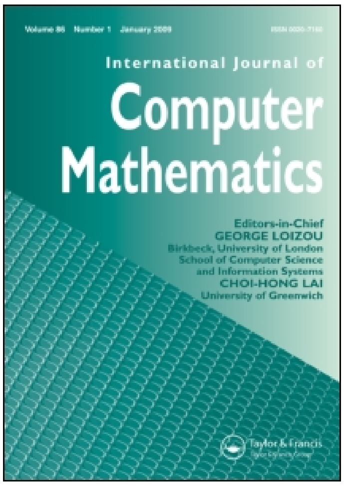

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The superconvergence of composite trapezoidal rule for Hadamard finite-part integral on a circle and its application

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The quadrature method for Hadamard finite-part integral on a circle is discussed and the emphasis is placed on the *pointwise superconvergence phenomenon* of the composite trapezoidal rule, *i.e.* when the singular point coincides with some *a priori* known points, the accuracy can be better than what is globally possible. The existence and uniqueness of the superconvergence points are proved and the correspondent superconvergence estimate is obtained. An indirect method is introduced and then applied to solve the integral equation of the second kind containing finite-part kernels, including that arising in the scattering theory. Some numerical results are also presented to confirm the theoretical results and to show the efficiency of the algorithms.

Keywords: finite-part integral; composite trapezoidal rule; superconvergence; finite-part integral equation of the second kind; scattering problem

2000 AMS Subject Classification: 65D30; 65D32; 45E99

1. Introduction

In recent decades, much attention has been paid to the evaluation of the finite-part integrals of the form

$$\neq\!\!\!\!\int_a^b \frac{f(t)}{(t-s)^{p+1}}\mathrm{d}t, \quad p = 1, 2, \cdots, \quad s \in (a,\ b), \tag{1}$$

where $\neq\!\!\!\!\int$ denotes a Hadamard finite-part integral (or hypersingular integral). The aim of this paper is to investigate a relatively less studied integral

$$I(c, s, f) := \neq\!\!\!\!\int_c^{c+2\pi} \frac{f(t)}{\sin^2 (t-s)/2}\mathrm{d}t, \quad s \in (c,\ c+2\pi), \tag{2}$$

where $f(t)$ is a 2π-periodic function and c an arbitrary constant. Integrals of this kind appear frequently in the formulation of certain classes of boundary value problems in a circular or an elliptic domain in terms of Hadamard finite-part integral equations [12,35,40,42].

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Equation (2) can be defined in a number of ways and these definitions are generally equivalent [42]. Here we propose the following new definition

$$\int\!\!\!\!\!\!=_{c}^{c+2\pi} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t = \lim_{\varepsilon\to 0}\left\{\int_{c}^{s-\varepsilon} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t + \int_{s+\varepsilon}^{c+2\pi} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t - \frac{8f(s)}{\varepsilon}\right\}. \tag{3}$$

$f(t)$ is said to be *finite-part integrable* with respect to the weight $\sin^{-2}[(t-s)/2]$ if the limit on the right-hand side of Equation (3) exists. This definition implies that $I(c,s,f)$ is a linear operator on f.

We recall that the finite-part integral (Equation (1)) with $p=1$ is defined by (see, *e.g.* [21, 32–34]).

$$\int\!\!\!\!\!\!=_{a}^{b} \frac{f(t)}{(t-s)^2}\mathrm{d}t = \lim_{\varepsilon\to 0}\left\{\int_{a}^{s-\varepsilon} \frac{f(t)}{(t-s)^2}\mathrm{d}t + \int_{s+\varepsilon}^{b} \frac{f(t)}{(t-s)^2}\mathrm{d}t - \frac{2f(s)}{\varepsilon}\right\}, \quad s\in(a,b). \tag{4}$$

Obviously, definition (3) is a natural extension of Equation (4). Compared with those definitions given by Yu [42], Equation (3) is relatively preliminary. The equivalence of Equation (3) with those definitions in [42] is beyond the aim and the scope of the present paper and will be given elsewhere. In the sequel, Equation (3) will be used as the starting point of our analysis. In addition, we shall employ a result to state that finite-part integrals of the form in Equation (2), defined by Equation (3), do appear in boundary element methods (BEMs).

THEOREM 1.1 *Let $\Omega\subset\mathbb{R}^2$ be the interior domain of the unit circle with its boundary denoted by $\partial\Omega$. Assume $u\in C^2(\bar{\Omega})$ is a solution of harmonic equation. Denote by u_0 and u_n the Dirichlet and Neumann boundary data of u on $\partial\Omega$, respectively. Then, it holds that*

$$u_n(\theta) = -\frac{1}{4\pi}\int\!\!\!\!\!\!=_{-\pi}^{\pi} \frac{u_0(\theta')}{\sin^2(\theta'-\theta)/2}\mathrm{d}\theta', \quad \theta\in(-\pi,\pi), \tag{5}$$

where the finite-part integral is defined by Equation (3).

Proof Since Ω is the interior domain of the unit circle, it is well known that u can be expressed by its Dirichlet boundary data u_0 via the following Poisson integral formula

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{(1-r^2)u_0(\theta')}{1+r^2-2r\cos(\theta'-\theta)}\mathrm{d}\theta', \quad 0\le r<1, \tag{6}$$

where r,θ denote the polar coordinates. Differentiating Equation (6) with respect to r yields

$$\frac{\partial u(r,\theta)}{\partial r} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{[-4r+2(1+r^2)\cos(\theta'-\theta)]u_0(\theta')}{[1+r^2-2r\cos(\theta'-\theta)]^2}\mathrm{d}\theta'. \tag{7}$$

By using the identity,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{-4r+2(1+r^2)\cos(\theta'-\theta)}{[1+r^2-2r\cos(\theta'-\theta)]^2}[u_0(\theta)+u_0'(\theta)\sin(\theta'-\theta)]\mathrm{d}\theta' = 0, \quad 0\le r<1,$$

we rewrite Equation (7) as

$$\frac{\partial u(r,\theta)}{\partial r} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{-4r+2(1+r^2)\cos(\theta'-\theta)}{[1+r^2-2r\cos(\theta'-\theta)]^2}[u_0(\theta')-u_0(\theta)-u_0'(\theta)\sin(\theta'-\theta)]\mathrm{d}\theta'. \tag{8}$$

Since $u \in C^2(\bar{\Omega})$, the integrand in Equation (8) is actually continuous and consequently,

$$u_n = \lim_{r \to 1-} \frac{\partial u(r,\theta)}{\partial r} = -\frac{1}{4\pi} \int_{-\pi}^{\pi} \frac{u_0(\theta') - u_0(\theta) - u_0'(\theta)\sin(\theta' - \theta)}{\sin^2(\theta' - \theta)/2} \mathrm{d}\theta'. \tag{9}$$

It is easy to see that the integrand function in Equation (9) is continuous except one removable discontinuity point at θ. Hence, by Equation (3), we get

$$\begin{aligned}
&\int_{-\pi}^{\pi} \frac{u_0(\theta') - u_0(\theta) - u_0'(\theta)\sin(\theta' - \theta)}{\sin^2(\theta' - \theta)/2} \mathrm{d}\theta' \\
&= \,\,{=}\!\!\!\!\!\!\int_{-\pi}^{\pi} \frac{u_0(\theta') - u_0(\theta) - u_0'(\theta)\sin(\theta' - \theta)}{\sin^2(\theta' - \theta)/2} \mathrm{d}\theta' \\
&= \,\,{=}\!\!\!\!\!\!\int_{-\pi}^{\pi} \frac{u_0(\theta')}{\sin^2(\theta' - \theta)/2} \mathrm{d}\theta' - u_0(\theta) \,\,{=}\!\!\!\!\!\!\int_{-\pi}^{\pi} \frac{1}{\sin^2(\theta' - \theta)/2} \mathrm{d}\theta' - u_0'(\theta) \,\,{=}\!\!\!\!\!\!\int_{-\pi}^{\pi} \frac{\sin(\theta' - \theta)}{\sin^2(\theta' - \theta)/2} \mathrm{d}\theta'.
\end{aligned}$$

Still by Equation (3),

$$\begin{aligned}
{=}\!\!\!\!\!\!\int_{c}^{c+2\pi} \frac{1}{\sin^2(t - s)/2} \mathrm{d}t &= \lim_{\varepsilon \to 0} \left\{ \int_{c}^{s-\varepsilon} \frac{1}{\sin^2(t - s)/2} \mathrm{d}t + \int_{s+\varepsilon}^{c+2\pi} \frac{1}{\sin^2(t - s)/2} \mathrm{d}t - \frac{8}{\varepsilon} \right\} \\
&= \lim_{\varepsilon \to 0} \left(4\cot\frac{\varepsilon}{2} - \frac{8}{\varepsilon} \right) = 0.
\end{aligned} \tag{10}$$

Similarly,

$${=}\!\!\!\!\!\!\int_{c}^{c+2\pi} \frac{\sin(t - s)}{\sin^2(t - s)/2} \mathrm{d}t = 0.$$

It follows that

$$\int_{-\pi}^{\pi} \frac{u_0(\theta') - u_0(\theta) - u_0'(\theta)\sin(\theta' - \theta)}{\sin^2(\theta' - \theta)/2} \mathrm{d}\theta' = \,\,{=}\!\!\!\!\!\!\int_{-\pi}^{\pi} \frac{u_0(\theta')}{\sin^2(\theta' - \theta)/2} \mathrm{d}\theta'. \tag{11}$$

Substituting Equation (11) into Equation (9) leads to Equation (5), which completes the proof. ■

Equation (5), which is usually referred to as the natural integral equation for harmonic problem, is of precisely the form obtained by Yu [37,38,42]. The difference is that we obtain this equation through a preliminary approach. Generally speaking, the natural integral equations, such as Equation (5), have not so much importance themselves, for they are seldom used directly for solving boundary value problems (BVPs). However, for some BVPs in unbounded domains, by introducing a circle or an ellipse as an artificial boundary and by employing the correspondent natural integral equation on this artificial boundary, some domain decomposition methods as well as certain coupled algorithms can be naturally constructed (see, *e.g.* [12,42,39,20,43,9,10,27,36,24,18]).

Since such an equation leads to non-integrable kernels only defined as finite parts, they are quite difficult to approximate. Nedelec [23] introduced a variational formulation which avoids this difficulty and then used stable finite element approximations. Han and Wu [13] reduced an original exterior problem to an equivalent boundary value problem on a bounded domain with integral conditions by introducing an artificial boundary as a circle, and then solving them by a finite element method. However, there also exist several direct methods, such as collocation methods, to solve such equations. Kress [19] described a fully discrete method based on trigonometric interpolation for the numerical solution of the finite-part equation arising from the scattering problem. Here, we will also consider how to solve certain integral equations containing finite-part kernels based on the composite trapezoidal rule and its superconvergence result which will be presented in the following text. Before doing these, we first suggest the corresponding quadrature method to compute Equation (2).

Numerous works have been devoted in developing efficient quadrature formulae for evaluating Equation (1), such as the Gaussian method [25,16,30,15,29], Newton–Cotes method [21,8,28,32,41,33,44,34], transformation method [6,11], global interpolation method [14,17,19] and so on. Relatively speaking, the quadrature method for (2) has been less studied.

Among the quadrature rules of Newton–Cotes type, the composite trapezoidal rule is the easiest one for implementation. It is well known that the convergence rate of the composite trapezoidal rule for Riemann integrals is $O(h^2)$. However, the same convergence rate cannot be expected for Hadamard finite-part integrals due to the hypersingularity of the integrand. For example, the convergence rate of the composite trapezoidal rule for finite-part integral (1) with $p = 1, 2$ is only $O(h^{2-p})$, p order lower than its counterpart for Riemann integrals (cf. [21,33]).

The superconvergence phenomenon has been extensively investigated for finite element method, spline approximation and collocation method (see, *e.g.* [31,3,2]). The *pointwise superconvergence phenomenon* of the Newton–Cotes methods, *i.e.* when the singular point s coincides with some *a priori* known points, the convergence rate of the rule is higher than what is globally possible, was explored in [32–34,44] for finite-part integral (1) on the interval with $p = 1, 2$.

In this paper we are concerned with the composite trapezoidal rule for Equation (2) and the emphasis is placed on its pointwise superconvergence phenomenon. We find that the pointwise superconvergence phenomenon also appears in the evaluation of Equation (2) by the composite trapezoidal rule. It is interesting to note that the local coordinates of our superconvergence points are the same as that for Equation (1) (cf. [33]). However, the superconvergence phenomenon on a circle exhibits some different nature, *e.g.* it occurs in every subinterval of the partition, while for the composite trapezoidal and Simpson's rules for Equation (1), this phenomenon only occurs in those subintervals that are not very close to the ending points of the integral interval [33]. Furthermore, an indirect method based on the superconvergence result is also suggested so that one can avoid the selection of the singular point. This is very important when solving the integral equation of the second kind with finite-part kernels, since in this case, the collocation points are needed to coincide with the nodal ones. In this paper, the superconvergence result is applied to solve the finite-part integral equation arising from the scattering theory.

The rest of this paper is organized as follows. The composite trapezoidal rule for Equation (2) is briefly introduced and then, the main result of this paper is stated in the next section. Then in Section 3 the proof of the main result is obtained. An indirect method is suggested and the corresponding algorithms are presented to certain integral equations in Section 4. Some numerical examples are obtained to show the validity of the theoretical analysis and the efficiency of the algorithms in Section 5 and concluding remarks are made in the last section.

2. The composite trapezoidal rule and its superconvergence

Let $c = t_0 < t_1 < \cdots < t_{n-1} < t_n = c + 2\pi$ be a uniform partition of the interval $[c,\ c + 2\pi]$ with mesh size $h = 2\pi/n$. Denote by $f_L(t)$ the piecewise linear interpolant of $f(t)$, defined by

$$f_L(t) = \frac{1}{h}[f(t_i)(t - t_{i-1}) + f(t_{i-1})(t_i - t)], \quad t \in [t_{i-1}, t_i], \quad 1 \le i \le n. \tag{12}$$

Replacing $f(t)$ in Equation (2) with $f_L(t)$ and making use of $f(t_0) = f(t_n)$, we obtain the composite trapezoidal rule

$$I(c, s, f_L) = \mathop{=\!\!\!\!\!\!\int}_c^{c+2\pi} \frac{f_L(t)}{\sin^2(t-s)/2}\mathrm{d}t = \sum_{i=1}^{n} \omega_i(s) f(t_i), \tag{13}$$

where $\omega_i(s)(1 \le i \le n)$ denote the Cotes coefficients. By Equation (3) and through direct calculations, we have

$$\omega_i(s) = \frac{4}{h} \ln \left| \frac{1-\cos(t_i - s)}{\cos h - \cos(t_i - s)} \right|. \tag{14}$$

For this quadrature rule, we have the error estimate as follows.

THEOREM 2.1 *Assume that $f(t) \in C^2[c, c+2\pi]$ and $f(c) = f(c+2\pi)$. Let $I(c, s, f_L)$ be computed by Equations* (13) *and* (14) *with a uniform mesh. Then, for $s \ne t_i (0 \le i \le n)$, there exists a positive constant C, independent of h and s, such that*

$$|I(c, s, f) - I(c, s, f_L)| \le C\gamma^{-1}(h, s)h, \tag{15}$$

where

$$\gamma(h, s) = \min_{0 \le i \le n} \frac{|s - t_i|}{h}. \tag{16}$$

Proof Let $E_L(t) = f(t) - f_L(t)$ and define

$$\kappa_s(t) = \begin{cases} \dfrac{(t-s)^2}{\sin^2(t-s)/2}, & t \ne s, \\ 4, & t = s. \end{cases} \tag{17}$$

Then, from Equations (3) and (4), we see that

$$I(c, s, f) - I(c, s, f_L) = \mathop{=\!\!\!\!\!\!\int}_c^{c+2\pi} \frac{E_L(t)}{\sin^2(t-s)/2}\mathrm{d}t = \mathop{=\!\!\!\!\!\!\int}_c^{c+2\pi} \frac{E_L(t)\kappa_s(t)}{(t-s)^2}\mathrm{d}t, \quad s \in (c,\ c+2\pi).$$

Now we split the error into two parts

$$I(c, s, f) - I(c, s, f_L) = 4 \mathop{=\!\!\!\!\!\!\int}_c^{c+2\pi} \frac{E_L(t)}{(t-s)^2}\mathrm{d}t + \mathop{=\!\!\!\!\!\!\int}_c^{c+2\pi} \frac{E_L(t)[\kappa_s(t) - 4]}{(t-s)^2}\mathrm{d}t. \tag{18}$$

The first part can be directly estimated by Theorem 3 in [28], *i.e.*

$$\left| \mathop{=\!\!\!\!\!\!\int}_c^{c+2\pi} \frac{E_L(t)}{(t-s)^2}\mathrm{d}t \right| \le C \min\{\gamma^{-1}(h, s), |\ln \gamma(h, s)| + |\ln h|\}h. \tag{19}$$

As for the second part, we observe that $[\kappa_s(t) - 4](t-s)^{-2}$ is non-negative and only has a removable discontinuity at $t = s$. So the correspondent finite-part integral degenerates to a Riemann integral and consequently,

$$
\begin{aligned}
&\left| \fint_{c}^{c+2\pi} \frac{E_L(t)[\kappa_s(t)-4]}{(t-s)^2}\mathrm{d}t \right| \\
&\quad \leq \max_{t\in[c,\ c+2\pi]}\{|E_L(t)|\} \int_{c}^{c+2\pi} \frac{\kappa_s(t)-4}{(t-s)^2}\mathrm{d}t \\
&\quad = \max_{t\in[c,\ c+2\pi]}\{|E_L(t)|\} \fint_{c}^{c+2\pi} \frac{\kappa_s(t)-4}{(t-s)^2}\mathrm{d}t \\
&\quad = \max_{t\in[c,\ c+2\pi]}\{|E_L(t)|\} \left\{ \fint_{c}^{c+2\pi} \frac{1}{\sin^2(t-s)/2}\mathrm{d}t - \fint_{c}^{c+2\pi} \frac{4}{(t-s)^2}\mathrm{d}t \right\} \\
&\quad = \frac{8\pi}{(c+2\pi-s)(s-c)} \max_{t\in[c,\ c+2\pi]}\{|E_L(t)|\} \\
&\quad \leq C\gamma^{-1}(h,s)h,
\end{aligned} \tag{20}
$$

where Equation (10) and the interpolation error estimate

$$\max_{t\in[c,\ c+2\pi]}\{|E_L(t)|\} \leq Ch^2$$

have been used. Now Equation (15) follows from Equations (18), (19) and (20). ∎

It is evident that Equation (15) achieves its optimal bound $O(h)$ when the singular point s is located near the centre of a subinterval. However, we find that when s coincides with some special points, the convergence rate can be higher than $O(h)$. We present the main result in the following theorem and the proof will be given in the next section.

THEOREM 2.2 *Let $I(c,\ s,\ f_L)$ be computed by Equations* (13) *and* (14) *with a uniform mesh. Assume that $f(t)$ is periodic with period 2π and*

$$s = t_{m-1} + \frac{h}{2} \pm \frac{h}{3}, \quad 1 \leq m \leq n. \tag{21}$$

Then there exists a positive constant C, independent of h and s, such that

$$|I(c,s,f) - I(c,s,f_L)| \leq \begin{cases} Ch^{1+\alpha}, & f(t) \in C^{2+\alpha}(-\infty,+\infty), \\ C|\ln h|h^2, & f(t) \in C^{3}(-\infty,+\infty), \end{cases} \tag{22}$$

where $0 < \alpha < 1$.

When $f(t) \in C^{2+\alpha}(-\infty,+\infty)$, Theorem 2.2 implies that the composite trapezoidal rule has its superconvergence at some special points. These special points are distributed in every subinterval of the partition and furthermore, they have the same local coordinates which are independent of the subintervals as well as the partition parameter h. The superconvergence for finite element method and collocation method usually arises from mesh points or Gaussian points. However, it is interesting to note that the superconvergence points in the composite trapezoidal rule for Equation (2) are neither mesh points nor Gaussian points.

3. The proof of the main result

We need some lemmas and additional notations for the proof of Theorem 2.2. Throughout this section, C will denote a generic constant that is independent of h and s and it may have different values in different places. In addition, we assume $s \in (t_{m-1}, t_m)$ for some m and let $s = t_{m-1} + (\tau + 1)h/2$ with $\tau \in (-1, 1)$ denoting its local coordinate. By Equation (16),

$$\gamma(h, s) = \begin{cases} \dfrac{1+\tau}{2}, & \tau \le 0, \\ \dfrac{1-\tau}{2}, & \tau > 0. \end{cases} \tag{23}$$

Define

$$⨎_{t_{m-1}}^{t_m} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t = \lim_{\varepsilon\to 0}\left\{\int_{t_{m-1}}^{s-\varepsilon} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t + \int_{s+\varepsilon}^{t_m} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t - \frac{8f(s)}{\varepsilon}\right\} \tag{24}$$

and

$$\mathcal{I}_{n,i}(s) = \begin{cases} \displaystyle\int_{t_{i-1}}^{t_i} \frac{(t-t_{i-1})(t-t_i)}{\sin^2(t-s)/2}\mathrm{d}t, & i \ne m, \\ \displaystyle⨎_{t_{m-1}}^{t_m} \frac{(t-t_{m-1})(t-t_m)}{\sin^2(t-s)/2}\mathrm{d}t, & i = m. \end{cases} \tag{25}$$

LEMMA 3.1 *Assume* $s = t_{m-1} + (\tau+1)h/2$ *with* $\tau \in (-1, 1)$. *Let* $\mathcal{I}_{n,i}(s)$ *be defined by Equation* (25). *Then it holds that*

$$\begin{aligned} \mathcal{I}_{n,i}(s) = &-4h\sum_{k=1}^{\infty}\frac{1}{k}\left\{\cos[k(t_i - s)] + \cos[k(t_{i-1}-s)]\right\} \\ &+ 8\sum_{k=1}^{\infty}\frac{1}{k^2}\left\{\sin[k(t_i-s)] - \sin[k(t_{i-1}-s)]\right\}. \end{aligned} \tag{26}$$

Proof For $i = m$,

$$\begin{aligned} \mathcal{I}_{n,m}(s) &= \lim_{\varepsilon\to 0}\left\{\left(\int_{t_{m-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{t_m}\right)\frac{(t-t_{m-1})(t-t_m)}{\sin^2(t-s)/2}\mathrm{d}t - \frac{8(s-t_{m-1})(s-t_m)}{\varepsilon}\right\} \\ &= \lim_{\varepsilon\to 0}\left\{2\left(\int_{t_{m-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{t_m}\right)(2t - t_m - t_{m-1})\cot\frac{t-s}{2}\mathrm{d}t\right\} \\ &= 4h\ln\left|\sin\frac{t_{m-1}-s}{2}\sin\frac{t_m - s}{2}\right| - \lim_{\varepsilon\to 0}\left\{8\left(\int_{t_{m-1}}^{s-\varepsilon} + \int_{s+\varepsilon}^{t_m}\right)\ln\left|\sin\frac{t-s}{2}\right|\mathrm{d}t\right\}. \end{aligned}$$

Analogously, for $i \ne m$, using integration by parts on the correspondent Riemann integral, we have

$$\mathcal{I}_{n,i}(s) = 4h\ln\left|\sin\frac{t_i - s}{2}\sin\frac{t_{i-1}-s}{2}\right| - 8\int_{t_{i-1}}^{t_i}\ln\left|\sin\frac{t-s}{2}\right|\mathrm{d}t, \quad i \ne m. \tag{27}$$

Now, by using the following well-known identity (see, *e.g.* [26]),

$$\ln\left(2\sin\frac{t}{2}\right) = -\sum_{k=1}^{\infty}\frac{1}{k}\cos kt, \quad t \in (0, 2\pi),$$

we obtain Equation (26). ∎

LEMMA 3.2 *Under the same assumptions of Lemma* 3.1*, it holds that*

$$\sum_{i=1}^{n} \mathcal{I}_{n,i}(s) = 8h \ln\left(2\cos\frac{\tau\pi}{2}\right). \tag{28}$$

Proof By Equation (26), we have

$$\begin{aligned}
\sum_{i=1}^{n} \mathcal{I}_{n,i}(s) &= -4h \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{n} \{\cos[k(t_i - s)] + \cos[k(t_{i-1} - s)]\} \\
&\quad + 8 \sum_{k=1}^{\infty} \frac{1}{k^2} \sum_{i=1}^{n} \{\sin[k(t_i - s)] - \sin[k(t_{i-1} - s)]\} \\
&= -8h \sum_{k=1}^{\infty} \frac{1}{k} \sum_{i=1}^{n} \cos[k(t_i - s)] \\
&= -8h \sum_{j=1}^{\infty} \frac{1}{j} \cos[nj(t_1 - s)] \\
&= -8h \sum_{j=1}^{\infty} \frac{1}{j} \cos[j(1+\tau)\pi] \\
&= 8h \ln\left[2\sin\frac{(1+\tau)\pi}{2}\right] = 8h \ln\left(2\cos\frac{\tau\pi}{2}\right),
\end{aligned}$$

where

$$\sum_{i=1}^{n} \cos[k(t_i - s)] = \begin{cases} n\cos[k(t_1 - s)], & k = nj, \\ 0, & \text{else} \end{cases}$$

has been used. ■

LEMMA 3.3 *Under the same assumptions of Lemma* 3.1*, it holds that*

$$\left| \sum_{i=1, i\neq m}^{n} \frac{f''(\eta_i) - f''(s)}{2} \mathcal{I}_{n,i}(s) \right| \le \begin{cases} C\varrho(h,s,c)h^{1+\alpha}, & f(t) \in C^{2+\alpha}[c, c+2\pi], \\ C\varrho(h,s,c)h^2|\ln h|, & f(t) \in C^3[c, c+2\pi], \end{cases} \tag{29}$$

where $\eta_i \in [t_{i-1}, t_i]$, $0 < \alpha < 1$ *and*

$$\varrho(h,s,c) = \max_{c \le t \le c+2\pi} \{\kappa_s(t)\} \gamma^{-2}(h,s). \tag{30}$$

Proof We observe from Equation (27) that $\mathcal{I}_{n,i}(s)(i \neq m)$ is actually the error of the trapezoidal rule for certain Riemann integral on $[t_{i-1}, t_i]$. Thus, there exists $\tilde{t}_i \in (t_{i-1}, t_i)$, such that

$$\mathcal{I}_{n,i}(s) = -\frac{h^3}{6\sin^2(\tilde{t}_i - s)/2}, \quad i \neq m,$$

which leads to

$$\begin{aligned}\left|\sum_{i=1,i\neq m}^{n}\frac{f''(\eta_i)-f''(s)}{2}\mathcal{I}_{n,i}(s)\right| &\leq \sum_{i=1,i\neq m}^{n}\frac{h^3|\eta_i-s|^\alpha}{12(\tilde{t}_i-s)^2}\frac{(\tilde{t}_i-s)^2}{\sin^2(\tilde{t}_i-s)/2}\\ &\leq \max_{c\leq t\leq c+2\pi}\{\kappa_s(t)\}\sum_{i=1}^{m-1}\frac{h^3|t_{i-1}-s|^\alpha}{12(t_i-s)^2}\\ &\quad+\max_{c\leq t\leq c+2\pi}\{\kappa_s(t)\}\sum_{i=m+1}^{n}\frac{h^3|t_i-s|^\alpha}{12(t_{i-1}-s)^2},\end{aligned}\tag{31}$$

where $\kappa_s(t)$ is defined in Equation (17). Noting $s=t_{m-1}+(\tau+1)h/2(-1<\tau<1)$, we have

$$\begin{aligned}\sum_{i=1}^{m-1}\frac{h^3|t_{i-1}-s|^\alpha}{12(t_i-s)^2} &\leq \sum_{i=1}^{m-1}\frac{h^{3+\alpha}+h^3|t_i-s|^\alpha}{12(t_i-s)^2}\\ &\leq \frac{h^{1+\alpha}}{12}\sum_{i=1}^{m-1}\frac{1+|i-m+1-(1+\tau)/2|^\alpha}{(i-m+1-(1+\tau)/2)^2}\\ &\leq \begin{cases}\dfrac{Ch^{1+\alpha}}{(1+\tau)^2}, & 0<\alpha<1,\\[2ex] \dfrac{Ch^2|\ln h|}{(1+\tau)^2}, & \alpha=1.\end{cases}\end{aligned}\tag{32}$$

Similarly,

$$\sum_{i=m+1}^{n}\frac{h^3|t_i-s|^\alpha}{12(t_{i-1}-s)^2}\leq\begin{cases}\dfrac{Ch^{1+\alpha}}{(1-\tau)^2}, & 0<\alpha<1,\\[2ex] \dfrac{Ch^2|\ln h|}{(1-\tau)^2}, & \alpha=1.\end{cases}\tag{33}$$

Now, putting Equations (31), (32) and (33) together and making use of Equation (23) we get Equation (29). The proof is then complete. ■

LEMMA 3.4 *Assume $f(c)=f(c+2\pi)$ and let $I(c,s,f_L)$ be computed by Equations* (13) *and* (14) *with a uniform mesh. Then*

$$I(c,s,f)-I(c,s,f_L)=4hf''(s)\ln\left(2\cos\frac{\tau\pi}{2}\right)+\mathcal{R}_L(s),\tag{34}$$

where

$$|\mathcal{R}_L(s)|\leq\begin{cases}C\varrho(h,s,c)h^{1+\alpha}, & f(t)\in C^{2+\alpha}[c,c+2\pi],\\ C\varrho(h,s,c)h^2|\ln h|, & f(t)\in C^3[c,c+2\pi],\end{cases}\tag{35}$$

and $0<\alpha<1$.

Proof First, we recall from the property of Lagrange interpolation that there exists $\xi_i \in (t_{i-1}, t_i)$, which may depend upon t, such that

$$f(t) - f_L(t) = \frac{f''(\xi_i)}{2}(t - t_{i-1})(t - t_i), \quad t \in [t_{i-1}, t_i].$$

Then, by the mean value theorem of integration and Lemma 3.2, we have

$$\begin{aligned}\left(\int_c^{t_{m-1}} + \int_{t_m}^{c+2\pi}\right) \frac{f(t) - f_L(t)}{\sin^2(t-s)/2}\mathrm{d}t &= \sum_{i=1, i\neq m}^{n} \int_{t_{i-1}}^{t_i} \frac{f''(\xi_i)(t - t_{i-1})(t - t_i)}{2\sin^2(t-s)/2}\mathrm{d}t \\ &= \sum_{i=1, i\neq m}^{n} \frac{f''(\eta_i)}{2}\mathcal{I}_{n,i}(s) \\ &= \sum_{i=1, i\neq m}^{n} \frac{f''(\eta_i) - f''(s)}{2}\mathcal{I}_{n,i}(s) - \frac{f''(s)}{2}I_{n,m}(s) \\ &\quad + 4hf''(s)\ln\left(2\cos\frac{\tau\pi}{2}\right), \end{aligned} \tag{36}$$

where $\eta_i \in [t_{i-1}, t_i]$. Second, by setting

$$\mathcal{E}_m(t) = f(t) - f_L(t) - \frac{f''(s)}{2}(t - t_{m-1})(t - t_m), \quad t \in [t_{m-1}, t_m]$$

and through a similar derivation of Equation (18), we get

$$\begin{aligned}⨎_{t_{m-1}}^{t_m} \frac{f(t) - f_L(t)}{\sin^2(t-s)/2}\mathrm{d}t &= ⨎_{t_{m-1}}^{t_m} \frac{\mathcal{E}_m(t)}{\sin^2(t-s)/2}\mathrm{d}t + \frac{f''(s)}{2}\mathcal{I}_{n,m}(s) \\ &= 4⨎_{t_{m-1}}^{t_m} \frac{\mathcal{E}_m(t)}{(t-s)^2}\mathrm{d}t + ⨍_{t_{m-1}}^{t_m} \frac{\mathcal{E}_m(t)[\kappa_s(t) - 4]}{(t-s)^2}\mathrm{d}t + \frac{f''(s)}{2}\mathcal{I}_{n,m}(s). \end{aligned} \tag{37}$$

Putting Equations (36) and (37) together yields Equation (34) with

$$\mathcal{R}_L(s) = 4\mathcal{R}_L^{(1)}(s) + \mathcal{R}_L^{(2)}(s) + \mathcal{R}_L^{(3)}(s), \quad \mathcal{R}_L^{(1)}(s) = ⨎_{t_{m-1}}^{t_m} \frac{\mathcal{E}_m(t)}{(t-s)^2}\mathrm{d}t,$$

$$\mathcal{R}_L^{(2)}(s) = ⨍_{t_{m-1}}^{t_m} \frac{\mathcal{E}_m(t)[\kappa_s(t) - 4]}{(t-s)^2}\mathrm{d}t, \quad \mathcal{R}_L^{(3)}(s) = \sum_{i=1, i\neq m}^{n} \frac{f''(\eta_i) - f''(s)}{2}\mathcal{I}_{n,i}(s).$$

Now we estimate $\mathcal{R}_L(s)$ term by term. Note that $f(t) \in C^{2+\alpha}[c, c+2\pi](0 < \alpha \leq 1)$ implies

$$|\mathcal{E}_m^{(i)}(t)| \leq Ch^{2-i+\alpha}, \quad i = 0, 1, 2.$$

Then, by using the identity (cf. [16,28,22,7])

$$⨎_a^b \frac{f(t)}{(t-s)^2}\mathrm{d}t = \frac{(b-a)f(s)}{(b-s)(s-a)} + f'(s)\ln\frac{b-s}{s-a} + \int_a^b \frac{f(t) - f(s) - f'(s)(t-s)}{(t-s)^2}\mathrm{d}t,$$

we have

$$\begin{aligned}|\mathcal{R}_L^{(1)}(s)| &\leq \left|\frac{h\mathcal{E}_m(s)}{(t_m - s)(s - t_{m-1})}\right| + \left|\mathcal{E}_m'(s)\ln\frac{t_m - s}{s - t_{m-1}}\right| + \left|\int_{t_{m-1}}^{t_m} \frac{1}{2}\mathcal{E}_m''(\sigma(t))\mathrm{d}t\right| \\ &\leq C\gamma^{-1}(h,s)h^{1+\alpha},\end{aligned}$$

where $\sigma(t) \in (t_{m-1}, t_m)$. As for the second term, by an argument similar to that of Equation (20), we see that

$$\begin{aligned}
\left|\mathcal{R}_L^{(2)}(s)\right| &\leq \max|\mathcal{E}_m(t)| \int_{t_{m-1}}^{t_m} \frac{\kappa_s(t)-4}{(t-s)^2}\mathrm{d}t \\
&= \max|\mathcal{E}_m(t)| \;⨎_{t_{m-1}}^{t_m} \frac{\kappa_s(t)-4}{(t-s)^2}\mathrm{d}t \\
&= \max|\mathcal{E}_m(t)| \left\{ ⨎_{t_{m-1}}^{t_m} \frac{1}{\sin^2(t-s)/2}\mathrm{d}t - ⨎_{t_{m-1}}^{t_m} \frac{4}{(t-s)^2}\mathrm{d}t \right\} \\
&= \max|\mathcal{E}_m(t)| \left\{ -2\cot\frac{s-t_{m-1}}{2} - 2\cot\frac{t_m-s}{2} + \frac{4h}{(t_m-s)(s-t_{m-1})} \right\} \\
&\leq C\gamma^{-1}(h,s)h^{1+\alpha}. \qquad (38)
\end{aligned}$$

The third term $\mathcal{R}_L^{(3)}(s)$ can be estimated directly by Lemma 3.3. Putting these estimates together leads to Equation (35). ■

We must point out that in Lemma 3.4, we actually obtain the error expansion of the trapezoidal rule (Equation (13)) and moreover, get the explicit expression of the first-order term. Thanks to this error expansion, the finding and the proof of the uniqueness of the superconvergence points become very easy, which is quite different from the case where finite-part integral (1) with $p = 1, 2$ is involved (cf. [32,33]). Besides, if the second-order derivative of $f(t)$ at s can be evaluated, then by adding the first term in the right-hand side of Equation (34) into the trapezoidal rule (13), a modified trapezoidal rule with approximate second-order accuracy is obtained.

LEMMA 3.5 *Assume that $f(t)$ is a periodic function with period 2π. Assume further that $f(t)$ is finite-part integrable with respect to the weight* $\sin^{-2}[(t-s)/2]$. *Then*

$$⨎_c^{c+2\pi} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t = ⨎_{\tilde{c}}^{\tilde{c}+2\pi} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t \qquad (39)$$

holds for any $s \in (c,\ c+2\pi)$ and $\tilde{c} \in (s-2\pi, s)$.

Proof We just prove the case $c \leq \tilde{c} < s$ since the argument for $s - 2\pi < \tilde{c} < c$ is analogous. By definition (3), we have

$$\begin{aligned}
⨎_c^{c+2\pi} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t &= \lim_{\varepsilon\to 0}\left\{ \int_c^{s-\varepsilon} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t + \int_{s+\varepsilon}^{c+2\pi} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t - \frac{8f(s)}{\varepsilon} \right\} \\
&= \lim_{\varepsilon\to 0}\left\{ \int_{\tilde{c}}^{s-\varepsilon} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t + \int_{s+\varepsilon}^{\tilde{c}+2\pi} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t - \frac{8f(s)}{\varepsilon} \right\} \\
&\quad + \int_c^{\tilde{c}} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t - \int_{c+2\pi}^{\tilde{c}+2\pi} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t \\
&= ⨎_{\tilde{c}}^{\tilde{c}+2\pi} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t + \int_c^{\tilde{c}} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t \\
&\quad - \int_c^{\tilde{c}} \frac{f(2\pi+t')}{\sin^2(2\pi+t'-s)/2}\mathrm{d}t' \\
&= ⨎_{\tilde{c}}^{\tilde{c}+2\pi} \frac{f(t)}{\sin^2(t-s)/2}\mathrm{d}t,
\end{aligned}$$

which completes the proof. ■

Proof of Theorem 2.2 Recall that $f_L(t)$, defined by Equation (12), is the linear interpolant of $f(t)$ on $[c, c+2\pi]$. We extend $f_L(t)$ to $(-\infty, \infty)$ to obtain a 2π-periodic function and denote the resulting function still by $f_L(t)$. Clearly, $f_L(t)$ becomes the linear interpolant of $f(t)$ on $(-\infty, +\infty)$. By Lemma 3.5, it holds for any $s = t_{m-1} + (\tau+1)h/2(-1 < \tau < 1,\ 1 \le m \le n-1)$ and $\tilde{c} \in (s - 2\pi, s)$ that

$$I(c, s, f) - I(c, s, f_L) = I(\tilde{c}, s, f) - I(\tilde{c}, s, f_L). \tag{40}$$

We first consider the case where $m > [n/2]$ and choose $\tilde{c} = t_{m-1-[n/2]}$. Let $\tilde{c} = \tilde{t}_0 < \tilde{t}_1 < \cdots < \tilde{t}_{n-1} < \tilde{t}_n = \tilde{c} + 2\pi$ be a uniform partition of $[\tilde{c},\ \tilde{c}+2\pi]$ with mesh size h. It is evident that $f_L(t)$ is still the linear interpolant of $f(t)$ on this new partition and consequently, the result of Lemma 3.4 holds, which leads to

$$I(\tilde{c}, s, f) - I(\tilde{c}, s, f_L) = 4hf''(s)\ln\left(2\cos\frac{\tau\pi}{2}\right) + \mathcal{R}_L(s), \tag{41}$$

where

$$|\mathcal{R}_L(s)| \le \begin{cases} C\varrho(h, s, \tilde{c})h^{1+\alpha}, & f(t) \in C^{2+\alpha}[\tilde{c}, \tilde{c}+2\pi], \quad 0 < \alpha < 1, \\ C\varrho(h, s, \tilde{c})h^2|\ln h|, & f(t) \in C^3[\tilde{c}, \tilde{c}+2\pi] \end{cases} \tag{42}$$

with $\varrho(h, s, \tilde{c})$ defined by Equation (30). Now, by the assumption (21), we find that

$$s = t_{m-1} + \frac{h}{2} \pm \frac{h}{3} = \tilde{t}_{[n/2]} + \frac{h}{2} \pm \frac{h}{3}, \tag{43}$$

which implies that the local coordinate of s is $\tau = \pm 2/3$. Then we are led to the results that the first term in the right hand-side of Equation (41) vanishes and moreover, $\gamma(h, s) = 1/6$ by Equation (23). Therefore, from Equations (17), (30) and (43), we deduce that

$$\varrho(h, s, \tilde{c}) = 36 \max_{\tilde{c} \le t \le \tilde{c}+2\pi}\{\kappa_s(t)\} \le 36 \max_{|\theta| \le 11\pi/6}\left\{\frac{\theta^2}{\sin^2\theta/2}\right\} \le C. \tag{44}$$

Incorporating this result with Equations (41) and (42) yields

$$|I(\tilde{c}, s, f) - I(\tilde{c}, s, f_L)| \le \begin{cases} Ch^{1+\alpha}, & f(t) \in C^{2+\alpha}[\tilde{c}, \tilde{c}+2\pi], \quad 0 < \alpha < 1, \\ Ch^2|\ln h|, & f(t) \in C^3[\tilde{c}, \tilde{c}+2\pi]. \end{cases}$$

Now Equation (22) follows directly from Equation (40). ■

We observe from Equation (41) that, the first-order term vanishes if and only if $\tau = \pm 2/3$. Thus, the uniqueness of the superconvergence points is verified immediately. We state our result next.

THEOREM 3.6 *For the composite trapezoidal rule defined by Equations* (13) *and* (14)*, there exist only two superconvergence points in each subinterval* $[t_{m-1}, t_m](1 \le m \le n)$*, i.e.* $s = t_{m-1} + h/2 \pm h/3$*, at which the superconvergence estimate* (22) *holds.*

4. Some applications

4.1 *An indirect method*

The superconvergence result of the composite trapezoidal rule for evaluating Equation (2) was discussed in the previous section, and it is natural to apply them for solving integral equation. From Equations (15) and (16), we can see that the singular point should be different from the nodal ones. But in many cases, *e.g.* for solving integral equation of the second kind by the collocation method, we always hope the collocation points coincide with the nodal ones, and therefore the aforementioned quadrature method cannot be directly applied. Here, to avoid the selection of the singular point, we suggest an indirect method for evaluating finite-part integral (2) based on the superconvergence result of the composite trapezoidal rule:

$$I^*(c, s, f_L) := \frac{1}{s_2 - s_1}[(s - s_1)I(c, s_2, f_L) + (s_2 - s)I(c, s_1, f_L)], \tag{45}$$

where s_1 and s_2 are two superconvergence points nearest to s such that $s_1 \leq s \leq s_2$.

THEOREM 4.1 *Suppose that the* 2π*-periodic function* $f(t) \in C^4(-\infty, \infty)$ *and the mesh is uniform. Let* $I^*(c, s, f_L)$ *be computed by Equation* (45) *with* s_1 *and* s_2 *being two superconvergence points nearest to* s *and* $s_1 \leq s \leq s_2$*. Then*

$$|I(c, s, f) - I^*(c, s, f_L)| \leq Ch^2. \tag{46}$$

Proof If $s_1 = s$ or $s_2 = s$, then Equation (46) is obvious by Theorem 2.2. Now consider the general case where $s_1 < s < s_2$. Set

$$I^*(c, s, f) := \frac{1}{s_2 - s_1}[(s - s_1)I(c, s_2, f) + (s_2 - s)I(c, s_1, f)].$$

On the one hand, by Theorem 2.2, we have

$$|I^*(c, s, f_L) - I^*(c, s, f)| \leq |I(c, s_2, f_L) - I(c, s_2, f)| + |I(c, s_1, f_L) - I(c, s_1, f)| \leq Ch^2. \tag{47}$$

On the other hand, since $f(t) \in C^4(-\infty, \infty)$, $I(c, s, f)$, being function of s, belongs to $C^2(-\infty, \infty)$ (see [8]). Note that $I^*(c, s, f)$ is actually the linear interpolant of $I(c, s, f)$ with respect to s. Thus

$$|I(c, s, f) - I^*(c, s, f)| \leq \max_{s \in [s_1, s_2]} |\frac{d^2}{ds^2} I(c, s, f)|h^2 \leq Ch^2. \tag{48}$$

Finally, we can get the result by Equations (47), (48) and the triangle inequality. ■

4.2 *Solving integral equation of the second kind containing finite-part kernels*

A natural application is to solve finite-part integral equation by using the composite trapezoidal rule and its superconvergence result. Finite-part integral equations often arise in numerical analyses of partial differential equation and many physical problems, such as in fracture mechanics, elasticity problems, acoustics as well as electromagnetic scattering [12,42,19,1,4,5], and there exist several numerical methods to solve them.

To make our idea clear, a simple model problem will be first discussed so as to show the convergence rate of the numerical schemes suggested before. Then, we will apply them to the scattering problem in the next subsection.

Here, we consider how to solve the following integral equation of the second kind containing finite-part and logarithm kernels:

$$\neq\!\!\!\!\!\!\int_{-\pi}^{\pi} \frac{\varphi(t)}{\sin^2(t-s)/2}\,\mathrm{d}t + \int_{-\pi}^{\pi} \ln 4\sin^2\frac{t-s}{2}\varphi(t)\,\mathrm{d}t + \pi\varphi(s) = g(s), \quad s \in [-\pi, \pi], \tag{49}$$

where $g(s)$ is a known 2π-periodic term and $\varphi(t)$ is to be determined.

ALGORITHM 4.1 Note that $\{s_i : s_i = t_i + (\tau+1)h/2\}_{i=0}^{n-1}$ with $\tau = \pm 2/3$ are the superconvergence points, we can use the composite trapezoidal rule by choosing $\{s_i\}_{i=0}^{n-1}$ as the collocation points to evaluate the finite-part term and then get the following linear systems:

$$\sum_{j=1}^{n}\left[\omega_j(s_i) + h\ln 4\sin^2\frac{t_j - s_i}{2}\right]\phi(t_j) + a(t_i)\phi(t_i) = f(s_i), \quad i = 1, \cdots, n. \tag{50}$$

Here, we use the classical composite trapezoidal rule for the evaluation of the logarithmic integrals.

ALGORITHM 4.2 Algorithm 4.1 has a drawback that one cannot approximate the third term in the left-hand of Equation (49) as good as possible. Numerical experiments show that even if we choose the superconvergence points as collocation points, the convergence rate of Algorithm 4.1 is always $O(h)$, which is the same as that of the case without the choice of the superconvergence ones. Here we suggest a more natural algorithm based on the previous indirect method which can exactly approximate the third term of Equation (49). Numerical experiments show that its accuracy can achieve a convergence rate one order higher than that of Algorithm 4.1. Using the indirect formula (45) for evaluating the finite-part term and the corresponding indirect form of the classical trapezoidal rule for the logarithm term in the left-hand of the integral equation (49), we get the following system:

$$\sum_{j=1}^{n}\frac{1}{(s_{i2}-s_{i1})}\left[\hat{\omega}_j(s_{i2})(t_i - s_{i1}) + \hat{\omega}_j(s_{i1})(s_{i2} - t_i)\right]\phi(t_j) + a(t_i)\phi(t_i) = f(t_i), \quad i = 1, \cdots, n. \tag{51}$$

where

$$\hat{\omega}_j(s) = \omega_j(s) + h\ln 4\sin^2\frac{t_j - s}{2},$$

$\omega_j(s)$ is defined by Equation (14) and s_{i1}, s_{i2} are two superconvergence points nearest to t_i and $s_{i1} \le t_i \le s_{i2}$.

4.3 *Application in the scattering theory*

The mathematical treatment of the scattering of time-harmonic electromagnetic waves by an infinitely long cylindrical obstacle with a simply connected bounded cross-section $D \subset \mathbb{R}^2$ leads to exterior boundary value problems for the Helmholtz equation:

$$\Delta u + k^2 u = 0 \quad \text{in} \quad \mathbb{R}^2/\bar{D} \tag{52}$$

with $k > 0$. In the subsequent analysis we denote the boundary of D by Γ, the outward unit normal to Γ by ν and the boundary Γ is assumed to be C^2. The total field u can be decomposed as $u = u^i + u^s$, where u^i is the incident field, which is assumed to be an entire solution to the Helmholtz equation, and u^s is the unknown scattered field, which is required to satisfy the Sommerfeld radiation condition

$$\lim_{r\to\infty} \sqrt{r}\left(\frac{\partial u^s}{\partial r} - iku^s\right) = 0, \quad r = |x|, \tag{53}$$

uniformly in all directions. In [19], Kress considered the scattering problem as a special case of the following exterior Neumann problem. Given a function $g \in C^{0,\alpha}(\Gamma)$, $0 < \alpha < 1$, find a solution $u \in C^2(\mathbb{R}^2\backslash\bar{D}) \bigcap C^{1,\alpha}(\mathbb{R}^2\backslash D)$ to the Helmholz equation which satisfies the Sommerfeld radiation and the boundary condition

$$\frac{\partial u}{\partial \nu} = g, \quad \text{on } \Gamma. \tag{54}$$

In order to arrive at a uniquely solvable integral equation, the solution to the exterior Neumann problem was obtained in the form of a combined double- and single-layer potential

$$u(x) = \int_\Gamma \left\{\frac{\partial \Phi(x,y)}{\partial \nu(y)} - i\eta\Phi(x,y)\right\}\varphi(y)\mathrm{d}s(y), \quad x \in \mathbb{R}^2/\bar{D}, \tag{55}$$

with unknown density $\varphi \in C^{1,\alpha}(\Gamma)$ and some real coupling parameter η, where $\Phi(x,y) = \frac{1}{4}iH_0^{(1)}(k|x-y|)$ is the fundamental solution to the Helmholtz function in $\mathbb{R}^2$ and $H_0^{(1)}$ is the Hankel function of order zero and of the first kind. Then for the jump relations for single- and double-layer potentials it follows that Equation (55) solves the exterior Neumann problem provided the density is a solution of the integral equation

$$T\varphi - i\eta K'\varphi + i\eta\varphi = 2g, \tag{56}$$

where K' and T denote the integral operators defined by

$$(K^{'}\varphi)(x) := 2\int_\Gamma \frac{\partial\Phi(x,y)}{\nu(x)}\varphi(y)\mathrm{d}s(y), \quad x \in \Gamma,$$

$$(T\varphi)(x) := 2\frac{\partial}{\partial\nu(x)}\int_\Gamma \frac{\partial\Phi(x,y)}{\nu(y)}\varphi(y)\mathrm{d}s(y), \quad x \in \Gamma.$$

Similar to the procedure of [19], we describe the parametrization of the integral equation (56) as follows. From now on, we assume that the boundary curve Γ is analytic and given through

$$\Gamma = \{x(t) = (x_1(t), x_2(t)) : 0 \le t \le 2\pi\},$$

where $x : \mathbb{R} \to \mathbb{R}^2$ is analytic and 2π-periodic with $|x'(t)| > 0$ for all t, such that the orientation of Γ is counterclockwise. Using $H_1^{(1)} = -H_0^{(1)'}$, where $H_1^{(1)}$ denotes the Hankel function of order one and of the first kind, note that $\mathrm{d}s(y) := \sqrt{x_1'(\tau)^2 + x_2'(\tau)^2}\mathrm{d}\tau = |x'(\tau)|\mathrm{d}\tau$ and

$$\frac{\partial\Phi(x,y)}{\partial\nu(x)} = \frac{ik}{4}\frac{H_1^{(1)}(k|x(t)-x(\tau)|)}{|x(t)-x(\tau)|}\nu(x(t))\cdot[x(\tau)-x(t)].$$

Thus, the kernel H in

$$(K'\varphi)(x) = \frac{1}{|x'(t)|}\int_0^{2\pi} H(t,\tau)\phi(x(\tau))\mathrm{d}\tau$$

is given by

$$H(t,\tau) := \frac{ik}{2}n(t)\cdot[x(\tau)-x(t)]\frac{H_1^{(1)}(k|x(t)-x(\tau)|)}{|x(t)-x(\tau)|}|x'(\tau)|, \tag{57}$$

where $n(t) := |x'(t)|\nu(x(t)) = (x_2'(t), -x_1'(t))$.

By direct calculation, we obtain that

$$\frac{\partial \Phi(x,y)}{\partial \nu(y)} = \frac{ik}{4}\frac{H_1^{(1)}(k|x-y|)}{|x-y|}\nu(y)\cdot(x-y),$$

$$\frac{\partial^2 \Phi(x,y)}{\partial \nu(x)\partial \nu(y)} = \frac{ik}{4}H_0^{(1)}(k|x-y|)\frac{k\nu(x)\cdot(x-y)\nu(y)\cdot(x-y)}{|x-y|^2}$$
$$+\frac{ik}{4}H_1^{(1)}(k|x-y|)\left[\frac{\nu(x)\cdot\nu(y)}{|x-y|} - 2\frac{\nu(x)\cdot(x-y)\nu(y)\cdot(x-y)}{|x-y|^3}\right].$$

Thus, the kernel M in

$$(T\varphi)(x) = \frac{1}{|x'(t)|}\int_0^{2\pi} M(t,\tau)\varphi(x(\tau))\mathrm{d}\tau$$

is given by

$$M(t,\tau) = \frac{ik}{2}\left\{k\frac{H_0^{(1)}(k|x(t)-x(\tau)|)}{|x(t)-x(\tau)|^2}n(t)\cdot[x(t)-x(\tau)]n(\tau)\cdot[x(t)-x(\tau)]\right.$$
$$-2\frac{H_1^{(1)}(k|x(t)-x(\tau)|)}{|x(t)-x(\tau)|^3}n(t)\cdot[x(t)-x(\tau)]n(\tau)\cdot[x(t)-x(\tau)]$$
$$\left.+\frac{H_1^{(1)}(k|x(t)-x(\tau)|)}{|x(t)-x(\tau)|}n(t)\cdot n(\tau)\right\}. \tag{58}$$

We then rewrite Equation (58) as follows:

$$M(t,\tau) = \frac{1}{4\pi\sin^2(t-\tau)/2} + M_1(t,\tau), \tag{59}$$

where

$$M_1(t,\tau) = M(t,\tau) - \frac{1}{4\pi\sin^2(t-\tau)/2}.$$

If we now piece Equations (57) and (59) together, we see that the parameterized integral Equation (56) is of the form

$$\frac{1}{4\pi}\,⨎_0^{2\pi}\frac{1}{\sin^2(t-\tau)/2}\phi(\tau)\mathrm{d}\tau + \int_0^{2\pi}K(t,\tau)\phi(\tau)\mathrm{d}\tau + a(t)\phi(t) = f(t) \tag{60}$$

for the unknown function $\phi(t) := \varphi(x(t))$ and the right-hand side given by $f(t) := 2|x'(t)|g(x(t))$. We have the set $a(t) := i\eta|x'(t)|$ and the kernel

$$K(t,\tau) = M_1(t,\tau) - i\eta H(t,\tau),$$

which are 2π-periodic functions.

ALGORITHM 4.3 Since Equation (60) is also an integral equation of the second kind containing finite-part kernels, and $K(t,\tau)$ contains only continuous and logarithmic kernels, we can suggest an indirect algorithm similar to Algorithm 4.2 and then get the following system:

$$\sum_{j=1}^{n} \frac{1}{(s_{i2}-s_{i1})} \left[\tilde{\omega}_j(s_{i2})(t_i - s_{i1}) + \tilde{\omega}_j(s_{i1})(s_{i2} - t_i)\right] \phi(t_j) + a(t_i)\phi(t_i) = f(t_i), \quad i = 1, \ldots, n. \tag{61}$$

where

$$\tilde{\omega}_j(s) = \frac{1}{4\pi}\omega_j(s) + hK(s, t_j),$$

and s_{i1}, s_{i2} are coincident with that of Algorithm 4.2. The main advantage of Algorithm 4.3 over the method suggested in [19] is that no complicated analysis needs to be done when constructing the numerical scheme, and it is very easy to implement.

5. Numerical examples

In this section, computational results are reported by five examples to confirm our theoretical analysis and to show the efficiency of the algorithms. Throughout, the computation is performed in double precision.

Example 5.1 We consider the finite-part integral (2) with $f(t) = 1 + 2\cos t + 2\cos 2t$ and $c = -\pi$. Obviously the integrand function $f(t)$ is smooth enough and by a direct calculation,

$$⨎_{-\pi}^{\pi} \frac{1 + 2\cos t + 2\cos 2t}{\sin^2(t-s)/2} \mathrm{d}t = -8\pi(\cos s + 2\cos 2s), \tag{62}$$

where $s \in (-\pi, \pi)$. Here we adopt a uniform mesh with an odd n. Numerical results are presented in Table 1 for a dynamic singular point $s = t_{[n/4]} + (1+\tau)h/2$ and in Table 2 for $s = t_{n-1} + (1+\tau)h/2$. In the first case, singular point s is not close to the ending-points of the interval $(-\pi, \pi)$ while in the second it approaches the ending-point π as h goes to zero. We can see from Tables 1 and 2 that in both cases errors at the superconvergence points, $\tau = \pm 2/3$, are $O(h^2)$ which is in good agreement with our theoretical analysis, while errors at the non-superconvergence points are $O(h)$.

Example 5.2 Now we consider an example with a fixed singular point. We still use Equation (62) and set $s = 0$ or 1. Here two mesh strategies are adopted. In the first one (Mesh I), s is always placed at the midpoint of some subinterval and in the second (Mesh II), s is placed at a point with local coordinate $\tau = -2/3$. Then s is a superconvergence point on Mesh II. Both meshes are uniform except two subintervals near the ending points, which may have smaller or longer mesh size. Numerical results are presented in Table 3. We find that the accuracy of the composite

Table 1. Errors of the composite trapezoidal rule with $s = t_{[n/4]} + (1+\tau)h/2$.

n	$\tau = 0$	$\tau = 1/2$	$\tau = 2/3$	$\tau = -2/3$
255	5.57443E-1	2.84096E-1	1.09599E-2	9.56223E-3
511	2.75461E-1	1.39069E-1	2.72896E-3	2.35978E-3
1023	1.36915E-1	6.87914E-2	6.80860E-4	5.86067E-4
2047	6.82535E-2	3.42102E-2	1.70041E-4	1.46031E-4
4095	3.40758E-2	1.70587E-2	4.24920E-5	3.64483E-5
h^α	1.008	1.015	2.003	2.009

Table 2. Errors of the composite trapezoidal rule with $s = t_{n-1} + (1+\tau)h/2$.

n	$\tau = 0$	$\tau = 1/2$	$\tau = 2/3$	$\tau = -2/3$
255	4.18578E-1	2.13717E-1	8.83328E-3	9.11517E-3
511	2.06737E-1	1.04474E-1	2.20820E-3	2.24373E-3
1023	1.02724E-1	5.16381E-2	5.52016E-4	5.56476E-4
2047	5.11998E-2	2.56689E-2	1.37996E-4	1.38554E-4
4095	2.55593E-2	1.27969E-2	3.44929E-5	3.45690E-5
h^α	1.008	1.016	2.000	2.011

Table 3. Errors of the composite trapezoidal rule on different meshes.

	$s = 0$		$s = 1$	
n	Mesh I	Mesh II	Mesh I	Mesh II
255	6.94536E-1	1.14271E-2	1.57305E-1	9.94010E-3
511	3.43755E-1	2.84778E-3	7.75009E-2	2.39413E-3
1023	1.71000E-1	7.10813E-4	3.85158E-2	6.11856E-4
2047	8.52810E-2	1.77555E-4	1.91925E-2	1.53004E-4
4095	4.25857E-2	4.43654E-5	9.57990E-3	3.83101E-5
h^α	1.007	2.002	1.009	2.005

trapezoidal rule on Mesh II is $O(h^2)$, while the accuracy on Mesh I is only $O(h)$, which both agree well with the theoretical analysis.

Example 5.3 We consider the finite-part integral Equation (49) with right-hand term $g(t) = -2\pi(5\cos t + 8\cos 2t)$. Its exact solution is $\varphi(t) = 2\cos t + 2\cos 2t$. In Algorithm 4.1, we adopt a uniform mesh and get the linear system (Equation (50)) with two set of collocation points:

$$\mathcal{S}_1 = \{t_i + h/2 + h/3, \quad 0 \le i \le n-1\},$$
$$\mathcal{S}_2 = \{t_i + h/2,\ 0 \le i \le n-1\}.$$

Obviously, $\mathcal{S}_1$ consists of the superconvergence points, and $\mathcal{S}_2$ does not. Although the result for S_1 is much better than that of S_2, the convergence rate is always $O(h)$ whether we choose $\mathcal{S}_1$ or $\mathcal{S}_2$. This may be due to the coarse approximation to the third term in the left-hand of Equation (60). In Algorithm 4.2, we adopt a uniform mesh and get the linear system (Equation (51)) with the collocations points as nodal ones and two sets of s_{i1} and s_{i2}:

$$\mathcal{S}_3 = \{s_{i1} = t_{i-1} + h/2 + h/3,\ \ s_{i2} = t_i + h/2 - h/3, \quad 1 \le i \le n\},$$
$$\mathcal{S}_4 = \{s_{i1} = t_i - h/2,\ \ s_{i2} = t_i + h/2, \quad 1 \le i \le n\}.$$

Obviously, $\mathcal{S}_3$ consists of the superconvergence points, and $\mathcal{S}_4$ contains non-superconvergence ones. Numerical results in Table 4 show that the accuracy for Algorithm 4.2 is much improved if we choose $\mathcal{S}_3$.

Example 5.4 Now we consider an example with less regularity. Let

$$\varphi(t) = |t(t^2 - \pi^2)|^3, \quad t \in [-\pi, \pi],$$

and we extend it to a periodic function, still denoted by $\varphi(t)$, with period 2π by taking $\varphi(t) = \varphi(t + 2\pi)$. Obviously, $\varphi(t) \in C^3[c, c + 2\pi]$, c is an arbitrary constant. For convenience, we first

Table 4. L_∞ error for Example 5.3.

	Algorithm 4.1		Algorithm 4.2	
n	$\mathcal{S}_1$	$\mathcal{S}_2$	$\mathcal{S}_3$	$\mathcal{S}_4$
16	0.3409817	0.8333414	0.1368259	1.200549
32	0.1537028	0.3624361	3.5379302E-2	0.4243268
64	7.1390398E-2	0.1669393	9.0258289E-3	0.1786756
128	3.4252074E-2	8.0446385E-2	2.2817904E-3	8.1812330E-2
h^α	1.105	1.124	1.969	1.292

Table 5. L_∞ error for Example 5.4.

n	Algorithm 4.2	Kress' method
32	19.83552	0.7372616
64	5.007455	0.1026070
128	1.244693	2.3605831E-02
256	0.3188093	1.4028244E-02
h^α	1.986	1.905

choose $\varphi(t) = |t(t^2 - \pi^2)|^3$ as the exact solution of Equation (49), and then use the trapezoidal rule (45) by using a mesh with size small enough to get the right-hand term. Finally, we obtain the approximation solutions by Algorithm 4.2 and Kress' method, respectively. From [19], we know that if the exact solution $\varphi(t)$ is analytic, the error decreases exponentially. But for the less regular $\varphi(t)$, its accuracy will descend, which can be seen from Table 5. Moreover, also from Table 5, we can see that even for $\varphi(t) \in C^3[c, c + 2\pi]$, the convergence rate of Algorithm 4.2 can achieve $O(h^2)$, which shows that our method is competitive with Kress' method in this special case. Thus, it is still meaningful for us to study the superconvergence result of the trapezoidal rule for the integral equation containing finite-part kernels.

Example 5.5 Here we consider the scattering of a plane wave u^i by a sound-hard cylinder with a non-convex kite-shaped cross-section with boudary Γ described by the parametric representation

$$x(t) = (\cos t + 0.65\cos 2t - 0.65, 1.5\sin t), \quad 0 \le t \le 2\pi.$$

The incident wave is given by a smooth function $u^i(x) = e^{ikd\cdot x}$ where d denotes a unit vector giving the direction of propagation. For the scattered wave u^s we have to solve an exterior Neumann problem with boundary values $g = -\partial u^i/\partial\nu$ on Γ. The far-field pattern u_∞ is defined by the asymptotic behaviour of the scattered wave

$$u^s(x) = \frac{e^{ik|x|}}{\sqrt{|x|}}\left\{u_\infty(\hat{x}) + O\left(\frac{1}{\sqrt{|x|}}\right)\right\}, \quad |x| \to \infty,$$

uniformly for all direction $\hat{x} := x/|x|$. From the asymmptotics for the Hankel function for large argument, we see that the far-field pattern of the combined double- and single-layer potential is given by

$$u_\infty(\hat{x}) = \frac{e^{-i\pi/4}}{\sqrt{8\pi k}}\int_\Gamma \{k\hat{x}\cdot\nu(y) + \eta\}e^{-ik\hat{x}\cdot y}\varphi(y)\,\mathrm{d}s(y). \tag{63}$$

After solving the integral equation (60) numerically by Algorithm 4.3, the integral (63) is evaluated by the classical trapezoidal rule. Tables 6 and 7 give some approximate values for the far-field

Table 6. Numerical results for Example 5.5 by Algorithm 4.3 with $\mathcal{S}_4$.

	n	$\mathrm{Re}u_\infty(d)$	$\mathrm{Im}u_\infty(d)$	$\mathrm{Re}u_\infty(-d)$	$\mathrm{Im}u_\infty(-d)$
$k=1$	32	0.1101255	0.1359606	−1.140504	−0.505935
	64	0.1340470	0.1686133	−1.117780	−0.507786
	128	0.1436574	0.1810827	−1.109203	−0.508598
	256	0.1478166	0.1865426	−1.105564	−0.508927
$k=3$	32	−9.2281945E-2	0.6230944	−1.857214	−0.854712
	64	−7.0194438E-2	0.6906148	−1.721669	−0.821593
	128	−5.1378526E-2	0.7057285	−1.672501	−0.820651
	256	−4.3204948E-2	0.7094656	−1.653001	−0.821785
$k=5$	32	−0.4244972	−0.4010827	−2.308823	−1.309626
	64	−0.3705059	−0.3258927	−2.116996	−1.254386
	128	−0.3225423	−0.3033766	−2.017308	−1.258708
	256	−0.2998221	−0.2989495	−1.978207	−1.266769

Table 7. Numerical results for Example 5.5 by Algorithm 4.3 with $\mathcal{S}_3$.

	n	$\mathrm{Re}u_\infty(d)$	$\mathrm{Im}u_\infty(d)$	$\mathrm{Re}u_\infty(-d)$	$\mathrm{Im}u_\infty(-d)$
$k=1$	32	0.1419960	0.1848495	−1.108783	−0.508948
	64	0.1490904	0.1898673	−1.103974	−0.509103
	128	0.1509149	0.1911187	−1.102754	−0.509165
	256	0.1513736	0.1914326	−1.102448	−0.509184
$k=3$	32	−8.913858E-2	0.6941959	−1.705979	−0.834866
	64	−5.004480E-2	0.7081655	−1.654401	−0.824769
	128	−3.991257E-2	0.7105141	−1.641329	−0.823619
	256	−3.733771E-2	0.7110443	−1.638012	−0.823419
$k=5$	32	−0.3839110	−0.3381853	−2.122159	−1.328245
	64	−0.3114551	−0.3022698	−1.993552	−1.277937
	128	−0.2885902	−0.2987530	−1.959255	−1.275653
	256	−0.2826746	−0.2982955	−1.950472	−1.275797

patterns $u_\infty(d)$ and $u_\infty(-d)$ in the forward direction d and backward direction $-d$. The direction of the incident wave is $d = (1, 0)$, and the coupling parameter is recommended in [19] to be chosen as $\eta = k$. Since the incident wave is analytic and the error of Kress' method decreases exponentially, the numerical result in Table 8 (presented in [19]) can be viewed as, to some extent, the exact solution. Comparing the results in Tables 6 and 7 with that in Table 8, we can see that the accuracy of Algorithm 4.3 is much improved if we choose the superconvergence points.

Table 8. Numerical results for Kress' method.

	n	$\mathrm{Re}u_\infty(d)$	$\mathrm{Im}u_\infty(d)$	$\mathrm{Re}u_\infty(-d)$	$\mathrm{Im}u_\infty(-d)$
$k=1$	8	0.13973626	0.18093027	−1.11011577	−0.50681485
	16	0.15158507	0.19159181	−1.10228822	−0.50925214
	32	0.15153740	0.19153454	−1.10234229	−0.50918721
	64	0.15153740	0.19153454	−1.10234230	−0.50918720
$k=3$	8	0.11163356	0.91056703	−1.67222931	−0.86694951
	16	−3.641571E-2	0.71129456	−1.63684382	−0.82343826
	32	−3.646654E-2	0.71122115	−1.63689151	−0.82335680
	64	−3.646654E-2	0.71122115	−1.63689151	−0.82335679
$k=5$	8	0.53567309	0.12509154	−1.92379981	−1.40068649
	16	−0.27473745	−0.29834046	−1.95374230	−1.27549747
	32	−0.28067233	−0.29817977	−1.94749252	−1.27590706
	64	−0.28067233	−0.29817977	−1.94749251	−1.27590706

6. Concluding remarks

We have proved the superconvergence estimate of the composite trapezoidal rule for the finite-part integral (2) and also obtained the uniqueness of the superconvergence points. The theoretical result is confirmed by the numerical one. Although our result is obtained for the uniform mesh, Example 5.2 indicates that this result is still valid for certain non-uniform mesh, say, local uniform mesh in the neighbourhood of the singular point combined with quasi-uniform mesh in the remaining part of the interval. Numerical results in the previous section also indicate that when the singular point coincides with a superconvergence point, the accuracy can be one order higher than that when the non-superconvergence point is involved. So a natural question raised is that whether it is possible to remove the factor $|\ln h|$ from estimate (22). The answer is certainly positive. Actually, we can prove that the superconvergence rate can be $O(h^2)$ provided that $f(t) \in C^{3+\alpha}(-\infty; +\infty)(0 < \alpha < 1)$. Here we omit this part of the argument just because it involves tedious details and its presentation will spoil the structure of the present paper and make our main idea obscure.

We have also suggested an indirect method which can avoid the selection of the singular point, and applied it to solve the integral equation of the second kind containing finite-part kernels, including that arising in the scattering theory. Numerical experiments show their efficiency in the less regular case. Theoretical analysis of these global error bounds for finite-part integral equations is beyond the aim and scope of the present paper and will be given elsewhere.

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