

边界积分方程中近奇异积分计算的一种变量替换法¹⁾

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摘要 准确估计近奇异边界积分是边界元分析中一项很重要的课题, 其重要性仅次于对奇异积分的处理. 近年来已发展了许多方法, 都取得了一定程度的成功, 但这个问题至今仍未得到彻底的解决. 基于一种新的变量变换的思想和观点, 提交了一种通用的积分变换法, 它非常有效地改善了被积函数的震荡特性, 从而消除了积分的近奇异性, 在不增加计算量的情况下, 极大地改进了近奇异积分计算的精度. 数值算例表明, 其算法稳定, 效率高, 并可达到很高的计算精度, 即使区域内点非常地靠近边界, 仍可取得很理想的结果.

关键词 边界元法, 近奇异积分, 变换法, 位势问题, 边界层效应

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引 言

边界元法实施中的一个关键的方面是核积分的准确计算. 当配置点属于积分单元时, 这些核是弱奇异的, 强奇异的, 甚至是超奇异的函数, 对此已有许多有效的处理方法^[1~11]. 另一个重要的问题是配置点接近但不在积分单元时的核积分计算. 理论上, 这样的积分是非奇异的, 但从计算的角度看, 当配置点很接近积分单元时, 被积函数在积分区间内变化非常迅速, 而且配置点越接近积分单元变化越迅速, 通过标准的 Gauss 公式不能准确计算, 称之为近奇异积分. 实践表明, 计算点充分远离边界曲线时, 我们甚至可以获得超收敛的结果, 然而, 随着计算点接近积分单元, 由标准的 Gauss 求积过程, 如果忽略被积函数这种病态行为, 解的精度会迅速降低, 甚至无精度可言, 即所谓的“边界层效应”.

精确计算近奇异积分在一些工程问题中成为关键要素, 例如裂纹问题中裂尖具有较小的张开位移、接触问题中两个接触体非常靠近的情形及薄体或薄壳结构问题等.

边界元法中, 准确估计近奇异积分的重要性仅次于奇异积分, 近年来引起许多学者的关注, 已发展了许多数值处理技术和方法, 大致可分为“间接算法”^[7~14]和“直接算法”^[8,15~30]两大类. 前者主

要是通过建立新的规则化边界积分方程, 来间接或避免计算近奇异积分, 如虚边界元法^[8]、刚体位移法、简单解法等^[7~14]. 简单解法和刚体位移法得益于奇异积分的规则化思想和方法, 利用密度函数中的零因子消除核函数分母中的近零因子; “直接算法”是对近奇异积分的直接计算或近似, 通常有区间分割法^[15]、特别的 Gaussian 积分法^[16]、解析或半解析法^[17~21]及变换法^[22~30]等. 区间分割法似乎是一种比较有效的方法, 但分割区间的数量密切依赖于计算点与边界的距离, 计算点越靠近边界, 分割区间的数量越多; 特别的 Gaussian 积分法需根据被积函数的形式确定求积系数和积分点, 是件非常困难的事; 解析法计算近奇异积分比计算奇异积分似乎更困难, 对于曲线单元一般来说是难以实现的. 半解析法主要是通过“加减法”分离近奇异部分, 分离出的部分采用解析计算, 规则化部分采用标准的 Gauss 求积公式计算, 需指出的是规则化部分仍然保留着弱近奇异性; 目前最盛行的“直接算法”是各种非线性变换法. 如: 二次多项式变换^[22]、双三次变换^[23]、Sigmoidal 变换^[24]和 Semi-sigmoidal 变换^[25]、优化坐标变换^[26]、衰减映射法^[27~28]、有理变换^[29]、距离变换^[30]等.

本文基于尽可能拉近运算因子间的数量级或变化尺度的思想, 构造了通用的变量替换, 有效地改

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善了近奇异核的震荡特性, 使近奇异积分的计算结果达到了很高的精度.

1 等价规则化边界积分方程中的近奇异积分

本文总设 Ω 是 R^2 中的一个有界区域, Ω^c 是其补域, Γ 是它们的边界. 文献 [31] 给出等价间接变量规则化边界积分方程

$$\int_{\Gamma} \phi(x) d\Gamma + \int_{\hat{\Omega}} f(x) d\hat{\Omega} = 0 \tag{1}$$

$$u(\mathbf{y}) = \int_{\Gamma} \phi(x) u^*(x, \mathbf{y}) d\Gamma_x + \int_{\hat{\Omega}} f(x) u^*(x, \mathbf{y}) d\hat{\Omega}_x + C, \mathbf{y} \in \Gamma \tag{2}$$

$$\begin{aligned} \nabla u(\mathbf{y}) = \hat{S}\phi(\mathbf{y}) + \int_{\Gamma} [\phi(x) - \phi(\mathbf{y})] \nabla u^*(x, \mathbf{y}) d\Gamma - \\ \phi(\mathbf{y}) \left\{ \int_{\Gamma} [\hat{\mathbf{t}}(x) - \hat{\mathbf{t}}(\mathbf{y})] \nabla_x u^*(x, \mathbf{y}) \cdot \hat{\mathbf{t}}(x) d\Gamma + \right. \\ \left. \int_{\Gamma} [\hat{\mathbf{n}}(x) - \hat{\mathbf{n}}(\mathbf{y})] \nabla_x u^*(x, \mathbf{y}) \cdot \hat{\mathbf{n}}(x) d\Gamma \right\} + \\ \int_{\hat{\Omega}} f(x) \nabla u^*(x, \mathbf{y}) d\hat{\Omega}, \mathbf{y} \in \Gamma \end{aligned} \tag{3}$$

内点积分方程为

$$u(\mathbf{y}) = \int_{\Gamma} \phi(x) u^*(x, \mathbf{y}) d\Gamma_x + \int_{\hat{\Omega}} f(x) u^*(x, \mathbf{y}) d\hat{\Omega}_x + C, \mathbf{y} \in \hat{\Omega} \tag{4}$$

$$\begin{aligned} \nabla u(\mathbf{y}) = \int_{\Gamma} \phi(x) \nabla_y u^*(x, \mathbf{y}) d\Gamma_x + \\ \int_{\hat{\Omega}} f(x) \nabla_y u^*(x, \mathbf{y}) d\hat{\Omega}_x, \mathbf{y} \in \hat{\Omega} \end{aligned} \tag{5}$$

方程 (1)~(5) 中, $\phi(x)$ 为待定密度函数, $f(x)$ 为体函数. 对内域问题, $\hat{\Omega} = \Omega, \hat{S} = 1, \hat{\mathbf{t}}(x), \hat{\mathbf{n}}(x)$ 分别是区域 Ω 边界 Γ 在 x 点处的单位切、外法向量; 对外域问题, $\hat{\Omega} = \Omega^c, \hat{S} = 0, \hat{\mathbf{t}}(x), \hat{\mathbf{n}}(x)$ 分别是区域 Ω^c 边界 Γ 在 x 点处的单位切、外法向量.

对方程 (4) 和 (5), 当内点 \mathbf{y} 充分远离边界 Γ 时, 无需作任何变换, 由常规的高斯求积法就可算得理想的结果. 然而, 当点 \mathbf{y} 趋近边界时, 直接由常规法计算, 结果的精度会迅速降低. 随着 \mathbf{y} 进一步靠近边界 Γ , 结果会无精度可言, 即“边界层效应”, 原因在于基本解核存在近奇异性. 这些近奇异积分可以写为

$$I_1 = \int_{\Gamma} \psi(x) \ln r^2 d\Gamma, \quad I_2 = \int_{\Gamma} \psi(x) \frac{1}{r^{2\alpha}} d\Gamma \tag{6}$$

这里 $\psi(x)$ 是规则函数, $\alpha > 0$.

(1) 边界 Γ 的线性单元逼近

设 $\mathbf{x}^1 = (x_1^1, x_2^1), \mathbf{x}^2 = (x_1^2, x_2^2)$ 是线性几何单元 Γ_j 的两个端点, 那么单元 Γ_j 可表示为

$$x_k(\xi) = N_1(\xi)x_k^1 + N_2(\xi)x_k^2, \quad k = 1, 2 \tag{7}$$

其中 $N_1(\xi) = (1 - \xi)/2, N_2(\xi) = (1 + \xi)/2, -1 \leq \xi \leq 1$. 设 $s_i = x_i^2 - x_i^1, w_i = y_i - (x_i^2 + x_i^1)/2$, 有

$$r_{,i} = \frac{r_i}{r} = \frac{y_i - x_i}{r} = \frac{-s_i \xi / 2 + w_i}{r} \tag{8}$$

$$r^2 = A\xi^2 + B\xi + E = L^2[(\xi - \eta)^2 + d^2] \tag{9}$$

其中 $A = s_i s_i / 4, B = s_i w_i, E = w_i w_i, \eta = \frac{B}{2A}, L = \sqrt{A}, d = \frac{1}{2A} \sqrt{4AE - B^2}$.

假设 $\sqrt{E} < \sqrt{A}$, 即点 \mathbf{y} 到单元中点的距离小于单元长度的一半时, 由 hölder 不等式可得

$$|\eta| = \left| \frac{s_i w_i}{2A} \right| \leq \left| \frac{\sqrt{s_i s_i} \sqrt{w_i w_i}}{2A} \right| = \left| \frac{\sqrt{E}}{\sqrt{A}} \right| < 1$$

因此式 (6) 中的积分 I_1 和 I_2 可在点 η 分成两部分

$$I_1 = \left\{ \int_{-1}^{\eta} + \int_{\eta}^1 \right\} g(\xi) \ln[(\xi - \eta)^2 + d^2] d\xi + \ln L^2 \int_{-1}^1 g(\xi) d\xi \tag{10}$$

$$I_2 = \left\{ \int_{-1}^{\eta} + \int_{\eta}^1 \right\} \frac{g(\xi)}{L^{2\alpha} [(\xi - \eta)^2 + d^2]^{\alpha}} d\xi \tag{11}$$

这里 $g(\cdot)$ 是规则函数.

(2) 边界 Γ 的“圆弧单元”逼近

“圆弧单元”是文献 [10] 提出的, 对于圆弧边界, 这样的插值逼近几乎是精确的. 设圆弧单元 Γ_j 的两个端点坐标是 $(R, \theta_1), (R, \theta_2)$, 那么 Γ_j 可表示为

$$x_1 = R \cos \theta, \quad x_2 = R \sin \theta,$$

$$\theta = N_1(\xi)\theta_1 + N_2(\xi)\theta_2, \quad -1 \leq \xi \leq 1$$

对域内点 $\mathbf{y} = (R_0 \cos \theta_0, R_0 \sin \theta_0)$, 不妨设 $\theta_1 < \theta_0 < \theta_2$, 则有

$$\theta_0 = N_1(\eta)\theta_1 + N_2(\eta)\theta_2 \quad (-1 < \eta < 1)$$

$$r^2 = |\mathbf{x} - \mathbf{y}|^2 = 4RR_0(\sin^2 \gamma + d^2)$$

其中 $\gamma = \beta(\xi - \eta), \beta = \frac{\theta_2 - \theta_1}{4}, d = \frac{R - R_0}{2\sqrt{RR_0}}$.

将式 (6) 中的积分 I_1 和 I_2 在点 η 分成两部分

$$I_1 = \left\{ \int_{-1}^{\eta} + \int_{\eta}^1 \right\} g(\xi) \ln(\sin^2 \gamma + d^2) d\xi + \ln L^2 \int_1^1 g(\xi) d\xi \quad (12)$$

$$I_2 = \frac{1}{L^{2\alpha}} \left\{ \int_{-1}^{\eta} + \int_{\eta}^1 \right\} \frac{g(\xi)}{(\sin^2 \gamma + d^2)^\alpha} d\xi \quad (13)$$

这里 $L = 2\sqrt{RR_0}$, $g(\cdot)$ 是由形函数及雅可比 (在式 (13) 中还包含 r_i) 组成的规则函数.

2 近奇异积分的变换

对式 (10) 右端前两个积分分别作变量替换

$$\left. \begin{aligned} \xi &= \eta - k_1 d(e^{k(1+t)} - 1) \\ \xi &= \eta + k_2 d(e^{k(1+t)} - 1) \end{aligned} \right\} \quad (14)$$

其中 $k_1 = 1 + \eta$, $k_2 = 1 - \eta$, $k = \frac{1}{2} \ln \left(1 + \frac{1}{d} \right)$. 有

$$\begin{aligned} I_1 &= k_1 k d \ln d^2 \int_{-1}^1 g_1(t) e^{k(1+t)} dt + k k_1 d \cdot \\ &\int_{-1}^1 g_1(t) e^{k(1+t)} \ln[k_1^2 (e^{k(1+t)} - 1)^2 + 1] dt + \\ &k_2 k d \ln d^2 \int_{-1}^1 g_2(t) e^{k(1+t)} dt + k k_2 d \cdot \\ &\int_{-1}^1 g_2(t) e^{k(1+t)} \ln[k_2^2 (e^{k(1+t)} - 1)^2 + 1] dt + \\ &\ln L^2 \int_{-1}^1 g(\xi) d\xi \end{aligned} \quad (15)$$

其中

$$\begin{aligned} g_1(t) &= g[\eta - k_1 d(e^{k(1+t)} - 1)] \\ g_2(t) &= g[\eta + k_2 d(e^{k(1+t)} - 1)] \end{aligned}$$

对式 (11) 右端两个积分, 同样作式 (14) 的变换

$$\begin{aligned} I_2 &= \frac{k k_1 d^{1-2\alpha}}{L^{2\alpha}} \int_{-1}^1 \frac{g_1(t) e^{k(1+t)} dt}{[k_1^2 (e^{k(1+t)} - 1)^2 + 1]^\alpha} + \\ &\frac{k k_2 d^{1-2\alpha}}{L^{2\alpha}} \int_{-1}^1 \frac{g_2(t) e^{k(1+t)} dt}{[k_2^2 (e^{k(1+t)} - 1)^2 + 1]^\alpha} \end{aligned} \quad (16)$$

这里 $g_1(t), g_2(t)$ 与前面的相同.

现在考虑式 (12) 和 (13), 有

$$\begin{aligned} I_1 &= \ln L^2 \int_{-1}^1 g(\xi) d\xi + \int_0^{t_1} g_1(t) \ln(t^2 + d^2) dt + \\ &\int_0^{t_2} g_2(t) \ln(t^2 + d^2) dt \end{aligned} \quad (17)$$

$$I_2 = \int_0^{t_1} \frac{g_1(t) dt}{h(t^2 + d^2)^\alpha} + \int_0^{t_2} \frac{g_2(t) dt}{h(t^2 + d^2)^\alpha} \quad (18)$$

在式 (17) 和 (18) 中

$$h = L^{2\alpha}, \quad t_1 = \sin[\beta(1 + \eta)]$$

$$t_2 = \sin[\beta(1 - \eta)]$$

$$g_1(t) = \frac{1}{\beta\sqrt{1-t^2}} g\left(\eta - \frac{1}{\beta} \arcsin t\right)$$

$$g_2(t) = \frac{1}{\beta\sqrt{1-t^2}} g\left(\eta + \frac{1}{\beta} \arcsin t\right)$$

只要 $\theta_2 - \theta_1 \leq \pi/4$, 就有 $|\beta(1 \pm \eta)| < \pi/8$, 故

$$|t_1|, |t_2| = |\sin[\beta(1 \pm \eta)]| \leq \sin \frac{\pi}{8} \ll 1$$

所以 $g_1(t), g_2(t)$ 均为规则函数.

因此可对式 (17) 等号右边第 2,3 个积分及式 (18) 中的积分同样作式 (14) 的变换.

3 数值算例

例 1 一正方形截面的无限长棱柱, 边界上的温度条件如图 1 所示.

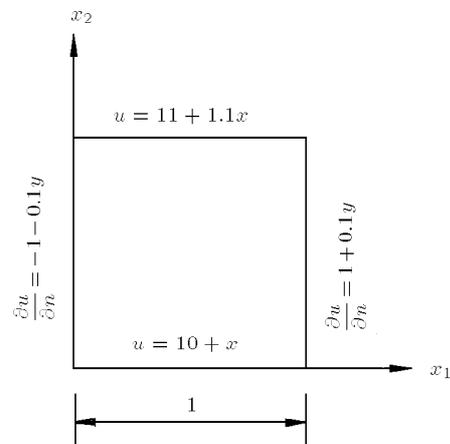


图 1 正方形截面的边界条件

Fig.1 Boundary conditions of the square section

将边界离散成 20 个均匀线性单元, 边界函数采用线性不连续插值近似^[10], 表 1 和表 2 分别列出近边界内点的位势和位势梯度 q_{x_1} 的数值解. 图 2, 图 3 和图 4, 图 5 描绘了随边界单元的逐步加密, 近边界点 $A = (0.5, 0.999999999)$, $B = (10^{-10}, 0.5)$, $C = (0.5, 10^{-10})$, $D = (0.999999999, 0.5)$ 的温度 u 和通量 q_{x_1} 数值解的相对误差 (%) 的变化, 即收敛曲线.

表 1 域内点的温度 u 的数值解

Table 1 Temperatures u of the internal points close to the boundary

x_1	x_2	Exact $\times 10^2$	Transform $\times 10^2$	Relative error (%)
0.100 000 000 0	0.999 999 999 9	0.111 100 000 0	0.111 092 824 0	$0.645 907 603 0 \times 10^{-2}$
0.500 000 000 0	0.999 999 999 9	0.115 500 000 0	0.115 500 768 1	$-0.665 057 562 7 \times 10^{-3}$
0.700 000 000 0	0.999 999 999 9	0.117 700 000 0	0.117 706 012 4	$-0.510 820 756 6 \times 10^{-2}$
$0.100 000 000 0 \times 10^{-9}$	0.300 000 000 0	0.103 000 000 0	0.103 025 555 4	$-0.248 110 553 8 \times 10^{-1}$
$0.100 000 000 0 \times 10^{-9}$	0.500 000 000 0	0.105 000 000 0	0.105 012 382 0	$-0.117 923 569 3 \times 10^{-1}$
$0.100 000 000 0 \times 10^{-9}$	0.900 000 000 0	0.109 000 000 0	0.109 005 943 7	$-0.545 296 328 7 \times 10^{-2}$
0.300 000 000 0	$0.100 000 000 0 \times 10^{-9}$	0.103 000 000 0	0.102 994 427 3	$0.541 038 756 1 \times 10^{-2}$
0.500 000 000 0	$0.100 000 000 0 \times 10^{-9}$	0.105 000 000 0	0.104 999 231 9	$0.731 563 319 1 \times 10^{-3}$
0.999 999 999 9	0.100 000 000 0	0.111 100 000 0	0.111 096 514 3	$0.313 741 901 3 \times 10^{-2}$
0.999 999 999 9	0.500 000 000 0	0.115 500 000 0	0.115 487 618 0	$0.107 203 244 8 \times 10^{-1}$
0.999 999 999 9	0.700 000 000 0	0.117 700 000 0	0.117 672 833 9	$0.230 808 117 6 \times 10^{-1}$

表 2 域内点的通量 q_{x_1} 的数值解

Table 2 Fluxes q_{x_1} of the internal points close to the boundary

x_1	x_2	Exact	Transform	Relative error (%)
0.100 000 000 0	0.999 999 999 9	1.100 000 000	1.101 357 854	-0.123 441 308 9
0.500 000 000 0	0.999 999 999 9	1.100 000 000	1.100 189 324	$-0.172 112 831 4 \times 10^{-1}$
0.700 000 000 0	0.999 999 999 9	1.100 000 000	1.101 608 705	-0.146 245 953 3
$0.100 000 000 0 \times 10^{-9}$	0.300 000 000 0	1.030 000 000	1.032 313 829	-0.224 643 636 2
$0.100 000 000 0 \times 10^{-9}$	0.500 000 000 0	1.050 000 000	1.050 515 573	$-0.491 022 010 2 \times 10^{-1}$
$0.100 000 000 0 \times 10^{-9}$	0.900 000 000 0	1.090 000 000	1.087 672 439	0.213 537 696 7
0.300 000 000 0	$0.100 000 000 0 \times 10^{-9}$	1.000 000 000	1.001 483 834	-0.148 383 389 0
0.500 000 000 0	$0.100 000 000 0 \times 10^{-9}$	1.000 000 000	1.000 163 935	$-0.163 934 627 9 \times 10^{-1}$
0.999 999 999 9	0.100 000 000 0	1.010 000 000	1.004 823 758	0.512 499 197 7
0.999 999 999 9	0.500 000 000 0	1.050 000 000	1.050 515 573	$-0.491 022 010 2 \times 10^{-1}$
0.999 999 999 9	0.700 000 000 0	1.070 000 000	1.072 487 277	-0.232 455 761 9

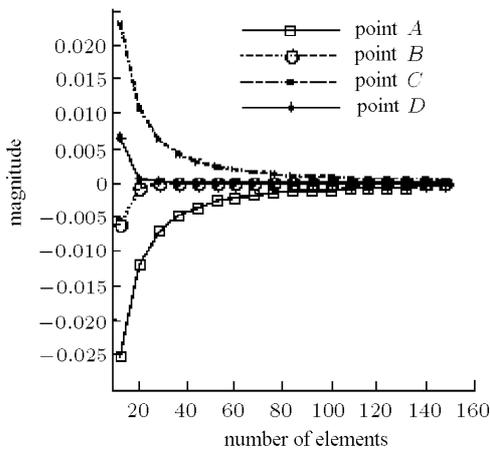


图 2 A, B, C, D 点上温度解收敛曲线

Fig.2 The convergence curve of computed temperatures at points A, B, C, D

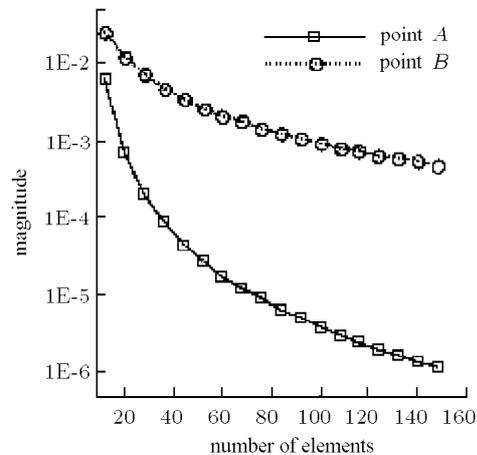


图 3 A, B 点上温度解收敛曲线

Fig.3 The convergence curve of computed temperatures at points A, B

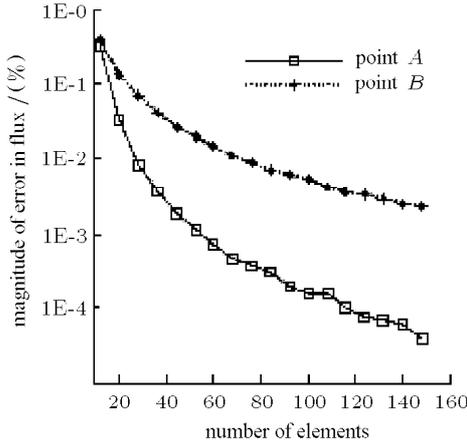


图 4 点 A 和 B 上通量解收敛曲线

Fig.4 The convergence curve of computed fluxes at points A and B

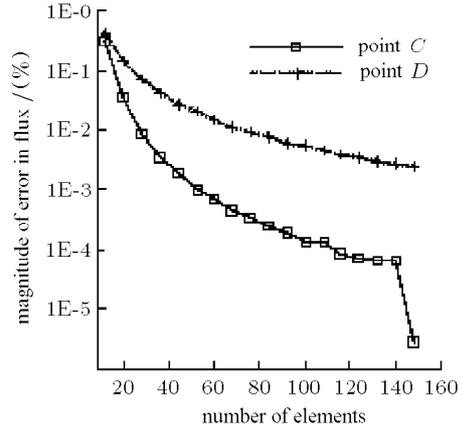


图 5 点 C 和 D 上通量解收敛曲线

Fig.5 The convergence curve of computed fluxes at points C and D

例 2 如图 6 所示，给定半径为 1 的无限长圆柱表面的温度分布 u_0 ，在 A 和 B 两点是不连续的。在稳定状态下，问题的解为 [1,8]

$$\frac{u(r, \theta)}{u_0} = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{2}{n\pi} \left(\frac{r}{R}\right)^n \sin n\theta, \quad n = 1, 3, \dots$$

表 3 列出了在 24 个不连续线性单元下，变换和不作变换近边界点处位势数值解与解析解的比较。

例 3 厚壁圆筒的热传导，外边界 Γ_1 和内边界 Γ_2 分别是半径为 2m 和 1m 的圆周，如图 7 所示。

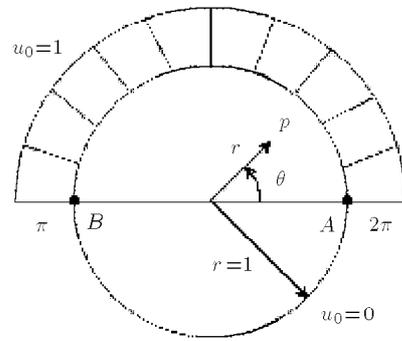


图 6 圆柱表面的温度分布

Fig.6 The boundary temperature distributions of the column

表 3 域内点及近边界点的温度 u 的数值解

Table 3 The numerical results of temperature u at interior points close to the boundary

Coordinates		Conventional (no transform)		Present	
r ($\theta = 37.5^\circ$)	Exact	Numerical	Relative error /%	Numerical	Relative error /%
0.960 000 0	0.978 681 0	0.965 999 9	1.295 734	0.978 732 7	$-0.528\ 756\ 0 \times 10^{-2}$
0.990 000 0	0.994 745 3	0.543 805 1	45.332 23	0.995 022 6	$-0.278\ 748\ 4 \times 10^{-1}$
0.999 000 0	0.999 611 9	0.270 748 8	72.914 60	0.999 846 7	$-0.234\ 894\ 5 \times 10^{-1}$
0.999 900 0	1.000 281	0.248 315 3	75.175 44	1.000 328	$-0.466\ 174\ 7 \times 10^{-2}$
0.999 990 0	1.000 360	0.246 142 1	75.394 64	1.000 376	$-0.160\ 771\ 4 \times 10^{-2}$
0.999 999 0	1.000 368	0.245 925 5	75.416 49	1.000 380	$-0.128\ 041\ 6 \times 10^{-2}$
0.999 999 9	1.000 368	0.245 903 8	75.418 67	1.000 381	$-0.124\ 393\ 9 \times 10^{-2}$

圆筒的表面温度分布是：在 Γ_1 上， $u = 0^\circ\text{C}$ ；在 Γ_2 上， $u = 10^\circ\text{C}$ 。计算点 A(1,0) 附近近边界点的温度和 x_1 方向热流，由对称性，在 1/2 部分边界划分单元，采用了线性元和圆弧元两种计算方案。在两种方案中边界划分都为内边界 18 个单元，外边界 34 个单元，截面处分别 4 个单元；边界函数都采用线

性不连续插值近似。

计算结果及与文献 [19] 的计算结果比较见图 8~图 11。其中接近度是指计算点到邻近边界单元的最小距离与单元长度比值的两倍。

由表 1 和表 2 可以看出，所取内点均匀环绕边界一周，而且非常靠近边界，与边界的距离小到

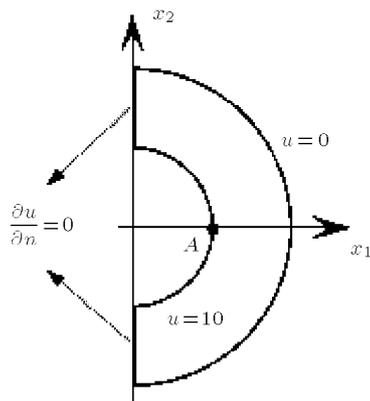


图 7 厚壁圆筒边界条件

Fig.7 The boundary temperature distributions of the cylinder

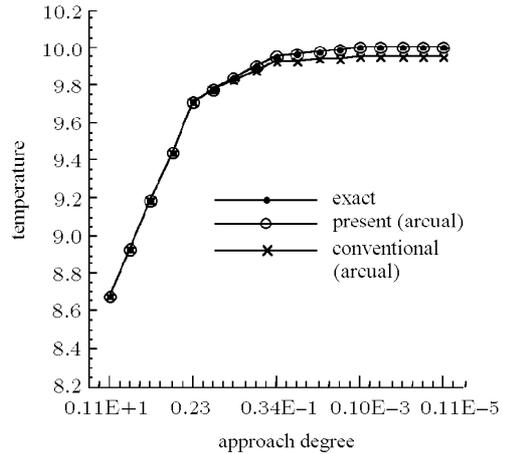


图 10 近边界点温度

Fig.10 Temperatures at interior points near the boundary

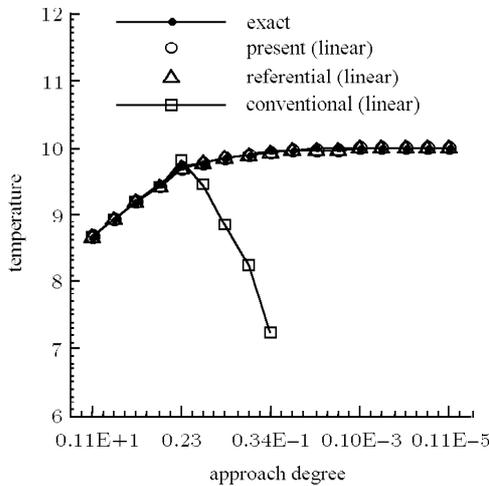


图 8 近边界点温度

Fig.8 Temperatures at interior points near the boundary

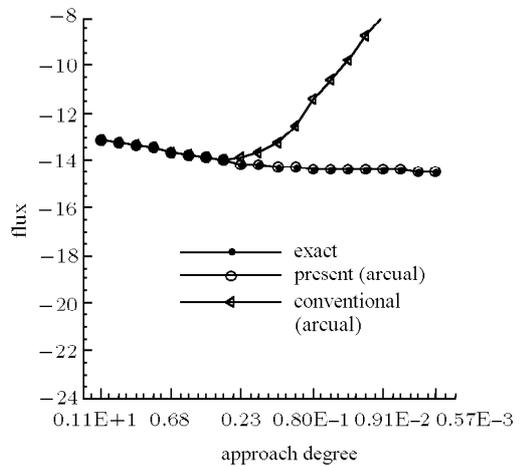


图 11 近边界点热流

Fig.11 Fluxes at interior points near the boundary

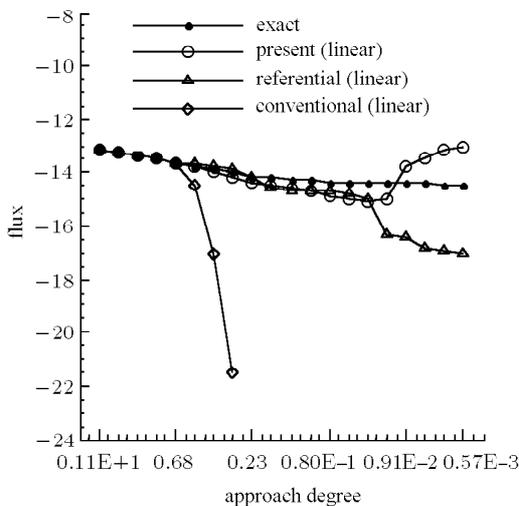


图 9 近边界点热流

Fig.9 Fluxes at interior points near the boundary

10^{-10} , 其上的温度和通量数值结果与精确解非常接近, 表明本文方法具有一般性, 不受内点位置选择以及内点与边界的距离的限制. 表 1 和表 2 中相对误差 (%) 列数据大都非常小, 只有个别靠近边界角点处内点的计算结果误差相对稍大些, 这纯属正常. 图 2~ 图 5 的收敛曲线表明, 对于非常靠近边界的不同内点, 本文方法计算出的位势和通量都具有较好的收敛性. 其中图 3~ 图 5 的纵轴采取的是对数轴以更明显地描绘本文方法计算结果的收敛效果.

表 3 的结果表明, 当内点靠近邻近边界的距离为 0.96 时, 常规方法的计算结果已开始明显偏离精确解, 随着内点更进一步地靠近边界, 不作变换的常规方法的计算结果则完全失真. 而本文方法在 $r = 0.999999$, 即当内点到邻近边界的距离为 10^{-7} 时, 其计算结果与精确解仍然相当吻合. 从表 3 中的

相对误差列我们可以看到, 尽管内点很接近边界, 本文计算结果的精度仍然相当高, 并且非常稳定.

图 8 和图 9 比较了本文方法与文献 [19] 例 1 方法的温度和热流计算结果. 图 8 中曲线表明, 本文方法温度解与文献温度解相比精确解都非常接近; 而由图 9 中曲线可以看出, 在接近度小于 0.80×10^{-1} 时, 虽然本文方法热流解与文献热流解相对精确解都有所偏离, 但是可以明显看出本文方法结果较文献结果更加靠近精确解, 所以本文变换方法更好地解决了近边界点位势问题.

图 10 和图 11 表明, 本文圆弧单元变换法可以更好地计算近边界点的位势和梯度, 其计算结果几乎与精确解重合, 不但计算温度值与精确解几乎一致, 而且计算热流值也是相当精确, 完全消除了位势问题的边界层效应. 图 11 中的常规方法 (不作变换) 计算结果曲线, 因为随着接近度的逐渐减小其结果偏离精确值太大而没有全部画出.

4 结 论

本文针对边界元分析中出现的近奇异边界积分, 提出了一种新的变量变换法. 利用本文提出的变换, 可以非常有效地改善近奇异核的特性, 使近奇异积分的计算可以通过普通的高斯求积公式精确地完成, 成功地克服了“边界层效应”. 位势算例充分表明了本文方法的可行性和有效性. 本文所提出的变换法具有一般性, 它可以应用到边界元法中的其它问题, 这将在以后的工作中进一步讨论.

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THE EVALUATION OF NEARLY SINGULAR INTEGRALS IN THE BOUNDARY INTEGRAL EQUATIONS WITH VARIABLE TRANSFORMATION¹⁾

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Abstract The numerical solution of boundary value problems using boundary integral equations demands the accurate computation of the integral of the kernels, which occur as the nearly singular integrals when the collocation point is close to the element of integration but not on the element in boundary element method (BEM). Such integrals are difficult to compute by standard quadrature procedures, since the integrand varies very rapidly within the integration interval, more rapidly the closer the collocation point is to the integration element. Practice shows that we can even obtain the results of superconvergence for the computed point far enough from the boundary; however, using standard quadrature procedures, which neglect the pathological behavior of the integrand as the computed point approaches the integration element, will lead to a degeneracy of accuracy of the solution, even no accuracy, which is the so-called “boundary layer effect”. To avoid the “boundary layer effect”, the accurate computation of the nearly singular boundary integrals would be more crucial to some of the engineering problems, such as the crack-like and thin or shell-like structure problems.

The importance of the accurate evaluation of nearly singular integrals is considered to be next to the singular boundary integrals in BEM, and great attentions have been attracted and many numerical techniques have been proposed for it in recent years. These developed methods can be divided on the whole into two categories: “indirect algorithms” and “direct algorithms”, which have obtained varying degree of success, but the problem of the nearly singular integrals has not been completely resolved so far. In this paper, a new efficient transformation is proposed based on a new idea of transformation with variables. The proposed transformation can remove the nearly singularity efficiently by smoothing out the rapid variations of the integrand of nearly singular integrals, and improve the accuracy of numerical results of nearly singular integrals greatly without increasing the computational effort. Numerical examples of potential problem with their satisfactory results in both curved and straight elements are presented, showing encouragingly the high efficiency and stability of the suggested approach, even when the internal point is very close to the boundary. The suggested algorithm is general and can be applied to other problems in BEM.

Key words BEM, nearly singular integrals, transformation, potential problem, boundary layer effect

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