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The method of fundamental solutions with dual reciprocity for three-dimensional thermoelasticity under arbitrary body forces

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Abstract

Purpose – The purpose of this paper is to develop a meshless numerical method for three-dimensional isotropic thermoelastic problems with arbitrary body forces.

Design/methodology/approach – This paper combines the method of fundamental solutions (MFS) and the dual reciprocity method (DRM) as a meshless numerical method (MFS-DRM) to solve three-dimensional isotropic thermoelastic problems with arbitrary body forces. In the DRM, the arbitrarily distributed temperature and body force are approximated by polyharmonic splines with augmented polynomial basis, whose particular solutions and the corresponding tractions are reviewed and given explicitly. The MFS is then applied to solve the complementary solution. Numerical experiments of Dirchlet, Robin, and peanut-shaped-domain problems are carried out to validate the method.

Findings – In literature, it is commented that the Gaussian elimination can be used reliably to solve the MFS equations for non-noisy boundary conditions. For noisy boundary conditions, the truncated singular value decomposition (TSVD) is more accurate than the Gaussian elimination. In this paper, it was found that the particular solutions obtained by the DRM act like noises and the use of TSVD improves the accuracy.

Originality/value – It is the first time that the MFS-DRM is derived to solve three-dimensional isotropic thermoelastic problems with arbitrary body forces.

Keywords Elasticity, Gaussian processes

Paper type Research paper

1. Introduction

Recently, meshless numerical methods have composed a vital research field in the computing society. Among these methods, a large category is to approximate the thought functions by radial basis functions (RBFs). Both governing equations and boundary conditions are approximated by the RBFs for domain-type methods (Kansa, 1990). On the other hand, the method of fundamental solutions (MFS) is a boundary-type meshless numerical method, in which the desired solution is represented by a series of fundamental solutions with sources located outside the computational domain. In the MFS, the fundamental solutions are taken as the RBFs that satisfy governing equations analytically, thus only boundary conditions should be collocated. The MFS was first proposed by Kupradze and Aleksidze (1964), and the mathematical foundations of the method were then established by Mathon and Johnston (1977) and Bogomolny (1985). Thereafter, the MFS was successfully applied to the elliptic

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boundary value problems (Fairweather and Karageorghis, 1998), the scattering and radiation problems (Fairweather *et al.*, 2003), the evaluations of eigenvalues (Karageorghis, 2001; Tsai *et al.*, 2006b), the Poisson's equation (Golberg, 1995), and the Stokes flow problems (Alves and Silvestre, 2004).

The MFS was also applied to solve elastostatic problems. Redekop (1982) applied the MFS to solve planar elastic problems. On the other hand, Redekop and Thompson (1983) and Karageorghis and Fairweather (2000) utilized the MFS for axisymmetric problems. For three-dimensional problems, Redekop and Cheung (1987) obtained solutions of exterior problems by the MFS. On the other hand, Poullikas *et al.* (2002) recently considered the source locations of fundamental solutions also as unknowns and utilized non-linear least-squares algorithms to solve the resulted algebraic systems.

Although the MFS can reduce the dimensionalities compared to the domain-type meshless numerical methods, its use is unfortunately limited to homogeneous solutions of partial differential equations. In the cases where the non-homogeneous terms are known functions, exact particular solutions can often be calculated. Fam and Rashed (2005) recently applied the MFS with analytical particular solutions for three-dimensional structures with body force. However in other cases the non-homogeneous terms should also be approximated by the RBFs. This method was named as dual reciprocity method (DRM) in the boundary element method society (Nardini and Brebbia, 1982), and was combined with the MFS as a meshless numerical method to solve Poisson's equation (Golberg, 1995). Recently, the combination of MFS and DRM (MFS-DRM) was also utilized to solve two-dimensional thermoelasticity with general body forces (Medeiros *et al.*, 2004). In this paper, we extend the MFS-DRM to three-dimensional thermoelasticity with arbitrary body forces, in which the DRM is based on the augmented polyharmonic splines (Duchon, 1976), whose particular solutions were summarized in (Cheng *et al.*, 2001).

Although the convergent property of MFS were established mathematically (Mathon and Johnston, 1977; Bogomolny, 1985), the ill-conditioning and the locations of source points are numerically problematic. Traditionally, the ill-conditioning was mitigated by the singular value decomposition as illustrated by Ramachandran (2002). Recently, Chen *et al.* (2006a) reviewed the issue and commented that the Gaussian elimination could be used reliably to solve the MFS equations for non-noisy boundary conditions. For noisy boundary conditions, they suggested the use of truncated singular value decomposition (TSVD) by choosing a sufficiently large amount of collocations and then cutting off half of the singular values. However, most of the previous studies considered ranks less than 100, in which the ill-conditioning was not critical and most equation solvers could be utilized safely to obtain accurate solutions according to the author's experiences. In this paper, we study issues of practically implementing the MFS-DRM to three-dimensional thermoelasticity, in which both Gaussian elimination and TSVD are considered. It is concluded that the particular solutions obtained by the DRM act like noises and the use of TSVD improves the accuracy. Alternatively, readers can also consider the recent modifications of MFS in which the sources are located on the boundary to avoid the ill-conditioning (Chen *et al.*, 2006b, c; Young *et al.*, 2005, 2007).

A brief outline of the paper is as follows. We introduce the formulations of MFS-DRM for solving thermoelasticity with body forces in section 2. In section 3, some numerical experiments are preformed and the issues of practically implementing the MFS-DRM are stated. Finally, the conclusions are summarized in section 4.

2. Formulations of the MFS-DRM

2.1 Governing equations

Consider an isotropic material in domain Ω, the governing equations of thermoelasticity with body force, b_i, are

$$\sigma_{ij,j} = -b_i \tag{1}$$

and the constitutive equation

$$\sigma_{ij} = \frac{2G\nu}{1-2\nu}\delta_{ij}e_{kk} + 2Ge_{ij} - m\delta_{ij}T \tag{2}$$

with

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{3}$$

where σ_{ij} is the stress tensor, e_{ij} is the strain tensor, u_i is the displacement vector, T is the temperature, δ_{ij} is the Kronecker delta, G is the shear modulus, ν is Poisson's ratio, and $m = 2G\alpha_T(1+\nu)/(1-2\nu)$ is the thermoelastic constant, with α_T the coefficient of linear thermal expansion. The above equations can be combined to give

$$Gu_{i,jj} + \frac{G}{1-2\nu}u_{j,ji} = mT_{,i} - b_i \tag{4}$$

To have a well-posed boundary value problem, on each part of the boundary either the displacement or the traction boundary condition is prescribed as

$$u_i = \bar{u}_i \quad \text{on } \Gamma_u \tag{5a}$$

$$t_i = \bar{t}_i \quad \text{on } \Gamma_t \tag{5b}$$

where $\Gamma_u + \Gamma_t = \Gamma$ is the boundary of the solution domain Ω, $\bar{u}_i$ and $\bar{t}_i$ are prescribed boundary data, and

$$t_i = \sigma_{ij}n_j \tag{6}$$

is the boundary traction, with n_j denoting the boundary outward normal.

In the formulation of the MFS-DRM, the principle of superposition is applied to decompose the displacement u_i into two parts, the particular solution u_i^p and the complementary solution u_i^c as follows:

$$u_i = u_i^p + u_i^c \tag{7}$$

in which the particular solution satisfies

$$Gu^p_{i,jj} + \frac{G}{1-2\nu}u^p_{j,ji} = mT_{,i} - b_i \tag{8}$$

without the need of fulfilling any boundary condition. Thus, the complementary solution u_i^c is governed by

$$Gu_{i,jj}^c + \frac{G}{1-2\nu} u_{j,ji}^c = 0 \tag{9}$$

with

$$u_i^c = \bar{u}_i - u_i^p \quad \text{on } \Gamma_u \tag{10a}$$

$$t_i^c = \bar{t}_i - t_i^p \quad \text{on } \Gamma_t \tag{10b}$$

In the MFS-DRM formulation, the particular solution is first obtained by the DRM described below, and the complementary solution is then solved by the MFS. As a result, the displacement can be evaluated by using Equation (7).

2.2 Dual reciprocity method

Now, we are in a position to introduce the DRM. Typically, we consider the second order augmented polyharmonic spline, r^3, which is the lowest order with regular temperature particular solutions. Particular solutions of higher orders can be found in the literature (Cheng *et al.*, 2001). First of all the temperature T is approximated by

$$T(\mathbf{x}; A^1, \ldots, A^{10}, B^1, \ldots, B^M) \cong \sum_{j=1}^{10} A^j p^j(\mathbf{x}) + \sum_{j=1}^{M} B^j r_j^3 \tag{11}$$

with $p^j(\mathbf{x}) = \{1, x, y, z, x^2, y^2, z^2, xy, yz, zx\}$.

Where $\mathbf{x} = (x, y, z)$ is the position vector, and $r_j = \|\mathbf{x} - \mathbf{x}_j\|$ is the Euclidean distance form point $\mathbf{x}_j = (x_j, y_j, z_j)$. In addition, A^j and B^j are $M + 10$ unknown coefficients which can be determined by collocation and constraint conditions as follows

$$T(\mathbf{x}_i) = \sum_{j=1}^{10} A^j p^j(\mathbf{x}_i) + \sum_{j=1}^{M} B^j r_{ij}^3 \quad i = 1, 2, \ldots, M \tag{12a}$$

$$\sum_{j=1}^{M} B^j p^i(\mathbf{x}_j) = 0 \text{ for } i = 1, 2, \ldots, 10 \tag{12b}$$

where $r_{ij} = \|\mathbf{x}_i - \mathbf{x}_j\|$. Similarly, the body forces b_i are approximated by

$$b_i(\mathbf{x}; C_i^1, \ldots, D_i^{10}, D_i^1, \ldots, D_i^M) \cong \sum_{j=1}^{10} C_i^j p^j(\mathbf{x}) + \sum_{j=1}^{M} D_i^j r_j^3 \tag{13}$$

where the $3(M + 10)$ unknown coefficients C_i^j and D_i^j can also be obtain in a same way.

Then, the particular solution u_i^p are approximated by

$$u_i^p(\mathbf{x}) \cong \sum_{j=1}^{10} A^j \tilde{P}_i^j(\mathbf{x}) + \sum_{j=1}^{M} B^j \tilde{F}_i(r_j) - \sum_{k=1}^{3}\sum_{j=1}^{10} C_k^j \bar{P}_{ik}^j(\mathbf{x}) - \sum_{k=1}^{3}\sum_{j=1}^{M} D_k^j \bar{F}_{ik}(r_j) \quad (14)$$

in which $\tilde{P}_i^j(\mathbf{x}), \tilde{F}_i(r_j), \bar{P}_{ik}^j(\mathbf{x})$ and $\bar{F}_{ik}(r_j)$ are governed by

$$G\tilde{P}_{i,kk}^j + \frac{G}{1-2\nu}\tilde{P}_{k,ki}^j = mp_{,i}^j \quad (15a)$$

$$G\tilde{F}_{i,kk} + \frac{G}{1-2\nu}\tilde{F}_{k,ki} = m(r_j^3)_{,i} \quad (15b)$$

$$G\bar{P}_{il,kk}^j + \frac{G}{1-2\nu}\bar{P}_{kl,ki}^j = \delta_{il}p^j \quad (15c)$$

$$G\bar{F}_{il,kk} + \frac{G}{1-2\nu}\bar{F}_{kl,ki} = \delta_{il}r_j^3 \quad (15d)$$

Using Equations (2) and (6), the corresponding tractions can be obtained

$$t_i^p(\mathbf{x}) \cong \sum_{j=1}^{10} A^j \tilde{Q}_i^j(\mathbf{x}) + \sum_{j=1}^{M} B^j \tilde{S}_i(r_j) - \sum_{k=1}^{3}\sum_{j=1}^{10} C_k^j \bar{Q}_{ik}^j(\mathbf{x}) - \sum_{k=1}^{3}\sum_{j=1}^{M} D_k^j \bar{S}_{ik}(r_j) \quad (16)$$

In Equations (14) and (16), $\tilde{P}_i^j(\mathbf{x}), \tilde{F}_i(r), \bar{P}_{ik}^j(\mathbf{x}), \bar{F}_{ik}(r), \tilde{Q}_i^j(\mathbf{x}), \tilde{S}_i(r), \bar{Q}_{ik}^j(\mathbf{x})$ and $\bar{S}_{ik}(r)$ have been derived in (Cheng *et al.*, 2001) and are summarized with corrections of typos in Appendix.

It should be noticed that the convergence of Equation (11) and the solvability of the resulted linear equations from Equation (12) have been mathematically investigated by Duchon (1976). However, few theoretical statements can be addressed for the convergence of Equations (14) and (16). Therefore, numerical validations are performed in this study.

2.3 Method of fundamental solutions

After the particular solution is solved, the boundary value problem (Equations (9) and (10)) becomes well-posed. Thus, the complementary solution can be approximated by the well-known MFS. In the spirits of MFS, the complementary solution is represented approximately by

$$u_i^c(\mathbf{x}; E_1^1, \ldots, E_1^L, E_2^1, \ldots, E_2^L, E_3^1, \ldots, E_3^L, \mathbf{s}_1, \ldots, \mathbf{s}_L) \cong \sum_{k=1}^{3}\sum_{j=1}^{N} E_k^j U_{ik}^*(\mathbf{x}, \mathbf{s}_j) \quad (17)$$

where

$$U_{ij}^{*}(\mathbf{x},\mathbf{s}) = \frac{(3-4\nu)\delta_{ij} + r_{,i}r_{,j}}{16\pi G(1-\nu)r} \tag{18}$$

is the fundamental solution defined by

$$GU_{ij,kk}^{*}(\mathbf{x},\mathbf{s}) + \frac{G}{1-2\nu}U_{kj,ki}^{*}(\mathbf{x},\mathbf{s}) = -\delta_{ij}\delta(\mathbf{x}-\mathbf{s}) \tag{19}$$

with $\delta(\mathbf{x}-\mathbf{s})$ the Dirac delta function. Then, the corresponding traction can be obtained by using Equations (2) and (6) as follows:

$$t_i^c(\mathbf{x};E_1^1,\ldots,E_1^L,E_2^1,\ldots,E_2^L,E_3^1,\ldots,E_3^L,\mathbf{s}_1,\ldots,\mathbf{s}_L) \cong \sum_{k=1}^{3}\sum_{j=1}^{N} E_k^j T_{ik}^{*}(\mathbf{x},\mathbf{s}_j) \tag{20}$$

with

$$T_{ij}^{*}(\mathbf{x},\mathbf{s}) = -\frac{(1-2\nu)r_{,k}n_k\delta_{ij} + 3r_{,i}r_{,j}r_{,k}n_k + (1-2\nu)(r_{,i}n_j - r_{,j}n_i)}{8\pi(1-\nu)r^2} \tag{21}$$

It is easily verified that Equation (17) satisfies the governing equations in Equation (9) analytically. To determine the unknowns, E_k^j and $\mathbf{s}_j$, boundary conditions in Equation (10) should be fulfilled in suitable ways. Traditionally, the N source points $\mathbf{s}_j$ can be treated either as unknown or *a priori* known. In which the first case results in a cumbersome non-linear optimization with $6N$ unknowns, E_k^j and $\mathbf{s}_j$ (Poullikas *et al.*, 2002). On the other hand, if the source points are considered as *a priori* known, the boundary conditions are simply collocated at $N = N_1 + N_2$ boundary field points $\mathbf{x}_l$. It results in a linear equations system as follows:

$$\bar{u}_i^c(\mathbf{x}_l) - u_i^p(\mathbf{x}_l) = \sum_{k=1}^{3}\sum_{j=1}^{N} E_k^j U_{ik}^{*}(\mathbf{x}_l,\mathbf{s}_j) \text{ for } l = 1,2,\ldots,N_1 \tag{22a}$$

$$\bar{t}_i^c(\mathbf{x}_l) - t_i^p(\mathbf{x}_l) = \sum_{k=1}^{3}\sum_{j=1}^{N} E_k^j T_{ik}^{*}(\mathbf{x}_l,\mathbf{s}_j) \text{ for } l = N_1+1, N_1+2,\ldots,N_1+N_2 \tag{22b}$$

where $u_i^p(\mathbf{x}_l)$ and $t_i^p(\mathbf{x}_l)$ are given by Equations (14) and (16), respectively. In Equation (22), there are $3N$ equations with $3N$ unknowns, E_k^j, and thus can be solved, in which the solvability was discussed by Bogomolny (1985). In this paper, we typically locate the boundary field points uniformly and place the source points stipulated out as depicted in Figure 1 (Tsai *et al.*, 2006a).

Once the complementary and particular solutions are obtained, we can get the desired solution by using Equation (7).

3. Numerical results

In order to validate the proposed MFS-DRM formulation, two numerical experiments with Dirchlet and Robin boundary conditions are first considered. Then, the method is applied to two problems of peanut-shaped domain and heated hollow ball. In all the

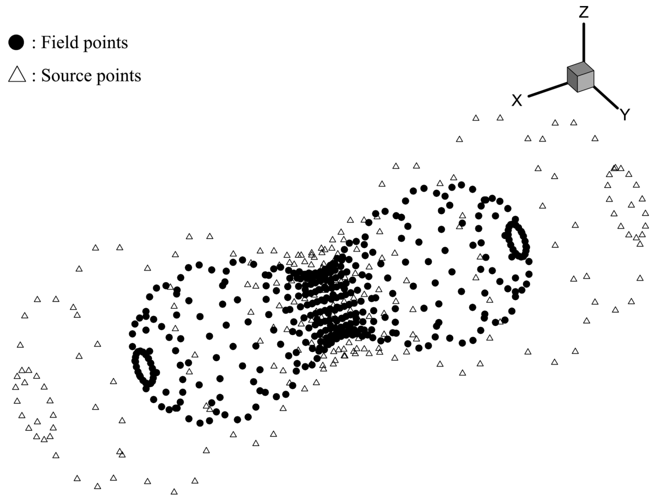

Figure 1.
Schematic diagram of the source and field points

four numerical experiments, both homogeneous and non-homogeneous cases are considered. In addition, both the Gaussian elimination and the TSVD (Chen *et al.*, 2006a) are utilized to solve the MFS equations in Equation (22). Typically, half of the singular values are ignored in the TSVD as suggested by Chen *et al.* (2006a). From these results, it can be concluded that the Gaussian elimination can obtain accurate solutions for homogeneous cases with non-noisy boundary conditions and the TSVD performs better for non-homogeneous cases in which the particular solutions obtained by the DRM act like noises to the MFS equations (Equation (22)).

In the results, the normalized root-mean-square error is defined as

$$\frac{\sqrt{\left(\sum_{j=1}^{\bar{N}}\sum_{i=1}^{3}(u_{i,numerical}(\mathbf{x}_j) - u_{i,exact}(\mathbf{x}_j))^2\right)/3\bar{N}}}{\operatorname*{Max}_{i,j}|u_{i,exact}(\mathbf{x}_j)|} \tag{23}$$

where $u_{i,numerical}(\mathbf{x}_j)$ is the numerical solutions obtained by the MFS-DRM (Equations (7), (14) and (17)) at $\mathbf{x}_j$, $u_{i,exact}(\mathbf{x}_j)$ is the corresponding exact solution, and $\bar{N}$ is the number of total nodes considered.

Besides, the material considered in these numerical experiments is the structural steel ASTM-A36 with density $\rho = 7{,}850$ kg/m^3, Young's modulus $E = 200$ GPa, Poisson's ratio $\nu = 0.29$, and coefficient of linear thermal expansion $\alpha_T = 1.2 \times 10^{-5}/^\circ$C. And, the gravitational acceleration $g = 9.8$ m/s^2 is assumed.

3.1 Dirchlet boundary condition

We consider both homogeneous and non-homogeneous cases in this numerical experiment. For the homogeneous case, we consider the solutions of Equation (4) in a cube of 2 m $\times$ 2 m $\times$ 2 m with center at (0, 0, 0), in which $T = 0$ and $b_i = 0$ and it is subjected to Dirichlet boundary conditions $u_1 = x, u_2 = y$ and $u_3 = z$. On the other hand, $T = E(x^2 + y^3 + z^4)/m, b_1 = 2Ex, b_2 = 3Ey^2$ and $b_3 = 3Ez^3$ for the non-homogeneous case. The exact solutions of these two cases are both $u_1 = x, u_2 = y, u_3 = z$.

Table I gives the normalized root-mean-square errors for different numbers of ranks. For the homogeneous case, it is clear to notice that the MFS can obtain excellent solutions almost up to machine error even for rank = 2,598 and the Gaussian elimination is able to solve the resulted algebraic linear equations system accurately and stably. In addition, the TSVD did not improve the accuracy as stated by Chen *et al.* (2006a). On the other hand, it is interesting for the non-homogeneous case that the particular solutions obtained by the DRM act as noises to the right hand side of the MFS equations. Thus, the TSVD can improve the accuracy as compared with the Gaussian elimination.

3.2 Robin boundary condition

Then we modify the previous problem by imposing traction boundary conditions on $|z| = 1$ and formally solve the homogeneous and non-homogeneous cases by the MFS and the MFS-DRM, respectively. The exact solution is the same as the previous case. Tables II addresses the normalized root-mean-square errors of the homogeneous and non-homogeneous cases. For the homogeneous Robin case, the solutions obtained by the Gaussian elimination are excellent although slightly worse than the Dirichlet problem. Similarly, the TSVD fails to obtain accurate solutions for the homogeneous Robin problem. On the other hand, it is observed that the MFS-DRM does not give accurate solutions for both the Gaussian elimination and TSVD.

To circumvent the problem, Balakrishnan and Ramachandran (1999) claimed that the sources should be located close to the boundary for Neumann condition and away from the boundary for the Dirchlet condition. In this work, we on the other hand rescale the Young's modulus to $E = 200$. In other hand, we use GPa in stead of Pa as the unit for stresses. The resulted normalized root-mean-square errors of the homogeneous and non-homogeneous cases are addressed in Table III. Compared with the results without rescaling in Table II, the accuracies are significantly improved especially for the non-homogeneous cases. The results in Table III also support the major declarations of the present paper that the Gaussian elimination can obtain accurate solutions for homogeneous problems and the TSVD can remedy the interference of noises and ill-conditioning for non-homogeneous problems.

Table I. Normalized root-mean-square errors of the homogeneous (up) and non-homogeneous (down) Dirichlet problems

Rank	654	1,158	1,806	2,598
Gaussian	3.25E-08	7.01E-10	4.97E-13	1.08E-14
TSVD	7.59E-07	1.24E-09	3.72E-11	2.58E-09
Rank	654	1,158	1,806	2,598
Gaussian	8.94E-04	3.42E-04	2.36E-04	8.61E-05
TSVD	8.74E-04	3.18E-04	1.42E-04	7.15E-05

Table II. Normalized root-mean-square errors of the homogeneous (up) and non-homogeneous (down) Robin problems without rescaling

Rank	654	1,158	1,806	2,598
Gaussian	3.93E-08	2.66E-08	9.25E-12	1.53E-10
TSVD	1.32E-06	2.09E-08	3.99E-08	4.51E-05
Rank	654	1,158	1,806	2,598
Gaussian	2.11E-03	1.64E-02	1.52E-02	3.58E-02
TSVD	1.93E-03	6.64E-04	3.42E-04	7.17E-03

In order to have a better understanding of the rescaling method, we also give the detailed accuracies for rank = 2,598 in Table IV. It is clear that the rescaling method does provide the flexibility of locating sources for Robin problems. Roughly, the value of Young's modulus should be rescaled to be in the same order of the displacement. Further researches should be undertaken for the optimal choice of scaling.

3.3 Thermoelasticity with body force in a peanut-shaped domain

In order to demonstrate the flexibility of the proposed numerical method to treat irregular domains, three-dimensional peanut shaped computational domains (Figure 1) are also considered. The same problem with Dirchlet boundary condition on the boundary of the peanut shaped domain is considered. The exact solution is the same as the previous cases. The homogeneous and non-homogeneous cases of this problem are solved formally by the MFS and MFS-DRM and the normalized root-mean-square errors for different numbers of ranks are stated in Table V. The results are nice, and perform similarly to the previous cases.

3.4 A heated hollow ball

Finally, we consider a problem of heated hollow ball. The radius of inner hole is a and the radius of outer ball is b. The temperature at $r = a$ and $r = b$ are $T = 0$ and $T = \bar{T}$,

Rank	654	1,158	1,806	2,598
Gaussian	3.93E-08	7.76E-09	1.22E-12	1.88E-14
TSVD	1.32E-06	3.38E-09	4.91E-10	6.93E-08
Rank	654	1,158	1,806	2,598
Gaussian	2.11E-03	3.66E-03	3.99E-04	3.45E-04
TSVD	1.92E-03	6.64E-04	2.92E-04	1.50E-04

Table III. Normalized root-mean-square errors of the homogeneous (up) and non-homogeneous (down) Robin problems with rescaling

Unit of stress	Pa	kPa	MPa	GPa
Gaussian	1.53E-10	1.22E-12	3.65E-13	1.88E-14
TSVD	4.51E-05	3.94E-05	1.38E-05	6.93E-08
Unit of stress	Pa	kPa	MPa	GPa
Gaussian	3.58E-02	1.83E-02	1.59E-03	3.45E-04
TSVD	7.17E-03	1.83E-04	1.70E-04	1.50E-04

Table IV. Normalized root-mean-square errors of the homogeneous (up) and non-homogeneous (down) Robin problems by using different units of stress

Rank	198	438	774	1,206
Gaussian	4.13E-06	5.52E-08	3.69E-09	1.18E-10
TSVD	2.31E-0 4	2.78E-06	3.47E-07	6.44E-07
Rank	198	438	774	1,206
Gaussian	1.19E-03	6.38E-04	9.20E-04	1.68E-03
TSVD	2.71E-03	2.20E-04	1.19E-04	5.86E-05

Table V. Normalized root-mean-square errors of the homogeneous (up) and non-homogeneous (down) peanut-shaped-domain problem respectively. On the other hand, we set up fixed boundary condition, $\mathbf{u} = \mathbf{0}$, in elasticity. The exact solution for this problem is

$$T = \frac{\bar{T}ab}{(a-b)r} - \frac{\bar{T}b}{(a-b)} \tag{24a}$$

$$u_r = \alpha\left(\frac{(a+b)r}{2(a^2+ab+b^2)} + \frac{a^2b^2}{2(a^2+ab+b^2)r^2} - \frac{1}{2}\right) \tag{24b}$$

where

$$\alpha = -\frac{\bar{T}abm}{(a-b)}\frac{1+2\nu}{(2+2\nu)G} \tag{25}$$

and u_r is the displacement in radial direction.

Table VI gives the normalized root-mean-square errors for the solutions obtained by the MFS-DRM. Accurate solutions are also observed for this practical problem.

4. Conclusions

The three-dimensional MFS-DRM formulation is introduced in this article, in which the augmented polyharmonic spline is adopted in the DRM and the corresponding particular solutions are reviewed with corrections for typos in the article of Cheng *et al.* (2001). Besides, in order to avoid the singularities for the thermal particular solution, second order polyharmonic spline was utilized. Three numerical experiments were carried out to validate the method. Both essential and mixed boundary conditions are considered. The method is also applied to a problem of peanut-shaped domain to demonstrate the flexibility to treat irregular domains. We also consider a practical problem of heated hollow ball. Overall, good agreements with the exact solutions are observed.

Furthermore, numerical issues of practical implementations are discussed. It is found that the Gaussian elimination is able to obtain accurate solutions for homogeneous cases with non-noisy boundary conditions and the TSVD performs better for non-homogeneous cases in which the particular solutions obtained by the DRM act like noises to the constant terms of MFS equations. For the Robin problems, the rescaling of Young's modulus significantly improves the accuracy.

Overall, the purpose of present work is to develop the fundamental meshless MFS-DRM framework for thermoelasticity with arbitrary body forces. The convergence is numerically established. It also provides the base for further applications to unsteady problems as was done by the dual reciprocity boundary element method. This will be our further researches.

Table VI. Normalized root-mean-square errors for the problem of heated hollow ball

Rank	516	768	1,068	1,416
NRMSE	1.44E-02	2.50E-03	5.12E-04	2.15E-04

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Appendix

The functions in Equations (14) and (16) are defined by

$$\tilde{P}_i^1(\mathbf{x}) = \frac{m(1-2\nu)\delta_{i1}x}{2G(1-\nu)} \tag{A1}$$

$$\tilde{P}_i^2(\mathbf{x}) = \frac{m(1-2\nu)\delta_{i1}x^2}{4G(1-\nu)} \tag{A2}$$

$$\tilde{P}_i^3(\mathbf{x}) = \frac{m(1-2\nu)\delta_{i2}y^2}{4G(1-\nu)} \tag{A3}$$

$$\tilde{P}_i^4(\mathbf{x}) = \frac{m(1-2\nu)\delta_{i3}z^2}{4G(1-\nu)} \tag{A4}$$

$$\tilde{P}_i^5(\mathbf{x}) = \frac{m(1-2\nu)\delta_{i1}x^3}{6G(1-\nu)} \tag{A5}$$

$$\tilde{P}_i^6(\mathbf{x}) = \frac{m(1-2\nu)\delta_{i2}y^3}{6G(1-\nu)} \tag{A6}$$

$$\tilde{P}_i^7(\mathbf{x}) = \frac{m(1-2\nu)\delta_{i3}z^3}{6G(1-\nu)} \tag{A7}$$

$$\tilde{P}_i^8(\mathbf{x}) = \frac{m(1-2\nu)x^2(3\delta_{i1}y+\delta_{i2}x)}{12G(1-\nu)} \tag{A8}$$

$$\tilde{P}_i^9(\mathbf{x}) = \frac{m(1-2\nu)y^2(3\delta_{i2}z+\delta_{i3}y)}{12G(1-\nu)} \tag{A9}$$

$$\tilde{P}_i^{10}(\mathbf{x}) = \frac{m(1-2\nu)z^2(3\delta_{i3}x+\delta_{i1}z)}{12G(1-\nu)} \tag{A10}$$

$$\tilde{F}_i(r_j) = \frac{m(1-2\nu)r_{j,i}r_j^4}{12G(1-\nu)} \tag{A11}$$

$$\bar{P}_{ik}^1(\mathbf{x}) = \frac{[2(1-\nu)\delta_{ik}-\delta_{1i}\delta_{1k}]x^2}{4G(1-\nu)} \tag{A12}$$

$$\bar{P}_{ik}^2(\mathbf{x}) = \frac{[2(1-\nu)\delta_{ik}-\delta_{1i}\delta_{1k}]x^3}{12G(1-\nu)} \tag{A13}$$

$$\bar{P}_{ik}^3(\mathbf{x}) = \frac{[2(1-\nu)\delta_{ik}-\delta_{2i}\delta_{2k}]y^3}{12G(1-\nu)} \tag{A14}$$

$$\bar{P}_{ik}^4(\mathbf{x}) = \frac{[2(1-\nu)\delta_{ik}-\delta_{3i}\delta_{3k}]z^3}{12G(1-\nu)} \tag{A15}$$

$$\bar{P}_{ik}^5(\mathbf{x}) = \frac{[2(1-\nu)\delta_{ik}-\delta_{1i}\delta_{1k}]x^4}{24G(1-\nu)} \tag{A16}$$

$$\bar{P}_{ik}^6(\mathbf{x}) = \frac{[2(1-\nu)\delta_{ik}-\delta_{2i}\delta_{2k}]y^4}{24G(1-\nu)} \tag{A17}$$

$$\bar{P}^{7}_{ik}(\mathbf{x}) = \frac{[2(1-\nu)\delta_{ik} - \delta_{3i}\delta_{3k}]z^4}{24G(1-\nu)} \tag{A18}$$

$$\bar{P}^{8}_{ik}(\mathbf{x}) = \frac{\{4[2(1-\nu)\delta_{ik} - \delta_{1i}\delta_{1k}]y - [\delta_{1i}\delta_{2k} + \delta_{1k}\delta_{2i}]x\}x^3}{48G(1-\nu)} \tag{A19}$$

$$\bar{P}^{9}_{ik}(\mathbf{x}) = \frac{\{4[2(1-\nu)\delta_{ik} - \delta_{2i}\delta_{2k}]z - [\delta_{2i}\delta_{3k} + \delta_{2k}\delta_{3i}]y\}y^3}{48G(1-\nu)} \tag{A20}$$

$$\bar{P}^{10}_{ik}(\mathbf{x}) = \frac{\{4[2(1-\nu)\delta_{ik} - \delta_{3i}\delta_{3k}]x - [\delta_{3i}\delta_{1k} + \delta_{3k}\delta_{1i}]z\}z^3}{48G(1-\nu)} \tag{A21}$$

$$\bar{F}_{ik}(r_j) = \frac{r_j^5[-5r_{j,i}r_{j,k} + (15-16\nu)\delta_{ik}]}{480G(1-\nu)} \tag{A22}$$

$$\tilde{Q}^{1}_{i}(\mathbf{x}) = \frac{m[\nu + \delta_{i1}(1-2\nu)]n_i}{(1-\nu)} \tag{A23}$$

$$\tilde{Q}^{2}_{i}(\mathbf{x}) = \frac{m[\nu + \delta_{i1}(1-2\nu)]n_i x}{(1-\nu)} \tag{A24}$$

$$\tilde{Q}^{3}_{i}(\mathbf{x}) = \frac{m[\nu + \delta_{i2}(1-2\nu)]n_i y}{(1-\nu)} \tag{A25}$$

$$\tilde{Q}^{4}_{i}(\mathbf{x}) = \frac{m[\nu + \delta_{i3}(1-2\nu)]n_i z}{(1-\nu)} \tag{A26}$$

$$\tilde{Q}^{5}_{i}(\mathbf{x}) = \frac{m[\nu + \delta_{i1}(1-2\nu)]n_i x^2}{(1-\nu)} \tag{A27}$$

$$\tilde{Q}^{6}_{i}(\mathbf{x}) = \frac{m[\nu + \delta_{i2}(1-2\nu)]n_i y^2}{(1-\nu)} \tag{A28}$$

$$\tilde{Q}^{7}_{i}(\mathbf{x}) = \frac{m[\nu + \delta_{i3}(1-2\nu)]n_i z^2}{(1-\nu)} \tag{A29}$$

$$\tilde{Q}^{8}_{i}(\mathbf{x}) = \frac{m[\nu + \delta_{i1}(1-2\nu)]n_i xy}{(1-\nu)} + \frac{m(1-2\nu)[\delta_{i1}n_2 + \delta_{i2}n_1]x^2}{2(1-\nu)} \tag{A30}$$

$$\tilde{Q}_i^{9}(\mathbf{x}) = \frac{m[\nu + \delta_{i2}(1-2\nu)]n_i yz}{(1-\nu)} + \frac{m(1-2\nu)[\delta_{i2}n_3 + \delta_{i3}n_2]y^2}{2(1-\nu)} \tag{A31}$$

$$\tilde{Q}_i^{10}(\mathbf{x}) = \frac{m[\nu + \delta_{i3}(1-2\nu)]n_i zx}{(1-\nu)} + \frac{m(1-2\nu)[\delta_{i3}n_1 + \delta_{i1}n_3]z^2}{2(1-\nu)} \tag{A32}$$

$$\tilde{S}_i(r_j) = \frac{mr_j^3}{6(1-\nu)}[(1+4\nu)n_i + (3-6\nu)r_{j,i}\frac{\partial r_j}{\partial n}] \tag{A33}$$

$$\bar{Q}_{ik}^{1}(\mathbf{x}) = \frac{[(1-\nu)(\delta_{ik}n_1 + \delta_{1i}n_k) + \nu\delta_{1k}n_i - \delta_{1i}\delta_{1k}n_1]x}{(1-\nu)} \tag{A34}$$

$$\bar{Q}_{ik}^{2}(\mathbf{x}) = \frac{[(1-\nu)(\delta_{ik}n_1 + \delta_{1i}n_k) + \nu\delta_{1k}n_i - \delta_{1i}\delta_{1k}n_1]x^2}{2(1-\nu)} \tag{A35}$$

$$\bar{Q}_{ik}^{3}(\mathbf{x}) = \frac{[(1-\nu)(\delta_{ik}n_2 + \delta_{2i}n_k) + \nu\delta_{2k}n_i - \delta_{2i}\delta_{2k}n_2]y^2}{2(1-\nu)} \tag{A36}$$

$$\bar{Q}_{ik}^{4}(\mathbf{x}) = \frac{[(1-\nu)(\delta_{ik}n_3 + \delta_{3i}n_k) + \nu\delta_{3k}n_i - \delta_{3i}\delta_{3k}n_3]z^2}{2(1-\nu)} \tag{A37}$$

$$\bar{Q}_{ik}^{5}(\mathbf{x}) = \frac{[(1-\nu)(\delta_{ik}n_1 + \delta_{1i}n_k) + \nu\delta_{1k}n_i - \delta_{1i}\delta_{1k}n_1]x^3}{3(1-\nu)} \tag{A38}$$

$$\bar{Q}_{ik}^{6}(\mathbf{x}) = \frac{[(1-\nu)(\delta_{ik}n_2 + \delta_{2i}n_k) + \nu\delta_{2k}n_i - \delta_{2i}\delta_{2k}n_2]y^3}{3(1-\nu)} \tag{A39}$$

$$\bar{Q}_{ik}^{7}(\mathbf{x}) = \frac{[(1-\nu)(\delta_{ik}n_3 + \delta_{3i}n_k) + \nu\delta_{3k}n_i - \delta_{3i}\delta_{3k}n_3]z^3}{3(1-\nu)} \tag{A40}$$

$$\bar{Q}_{ik}^{8}(\mathbf{x}) = \frac{\begin{array}{c}3[(1-\nu)(\delta_{ik}n_1 + \delta_{1i}n_k) + \nu\delta_{1k}n_i - \delta_{1i}\delta_{1k}n_1]x^2y \\ +[(1-\nu)(\delta_{ik}n_2 + \delta_{2i}n_k) + \nu\delta_{2k}n_i - \delta_{2i}\delta_{1k}n_1 - \delta_{1i}\delta_{2k}n_1 - \delta_{1i}\delta_{1k}n_2]x^3\end{array}}{6(1-\nu)} \tag{A41}$$

$$\bar{Q}_{ik}^{9}(\mathbf{x}) = \frac{\begin{array}{c}3[(1-\nu)(\delta_{ik}n_2 + \delta_{2i}n_k) + \nu\delta_{2k}n_i - \delta_{2i}\delta_{2k}n_2]y^2z \\ +[(1-\nu)(\delta_{ik}n_3 + \delta_{3i}n_k) + \nu\delta_{3k}n_i - \delta_{3i}\delta_{2k}n_2 - \delta_{2i}\delta_{3k}n_2 - \delta_{2i}\delta_{2k}n_3]y^3\end{array}}{6(1-\nu)} \tag{A42}$$

$$\bar{Q}_{ik}^{10}(\mathbf{x}) = \frac{\begin{array}{c}3[(1-\nu)(\delta_{ik}n_3+\delta_{3i}n_k)+\nu\delta_{3k}n_i-\delta_{3i}\delta_{3k}n_3]z^2x \\ +[(1-\nu)(\delta_{ik}n_1+\delta_{1i}n_k)+\nu\delta_{1k}n_i-\delta_{1i}\delta_{3k}n_3-\delta_{3i}\delta_{1k}n_3-\delta_{3i}\delta_{3k}n_1]z^3\end{array}}{6(1-\nu)} \tag{A43}$$

$$\bar{S}_{ik}(r_j) = \frac{r_j^4[(7-8\nu)(\delta_{ik}(\partial r_j/\partial n)+r_{j,i}n_k)-(1-8\nu)r_{j,k}n_i-3r_{j,i}r_{j,k}(\partial r_j/\partial n)]}{48(1-\nu)} \tag{A44}$$

Please note the typos in Table II and Equation (90) of Cheng *et al.* (2001) corresponding to Equations (A23)-(A32), and (A44).

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