Engineering Fracture Mechanics 77 (2010) 753–757

Contents lists available at ScienceDirect

Engineering Fracture Mechanics

journal homepage: www.elsevier.com/locate/engfracmech

Note

A rigorous derivation for T-stress in line crack problem

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A R T I C L E I N F O

Article history:
Received 20 July 2009
Received in revised form 19 November 2009
Accepted 28 November 2009
Available online 2 December 2009

Keywords:
T-stress
Crack problem

A B S T R A C T

In this paper, a rigorous derivation for T-stress in line crack problem is presented. Similar to the edge crack case, this paper provides the T-stress dependence on loading with the Dirac delta function property.

1. Introduction

The T-stress evaluation was addressed by many researchers [1–3]. Most of studies were based on a semi-infinite crack with loading on the crack face. However, some confusing statement was found. For example, it was said in [3] that: "Fett [14] (now [14] becomes [1]) on the other hand lumped all nonsingular terms of Eq. (6) into the T term, which is obviously not correct". The present researcher found that the above-mentioned statement was incorrect. A perfect derivation for relevant equation was suggested [4].

In the semi-infinite crack case, the T-stress dependence on loading with Dirac delta function property was pointed out by Fett [1]. A perfect derivation for relevant equation was suggested recently [4]. However, no rigorous derivation for T-stress for a line crack in infinite plate was addressed from the stress solution in front of a crack tip. In this paper, a rigorous derivation for T-stress in line crack problem is presented. Similar to the edge crack case, this paper provides the T-stress dependence on loading with the Dirac delta function property.

Some researchers studied the hypersingular integral equation from the formulation of the dual integral equations [5]. It was pointed out that the integral equation based on the Somigliana identity is too slim to solve the general elastic crack problems. The authors suggested an additional integral equation, or so-called dual integral equation. On the basis of dual integral equation, a line crack problem was solved using the hypersingular integral equation. Recently, methods for evaluating T-stress were summarized [6].

2. Nomenclatures used

To facilitate reading the article, the following nomenclatures are introduced.

σ_x, σ_y, σ_{xy}	the stress components
$\phi(z)$, $\Phi(z) = \phi'(z)$, $\omega(z)$, $\Omega(z) = \omega'(z)$	the complex potentials used in the representation of elastic field

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0013-7944/$ - see front matter
doi:10.1016/j.engfracmech.2009.11.008

K_I, K_{II}	the mode I and II stress intensity factors
$f_{ij}(\theta)$, $g_{ij}(\theta)$ $(i, j = 1, 2)$	the angular distribution functions for singular stresses
r	the distance between a point and the crack tip
T, T_1, T_2	the T-stresses
q_c, p_c	two regular stresses at vicinity of crack tip
$P(t) + iQ(t)$	the loadings applied on the crack face
$I(s)$, $I_1(s)$, $I_2(s)$, $J(s)$	some intermediate integrals
F, P_x, P_y	some concentrated forces
i	a unit imaginary value

3. Analysis

In the complex variable function method of plane elasticity, the stresses (σ_x, σ_y, σ_{xy}) are expressed in terms of the complex potentials $\phi(z)$ and $\omega(z)$ such that [7]

$$\sigma_x - \sigma_y + 2i\sigma_{xy} = 2(\overline{\Phi(z)} - (z-\bar{z})\overline{\Phi'(z)} - \Omega(\bar{z})) \tag{1}$$

$$\sigma_x + i\sigma_{xy} = \Phi(z) + 2\overline{\Phi(z)} - (z-\bar{z})\overline{\Phi'(z)} - \Omega(\bar{z}) \tag{2}$$

where $\Phi(z) = \phi'(z)$, $\Omega(z) = \omega'(z)$, a bar over a function denotes the conjugated value for the function.

The stress distribution near a crack tip under the traction free crack face was early investigated by Williams [8]. A little modification for the Williams expansion is suggested below. It is assumed that the crack face has the following loadings (Fig. 1a)

$$\sigma_y^+ = \sigma_y^- = p_c, \quad \sigma_{xy}^+ = \sigma_{xy}^- = q_c \tag{3}$$

From Williams expansion, the stresses at the crack tip area can be expressed as

$$\begin{bmatrix} \sigma_x & \sigma_{xy} \\ \sigma_{xy} & \sigma_y \end{bmatrix} = \frac{K_I}{\sqrt{2\pi r}} \begin{bmatrix} f_{11}(\theta) & f_{12}(\theta) \\ f_{12}(\theta) & f_{22}(\theta) \end{bmatrix} + \frac{K_{II}}{\sqrt{2\pi r}} \begin{bmatrix} g_{11}(\theta) & g_{12}(\theta) \\ g_{12}(\theta) & g_{22}(\theta) \end{bmatrix} + \begin{bmatrix} T & q_c \\ q_c & p_c \end{bmatrix} \tag{4}$$

where the first two terms in the expansion form are singular at the crack tip, K_I, K_{II} denote the mode I and mode II stress intensity factors, respectively, and the functions $f_{ij}(\theta)$, $g_{ij}(\theta)$ represent the angular distributions of the crack tip stresses. In Eq. (4), the third term is finite and bounded. The term T is denoted as the T-stress and can be regarded as the stress acting parallel to the crack flanks. In Eq. (4) the term $O(r^{1/2})$ has been neglected for clarity. In addition, the angular distribution can be expressed as [8]

$$\begin{Bmatrix} f_{11} \\ f_{12} \\ f_{22} \end{Bmatrix} = \cos(\theta/2) \begin{Bmatrix} 1 - \sin(\theta/2)\sin(3\theta/2) \\ \sin(\theta/2)\cos(3\theta/2) \\ 1 + \sin(\theta/2)\sin(3\theta/2) \end{Bmatrix} \tag{5}$$

$$\begin{Bmatrix} g_{11} \\ g_{12} \\ g_{22} \end{Bmatrix} = \begin{Bmatrix} -\sin(\theta/2)[2 + \cos(\theta/2)\cos(3\theta/2)] \\ \cos(\theta/2)[1 - \sin(\theta/2)\sin(3\theta/2)] \\ \sin(\theta/2)\cos(\theta/2)\cos(3\theta/2) \end{Bmatrix} \tag{6}$$

From Eqs. (1)–(6), we will find two ways to define the T-stress. In the first way, substituting $r = s$ and $\theta = 0$ in Eqs. (4), (5) and (6) yields

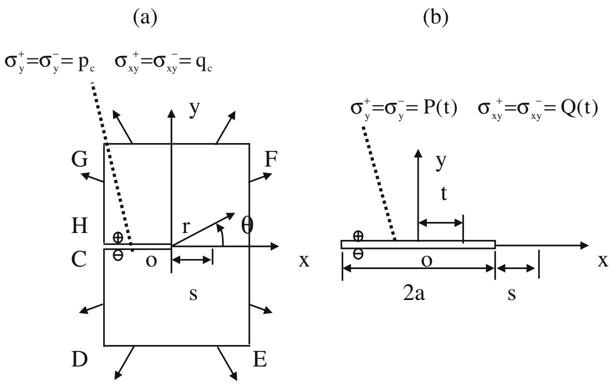

Fig. 1. (a) an edge crack and (b) a line crack in an infinite plate.

$$T = \lim_{s \to 0}(\sigma_x - \sigma_y) + p_c \tag{7}$$

From Eq. (7) we see that, in order to evaluate the T-stress, one must evaluate the stress components σ_x and σ_y at the position $x > s$, $y = 0$.

In the second way, substituting $r = s$ and $\theta = 0$ in Eqs. (4), (5) and (6) yields

$$T = \lim_{s \to 0}\left(\sigma_x - \frac{K_I}{\sqrt{2\pi s}}\right) \tag{8}$$

From Eq. (8) we see that, in order to evaluate the T-stress, one must evaluate the stress components σ_x at the position $x > s$, $y = 0$ and the K_I value in the problem.

In a semi-infinite crack, the $\sigma_x(s)$ component and the K_I value in Eq. (8) have been derived in an explicit form [9]. However, in the line crack, the $\sigma_x(s)$ component and the K_I value in Eq. (8) must be different with those in the semi-infinite crack. Therefore, we consider the boundary value problem shown by Fig. 1b. In the problem, the crack face has the following loading

$$(\sigma_y + \sigma_{xy})^+ = (\sigma_y + \sigma_{xy})^- = P(t) + iQ(t), \quad (|t| \leqslant a) \tag{9}$$

where the superscript + (−) denotes the upper (lower) edge of crack, respectively.

After some manipulations, the relevant solution for the complex potentials was obtained previously [9]

$$\Phi(z) = \Omega(z) = \frac{1}{2\pi X(z)} \int_{-a}^{a} \frac{\sqrt{a^2 - t^2}(P(t) - iQ(t))dt}{t - z} \tag{10}$$

where the function $X(z)$ is defined by

$$X(z) = \sqrt{z^2 - a^2}, \quad (\text{taking the branch} \lim_{z \to \infty} X(z)/z = 1) \tag{11}$$

Let us evaluate the T-stress at the right cark tip. In a point ($x = a + s$, $y = 0$, or $z = a + s + i0$) in the front position of the crack tip, we have $X(z) = \sqrt{(2a + s)s}$. Therefore, at that point the complex potential can be expressed as follows

$$\Phi(z) = \Omega(z) = \frac{1}{2\pi\sqrt{(2a+s)s}} \int_{-a}^{a} \frac{\sqrt{a^2 - t^2}(P(t) - iQ(t))dt}{t - (a+s)}, \quad (\text{at } z = a + s + i0) \tag{12}$$

From Eqs. (1) and (12) (now$z = \bar{z}$), we will find

$$\sigma_x - \sigma_y = 2\mathrm{Re}(\overline{\Phi(z)} - \Omega(z)) = 0, \quad (\text{at } z = a + s + i0) \tag{13}$$

Substituting Eq. (13) into (7), and substituting p_c in Eq. (7) by $P(a)$ yields

$$T = P(a) \tag{14}$$

In addition, we try to evaluate the T-stress at the right crack tip from Eq. (8). From Eq. (2) and (12) (now $z = \bar{z}$), we will find

$$\sigma_x = 2\mathrm{Re}\Phi(z) = -\frac{1}{\pi\sqrt{(2a+s)s}} \int_{-a}^{a} \frac{\sqrt{a^2 - t^2}P(t)dt}{a - t + s}, \quad (\text{at } z = a + s + i0) \tag{15}$$

Previously, we have the following stress intensity factor solution [9]

$$K_I = -\frac{1}{\sqrt{\pi a}} \int_{-a}^{a} \frac{\sqrt{a^2 - t^2}P(t)dt}{a - t}, \quad \text{and} \quad \frac{K_I}{\sqrt{2\pi s}} = -\frac{1}{\pi\sqrt{2as}} \int_{-a}^{a} \frac{\sqrt{a^2 - t^2}P(t)dt}{a - t} \tag{16}$$

By using Eqs. (8), (15) and (16), the T-stress can be evaluated by

$$T = \lim_{s \to 0}\left(\sigma_x - \frac{K_I}{\sqrt{2\pi s}}\right) = \frac{1}{\pi}\lim_{s \to 0} I(s) \tag{17}$$

where

$$I(s) = \frac{1}{\sqrt{s}} \int_{-a}^{a} \left(\frac{1}{\sqrt{2a}(a - t)} - \frac{1}{\sqrt{2a + s}(a - t + s)}\right)\sqrt{a^2 - t^2}P(t)dt \tag{18}$$

The integral $I(s)$ may be decomposed into

$$I(s) = I_1(s) + I_2(s) \tag{19}$$

where

$$I_1(s) = \frac{1}{\sqrt{s}} \int_{-a}^{a} \left(\frac{1}{\sqrt{2a}(a-t+s)} - \frac{1}{\sqrt{2a+s}(a-t+s)} \right) \sqrt{a^2-t^2} P(t) dt \quad (20)$$

$$I_2(s) = \frac{1}{\sqrt{s}} \int_{-a}^{a} \left(\frac{1}{\sqrt{2a}(a-t)} - \frac{1}{\sqrt{2a}(a-t+s)} \right) \sqrt{a^2-t^2} P(t) dt \quad (21)$$

The integral $I_1(s)$ can be rewritten as

$$I_1(s) = \sqrt{s} \int_{-a}^{a} \left(\frac{1}{\sqrt{2a+s}\sqrt{2a}\left(\sqrt{2a+s}+\sqrt{2a}\right)(a-t+s)} \right) \sqrt{a^2-t^2} P(t) dt \quad (22)$$

Therefore, we have

$$\lim_{s \to 0} I_1(s) = 0 \quad (23)$$

The integral $I_2(s)$ can be rewritten as

$$I_2(s) = \int_{-a}^{a} \left(\frac{\sqrt{s}}{\sqrt{2a}\sqrt{a-t}(a-t+s)} \right) \sqrt{a+t} P(t) dt \quad (24)$$

After making a substitution $t_1 = a - t$, the integral can be reduced to

$$I_2(s) = \int_{0}^{2a} \left(\frac{\sqrt{s}}{\sqrt{2a}\sqrt{t_1}(t_1+s)} \right) \sqrt{2a-t_1} P(a-t_1) dt_1 \quad (25)$$

Previously, an integral is defined as follows [3,4]

$$J(s) = \int_{0}^{h} \left(\frac{\sqrt{s}}{\sqrt{u}(u+s)} \right) f(u) du \quad (26)$$

where "h" is any positive value, and $f(u)$ is any continuous function. In addition, It has been proved that [4]

$$\lim_{s \to 0} J(s) = \pi f(0) \quad (27)$$

Therefore, from Eqs. (25), (26) and (27), we will find

$$\lim_{s \to 0} I_2(s) = \pi P(a) \quad (28)$$

Finally, from Eqs. (17), (19), (23) and (28), we have

$$T = P(a) \quad (29)$$

Same result was obtained in Eq. (14) previously. Eq. (29) means that the T-stress dependence on the applied loading possesses a property of the Dirac delta function.

One example for the usage of suggested equation is introduced below. A line crack in an infinite plate is applied by the concentrated forces (P_x,P_y) at the point z_o. In this case, the original problem shown by Fig. 2a can be considered a superposition of the basic field and the perturbation field, shown by Fig. 2b and c, respectively.

The T-stress at the right crack tip in the original problem can be expressed as

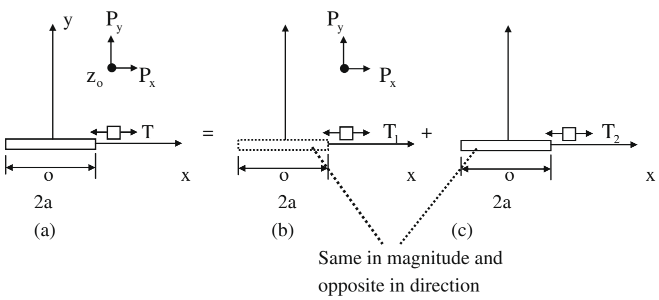

Fig. 2. A line crack with the concentrated force loading applied at the point z_o, (a) the original field, (b) the basic field, T_1 from continuous distribution of σ_x at prospective site of crack tip (no crack), (c) the perturbation field, T_2 equivalent to a value extracted from singular distribution of σ_x in front position of crack.

$$T = T_1 + T_2 \tag{30}$$

The component T_1 is derived form the basic field shown by Fig. 2b, and he component T_2 is derived form the perturbation field shown by Fig. 2c.

In the basic field, the σ_x component along the real axis can be obtained as follows [7]

$$\sigma_x = \mathrm{Re}[2\Phi(t) - t\Phi'(t) - \Psi(t)], \quad (\mathrm{t - real}) \tag{31}$$

where

$$\Phi(z) = \frac{F}{z - z_o}, \quad \Psi(z) = -\frac{\kappa \bar{F}}{z - z_o} + \frac{\bar{z}_o F}{(z - z_o)^2} \tag{32}$$

$$F = -\frac{P_x + iP_y}{2\pi(\kappa + 1)}, \quad \kappa = 3 - 4\nu \,(\nu - \mathrm{Poisson'sratio}) \tag{33}$$

Thus, the T_1 value can be evaluated immediately

$$T_1 = \mathrm{Re}[2\Phi(t) - t\Phi'(t) - \Psi(t)]|_{t=a} \tag{34}$$

In Eq. (34), T_1 is from continuous distribution of σ_x at prospective site of crack tip (no crack).

In the perturbation field, the applied boundary tractions are the same in magnitude and opposite in direction with those in the basic field. This portion is easy to evaluate, which is as follows [7]

$$\sigma_y + i\sigma_{xy} = P(t) + iQ(t) = -2\mathrm{Re}\Phi(t) - t\Phi'(t) - \Psi(t), \quad |t| \leqslant a \tag{35}$$

Therefore, from Eq. (29), we have

$$T_2 = P(a) = \mathrm{Re}[-2\Phi(t) - t\Phi'(t) - \Psi(t)]|_{t=a} \tag{36}$$

In Eq. (36), T_2 is equivalent to a value which is extracted from singular distribution of σ_x in front position of crack [6].

Finally, from Eqs. (34) and (36), the final result for T-stress is obtained

$$T = T_1 + T_2 = 2\mathrm{Re}\left(\frac{\kappa F}{a - \bar{z}_o} + \frac{(a - \bar{z}_o)F}{(a - z_o)^2}\right) \tag{37}$$

The result shown by Eq. (37) was obtained by using quite different method [2]. Clearly, this result can prove the correctness of the suggested method from other side.

If $P_x = 0$, $P_y = p_o a$ and $z_o = ih$, or a concentrated force applied at a point of y-axis, we will find

$$T = -\frac{p_o a}{2\pi(\kappa + 1)}\left(\frac{\kappa h}{a^2 + h^2} + \frac{h^3 - 3a^2 h}{(a^2 + h^2)^2}\right) \tag{38}$$

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