71.048	71.068	71.069	71.073	71.084	71.087	71.076
4	100.807	100.814	100.772	100.783	100.83	100.815	100.792
5	116.357	116.363	116.341	116.347	116.40	116.376	116.357
6	130.348	130.328	130.333	130.339	—	130.388	130.351
7	151.890	151.863	151.936	151.909	—	151.938	151.893
8	159.479	159.477	159.444	159.446	—	159.534	159.446
9	189.801	189.767	189.766	189.768	—	—	—
10	—	209.365	209.330	209.373	—	—	—

The first 10 eigenfrequencies.

of the eigenmode in the polar coordinates (r,θ) is

$$[AJ_n(kr)+BY_n(kr)+CI_n(kr)+DK_n(kr)]\cos(n\theta),$$

where A,B,C,D are free parameters. Substituting this expression in the boundary conditions we get the linear 4×4 linear system. For example, let us consider a plate with clamped inner and outer edges. In this case the matrix of the system has the form

$$\begin{pmatrix} J_n(kr_1) & Y_n(kr_1) & I_n(kr_1) & K_n(kr_1) \\ \mathrm{d}J_n(kr)/\mathrm{d}r|_{r=r_1} & \mathrm{d}Y_n(kr)/\mathrm{d}r|_{r=r_1} & \mathrm{d}I_n(kr)/\mathrm{d}r|_{r=r_1} & \mathrm{d}K_n(kr)/\mathrm{d}r|_{r=r_1} \\ J_n(kr_2) & Y_n(kr_2) & I_n(kr_2) & K_n(kr_2) \\ \mathrm{d}J_n(kr)/\mathrm{d}r|_{r=r_2} & \mathrm{d}Y_n(kr)/\mathrm{d}r|_{r=r_2} & \mathrm{d}I_n(kr)/\mathrm{d}r|_{r=r_2} & \mathrm{d}K_n(kr)/\mathrm{d}r|_{r=r_2} \end{pmatrix}.$$

Table 6
A square plate with SCSC boundary conditions

i	$N=34$	$N=38$	$N=42$	$N=46$	Liessa	DSC	GDQ
1	28.9508	28.9508	28.9508	28.95085	28.951	28.953	28.951
2	54.7431	54.7431	54.7431	54.74307	54.743	54.747	54.743
3	69.3270	69.3270	69.3270	69.32701	69.327	69.337	69.327
4	94.5854	94.5853	94.58527	94.58527	94.585	94.601	94.585
5	102.213	102.212	102.2164	102.2161	102.21	102.222	102.216
6	129.0955	129.0955	129.09554	129.09554	129.09	129.130	129.096
7	140.21	140.21	140.206	140.204	140.20	140.230	140.205
8	154.7757	154.7758	154.7756	154.7757	154.77	154.823	154.776
9	170.35	170.33	170.33	170.34	—	—	—
10	199.80	199.80	199.810	199.811	—	—	—

The first 10 eigenfrequencies.

Table 7
A square plate with the SCSF boundary conditions

i	$N=34$	$N=38$	$N=42$	$N=46$	Liessa	GDQ
1	12.68735	12.68735	12.68736	12.68736	12.687	12.687
2	33.06512	33.06509	33.06509	33.06509	33.065	33.065
3	41.70121	41.70199	41.70192	41.70193	41.702	41.702
4	63.01531	63.01524	63.01480	63.01483	63.015	63.016
5	72.39758	72.39756	72.39756	72.39756	72.398	72.400
6	90.62610	90.60705	90.61915	90.60988	—	—
7	103.16207	103.16174	103.16150	103.16164	—	—
8	—	111.88123	111.90593	111.90027	—	—
9	—	131.42869	131.42870	131.42870	—	—
10	—	152.70648	152.77072	152.78119	—	—

The first 10 different eigenfrequencies.

The eigenvalues are obtained by equating the determinant of this matrix to zero.

Some results of the calculations are placed in Table 11. The letters ‘C’, ‘S’ and ‘F’ correspond to the clamped, simply supported and free edge. For example, CS denotes that the boundary $r=r_1$ is clamped and the boundary $r=r_2$ is simply supported.

Example 6. In this example, doubly connected plate with the inner hole of minimal dimension is considered. We take the simply supported boundary conditions on the both edges (SS case). The geometry of the problem is the same as in Example 5. However, here we consider the case of very small inner holes. In particular, we take $r_1=10^{-1},10^{-2},10^{-3}$ with the same fixed $r_2=2$. The Kupradze type basis functions (21), (22) are unfit to approximate solution in a neighbourhood of the hole because when the singular points, say ζ_i, ζ_j of two sources are very close, the corresponding functions $\Phi_i(\mathbf{x})$, $\Phi_j(\mathbf{x})$ become indistinguishable and the collocation matrix has two identical columns. Here we use a combined basis which includes the trial functions (28), (29) with the singular points placed on an auxiliary circular contour outside the solution domain and a multipole expansion with the origin at the center of the hole. Thus, we look for an

Table 8
A square plate with the SFSF boundary conditions

i	$N=34$	$N=38$	$N=42$	$N=46$	Liessa	GDQ
1	9.63138488	9.63138485	9.63138485	9.63138485	9.631	9.631
2	16.13478	16.134777	16.1347769	16.1347769	16.135	16.135
3	36.725649	36.725644	36.725642	36.725642	36.726	36.726
4	38.9445	38.9449	38.944955	38.944958	38.945	38.945
5	46.739	46.738	46.73812	46.73815	46.738	46.738
6	70.7407	70.7405	70.7402	70.7401	—	—
7	75.2833	75.28338	75.28338	75.28338	—	—
8	88.1	87.98	87.9854	87.9854	—	—
9	96.48	96.03	96.04	96.037	—	—
10	111.0287	111.0253	111.0253	111.0253	—	—

The first 10 different eigenfrequencies.

Table 9
A square plate with the SSSF boundary conditions

i	$N=34$	$N=38$	$N=42$	$N=46$	Liessa	GDQ
1	11.6845364	11.6845367	11.6845367	11.6845367	11.685	11.685
2	27.75633	27.756344	27.756344	27.756344	27.756	27.756
3	41.1960	41.1965	41.196647	41.196651	41.197	41.197
4	59.066	59.065	59.06548	59.06550	59.066	59.066
5	61.8606	61.8606	61.8606	61.8606	61.861	61.861
6	90.35	90.289	90.301	90.295	—	—
7	94.4842	94.4838	94.4836	94.4836	—	—
8	108.980	108.913	108.927	108.914	—	—
9	115.6862	115.6857	115.68572	115.68573	—	—
10	134.96	145.570	145.634	145.647	—	—

The first 10 eigenfrequencies.

Table 10
A square plate with CCCS boundary conditions

i	$N=34$	$N=38$	$N=42$	$N=46$	Liessa	DSC	GDQ
1	31.82	31.822	31.824	31.825	31.829	31.828	31.826
2	63.35	63.34	63.337	63.335	63.347	63.338	63.331
3	71.05	71.07	71.07	71.07	71.084	71.087	71.076
4	100.81	100.81	100.77	100.78	100.83	100.815	100.792
5	116.36	116.36	116.34	116.35	116.40	116.376	116.357
6	130.35	130.33	130.333	130.339	130.37	130.388	130.351
7	151.89	151.86	151.94	151.91	—	151.938	151.893
8	159.48	159.48	159.444	159.446	—	159.534	159.476
9	189.80	189.767	189.766	189.768	—	—	—
10	—	209.37	209.33	209.37	—	—	—

The first 10 eigenfrequencies.

approximate solution in the form

$$w(\mathbf{x}|\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2) = w_p(\mathbf{x}) + \sum_{n=1}^{N} q_{1,n}\Phi_n^{(1)}(\mathbf{x}) + \sum_{n=1}^{N} q_{2,n}\Phi_n^{(2)}(\mathbf{x}) + \sum_{m=1}^{M} p_{1,m}\Psi_m^{(1)}(\mathbf{x}) + \sum_{m=1}^{M} p_{2,m}\Psi_m^{(2)}(\mathbf{x}).$$

The data presented in Table 12 correspond to the number of sources on the outer auxiliary circular contour $N=40$. The number of terms in multipole expansion M varies from $M=11(r_1=10^{-1})$ to $M=5(r_1=10^{-3})$. The exciting source is placed at the position $\zeta_{ext}=(5,5)$. One should take $\Delta k = 0.0005$ in the algorithm (A) to separate the very close eigenvalues: $k_1^{(ex)}=1.9129217513$ and $k_2^{(ex)}=1.9199981242$ (see data corresponding to $r_1=10^{-3}$).

Example 7. Consider the elliptic plate $\{x,y|\ x=0.5\cos t,\ y=\sin t,\ 0\leqslant t\leqslant 2\pi\}$ with the clamped edge. The results placed in Table 13 are obtained using the MFS sources posed on the circle with the radius 2.

Note that the technique presented in the paper can also be applied to problems of stability. As a simple example, let us consider the following eigenvalue problem arising in stability analysis of a homogeneous beam with simply supported endpoints:

$$\frac{d^4 w}{dx^4} + k^2\frac{d^2 w}{dx^2} = 0, \quad w(0) = \frac{d^2 w(0)}{dx^2} = w(1) = \frac{d^2 w(1)}{dx^2} = 0.$$

Splitting the operator of the problem

$$\frac{d^4}{dx^4} + k^2\frac{d^2}{dx^2} = \frac{d^2}{dx^2}\left(\frac{d^2}{dx^2} + k^2\right)$$

Table 11
Annular plate with different boundary conditions

CC		CS		CF	
$k_i^{(ex)}$	e_r	$k_i^{(ex)}$	e_r	$k_i^{(ex)}$	e_r
3.1401504659	3×10^{-12}	2.5443791172	2×10^{-11}	1.2022801831	2×10^{-10}
3.1951697814	2×10^{-8}	2.6168933961	1×10^{-9}	1.3046446271	2×10^{-8}
3.3866059374	1×10^{-7}	2.8584277218	4×10^{-9}	1.6390304939	3×10^{-7}
3.7421791759	2×10^{-7}	3.2737822106	1×10^{-9}	2.1464219799	2×10^{-6}
4.2273595379	3×10^{-7}	3.8005768866	9×10^{-9}	2.7106796721	4×10^{-6}
SS		SC		SF	
2.5443791172	2×10^{-11}	2.7388244961	6×10^{-12}	0.9218818071	2×10^{-11}
2.6168933961	1×10^{-9}	2.8419404702	1×10^{-7}	1.0654228033	3×10^{-7}
2.8584277218	4×10^{-9}	3.1546693532	3×10^{-7}	1.5367565717	6×10^{-8}
3.2737822106	1×10^{-9}	3.6318734200	6×10^{-7}	2.1189079314	8×10^{-7}
3.8005768866	9×10^{-9}	4.1880284517	1×10^{-6}	2.6366356355	4×10^{-11}
FF		FC		FS	
0.9278958281	4×10^{-7}	1.6509548406	2×10^{-11}	1.0792983840	2×10^{-11}
1.4469238878	4×10^{-11}	2.2423389731	1×10^{-6}	1.8520671184	2×10^{-7}
1.5078856902	2×10^{-7}	2.9555098845	2×10^{-7}	2.5441700655	3×10^{-8}
2.1140630213	7×10^{-9}	3.4284188267	8×10^{-12}	2.9129783079	1×10^{-11}
2.2162832243	8×10^{-8}	3.5787033018	2×10^{-6}	3.1653838093	2×10^{-7}

The relative errors in calculation of the first five eigenvalues. The number of the MFS sources $N = 40$; ε procedure with $\varepsilon = 10^{-6}$.

Table 12
Circular plate with a small hole

i	$r_1 = 0.1,\ N = 40,\ M = 11$		$r_1 = 0.01,\ N = 40,\ M = 7$		$r_1 = 0.001,\ N = 40,\ M = 5$	
	$k_i^{(ex)}$	e_r	$k_i^{(ex)}$	e_r	$k_i^{(ex)}$	e_r
1	1.8879730775	5×10^{-11}	1.9172920792	1×10^{-10}	1.9129217513	3×10^{-5}
2	1.9952521601	4×10^{-7}	1.9354740768	4×10^{-7}	1.9199981242	9×10^{-10}
3	2.5313745879	2×10^{-7}	2.5275748228	2×10^{-7}	2.5275378847	2×10^{-7}
4	3.1583578383	8×10^{-8}	3.1583129875	8×10^{-8}	3.1583129831	3×10^{-4}
5	3.4952779483	2×10^{-10}	3.5086948861	5×10^{-11}	3.5147880268	4×10^{-9}

Simply supported edges. The outer radius: $r_2 = 2$; the relative errors in calculation of the first five eigenvalues. $\varepsilon = 10^{-5}$.

Table 13
Elliptic plate with clamped edge

i	$N = 25$	$N = 30$	$N = 35$	$N = 40$
1	5.2323455	5.2323454	5.2323454	5.2323454
2	6.2847	6.284698	6.2846982	6.2846982
3	7.4817	7.481689	7.4816890	7.4816892
4	8.358108	8.358109	8.3581090	8.3581090
5	8.7749	8.77468	8.7746855	8.7746854
6	9.3835	9.383357	9.3833562	9.3833562
7	10.132	10.1315	10.131428	10.131428

The first seven eigenvalues. ε procedure with $\varepsilon = 10^{-6}$.

one gets two singular solutions:

$$\Phi_1(x,\xi) = \exp(\mathrm{i}k|x-\xi|), \quad \Phi_2(x,\xi) = |x-\xi|.$$

Applying the algorithm described above we get the results which are placed in Table 14. Here the exact solution is $k_i^{(ex)} = \mathrm{i}\pi$.

3.2. Eigenmodes

The algorithm described above is focused on the problem of finding eigenvalues or eigenfrequencies of free vibrations. Let us dwell in brief on the problem of calculation of the corresponding eigenmodes. The method of finding eigenmodes proposed in the paper, following the mechanical analogy, is based on the simple physical fact that when a mechanical system approaches to resonance then just the resonance (or eigen) mode is excited in the system.

So, if k_{ex} is an approximate eigenvalue which is found as an extremum of $F(k)$, then the corresponding solution

$w(\mathbf{x}, k_{ex})$ of (19) is close to the eigenmode. The modes presented in Fig. 4 correspond to the first and third eigenvalues of the problem described in Example 1. The figures are obtained using the scaled data $(x_i, \widehat{u}_i(k))$, $i = 1, \ldots, N_f$. Here

$$\widehat{u}_i(k) = \mathrm{Re}[w(x_i, k)]/U(k), \quad U(k) = \sqrt{\frac{1}{N_f}\sum_{i=1}^{N_f} \mathrm{Re}[w(x_i, k)]^2}.$$

The calculations were performed with $k = 3.926602312$ and 10.21017612.

The same method was applied in a 2D case. The first, fourth, seventh and 10th eigenmodes of free vibrations of the square plate with SCSF boundary conditions (see Table 7) are presented in Fig. 5.

Table 14
1D eigenproblem $w^{(4)} + k^2 w^{(2)} = 0$, $w(0) = w^{(2)}(0) = w(1) = w^{(2)}(1) = 0$

$k_i^{(ex)}$	$\varepsilon = 0.1$	$\varepsilon = 10^{-3}$	$\varepsilon = 10^{-6}$
π	3×10^{-6}	3×10^{-10}	1×10^{-13}
2π	2×10^{-7}	3×10^{-11}	3×10^{-12}
3π	4×10^{-8}	4×10^{-12}	9×10^{-13}
4π	1×10^{-8}	3×10^{-13}	8×10^{-13}
5π	5×10^{-9}	7×10^{-13}	2×10^{-12}
6π	2×10^{-9}	2×10^{-13}	3×10^{-12}
7π	1×10^{-9}	1×10^{-13}	4×10^{-13}
8π	8×10^{-10}	2×10^{-12}	1×10^{-13}
9π	5×10^{-10}	1×10^{-12}	3×10^{-13}
10π	3×10^{-10}	5×10^{-13}	2×10^{-12}

The relative errors in calculations of the eigenvalues.

4. Concluding remarks

In this paper, a new boundary method for the problem of free vibrations of plates is proposed. This is a mathematical model of physical measurements, when a mechanical or acoustic system is excited by an external source and resonance frequencies can be determined by the growth of amplitude of oscillations near these frequencies.

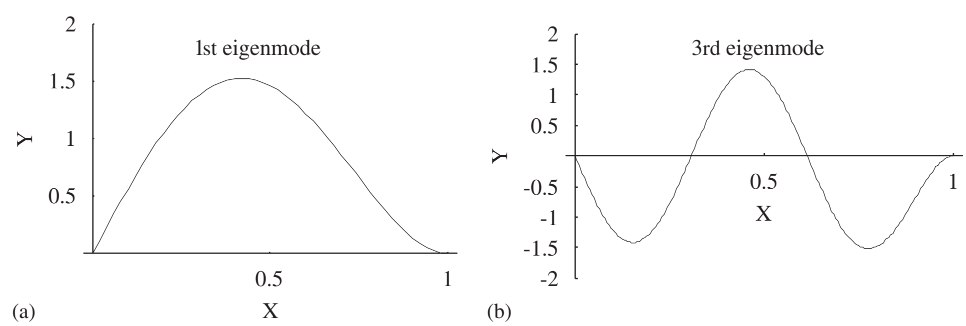

Fig. 4. The first and third eigenmodes for beam with SC boundary conditions.

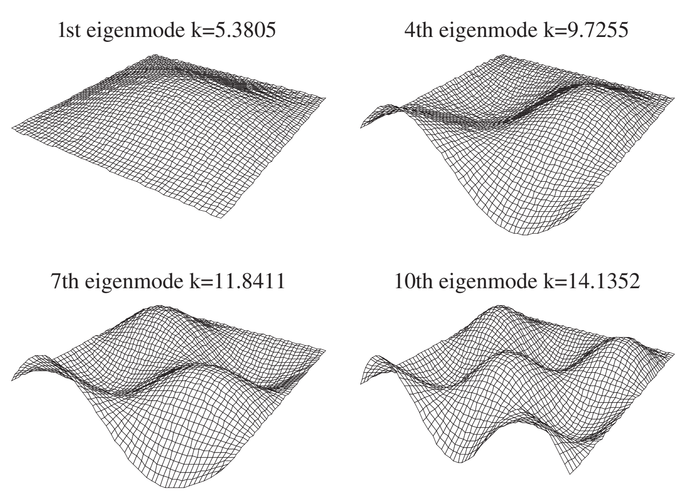

Fig. 5. The first, fourth, seventh and 10th eigenmodes for the square plate with SCSF boundary conditions.

The calculation of the eigenvalue problem is reduced to a sequence of inhomogeneous problems with the differential operator studied. The method shows a high precision in simply and doubly connected domains. The method is convenient for determining some first eigenvalues of the system which are often of the most interest from the point of view of engineering applications.

The method presented is based on the MFS solution of the problem. However, it can be combined with other boundary techniques. For instance, the boundary knot method (BKM) [35,36] seems to be perspective in this connection. An example of application of BKM to the Helmholtz eigenproblem is presented in [27]. Generally speaking, any effective numerical technique can be used as a solver of BVP (19). For example, the DSC and GDQ techniques mentioned above also can be used in the framework of this approach.

Acknowledgement

The author acknowledges the support of NATO Collaborative Linkage Grant under reference PST.CLG.980398.

Appendix A. The boundary conditions for plates in Cartesian and polar coordinates

Here we follow [1]. Let $\partial\Omega$ be a piecewise smooth curve with the outward normal vector $\mathbf{n} = (n_x, n_y)$. Let us define the moments of deflection M_n, the twisting moment M_{nt} and the intersecting force Q_n acting on the edge in the following way:

$$M_n = M_x n_x + M_y n_y - 2M_{xy} n_x n_y, \tag{A.1}$$

$$M_{nt} = M_{xy}(n_x^2 - n_y^2) + (M_x - M_y) n_x n_y, \tag{A.2}$$

$$Q_n = Q_x n_x + Q_y n_y, \tag{A.3}$$

where

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right), \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \upsilon \frac{\partial^2 w}{\partial x^2}\right),$$

$$M_{xy} = -M_{yx} = D(1-\upsilon)\frac{\partial^2 w}{\partial x \partial y}, \tag{A.4}$$

$$Q_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \quad Q_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right),$$

$$D = \frac{Eh^3}{12(1-v^2)} \tag{A.5}$$

and υ is Poisson's ratio. It is taken $\upsilon = 0.3$ throughout the paper.

The following three types of boundary conditions are considered:

clamped edge (C):

$$w = 0, \quad \frac{\partial w}{\partial n} = 0, \tag{A.6}$$

simply supported edge (S):

$$w = 0, \quad M_n = 0, \tag{A.7}$$

free edge (F):

$$M_n = 0, \quad Q_n - \frac{\partial M_{nt}}{\partial s} = 0. \tag{A.8}$$

Here the normal and tangential derivatives are denoted:

$$\frac{\partial}{\partial n} = n_x \frac{\partial}{\partial x} + n_y \frac{\partial}{\partial y}, \quad \frac{\partial}{\partial s} = n_x \frac{\partial}{\partial y} - n_y \frac{\partial}{\partial x}.$$

In particular, for Cartesian coordinate system one gets the following conditions:

clamped edge (C):

$$w|_{x=c} = \left.\frac{\partial w}{\partial x}\right|_{x=c} = 0, \quad w|_{y=c} = \left.\frac{\partial w}{\partial y}\right|_{y=c} = 0, \tag{A.9}$$

simply supported edge (S):

$$w|_{x=c} = \left.\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)\right|_{x=c} = 0,$$

$$w|_{y=c} = \left.\left(\frac{\partial^2 w}{\partial y^2} + \upsilon \frac{\partial^2 w}{\partial x^2}\right)\right|_{y=c} = 0$$

because $\partial w/\partial y|_{x=c} = \partial^2 w/\partial y^2|_{x=c} = 0$ and $\partial w/\partial x|_{y=c} = \partial^2 w/\partial x^2|_{y=c} = 0$ this condition can be simplified:

$$w|_{x=c} = \left.\frac{\partial^2 w}{\partial x^2}\right|_{x=c} = 0, \quad w|_{y=c} = \left.\frac{\partial^2 w}{\partial y^2}\right|_{y=c} = 0, \tag{A.10}$$

free edge (F):

$$\left.\left(\frac{\partial^2 w}{\partial x^2} + \upsilon \frac{\partial^2 w}{\partial y^2}\right)\right|_{x=c} = \left.\left(\frac{\partial^3 w}{\partial x^3} + (2-\upsilon)\frac{\partial^3 w}{\partial x \partial y^2}\right)\right|_{x=c} = 0,$$

$$\left.\left(\frac{\partial^2 w}{\partial y^2} + \upsilon \frac{\partial^2 w}{\partial x^2}\right)\right|_{y=c} = \left.\left(\frac{\partial^3 w}{\partial y^3} + (2-\upsilon)\frac{\partial^3 w}{\partial y \partial x^2}\right)\right|_{y=c} = 0. \tag{A.11}$$

Using the polar coordinates (r, θ) one gets the following conditions on the boundary of a circle $\{r = a, 0 \leqslant \theta \leqslant 2\pi\}$:

clamped edge (C):

$$w(r,\theta)|_{r=a} = \left.\frac{\partial w}{\partial r}\right|_{r=a} = 0, \tag{A.12}$$

simply supported edge (S):

$$w(r,\theta)|_{r=a} = \left.\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + v\frac{\partial^2 w}{\partial r^2}\right]\right|_{r=a} = 0 \tag{A.13}$$

and free edge (F):

$$\left.\left[\frac{1}{r}\frac{\partial w}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w}{\partial \theta^2} + v\frac{\partial^2 w}{\partial r^2}\right]\right|_{r=a} = \left[\frac{\partial^3 w}{\partial r^3} + \frac{1}{r}\frac{\partial^2 w}{\partial r^2} - \frac{1}{r^2}\frac{\partial w}{\partial r} + \frac{2-v}{r}\frac{\partial^3 w}{\partial \theta^2 \partial r} \left. - \frac{3-v}{r^3}\frac{\partial^2 w}{\partial \theta^2}\right]\right|_{r=a} = 0. \tag{A.14}$$

References

[1] Timoshenko S, Woinowsky-Kriger S. Theory of plates and shells. New York, Toronto, London: McGraw-Hill Book Company; 1959.

[2] Leissa AW. The free vibration of rectangular plates. J Sound Vib 1973;31:257–93.

[3] Leissa AW. Vibration of plates. New York: Acoustical Society of America, American Institute of Physics; 1993.

[4] Courant R. Variational methods for the solution of problems of equilibrium and vibrations. Bull Amer Math Soc 1943;43:1–23.

[5] Courant R, Hilbert D. Methods of mathematical physics. New York: Wiley Interscience; 1953.

[6] Morse PM, Feshbach H. Methods of theoretical physics. New York: McGraw-Hill; 1953.

[7] Golub GH, Loan CFV. Matrix computations. Baltimore, MD: Johns Hopkins University Press; 1996.

[8] Strang G. Linear algebra and its applications. New York: Academic Press; 1976.

[9] Bhat RB. Natural frequencies of rectangular plates using characteristic orthogonal polynomials in Rayleigh–Ritz method. J Sound Vib 1985;102:493–9.

[10] Liew KM, Lam KY. A set of orthogonal plate functions for vibration analysis of regular polygonal plates. Trans Amer Soc Mech Eng: J Vib Acoust 1991;113:182–6.

[11] Leung AYT, Chan JKW. Fourier p-element for analysis of beams and plates. J Sound Vib 1998;212:179–85.

[12] Strang G, Fix G. An analysis of the finite element method. Englewood Cliffs, NJ: Prentice-Hall; 1973.

[13] Zienkiewicz OC, Taylor RL. The finite element method. New York: McGraw-Hill; 1989.

[14] Bellman R, Kashef BG, Casti J. Differential quadrature: a technique for the rapid solution of non-linear partial differential equations. J Comput Phys 1972;10:40–52.

[15] Shu C, Du H. A generalized approach for implementing general boundary conditions in the GDQ free vibration analysis of plates. Int J Solids Struct 1997;34:837–46.

[16] Shu C, Du H. Implementation of clamped and simply supported boundary conditions in the GDQ free vibration analysis of beams and plates. Int J Solids Struct 1997;34:819–35.

[17] Wei GW. Vibration analysis by discrete singular convolution. J Sound Vib 2001;244:535–53.

[18] Ng CHW, Zhao YB, Wei GW. Comparison of discrete singular convolution and generalized differential quadrature for the vibration analysis of rectangular plates. Comput Methods Appl Mech Eng 2004;193:2483–506.

[19] Brebbia CA, Telles JCF, Wrobel LC. Boundary element techniques—theory and applications in engineering. Berlin: Springer; 1984.

[20] Fairweather G, Karageorghis A. The method of fundamental solutions for elliptic boundary value problems. Adv Comput Math 1998;9:69–95.

[21] Golberg MA, Chen CS. Discrete projection methods for integral equations. Southampton: Computational Mechanics Publications; 1997.

[22] Karageorghis A. The method of fundamental solutions for the calculation of the eigenvalues of the Helmholtz equation. Appl Math Lett 2001;14:837–42.

[23] Chen JT, Lin JH, Kuo SR, Chyuan SW. Boundary element analysis for the Helmholtz eigenvalues problems with a multiply connected domain. Proc R Soc London A 2001;457:2521–46.

[24] Chen JT, Liu LW, Hong HK. Spurious and true eigensolutions of Helmholtz BIEs and BEMs for a multiply connected problem. Proc R Soc London A 2003;459:1897–924.

[25] Chen JT, Chen IL, Chen KH, Lee YT, Yeh YT. A meshless method for free vibration analysis of circular and rectangular clamped plates using radial basis function. Eng Anal Bound Elem 2004;28: 535–45.

[26] Chen JT, Chen IL, Lee YT. Eigensolutions of multiply connected membranes using the method of fundamental solutions. Eng Anal Bound Elem 2005;29:166–74.

[27] Reutskiy SY. The method of fundamental solutions for eigenproblems with Laplace and biharmonic operators. CMC: Comput Mater Continua 2005;2:177–88.

[28] Reutskiy SY. The method of fundamental solutions for Helmholtz eigenvalue problems in simply and multiply connected domains. Eng Anal Bound Elem 2006;30:150–9.

[29] Reutskiy SY. The method of external sources (MES) for eigenvalue problems with Helmholtz equation. CMES: Comput Modeling Eng Sci 2006;12:27–39.

[30] Press WH, Teukolsky SA, Vetterling WT, Flannery BP. Numerical recipescipes in C++. 2nd ed. Cambridge: Cambridge University Press; 2002.

[31] Brent RP. Algorithms for minimization without derivatives. Englewood Cliffs, NJ: Prentice-Hall; 1973.

[32] Kupradze VD. On approximate solution of problems in mathematical physics. Russ Math Surv 1967;22:58–108.

[33] Vekua IN. New methods for solving elliptic equations. Amsterdam: North-Holland; 1967.

[34] Hafner C. The generalized multipole technique for computational electromagnetics. Boston: Artech House Books; 1990.

[35] Chen W. Symmetric boundary knot method. Eng Anal Bound Elem 2002;26:489–94.

[36] Chen W, Tanaka M. A meshless, exponential convergence, integration-free, and boundary-only RBF technique. Comput Math Appl 2002;43:379–91.