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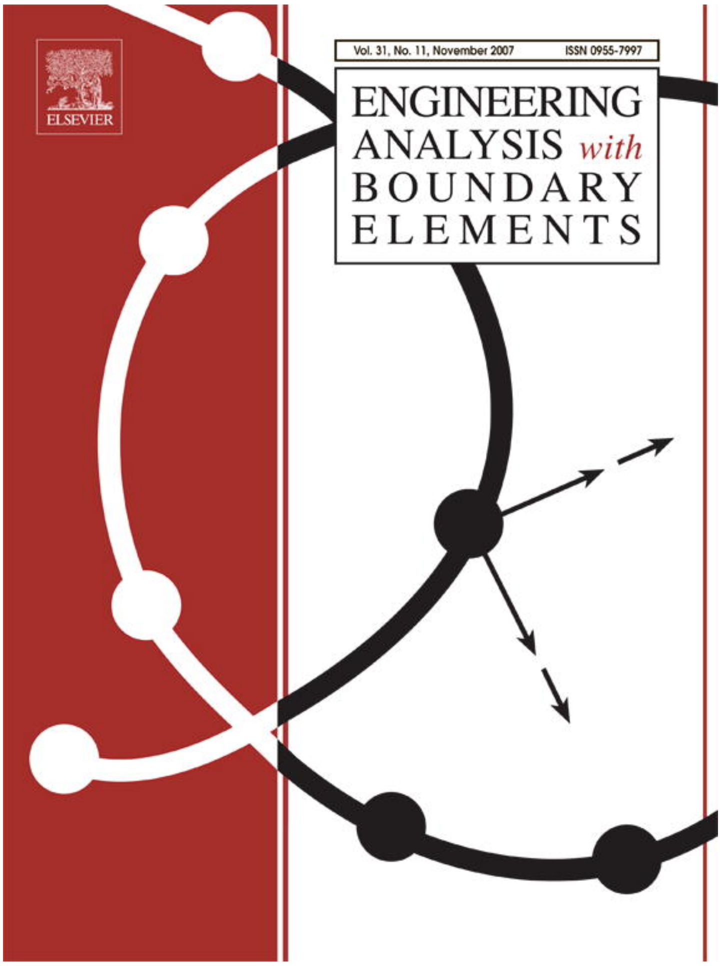

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The methods of external and internal excitation for problems of free vibrations of non-homogeneous membranes

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Abstract

In this paper a new numerical technique for problems of free vibrations of non-homogeneous membranes: $\nabla^2 w + k^2 q(\mathbf{x})w = 0$, $\mathbf{x} \in \Omega \subset \mathscr{R}^2$, $B[w] = 0, \mathbf{x} \in \partial\Omega$ is presented. The method is based on mathematically modelling of physical response of a system to excitation over a range of frequencies. The response amplitudes are then used to determine the resonant frequencies. Two versions of the method are described. The results of the numerical experiments justifying the method are presented.
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Keywords: Eigenvalues; Non-homogeneous membranes; Kansa's method

1. Introduction

The paper presents a new meshless numerical technique for problems of free vibrations of non-homogeneous membranes with continuously varying properties. We deal with the following 2D eigenvalue problem:

$$\nabla^2 w + k^2 q(\mathbf{x})w = 0, \quad \mathbf{x} \in \Omega \subset \mathscr{R}^2, \qquad B[w] = 0, \quad \mathbf{x} \in \partial\Omega. \tag{1}$$

Here, Ω is a simply or multiply connected domain with boundary $\partial\Omega$, and the density function $q>0$ is smooth enough in Ω. The boundary operator $B[...]$ specifies the boundary conditions. The problem of free vibration is to find such real k for which there exist non-null functions w verifying (1). This problem is important as a component in the design of many engineering devices: microphones, loudspeakers, pumps, compressors, pressure regulators, etc.

A general review of the dynamic aspects of membranes can be found in the review paper by Mazumdar [1]. It should be noted that the literature on the vibration of non-homogeneous membranes is not extensive. Masad [2] solved the problem mentioned above by the finite difference method and the perturbation method. Laura with co-authors [3] solved the same problem by the optimized Galerkin–Kantorovitch approach and the differential quadrature method. A closed form of exact solution of non-homogeneous membrane with the density function which varies linearly with respect to an edge $q = c + dx$ is found in [4]. An exact solution of non-homogeneous annular membrane with $q = c/r^2$ is also reported here. The fundamental frequencies of the circular membrane with the density which is a sinusoidal function of the radius are studied in [5]. In [6] four numerical techniques (1) the differential quadrature method; (2) the finite element technique; (3) the optimized and/or improved Rayleigh quotient method and (4) the Stodola Vianello iterative method are compared in application to the problem of free vibration of non-homogeneous annular membrane. The three density functions are considered: $q = 1 + \alpha r^\gamma, \gamma = \frac{1}{2}, 1, \frac{3}{2}$. However, only the first two axisymmetric vibration modes are calculated here. In [7] exact solutions for both the axisymmetric and antisymmetric modes of circular and annular membranes with any polynomial variation of the density are given using a power series solution. The data placed in this paper are used as benchmark problems in the paper presented. In [8] a hybrid method composed of differential transforms and

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the Kantorovitch method is introduced to solve the above-referenced problems.

In this paper the method proposed earlier for homogeneous membranes [9–12] is extended on to the general case. The method described here can be interpreted as a mathematical model of physical measurements when the resonant frequencies of a system are determined by the amplitude of response to some excitation.

Let $w_e(\mathbf{x})$ be a smooth enough function defined in the solution domain below named as the *exciting field*. If the *response field* w_r is a solution of the boundary value problem

$$\nabla^2 w_r + k^2 q(\mathbf{x}) w_r = -\nabla^2 w_e - k^2 q(\mathbf{x}) w_e,$$
$$B[w_r] = -B[w_e], \tag{2}$$

then the sum $w(\mathbf{x}, k) = w_r + w_e$ satisfies the initial problem (1). Let $F(k)$ be some norm of the solution w. This function of k has extremums at the eigenvalues and, under some conditions described below, can be used for their determining.

Generally, we do not impose any conditions on w_e. However, when $q = const$ (homogeneous membrane), the exciting field can be chosen in such a way that the right-hand side of (2) is equal to zero: $\nabla^2 w_e + k^2 q w_e = 0$. It can be taken in a simple analytic form, e.g., in the form of a travelling wave $w_e = \exp[ikq^{1/2}(\cos\theta x + \sin\theta y)]$. Here $0 \leqslant \theta \leqslant 2\pi$ is the angle of incidence i.e., the one between direction of the wave-propagation and the x-axis. Note that in this case the *response field* w_r satisfies the homogeneous equation too. And the PDE has the known fundamental solutions $\Phi(\mathbf{x} - \boldsymbol{\zeta}) = H_0^{(1)}(kq^{1/2}|\mathbf{x} - \boldsymbol{\zeta}|)$, where $H_0^{(1)}$ is the Hankel function. This admits of applying a very effective meshless numerical technique—the method of the fundamental solutions (MFS). This version of the method with an *external* exciting source is described in [9–12] in application to problems in homogeneous mediums. However, this technique loses its attraction when applied to inhomogeneous problems. As it is shown in Section 2, to find the *exciting field* w_e we have to solve a scattering problem. A new version of the method with an *internal* exciting source is presented in Section 3. Finally, in Section 4, we give the conclusion and some directions for developing the method suggested.

2. External excitation

2.1. Homogeneous problems

To illustrate the method presented in the simplest case let us consider the wave equation in homogeneous medium

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2} \tag{3}$$

with the Dirichlet conditions at the endpoints of the interval $[0, 1]$, i.e., $u(0, t) = u(1, t) = 0$. This equation describes free vibrations of the homogeneous string [13]. Considering the harmonic vibrations $u(x, t) = e^{ikt} w(x)$, we get the eigenvalue problem on the interval $[0, 1]$

$$\frac{d^2 w}{dx^2} + k^2 w = 0, \quad w(0) = w(1) = 0. \tag{4}$$

The well known solution is $k_n = n\pi$, $w_n = \sin(n\pi x)$, $n = 1, 2, \ldots, \infty$.

According to the method presented we consider the equation for the response field w_r,

$$\frac{d^2 w_r}{dx^2} + k^2 w_r = -\frac{d^2 w_e}{dx^2} - k^2 w_e. \tag{5}$$

We take the exciting field w_e in the form

$$w_e(x) = \exp(ikx), \tag{6}$$

which satisfies

$$\frac{d^2 w_e}{dx^2} + k^2 w_e = 0.$$

This exciting field corresponds to a travelling wave which propagates from $+\infty$ to $-\infty$. Another admissible variant is

$$w_e(x) = \exp(-ik|x - \varsigma_s|), \tag{7}$$

which corresponds to the point source placed at $x = \varsigma_s \notin [0, 1]$.

Under this condition we get the following BVP for w_r:

$$\frac{d^2 w_r}{dx^2} + k^2 w_r = 0, \quad w_r(0) = -w_e(0),$$
$$w_r(1) = -w_e(1). \tag{8}$$

The sum $w = w_e + w_r$ satisfies the initial BVP (4). Let us introduce the norm of the solution as

$$F(k) = \sqrt{\frac{1}{N}\sum_{n=1}^{N} |w(x_n)|^2}, \tag{9}$$

where the points x_n are randomly distributed in $[0, 1]$. In all the calculations presented in this section we use $N = 7$. This function characterizes the value of the response of the system to the outer excitation. We also use the dimensionless form of this function: $F_d(k) = F(k)/F(1)$. The graph of the function corresponding to (6) is depicted in Fig. 1. It

Fig. 1. The response curve for $\varepsilon = 0$, $\Delta k = 0$.

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demonstrates the quasi-random distribution near zero. Indeed, looking for the response field in the form

$$w_r = A_r \exp(\mathrm{i}kx) + B_r \exp(-\mathrm{i}kx),$$

one gets the linear system

$$A_r + B_r = -1,$$
$$A_r \exp(\mathrm{i}k) + B_r \exp(-\mathrm{i}k) = -\exp(\mathrm{i}k). \quad (10)$$

For $k \neq n\pi$ the system has the unique solution $A_r = -1$, $B_r = 0$. Thus, $w \equiv 0$ and $F(k) = 0$ with the precision error.

Now we describe two regularizing procedures which give a smooth response curve. We substitute BVP (8) by the following one:

$$\frac{\mathrm{d}^2 w_r}{\mathrm{d}x^2} + (k^2 + \mathrm{i}\varepsilon k)w_r = 0, \quad w_r(0) = -w_e(0),$$
$$w_r(1) = -w_e(1), \quad (11)$$

where $\varepsilon > 0$ is a small value. From the mathematical point of view this means that we shift the spectra of differential operator from the real axis. On the other hand, from the physical point of view, this means that the initial equation (3) is replaced by the equation $\partial_{tt}^2 u = \partial_{xx}^2 u - \varepsilon \partial_t u$ which describes vibrations of a homogeneous string with friction [13]. As a result, instead of (10) we get the system:

$$A_r + B_r = -1,$$
$$A_r \exp(\mathrm{i}k_\varepsilon) + B_r \exp(-\mathrm{i}k_\varepsilon) = -\exp(\mathrm{i}k), \quad (12)$$

where $k_\varepsilon = \sqrt{k^2 + \mathrm{i}\varepsilon k}$ and the branch $\mathrm{Re}(k_\varepsilon) > 0$ is taken. The system has a unique non-zero solution for all real k. In Figs. 2–4 we place the dimensionless function $F_d(k)$ corresponding to $\varepsilon = 10^{-15}$, 10^{-10} and 10^{-1}.

The value $\varepsilon = 10^{-15}$ is too small to regularize the solution. The response curve $F_d(k)$ has separate maximums at the positions of eigenvalues but still it is not smooth as it is shown in more details in the right part of Fig. 2. The value $\varepsilon = 10^{-10}$ provides a smooth curve with separated maximums at the positions of eigenvalues (Fig. 3). This admits of using the following simple algorithm [9–12]. First, we localize these maxima of $F_d(k)$ on the intervals $[a_i, b_i]$. Next, we solve the univariate optimization problem inside each one. In particular, we apply Brent's method based on a combination of parabolic interpolation and bisection of the function near to the extremum (see [15,14]).

In Table 1 we place the relative errors

$$e_r = \frac{|k_i - k_i^{(ex)}|}{k_i^{(ex)}} \quad (13)$$

in the calculation of the first five eigenvalues. The regularizing parameter ε coarsens the system. The peaks at the positions of the eigenvalues have decreased and become wider (Fig. 4). This worsens their localization and increases the errors in calculation of k_i. So, from the practical point of view, one should take the parameter ε as small as possible but large enough to get a smooth response curve $F_d(k)$.

Another regularizing procedure can be described in the following way. Let us introduce the constant shift Δk between the wave numbers of the exciting source and the

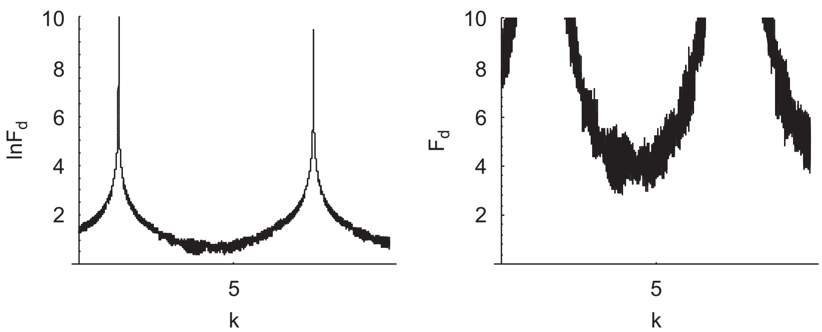

Fig. 2. The response curve; ε-procedure with $\varepsilon = 10^{-15}$.

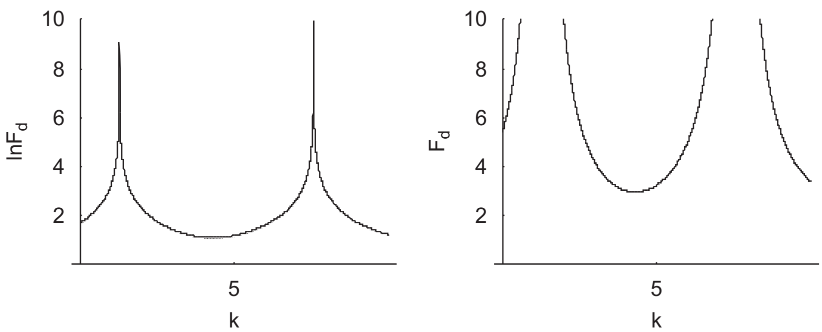

Fig. 3. The response curve; ε-procedure with $\varepsilon = 10^{-10}$.

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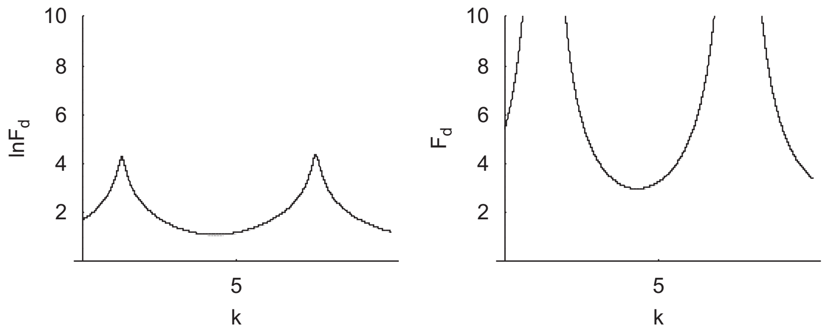

Fig. 4. The response curve; ε-procedure with $\varepsilon = 10^{-1}$.

Table 1
One dimensional eigenvalue problem

$k_i^{(ex)}$	$\varepsilon = 10^{-1}$	$\varepsilon = 10^{-10}$
π	1.2×10^{-4}	5.0×10^{-11}
2π	3.2×10^{-5}	5.8×10^{-11}
3π	1.4×10^{-5}	4.8×10^{-11}
4π	7.9×10^{-6}	3.3×10^{-11}
5π	4.6×10^{-6}	6.0×10^{-11}

The relative errors in calculations of the eigenvalues. ε-procedure.

studied mode, i.e., instead of (6), we take the exciting field in the form

$$w_e(x) = \exp(\mathrm{i}(k + \Delta k)x).$$

Now the linear system

$$A_r + B_r = -1,$$
$$A_r \exp(\mathrm{i}k) + B_r \exp(-\mathrm{i}k) = -\exp(\mathrm{i}(k + \Delta k))$$

(cf. (10)) provides non-zero solutions w for all k except the eigenvalues k_n when the system becomes degenerate. However, due to the iterative procedure of Brent's method and rounding errors we never solve the system with the *exact* k_n. We observe degeneration of the system as a considerable growth of the solution in a neighborhood of the eigenvalues.

The data corresponding to $\Delta k = 10^{-15}$ and 10^{-10} are presented in Figs. 5 and 6. The value $\Delta k = 10^{-15}$ is too small to regularize the solution. But the value $\Delta k = 10^{-10}$ yields a smooth curve. We call these two regularizing procedures described above the ε-procedure and the k-procedure.

Comparing these two procedures, it should be noted that they provide approximately the same precision in the calculations of eigenvalues. However, dealing with a real PDE and using the ε-procedure, we have to perform the calculations with complex variables. The use of the k-procedure provides the calculations with real variables only.

The same approach of an external excitation can be combined with an approximate solution of BVP (8) for the response field w_r. Numerous examples of application of this technique to different homogeneous eigenvalue problems can be found in [9–12].

2.2. Non-homogeneous problems

For the sake of simplicity we begin with the 1D eigenvalue problem:

$$\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} + k^2 q(x) w = 0, \quad w(0) = w(1) = 0, \tag{14}$$

where $q(x) > 0$ is smooth enough on $[0, 1]$.

Here, to obtain the exciting field w_e with a source placed outside $[0, 1]$, we have to use a more complex algorithm. To consider the equation for w_e we have to extend $q(x)$ from $[0, 1]$ onto a wider interval which contains the source of the field. In the previous subsection, solving the problem with the constant density function $q = 1$, we extend this constant from $[0, 1]$ onto $(-\infty, +\infty)$ automatically. And then we consider w_e in the form of the travelling wave (6) defined on the whole infinite interval $(-\infty, +\infty)$. Dealing with the inhomogeneous problem, we consider two different intervals with two different density functions: (i) the solution domain $[0, 1]$ with the continuously varying $q(x)$; (ii) the double connected domain $(-\infty, 0) \cup (1, +\infty)$, where we take $q = 1$ in order to have here a simple analytic solution in the form of a travelling wave.

The initial travelling wave $w_e^{(i)}(x) = \mathrm{e}^{\mathrm{i}kx}$ propagates from $+\infty$ and falls on the interface $x = 1$ between the regions with different density functions. This interface generates the reflected wave $\mathscr{R}\mathrm{e}^{-\mathrm{i}kx}$ which propagates from the interface $x = 1$ back to $+\infty$. Thus, the total exciting field in the domain $x > 1$ is the sum of the incident and reflected fields $w_e^{(+)}(x) = \mathrm{e}^{\mathrm{i}kx} + \mathscr{R}\mathrm{e}^{-\mathrm{i}kx}$. The exciting field inside the solution domain $w_e(x)$ satisfies the equation:

$$\frac{\mathrm{d}^2 w_e}{\mathrm{d}x^2} + k^2 q(x) w_e = 0, \quad x \in [0, 1]. \tag{15}$$

This field, partly passing through the second interface $x = 0$, generates the transmitted wave $w_e^{(-)}(x) = \mathscr{T}\mathrm{e}^{\mathrm{i}kx}$ which propagates from the second interface $x = 0$ to $-\infty$. The functions $w_e^{(-)}(x)$, $w_e(x)$, $w_e^{(+)}(x)$ and their derivatives should be matched at the interfaces $x = 0$

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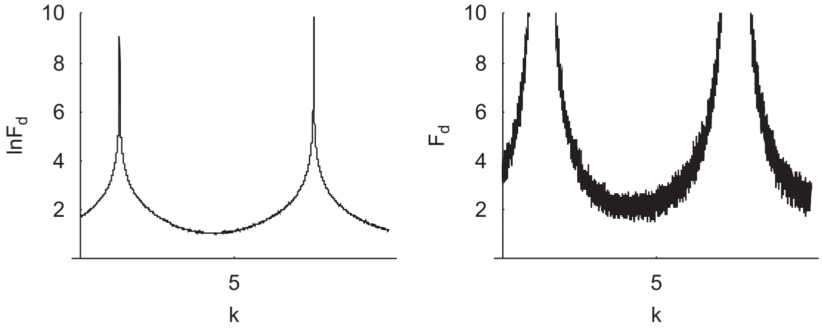

Fig. 5. The response curve; k-procedure with $\Delta k = 10^{-15}$.

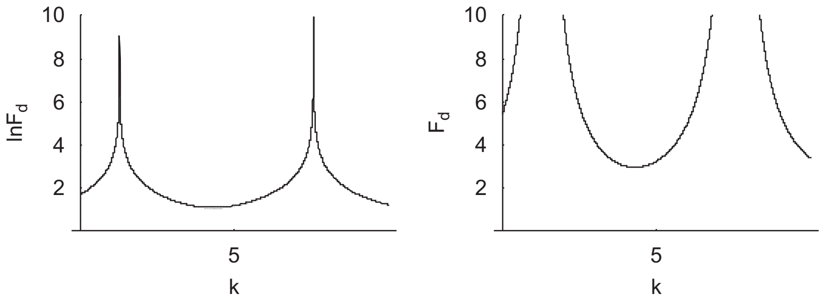

Fig. 6. The response curve; k-procedure with $\Delta k = 10^{-10}$.

and $x = 1$:

$$w_e(0^+) = w_e^{(-)}(0^-) = \mathscr{T},$$
$$\frac{dw_e}{dx}(0^+) = \frac{dw_e^{(-)}}{dx}(0^-) = ik\mathscr{T}, \tag{16}$$
$$w_e(1^-) = w_e^{(+)}(1^+) = e^{ik} + \mathscr{R}e^{-ik},$$
$$\frac{dw_e}{dx}(1^-) = \frac{dw_e^{(+)}}{dx}(1^+) = ike^{ik} - ik\mathscr{R}e^{-ik}. \tag{17}$$

The constant values $\mathscr{R}$ and $\mathscr{T}$ are the unknowns of the problem. Note that the coefficients $\mathscr{R}$ and $\mathscr{T}$ can be excluded from (16), (17) and the boundary conditions can be re-written in the form containing w_e and dw_e/dx only

$$\frac{dw_e}{dx}(0^+) = ikw_e(0^+),$$
$$\frac{dw_e}{dx}(1^-) = -ikw_e(1^-) + 2ike^{ik}. \tag{18}$$

Having w_e, we get the response field w_r as a solution of the BVP:

$$\frac{d^2w_r}{dx^2} + k^2q(x)w_r = 0, \quad w_r(0) = -w_e(0^+),$$
$$w_r(1) = -w_e(1^-). \tag{19}$$

The sum $w = w_e + w_r$ satisfies the initial BVP (14). We introduce the norm $F(k)$ like (9). Varying k, we get the response curve and calculate the eigenvalues as positions of maxima. However, without a regularizing procedure the response curve looks like the one shown in Fig. 1. To get a smooth response curve one should apply the regularizing procedures described in the previous subsection.

To solve (16)–(19) we use the asymmetric RBF collocation method (Kansa's method) [19,20]. This method is chosen as a truly meshless technique which can be extended easily onto the 2D case. According to this approach one looks for an approximate solution of the BVP

$$L[w] = f(\mathbf{x}), \ \ \mathbf{x} \in \Omega \subset R^d, \ \ d = 1, 2, 3;$$
$$B[w] = g(\mathbf{x}), \quad \mathbf{x} \in \partial\Omega \tag{20}$$

in the form of the linear combination:

$$w = \sum_{j=1}^{N+N_B} q_j \Psi(\|\mathbf{x} - \xi_j\|). \tag{21}$$

Here, ξ_j, $j = 1, \ldots, N$ are the points distributed inside the domain Ω and the points ξ_{N+i}, $i = 1, \ldots, N_B$ are placed on the boundary $\partial\Omega$; q_j are the free parameters of the problem. We use only the multiquadrics basis functions in this paper

$$\Psi(\|\mathbf{x} - \xi\|) = \sqrt{c^2 + \|\mathbf{x} - \xi\|^2}, \tag{22}$$

where c is the shape parameter.

The collocation with the right-hand side f at the interior points and with the boundary data at the boundary points leads to the linear system:

$$\begin{bmatrix} \sum_{j=1}^{N+N_B} q_j L[\Psi(\|\xi_{i_1} - \xi_j\|)] \\ \sum_{j=1}^{N+N_B} q_j B[\Psi(\|\xi_{i_2} - \xi_j\|)] \end{bmatrix} = \begin{bmatrix} f(\xi_{i_1}) \\ g(\xi_{i_2}) \end{bmatrix},$$
$$i_1 = 1, \ldots, N; \ \ i_2 = N+1, \ldots, N+N_B. \tag{23}$$

It should be noted that Kansa's method suffers from ill-conditioning with the growth of the number of the centers. The method can be optimized by choosing the shape

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parameter of RBFs, by an optimal distribution of the centers or by using a special preconditioner to solve the linear system. See, e.g., [21–23] for more details. A complete bibliography on the subject considered is given in the references of these papers. However, this is not the topic of this study.

Solving the system (16)–(17) we have $N_B = 2$ endpoints $x = 0$ and 1. So, we denote $\xi_{N+1} = 0$, $\xi_{N+2} = 1$ and the whole number of the RBF functions in (21) is $N + 2$. The number of the unknowns q_j is the same. Besides, we have two additional unknowns: the coefficients $\mathscr{R}$ and $\mathscr{T}$. So, we set up $N + 4$ equations. Let us denote

$$L(x,k)[\Psi(\|x-\xi\|)] = \left(\frac{d^2}{dx^2} + k^2 q(x)\right)[\Psi(\|x-\xi\|)].$$

The collocation at the inner points gives us the following N equations:

$$\sum_{j=1}^{N+2} q_j L(\xi_i, k)[\Psi(\|\xi_i - \xi_j\|)] = 0, \quad i = 1, \ldots, N.$$

The rest four equations follow from the boundary conditions:

$$\sum_{j=1}^{N+2} q_j \Psi(\|\xi_{N+1} - \xi_j\|) - \mathscr{T} = 0,$$

$$\sum_{j=1}^{N+2} q_j \frac{d}{dx}\Psi(\|\xi_{N+1} - \xi_j\|) - ik\mathscr{T} = 0,$$

$$\sum_{j=1}^{N+2} q_j \Psi(\|\xi_{N+2} - \xi_j\|) - \mathscr{R}e^{-ik} = e^{ik},$$

$$\sum_{j=1}^{N+2} q_j \frac{d}{dx}\Psi(\xi_{N+2}, \xi_j) + ik\mathscr{R}e^{-ik} = ike^{ik}.$$

The $(N+4)\times(N+4)$ system is solved by a standard procedure of the Gauss elimination. The response field w_r is found by the similar procedure using the Dirichlet boundary conditions (see (19)).

Example 1. The results presented in Table 2 and depicted in Fig. 7 correspond to the density function

$$q(x) = \tfrac{1}{4}(\alpha+2)^2[(\alpha^2+2\alpha)x+1]$$

Table 2
Relative errors in the solution of the eigenvalue problem $d^2_{xx}w + k^2 q(x)w = 0$, $w(0) = w(1) = 0$, $q(x) = \frac{1}{4}(\alpha+2)^2[(\alpha^2+2\alpha)x+1]$

$\alpha = 1$		$\alpha = 10$	
$k_i^{(ex)}$	$c = 0.25$	$k_i^{(ex)}$	$c = 0.2$
3.1965784	2.5×10^{-7}	2.9725880	5.3×10^{-5}
6.3123495	2.0×10^{-7}	6.1629512	2.8×10^{-4}
9.4444649	8.7×10^{-7}	9.3320059	7.4×10^{-4}
12.5812028	2.2×10^{-6}	12.4911068	1.5×10^{-3}
15.7198543	2.1×10^{-6}	15.6448046	2.4×10^{-3}

Kansa's method with $N = 50$. Regularization by the k-procedure with $\Delta k = 10^{-1}$.

with $\alpha = 1$. The left part of the figure corresponds to the non-regularized solution. The smooth response curve is obtained with the help of the k-procedure. In the table we place the relative errors (13) in the calculation of the first five eigenvalues. The exact eigenvalues $k_i^{(ex)} = \alpha\chi_i$, where χ_i are the roots of the equation [24]:

$$J_1(\chi)Y_1[(1+\alpha)\chi] - Y_1(\chi)J_1[(1+\alpha)\chi] = 0.$$

Here J_1 and Y_1 stand for the Bessel functions. The collocation points ξ_i, $i = 1, \ldots, N$ (and at the same time the centers of the RBF) are uniformly distributed on [0, 1]. Here $N = 50$ and the shape parameter $c = 0.2$.

We extend the method described above to 2D eigenvalue problems of the general type (1). Let us assume that the plane wave

$$w^{(inc)} = \exp(i(k_x x + k_y y)), \quad k_x^2 + k_y^2 = k^2$$

falls from the homogeneous space with $q = 1$ to the domain Ω with the density function $q(x, y)$. The exciting field is the solution of the problem:

$$\nabla^2 w_e^+ + k^2 q(\mathbf{x}) w_e^+ = 0, \quad \mathbf{x} = (x,y) \in \Omega \subset \mathscr{R}^2, \tag{24}$$

$$\nabla^2 w_e^- + k^2 w_e^- = 0, \quad (x,y) \in \mathscr{R}^2 \backslash \Omega. \tag{25}$$

The boundary conditions are

$$w_e^+ = w^{(inc)} + w_e^-, \quad \frac{\partial w_e^+}{\partial n} = \frac{\partial w^{(inc)}}{\partial n} + \frac{\partial w_e^-}{\partial n},$$
$$(x,y) \in \partial\Omega. \tag{26}$$

Here, $\partial/\partial n$ denotes the derivative in the direction of the normal vector $\mathbf{n} = (n_x, n_y)$ of $\partial\Omega$.

We also suppose that the exciting field w_e^- is an outgoing cylindrical wave at a large distance from Ω:

$$w_e^- \sim \frac{1}{\sqrt{\|\mathbf{x}\|}} \exp(-ik\|\mathbf{x}\|), \quad \|\mathbf{x}\| \to \infty,$$
$$\|\mathbf{x}\| = \sqrt{x^2+y^2}. \tag{27}$$

To get the solution w_e^- in the homogeneous exterior region $\mathscr{R}^2\backslash\Omega$ we use the method of fundamental solutions (MFS)—a very effective technique for problems in homogeneous mediums which has been developed recently. The description of MFS and the other references can be found in [16–18]. An approximate solution is looked for in the form of a linear combination of the fundamental solutions of (25):

$$w_e^-(\mathbf{x}) = \sum_{n=1}^{N^-} q_n^- H_0^{(2)}(k\|\mathbf{x} - \varsigma_n\|). \tag{28}$$

Here, $H_0^{(2)}$ stands for the Hankel function of the second kind and zero order. The source points ς_n are placed inside Ω and q_n^- are the free parameters. Note that $w_e^-(\mathbf{x})$ satisfies the radiation condition in infinity (27) with an arbitrary choice of q_n^-.

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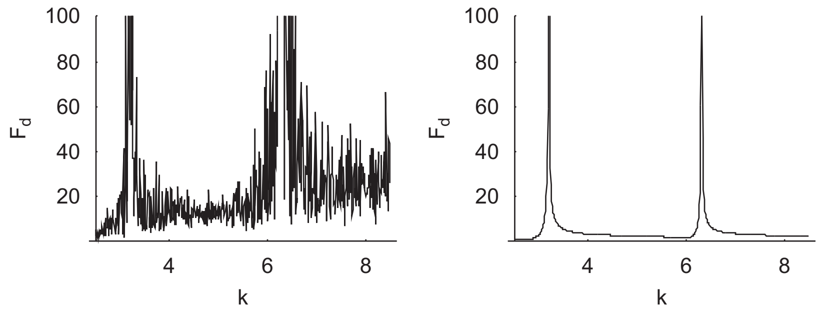

Fig. 7. The response curve for $q(x) = \frac{1}{4}(\alpha+2)^2[(\alpha^2+2\alpha)x+1]$. Kansa's method with $N = 50$. The k-procedure with $\Delta k = 0$ (left) and $\Delta k = 10^{-1}$ (right).

Table 3
Two-dimensional eigenvalue problem: $\nabla^2 w + k^2 q(x,y) w = 0$ in the disk with the radius $R = 1$

$q(x,y) = 1+\sqrt{x^2+y^2}$			$q(x,y) = 1+x^2+y^2$			$q(x,y) = 1+(x^2+y^2)^{3/2}$		
$k_i^{(ex)}$	k_i	e_r	$k_i^{(ex)}$	k_i	e_r	$k_i^{(ex)}$	k_i	e_r
2.0108	2.0107	2.7×10^{-5}	2.17358	2.17369	4.9×10^{-5}	2.2620	2.2621	5.8×10^{-5}
3.0678	3.0674	1.2×10^{-4}	3.30525	3.30528	9.2×10^{-6}	3.4633	3.4633	9.3×10^{-6}
4.0224	4.0222	4.9×10^{-5}	4.30647	4.30653	1.2×10^{-5}	4.5142	4.5141	2.4×10^{-5}
4.5549	4.5538	2.5×10^{-4}	4.84162	4.84164	2.1×10^{-6}	4.9993	4.9978	2.9×10^{-4}
4.9284	4.9279	9.3×10^{-5}	5.24699	5.24727	3.2×10^{-5}	5.4926	5.4927	1.9×10^{-5}

Dirichlet boundary conditions. Regularization by the ε-procedure with $\varepsilon = 10^{-3}$.

To approximate the solution in the non-homogeneous region Ω we use the RBF approximation described above

$$w_e^+(x,y) = \sum_{j=1}^{N^+} q_n^+ \Psi(\|\mathbf{x}-\xi_j\|). \tag{29}$$

Here, $N^+ = N_I + N_B$. The centers ξ_j, $j = 1,\dots,N_I$ are randomly distributed inside Ω and ξ_j, $j = N_I+1,\dots,N_I+N_B$ are placed on the boundary $\partial\Omega$.

Let $\mathbf{x}_i$, $i = 1,\dots,M_I$ be the collocation points distributed inside Ω, and $\mathbf{y}_i$, $i = 1,\dots,M_B$ are placed on the boundary $\partial\Omega$. The linear system is

$$\sum_{j=1}^{N^+} q_n^+ L(\mathbf{x}_i,k)[\Psi(\|\mathbf{x}_i-\xi_j\|)] = 0, \quad i = 1,\dots,M_I, \tag{30}$$

$$\sum_{j=1}^{N^+} q_n^+ \Psi(\|\mathbf{y}_i-\xi_j\|) - \sum_{n=1}^{N^-} q_n^- H_0^{(2)}(k\|\mathbf{y}_i-\varsigma_n\|) = w^{(inc)}(\mathbf{y}_i),$$
$$i = 1,\dots,M_B, \tag{31}$$

$$\sum_{j=1}^{N^+} q_n^+ \frac{\partial}{\partial n}\Psi(\|\mathbf{y}_i-\xi_j\|) - \sum_{n=1}^{N^-} q_n^- \frac{\partial}{\partial n} H_0^{(2)}$$
$$\times(k\|\mathbf{y}_i-\varsigma_n\|) = \frac{\partial}{\partial n} w^{(inc)}(\mathbf{y}_i), \quad i = 1,\dots,M_B. \tag{32}$$

The system has $N = N^+ + N^-$ unknowns and $M = M_I + 2M_B$ equations. We take M approximately twice as large as N. The overdetermined system is solved by the least squares procedure.

When the exciting field w_e is known, we get the response field w_r as a solution of the problem

$$\nabla^2 w_r + k^2 q(\mathbf{x}) w_r = 0, \quad \mathbf{x} \in \Omega,$$
$$w_r = -w_e^+, \quad \mathbf{x} \in \partial\Omega. \tag{33}$$

We look for w_r in the form

$$w_r(\mathbf{x}) = \sum_{j=1}^{N^+} q_n^r \Psi(\|\mathbf{x}-\xi_j\|). \tag{34}$$

The collocation inside Ω and on $\partial\Omega$ gives the following system for q_n^r:

$$\sum_{j=1}^{N^+} q_n^r L(\mathbf{x}_i,k)[\Psi(\|\mathbf{x}_i-\xi_j\|)] = 0, \quad i = 1,\dots,M_I, \tag{35}$$

$$\sum_{j=1}^{N^+} q_n^r \Psi(\|\mathbf{y}_i-\xi_j\|) = -w_e(\mathbf{y}_i), \quad i = 1,\dots,M_B. \tag{36}$$

The sum $w = w_r + w_e^+$ satisfies (1). It is used to compute the norm $F_d(k)$. We find the eigenvalues k_i as extremums of the response curve.

Example 2. The data presented in Table 3 correspond to the following density functions:

$$q_1(x,y) = 1+\sqrt{x^2+y^2}, \quad q_2(x,y) = 1+x^2+y^2,$$
$$q_3(x,y) = 1+(x^2+y^2)^{3/2}. \tag{37}$$

The solution domain Ω is the disk with the radius $R = 1$. The numbers of the centers and the collocation points

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are: $N_I = 100$, $N_B = 50$, $N^- = 50$, $M_I = 100$, $M_B = 100$. We compare the results of the calculations k_i with the data published in [6,7] and place the errors (13) in the table.

3. Internal excitation

In this section we consider the general case when $\nabla^2 w_e + k^2 q w_e \neq 0$ in the solution domain. So, this case can be treated as the one where the exciting source is placed inside the domain of the interest.

Let us consider the same simplest eigenproblem (4) on the interval $[0,1]$. In accordance with the technique described in Section 1 we take the *response field* w_r as a solution of the BVP:

$$\frac{d^2 w_r}{dx^2} + k^2 w_r = -\frac{d^2 w_e}{dx^2} - k^2 w_e, \quad x \in [0,1],$$
$$w_r(0) = -w_e(0), \quad w_r(1) = -w_e(1). \tag{38}$$

Note that here we can take any smooth enough function defined in $[0,1]$ as the *exciting field* $w_e(x)$. Obviously, the sum $w = w_e + w_r$ satisfies the initial BVP (14) with any choice of $w_e(x)$.

The rest part of the algorithm is the same as the one described in the previous section. Varying k, we get the response curve $F(k)$ (9) and calculate the eigenvalues as the positions of its maxima.

However, this initial form of the method is unfit for our goal. Indeed, a particular solution of (38) is $\widetilde{w}_r = -w_e$. Looking for the response field in the form

$$w_r = A_r \exp(ikx) + B_r \exp(-ikx) + \widetilde{w}_r(x),$$

we get the linear system for A_r, B_r:

$$A_r + B_r - w_e(0) = -w_e(0),$$
$$A_r \exp(ik) + B_r \exp(-ik) - w_e(1) = -w_e(1). \tag{39}$$

For $k \neq n\pi$ the system has the unique solution $A_r = 0$, $B_r = 0$. Thus, $w \equiv 0$ and $F(k) = 0$ with the precision error. In Fig. 8 we place the response curve corresponding to the exciting field

$$w_e(x) = 1 + x + x^2. \tag{40}$$

To get a smooth response curve we use the same two regularizing procedures. Applying the ε-procedure, we replace (38) by the following BVP:

$$\frac{d^2 w_r}{dx^2} + (k^2 + i\varepsilon k) w_r = -\frac{d^2 w_e}{dx^2} - k^2 w_e,$$
$$w_r(0) = -w_e(0), \quad w_r(1) = -w_e(1), \tag{41}$$

where $\varepsilon > 0$ is a small value. As an example, let us take the same exciting field $w_e(x)$ (40). The particular solution can also be taken in the same polynomial form

$$\widetilde{w}_r(x,k,\varepsilon) = -\frac{k^2}{k_\varepsilon^2} x^2 - \frac{k^2}{k_\varepsilon^2} x - \frac{2+k^2}{k_\varepsilon^2} + \frac{2k^2}{k_\varepsilon^4},$$
$$k_\varepsilon^2 = k^2 + i\varepsilon k. \tag{42}$$

Note that $\widetilde{w}_r \neq -w_e$. As a result, we get the following system instead of (39):

$$A_r + B_r - \widetilde{w}_r(0) = -w_e(0),$$
$$A_r e^{ik_\varepsilon} + B_r e^{-ik_\varepsilon} - \widetilde{w}_r(1) = -w_e(1). \tag{43}$$

The dimensionless response curves $F_d(k)$ depicted in Fig. 9 correspond to $\varepsilon = 10^{-15}$ (left) and $\varepsilon = 10^{-6}$ (right).

The value $\varepsilon = 10^{-15}$ is too small to regularize the solution. The response curve $F_d(k)$ has separate maximums at the positions of eigenvalues but is not smooth. The value $\varepsilon = 10^{-6}$ provides a smooth curve.

Using the k-procedure, we take the response field as the solution of the BVP

$$\frac{d^2 w_r}{dx^2} + k^2 w_r = -\frac{d^2 w_e}{dx^2} - (k + \Delta k)^2 w_e,$$
$$w_r(0) = -w_e(0), \quad w_r(1) = -w_e(1). \tag{44}$$

The particular solution is

$$\widetilde{w}_r(x,k,\Delta k) = -\frac{(k+\Delta k)^2}{k^2} x^2 - \frac{(k+\Delta k)^2}{k^2} x - \frac{2+(k+\Delta k)^2}{k^2} + \frac{2(k+\Delta k)^2}{k^4} \tag{45}$$

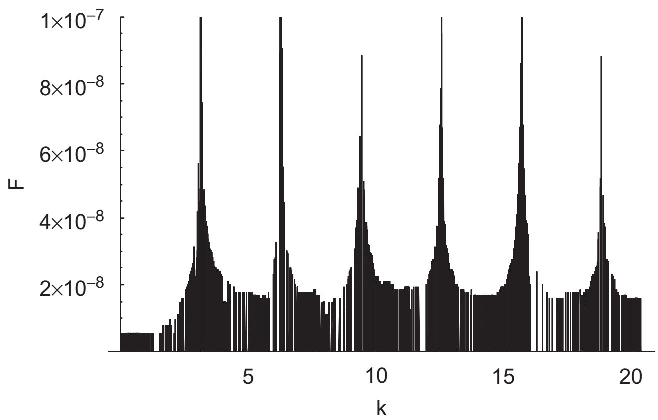

Fig. 8. The response curve $F(k)$. Internal excitation without regularization.

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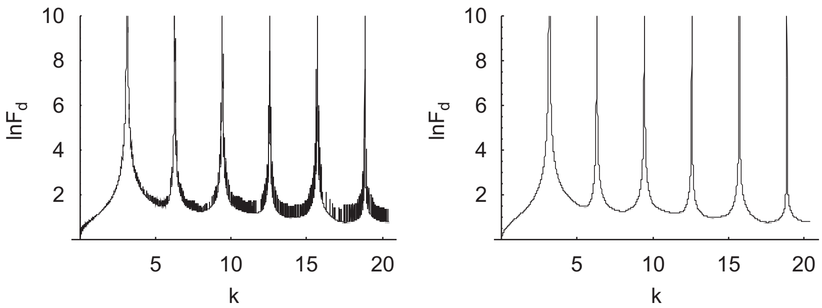

Fig. 9. The dimensionless response curve $F_d(k)$. Internal excitation. ε-procedure with $\varepsilon = 10^{-15}$ (left) and $\varepsilon = 10^{-6}$ (right).

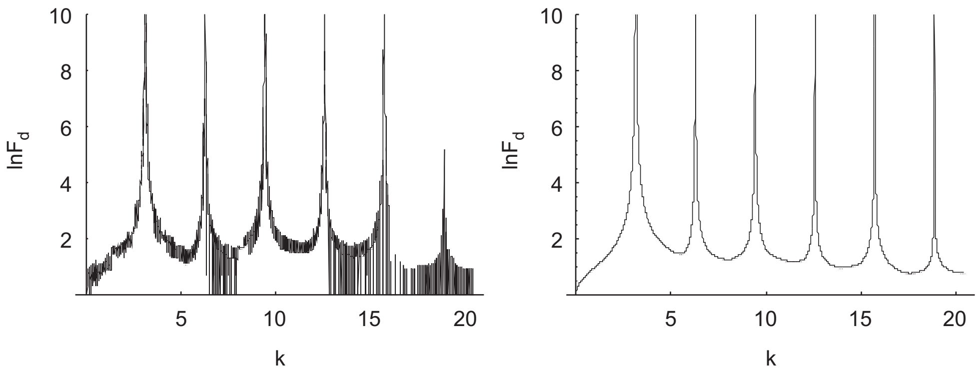

Fig. 10. The dimensionless response curve $F_d(k)$. Internal excitation. k-procedure with $\Delta k = 10^{-15}$ (left) and $\Delta k = 10^{-6}$ (right).

and again $\widetilde{w}_r \neq -w_e$. The linear system for A_r, B_r takes the form

$$A_r + B_r - \widetilde{w}_r(0) = -w_e(0),$$
$$A_r \exp(\mathrm{i}k) + B_r \exp(-\mathrm{i}k) - \widetilde{w}_r(1) = -w_e(1)$$

with $\widetilde{w}_r$ given in (45). The system has non-zero solutions for all k except the eigenvalues k_n when the system becomes degenerate. However, due to the iterative procedure of solution and rounding errors we never solve the system with the *exact* k_n. We observe degeneration of the system as a considerable growth of the solution in a neighborhood of the eigenvalues (Fig. 10).

3.1. Non-homogeneous string

Here we consider eigenvalue problem (14) for non-homogeneous string. We take particular solution $w_e(x)$ and get the response field w_r as a solution of the BVP

$$\frac{\mathrm{d}^2 w_r}{\mathrm{d}x^2} + k_\varepsilon^2 q(x) w_r = -\frac{\mathrm{d}^2 w_e}{\mathrm{d}x^2} - k^2 q(x) w_e,$$
$$w_r(0) = -w_e(0), \quad w_r(1) = -w_e(1). \tag{46}$$

The BVP is solved by Kansa's method described in the previous section. Using the sum $w = w_e + w_r$, we obtain the response curve $F_d(k)$ and calculate the eigenvalues as positions of extremums.

Example 3. The data presented in Table 4 correspond to the density functions

$$q_1(x) = 0.25(\alpha + 2)^2[(\alpha^2 + 2\alpha)x + 1] \quad \text{and}$$
$$q_2(x) = [\ln(1+\alpha)/\alpha]^2(1+\alpha)^{2x}.$$

The exact eigenvalues $k_i^{(ex)}$ are the roots of the equations

$$J_1(\chi)Y_1[(1+\alpha)\chi] - Y_1(\chi)J_1[(1+\alpha)\chi] = 0$$

and

$$J_0(\chi)Y_0[(1+\alpha)\chi] - Y_0(\chi)J_0[(1+\alpha)\chi] = 0$$

correspondingly (see [24]). The collocation points ξ_i, $i = 1, \ldots, N$ (and at the same time the centers of the RBF) are randomly distributed on $[0, 1]$. We take $N = 100$ RBFs with the shape parameter $c = 0.4$ and the particular solution $w_e(x) = 1 - x$. It is important to note that in this example we get a smooth response curve without regularization, i.e., we use $\varepsilon = 0$, $\Delta k = 0$. This can be explained by the fact that the errors introduced by the RBF approximation play a role of an intrinsic regularizing procedure.

3.2. Non-homogeneous membrane

The same technique can be applied in the 2D case. Similar to the 1D problem considered above, we get the

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Table 4
Eigenvalues of the problem: $d_{xx}^2 w + k^2 q(x)w = 0,\ w(0) = w(1) = 0$

$0.25(\alpha+2)^2[(\alpha^2+2\alpha)x+1]$		$[\ln(1+\alpha)/\alpha]^2(1+\alpha)^{2x}$	
$k_i^{(ex)}$	k_i	$k_i^{(ex)}$	k_i
3.1965784	3.1965708	3.1230309	3.1230303
6.3123495	6.3123495	6.2734357	6.2734366
9.4444649	9.4444647	9.4182075	9.4182103
12.5812028	12.5811999	12.5614232	12.5614245
15.7198543	15.7207882	15.7039979	15.7039696

Solution by Kansa's method with $N = 100$. Inner exciting source with the particular solution $w_e(x) = 1 - x$. $\alpha = 1$.

response field as a solution of the BVP

$$\nabla^2 w_r + k_\varepsilon^2 q(\mathbf{x})w_r = -\nabla^2 w_e - (k+\Delta k)^2 q(\mathbf{x})w_e,$$
$$\mathbf{x} \in \Omega \subset \mathscr{R}^2, \tag{47}$$

$$B[w_r] = -B[w_e], \quad \mathbf{x} \in \partial\Omega. \tag{48}$$

The BVP is solved by using the asymmetric RBF collocation method with the multiquadrics described in the previous section. The response curve is obtained using the sum $w = w_e + w_r$. In all the numerical examples presented in this subsection we use $N = 7$ points randomly distributed in Ω to calculate the response curve.

Example 4. Let us consider the same eigenvalue problem in the disk as the one in Example 2. The response field is looked for in the form of the linear combination of the multiquadrics

$$w_r(\mathbf{x}) = \sum_{j=1}^{N} q_n \Psi(\|\mathbf{x} - \xi_j\|). \tag{49}$$

We use $N = NI + 2NB$ centers. The first NI centers are randomly distributed inside the disk Ω; the next NB centers are taken on the boundary $\partial\Omega$; and the last NB centers are distributed on the auxiliary contour $\partial\Omega'$ placed at the distance dr from $\partial\Omega$ *outside* Ω.

We set $NI + NB$ collocation conditions

$$\sum_{j=1}^{N} q_n(\nabla^2 + k_\varepsilon^2 q(\mathbf{x}_i))[\Psi(\|\mathbf{x}_i - \xi_j\|)] = -(\nabla^2 + k^2 q(\mathbf{x}_i))[w_e(\mathbf{x}_i)],$$
$$i = 1, \ldots, NI + NB \tag{50}$$

to approximate (47) and NB conditions

$$\sum_{j=1}^{N} q_n B[\Psi(\|\mathbf{x}_i - \xi_j\|)] = -B[w_e(\mathbf{x}_i)],$$
$$i = NI + NB + 1, \ldots, NI + 2NB \tag{51}$$

to approximate (48). Here the first $NI + NB$ collocation points $\mathbf{x}_i$ coincide with the first $NI + NB$ centers and the last NB collocation points $\mathbf{x}_i$ are placed on the boundary $\partial\Omega$. So, we get $N \times N$ linear system which is solved by the Gauss elimination procedure.

The data shown in Tables 5–7 are obtained with $NI = NB = 100$, $dr = 0.05$. The shape parameter $c = 1$. The density functions are taken in the form

$$q_1(x,y) = 1 + \alpha(x^2+y^2)^{1/2}, \quad q_2(x,y) = 1 + \alpha(x^2+y^2),$$
$$q_3(x,y) = 1 + \alpha(x^2+y^2)^{3/2},$$

which admits of exact solutions [7]. The particular solution is taken in the form $w_e = 1 + xy$. It should be stressed that the algorithm is again self-regularizing and we perform all the calculations with $\varepsilon = 0$, $\Delta k = 0$.

Example 5. Here we consider the same circular non-homogeneous membrane as above with the density

Table 5
Eigenvalues of the problem $\nabla^2 w + k^2\alpha[1 + (x^2+y^2)^{1/2}]w = 0$ in the circle with the radius $R = 1$

$\alpha = 1$		$\alpha = 3$	
$k_i^{(ex)}$	k_i	$k_i^{(ex)}$	k_i
2.0108	2.0097	1.5832	1.5818
3.0678	3.0677	2.3334	2.3333
4.0224	4.0223	3.0129	3.0129
4.5549	4.5531	3.5835	3.5800
4.9284	4.9286	3.6582	3.6585

Dirichlet conditions. Inner exciting source with $w_e = 1 + xy$.

Table 6
Eigenvalues of the problem $\nabla^2 w + k^2\alpha[1 + (x^2+y^2)]w = 0$ in the circle with the radius $R = 1$

$\alpha = 1$		$\alpha = 3$	
$k_i^{(ex)}$	k_i	$k_i^{(ex)}$	k_i
2.173584	2.173587	1.8474	1.8475
3.30525	3.30524	2.6681	2.6681
4.30647	4.30654	3.3870	3.3872
4.84162	4.84153	4.0608	4.0622
5.24698	5.24725	4.0610	4.0611

Dirichlet conditions. Inner exciting source with $w_e = 1 + xy$.

Table 7
Eigenvalues of the problem $\nabla^2 w + k^2\alpha[1 + (x^2+y^2)^{3/2}]w = 0$ in the circle with the radius $R = 1$

$\alpha = 1$		$\alpha = 3$	
$k_i^{(ex)}$	k_i	$k_i^{(ex)}$	k_i
2.2620	2.2620	2.0258	2.0259
3.4633	3.4633	2.9338	2.9337
4.5142	4.5142	3.7023	3.7026
4.9993	4.9992	4.3370	4.3367
5.4926	5.4926	4.4107	4.4109

Dirichlet conditions. Inner exciting source with $w_e = 1 + xy$.

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function $q(x, y) = 1 + (x^2 + y^2)^{1/2}$ and with the different particular solutions

$$w_{e,1} = 1, \quad w_{e,2} = 1 + xy, \quad w_{e,3} = e^x \cos y,$$
$$w_{e,4} = x^3 - 3xy^2.$$

The other parameters of the problem are the same as in Example 4. The results of the calculations are placed in Table 8.

Example 6. We consider an annular domain Ω between two circles. The inner and outer radii of an annular domain are $r_1 = 0.5$ and $r_2 = 1$ correspondingly [6,7]. We use $N = NI + 2NB$ centers. The first NI centers are randomly distributed inside the disk Ω; the next $NB = NB_1 + NB_2$ centers are taken on the boundaries $\partial\Omega_1$ and $\partial\Omega_2$; and the rest NB centers are distributed on the auxiliary contours $\partial\Omega_1'$ and $\partial\Omega_2$ placed at the distance dr from $\partial\Omega_1$ and $\partial\Omega_2$ correspondingly *outside* Ω. Other details are the same as in Example 4.The data placed in Table 9 are obtained with: $NB = 150, \quad NB_1 = 100, \quad NB_2 = 50, \quad dr = 0.05, \quad c = 0.5, \quad w_e = 1 + xy$.

Table 8
Eigenvalues of the problem $\nabla^2 w + k^2[1 + (x^2 + y^2)^{1/2}]w = 0$ in the circle with the radius $R = 1$

1	$1 + xy$	$e^x \cos y$	$x^3 - 3xy^2$
2.00987	2.00987	2.00987	2.00987
3.06760	3.06758	3.06758	3.06757
4.02232	4.02232	4.02235	4.02232
4.55457	4.55457	4.55457	4.55456
4.92806	4.92830	4.92806	4.92806

Dirichlet conditions. Inner exciting source with different particular solutions.

Table 9
Two-dimensional eigenvalue problem: $\nabla^2 w + k^2 q(x, y)w = 0$ in the annular domain between the two circles $R_1 = 1$, $R_2 = 0.5$

$q(x,y) = 1 + \sqrt{x^2 + y^2}$		$q(x,y) = 1 + x^2 + y^2$		$q(x,y) = 1 + (x^2 + y^2)^{3/2}$	
$k_i^{(ex)}$	k_i	$k_i^{(ex)}$	k_i	$k_i^{(ex)}$	k_i
4.7198	4.7189	4.9782	4.9772	5.1953	5.1939
4.8298	4.8294	5.0934	5.0925	5.3150	5.3133
5.1443	5.1517	5.4224	5.4282	5.6564	5.6622

Dirichlet boundary conditions. Inner exciting source with $w_e = 1 + xy$.

Example 7. We consider the case when Ω is the unit square. The density functions of two kinds are considered:

$$q_1 = 1 + 0.1x, \quad q_2 = 1 + 0.1\sin(\pi x).$$

The results of the calculations are compared with the data taken from [8] where the lowest eigenvalues are computed by the a hybrid method composed of differential transforms and the Kantorovitch method. The results are placed in Tables 10 and 11.

Example 8. Through the paper we use Kansa's method only. However, any appropriate method, e.g., FE, FD, can be used as a BVPs solver in the framework of this technique. As an example, let us consider again the problem of Example 7 The simple square geometry of the solution domain allows to use the FD method. To approximate (47) we apply the following FD scheme

$$\begin{aligned}20w_{i,j} = {} & 4(w_{i+1,j} + w_{i,j+1} + w_{i,j-1} + w_{i-1,j}) \\ & + w_{i+1,j+1} + w_{i+1,j-1} + w_{i-1,j+1} + w_{i-1,j-1} - 6h^2 g_{i,j} \\ & - 0.5h^2(g_{i+1,j} + g_{i,j+1} + g_{i,j-1} + g_{i-1,j} - 4g_{i,j})\end{aligned}$$

which approximates the equation $\nabla^2 w = g(x, y)$ with the fourth order [25]. Here we write (47) in the form: $\nabla^2 w_r = -k^2 q(\mathbf{x})w_e - \nabla^2 w_e - k_\varepsilon^2 q(\mathbf{x})w_r$ and denote the right-hand side as $g(x, y)$. $h = 1/N$ is the mesh step; $w_{i,j} = w(x_i, y_j)$; $x_i = h(i - 1), y_j = h(j - 1)$, $i, j = 1, \ldots, N + 1$.

Table 10
Square membrane with the density function $q = 1 + 0.1x$

$NI/NB = 50/40$	$NI/NB = 100/100$	$NI/NB = 150/100$	$NI/NB = 200/100$	Ho&Chen
4.34249	4.33636	4.33547	4.33538	4.33538
6.87344	6.85448	6.85370	6.85369	–
8.68489	8.67274	8.67307	8.67253	–

Table 11
Square membrane with the density function $q = 1 + 0.1\sin(\pi x)$

$NI/NB = 50/40$	$NI/NB = 100/100$	$NI/NB = 150/100$	$NI/NB = 200/100$	Ho&Chen
4.27252	4.26643	4.26546	4.26540	4.26541
6.75445	6.74428	6.74370	6.74373	–
6.81122	6.79707	6.79769	6.79737	–

Inner excitation with the particular solution $w_e = 1 + xy$.

Inner excitation with the particular solution $w_e = 1 + xy$.

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Table 12
Square membrane with the density function $q = 1 + 0.1\sin(\pi x)$

i	$N = 20$	$N = 30$	$N = 40$	$N = 50$
1	4.2654	4.26540	4.265404	4.265404
2	6.7438	6.74388	6.743886	6.743888
3	6.7972	6.79732	6.797324	6.797326
4	8.5977	8.59768	8.597665	8.597662
5	9.5359	9.53645	9.536548	9.536574
6	9.6241	9.62472	9.624822	9.624849
7	10.9591	10.95915	10.95915	10.95915
8	12.4295	12.43227	12.43273	12.43285
9	12.5509	12.55375	12.55423	12.55436
10	12.9143	12.91364	12.91352	12.91349

Inner excitation with the particular solution $w_e = 1 + xy$. FD solution.

As a result, we write the system in the block tridiagonal form:

$$\widehat{\mathbf{A}}_j\mathbf{W}_{j+1} + \widehat{\mathbf{B}}_j\mathbf{W}_j + \widehat{\mathbf{C}}_j\mathbf{W}_{j-1} = \mathbf{F}_j,$$

where $\mathbf{W}_j = (w_{1,j}, w_{2,j}, \ldots, w_{N+1,j})^{\mathrm{T}}$ are the vectors of the unknowns; $\mathbf{F}_j = (f_{1,j}, f_{2,j}, \ldots, f_{N+1,j})^{\mathrm{T}}$ are the vectors of the right-hand side; $\widehat{\mathbf{A}}_j, \widehat{\mathbf{B}}_j, \widehat{\mathbf{C}}_j$ are $(N+1)\times(N+1)$ matrices. The system is solved by the sweep method.

In Table 12 we test a convergence of the eigenvalues of the membrane with the density function $q = 1 + 0.1\sin(\pi x)$. The number of mesh nodes varies from $N = 20$ to 50.

4. Concluding remarks

In this paper, a new numerical technique for the problem of free vibrations of inhomogeneous membranes with continuously varying properties is proposed. This is a mathematical model of physical measurements, when a mechanical or acoustic system is excited by some source, and resonant frequencies can be determined using the growth of the amplitude of oscillations near these frequencies. It is convenient for determining some first eigenvalues of the system which are often of the most interest from the point of view of engineering applications.

This technique is presented in two versions. The first one assumes that the exciting source is placed outside the solution domain. It is more convenient for homogeneous problems. In this case the exciting field w_e can be obtained in an analytic way. Besides, here it is natural to use some boundary technique for solving BVP for the response field w_r. As it is shown in [9–12], the resulting technique provides a high accuracy in determining natural frequencies of the homogeneous membranes and plates. However, applying this version of the method in non-homogeneous cases, one faces the scattering problem which should be solved to get w_e. It rather complicates the algorithm because one should solve the problem in an unbounded domain with some kind of radiation conditions in infinity.

The second version uses a given exciting field w_e and one should solve the BVP for w_r inside the solution domain only. This version is found more convenient for non-homogeneous problems.

The both versions of the method presented lead to a sequence of BVPs and can be combined with different solvers. In the paper we apply mainly the MFS and Kansa's method. Application of the FD method is shown in Example 8 of the previous section. However, it can be combined with any appropriate BVP solver. It seems possible to extend the same approach to eigenvalue problems with other differential equations, e.g., to problems of plates and shells vibration. This will be the subject of further investigations.

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