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Inverse Identification of Boundary Conditions for 3D Potential Problems by Using the Boundary Element Method

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Abstract Three-dimensional Cauchy inverse problems for the Laplace equation are considered. Both potential and flux conditions on a part of the boundary are solved based on the over-prescribed boundary data on the rest of the boundary. The boundary element method is applied to implement the numerical inverse analysis. The truncated singular value decomposition technique is employed to deal with the system equation. The number of the truncated singular value is obtained by the undulating-curve method. Numerical examples demonstrate its accuracy. The TSVD method has been found to produce reasonably accurate results for the temperature and less accurate numerical results for the flux.

Key words: BEM, inverse problems, Cauchy problem, potential, truncated singular value decomposition

INTRODUCTION

Inverse problems are very important in engineering and science. Many unmeasurable physical quantities need be determined from measurable and known data. There are different kinds of inverse problems. 1) Reconstruction inverse problem: the boundary conditions are unknown on all or part of the boundary. 2) Identification inverse problem: part of the domain or its boundary is unknown, such as the problems with crack, flaw, defect, cavity, erosion et al.. 3) Material properties inverse problem: the properties of the material, such as Young's modulus, Poisson's ratio, the linear coefficient of thermal expansion, are not known. 4) External source inverse problem: the loads or sources are unknown.5) Modelization inverse problem: the differential equation is not known. The boundary element method is becoming more and more important as a classic numerical method to solve different inverse problems[1]. Inverse problems are in general unstable in the sense that small noise in the input data may amplify significantly the errors in the solution. So the regularization techniques are very critic to the inverse numerical analysis [2].

It is often encountered that all the potentials and fluxes are known on a part of the boundary and no boundary data can be directly measured on the rest of the boundary in engineering for potential problems. This is the reconstruction inverse problem, also named as Cauchy inverse problem. It is also encountered that only potentials or fluxes are given on two different parts of the boundary, and no boundary conditions are given on the other part of the boundary. These problems are over-determined or under-determined. For under-determined problems, the measurable information of internal points is helpful to obtain all unknown boundary conditions.

The function specification method and the zeroth-order regularization procedure were employed by Kurpisz et al. to solve the inverse transient heat conduction problems [3]. The mathematical mechanism of the ill-posed Cauchy problem for Laplace equation was analyzed by Chen [4]. An iterative method was applied to solve Cauchy inverse problem for the Laplace equation by Lesnic et al. [5]. The sequential function specification method was used by Chantasiriwan to determine the unknown time-dependent boundary heat flux from temperature measurements inside a body or on its boundary [6]. The dual reciprocity boundary element method in conjunction with iterative regularization methods of conjugate gradient type was discussed by Singh et al. for the solution of time-dependent inverse heat conduction problems. The influence of measurement location, measurement error and element option were investigated [7]. An inverse problem for Laplace equations was recast into primary and adjoint boundary value problems by Hayashi et al.. The Dirichlet and Neumann data were specified on respective part of the boundary, while no data on the second part of the boundary were given and Robin condition was prescribed on the third part of the

boundary [8,9]. An iterative approach was introduced by Delvare et al. The Cauchy inverse problem for Laplace equation was reduced to a sequence of well-posed optimization problems under equality constraints [10].
Spatial regularization was applied by Zabaras et al. for the solution of boundary traction using the BEM. Numerical results showed that accurate and stable boundary solutions were achieved even in case with significant data error and with a limited number of internal sensors [11].The Cauchy inverse elasticity problem was investigated by Yeih et al., who analyzed its existence, uniqueness and stability of solutions and formulated a regularization method based on the fictitious boundary indirect method [12,13]. The problem of identifying unknown displacements on a part of the boundary was considered by Ohura et al. when surface tractions on the rest of the boundary and displacements at a finite number of points inside the domain were given. The inverse problem was recast using a variational approach to direct primary and adjoint boundary value problems[14].The uncertain boundary conditions were obtained by Lu et al. from the boundary and measured displacement/strain data. The over-determined algebraic system was solved by singular value decomposition [15]. The approximate solutions to the Cauchy problem in linear elasticity were determined by Marin et al. by using an alternating iterative BEM which reduces the problem to a sequence of well-posed boundary value problems [16]. Truncated singular value decomposition (TSVD) and Tikhonov regularization method were also proposed to solve the problem. The discrepancy principle or the L-curve method was used for the choice of the regularization parameter [17,18]. The iterative, conjugate gradient, TSVD, Tikhonov regularization methods were investigated and compared. The Tikhonov regularization method and the SVD had been found to produce reasonably accurate results for the displacement vector and less accurate numerical results for the traction vector [19]. A numerical iterative BEM for solving Cauchy problem in elasticity was developed by Ellabib et al. The accuracy was improved by the use of automatic selection of relaxation parameter. It was concluded that the approach produced accurate, convergent and stable solution with respect to increasing the number of elements and decreasing the amount of noise [20].
In this paper, the boundary potentials and fluxes on a part of boundary are inverse identification by using the over-specified boundary conditions on the remaining boundary in the BEM for 3-D cauchy potential inverse problems. The over-determined system equation is solved by singular value decomposition technique. The unknown potential and flux boundary conditions are accurately calculated.

BOUNDARY INTEGRAL EQUATIONS

Consider three-dimensional potential problem. For an interior point $\boldsymbol{y}$, the boundary integral equation can be given as

$$u(\boldsymbol{y}) = \int_{\Gamma} u^*(\boldsymbol{x},\boldsymbol{y}) q(\boldsymbol{x}) \mathrm{d}\Gamma - \int_{\Gamma} q^*(\boldsymbol{x},\boldsymbol{y}) u(\boldsymbol{x}) \mathrm{d}\Gamma \tag{1}$$

When the source point $\boldsymbol{y}$ locates on the boundary Γ , the boundary integral equation can be written as

$$C(\boldsymbol{y}) u(\boldsymbol{y}) = \int_{\Gamma} u^*(\boldsymbol{x},\boldsymbol{y}) q(\boldsymbol{x}) \mathrm{d}\Gamma - \int_{\Gamma} q^*(\boldsymbol{x},\boldsymbol{y}) u(\boldsymbol{x}) \mathrm{d}\Gamma \tag{2}$$

In the above equations, $u^*(\boldsymbol{x},\boldsymbol{y})$ is the fundamental solution of potential problems, and $q^*(\boldsymbol{x},\boldsymbol{y})$ is the gradient of potential with respect to an outward normal to the boundary. $C(\boldsymbol{y})$ is the boundary singular coefficient, which is determined by the boundary geometry characterization.
The potential gradients q_k at the interior point $\boldsymbol{y}$ with respect to directions x_k can be obtained by differentiating Eq. (1)

$$q_k(\boldsymbol{y}) = \int_{\Gamma} \frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial y_k} q(\boldsymbol{x}) \mathrm{d}\Gamma - \int_{\Gamma} \frac{\partial q^*(\boldsymbol{x},\boldsymbol{y})}{\partial y_k} u(\boldsymbol{x}) \mathrm{d}\Gamma \quad (k = 1,2,3) \tag{3}$$

The integral kernel functions in Eqs. (1~3) are given as follows

$$u^*(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{4\pi} \frac{1}{r} \tag{4}$$

$$q^*(\boldsymbol{x},\boldsymbol{y}) = -\frac{1}{4\pi} \frac{r_n}{r^3} \tag{5}$$

$$\frac{\partial u^*(\boldsymbol{x},\boldsymbol{y})}{\partial y_k} = \frac{1}{4\pi} \frac{r_k}{r^3} \quad , \quad k = 1,2,3 \tag{6}$$

$$\frac{\partial q^*(\boldsymbol{x},\boldsymbol{y})}{\partial y_k} = -\frac{1}{4\pi} \left(-\frac{n_k}{r^3} + \frac{3 r_k r_n}{r^5}\right) \quad , \quad k = 1,2,3 \tag{7}$$

where r is the distance between the source point $\boldsymbol{y}$ and arbitrary field point $\boldsymbol{x}$ on the boundary.
For a direct potential problem, the boundary conditions are sufficiently given, the unknown boundary conditions, such as potential and potential gradient with respect to boundary outward normal, can be calculated by discretizing Eq.(2). Gaussian elimination can give satisfactory results for the system equation. When the conditions for solving the direct problem are partially or entirely unknown then an inverse reconstruction problem must be formulated to determine the

unknowns from the specific or measured system response. Inverse problems are ill-posed and more difficult to solve than direct problems. It is well known that inverse problems are instable, i.e. the existence, uniqueness and stability of their solutions are not always guaranted. The Cauchy potential inverse problem is presented thereafter. The singular value decomposition is used to treat the over-determined system.

SINGULAR VALUE DECOMPOSITION

There exists a very powerful set of techniques for dealing with sets of equations or matrices that are either singular or else numerically very close to singular. In many cases where Gaussian elimination and LU decomposition fail to give satisfactory results, this set of techniques, known as singular value decomposition [21, 22], will diagnose precisely what the problem is. In some cases, SVD will not only diagnose the problem, it will also give a useful numerical answer.

SVD is based on the following theorem of linear algebra: Any $M \times N$ matrix **A** whose number of rows M is greater than or equal to its number of columns N, can be written as the product of an $M \times N$ column-orthogonal matrix **U**, an $N \times N$ diagonal matrix **W** with positive or zero elements (the *singular values*), and the transpose of an $N \times N$ orthogonal matrix **V**.

$$\mathbf{A} = \mathbf{U}\mathbf{W}\mathbf{V}^{\mathrm{T}} \tag{8}$$

where $\mathbf{U} = (\mathbf{u}_1, \mathbf{u}_2, ..., \mathbf{u}_N) \in \mathbf{R}^{M \times N}$ and $\mathbf{V} = (\mathbf{v}_1, \mathbf{v}_2, ..., \mathbf{u}_N) \in \mathbf{R}^{N \times N}$ are each orthogonal in the sense that their columns are orthonormal. $\mathbf{W} = \mathrm{diag}(w_1, w_2, ..., w_n)$, the singular values w_i are nonnegative and are typically written in a non-increasing order

$$w_1 \geq w_2 \geq \cdots \geq w_n \geq 0 \tag{9}$$

For Cauchy inverse problems, the system equation of discretized Eq. (2) is

$$\mathbf{A}\mathbf{x} = \mathbf{b} \tag{10}$$

where matrix $\mathbf{A}$ has M rows and N columns. M equals the number of the boundary nodes or the linear elements. N equals the number of the unknown boundary potential and normal flux conditions.

In the ideal setting, without perturbations and rounding errors, the treatment of ill-conditioned system Eq. (10) is easy, i.e. simply ignore the SVD components associated with zero singular values. Therefore, by applying SVD techniques, the minimum norm least square solution to Eq. (10) can be expressed as using the Moore-Penrose generalized inverse $\mathbf{A}^{+}$.

$$\mathbf{x} = \mathbf{A}^{+}\mathbf{b} = \sum_{i=1}^{\mathrm{rank}(\mathbf{A})} \frac{\mathbf{u}_i^T \mathbf{b}}{w_i} \mathbf{v}_i \tag{11}$$

In practice, $\mathbf{A}$ is never exactly mathematically rank deficient, but instead numerically rank deficient, i.e. it has one or more small nonzero singular values w_i for some i greater than k , $1 < k < N$. The very small singular values inevitably give rise to the error and undulation of the solution $\mathbf{x}$.

NUMERICAL EXAMPLES

Example 1: Heat conduction in a cube.

The length of the cube is 2 m, as shown in Fig. 1. The temperature and flux conditions on the 4 lateral surfaces and the upper surface are specified. But all boundary conditions on the bottom surface are unknown. The exact temperature solution for the problem is $u(x_1, x_2, x_3) = 100 - 50x_3$. In the BEM model, the boundary is discretized by 24 8-node quadratic elements, that is, each of the six surfaces of the cube has 4 elements. The results of the temperature on the bottom surface are illustrated in Fig.2.

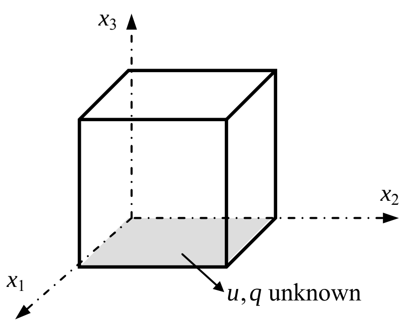


Fig. 1 Heat conduction in a cube

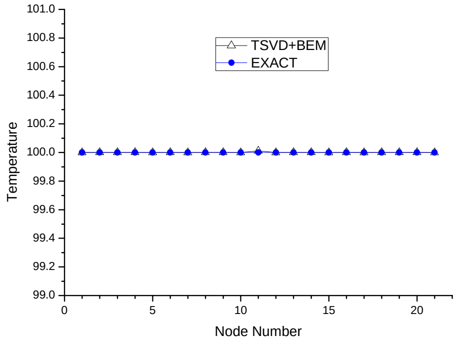
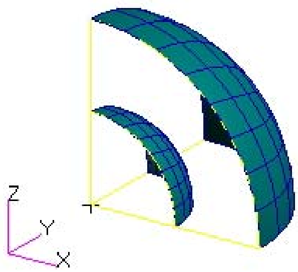


Fig. 2 Temperature at nodes on the bottom surface

Fig.2 shows that temperature results at nodes on the bottom surface of the cube agree with the exact solution. According to the numerical analysis, the flux results are also accurate. But if the data are perturbed by random error even with 1%, the SVD results become inaccurate to a great degree. The influence of the random measurement error to the stable and accurate solutions need be further analyzed.

Example 2 A hollow sphere subjected to spherically symmetric temperature.
The inner and outer radii r of the cylinder are 1 m and 2 m, respectively. Due to spherical symmetry, only one-eighth of the hollow sphere is considered in the analysis model. Both inner and outer surfaces are discretized, respectively, with 48 8-node quadrilateral elements, as shown in Fig.3. The exact temperature solution for the problem is $u(r)=200/r$ .The temperature and flux on the outer surface are over-prescribed. The temperature and flux on inner surface is unknown. The information of the singular values distribution and the Euclidean norm ||x|| of solutions are listed in Table 1. The results of the temperature on the inner surface are illustrated in Fig.4.



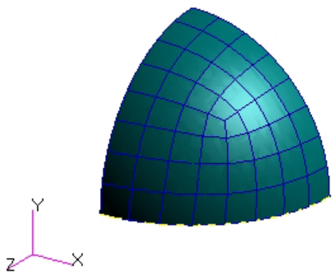


Fig. 3 Heat conduction in a hollow sphere

Table 1 The information of the undulating-curve method

SVs distribution	1e-1	1e-2	1e-3	1e-4	1e-5	1e-6	1e-7	1e-8	1e-9
Number of SVs	170	5	4	5	13	8	9	21	12
k	170	175	179	184	197	205	214	235	247
$\|\|x_k\|\|2$	2922.2	2926.3	2929.3	2935.6	2942. 6	2943.6	2962.9	5846.1	18413.2

For the problem, matrix $\mathbf{A}$ has 338 rows and 338 columns. There are 338 singular values. Table 1 shows that there are 247 singular values more than the magnitude 1e-10 and the 247 singular values are distributed in different orders of magnitude from 1E-1 to 1E-9. The Euclidean norm ||x|| increases from 2962.9 to 5846.1 when the magnitude of the singular values distribution decreases from 1e-7 to 1e-8, which means that some solutions deviate greatly from the exact solutions. The curve of the solution also undulates rapidly, which is unreasonable because the physical field is generally continued. Hence the truncated factor k can be selected as 214 according to the distribution of singular values and the change of the Euclidean norm. The method is named as the undulating-curve method [23].

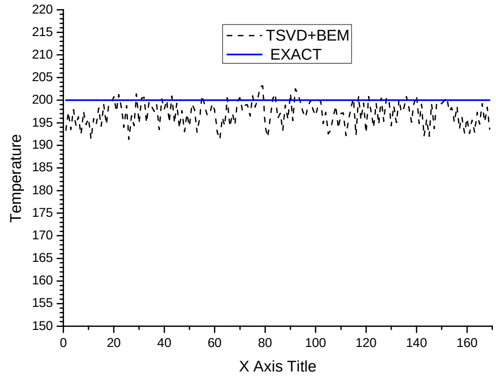


Fig. 4 Temperature at nodes on the inner surface

Fig.4 shows that temperature results at nodes on the inner surface of the sphere are close to the exact solutions. But the flux results are completely ineffective. The TSVD method has been found to produce reasonably accurate results for the temperature and less accurate numerical results for the flux.

CONCLUSION

Three-dimensional inverse problems for the Laplace equation are solved by using the boundary element method in conjunction with the truncated singular value decomposition. The boundary element method has a great advantage on solving this kind of inverse problems, because only the contour of the considered domain is divided into meshes and the unknown boundary physical quantities on the boundary nodes, including both potentials and fluxes, can be identified based on the given boundary data after the potential boundary integral equation is discretized for the numerical analysis. The truncated singular value decomposition technique is applied to deal with the over-determined system equation. The number of the truncated singular value is obtained by the undulating-curve method. Numerical examples demonstrate its accuracy. The TSVD method has been found to produce reasonably accurate results for the temperature and less accurate numerical results for the flux. The stable of the solution with respect to the random measure errors remains to be investigated. Future work will be concerned with developing an iterative algorithm for solving these problems.

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