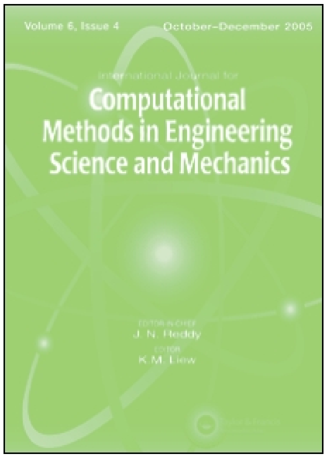

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Analysis of Laminates using Multiquadric Radial Basis Function

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Multiquadric radial basis function (MQRBF) is developed for static and dynamic analysis and to estimate natural frequency of laminated plate at various boundary conditions. MQRBF is applied for spatial discretization and Newmark implicit scheme is used for temporal discretization. The spatial discretization of the differential equations generates a greater number of algebraic equations than the unknown coefficients. The multiple linear regression analysis, which is based on the least square error norm, is employed to obtain the coefficients. Numerical results are compared with those obtained by other analytical methods.

Keywords Multiquadric Radial Basis Function, Cross-Ply Laminate, Natural Frequency, Finite Difference Method

NOMENCLATURE

D_{11}, D_{22}, D_{12}, D_{66}	Flexural rigidity of plates
D_{16}, D_{26}	Bending-twisting coupling stiffness terms
ρ	Mass density of the plate
M	Mass of the laminate.
a, b	Dimensions of plate
C_ν^*, C_ν	Viscous damping, dimensionless viscous damping
h	Thickness of plate
q, Q	Transverse load, dimensionless transverse load
t^*, t	Time, dimensionless time
ν	Poisson's ratio
w^*	Displacement in z^*direction
w	Dimensionless displacement in z direction
R	Aspect ratio (a/b)
ω	Natural frequency of vibration
$\rho_0(\rho h)$	Surface density

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INTRODUCTION

Wide applications of composite in high performance spacecrafts, aeronautics and submarines, etc., demand efficient computational algorithm to predict static and dynamic response of laminated structure. Currently, the exact solutions are available only for a few special cases [1]. As a result, analytical methods are extensively used for analysis of the laminated plates [2–5].

At present, the finite element method (FEM) is a universally acceptable technique for solving boundary and initial value problems. Various types of plate elements are developed and used in the finite element method [6, 7]. Although FEM is the most versatile and powerful technique, the method is not very efficient as it consumes more time in mesh generation than the execution. Recently, as an alternative to the finite element method, a number of meshless methods were developed that avoid the time-consuming mesh generation process. These methods include smooth particle hydrodynamics [8], element free Galerkin (EFG) [9], partition of unity method [10], wavelet Galerkin method [11], the method of finite sphere [12], meshless local Petrov-Galerkin (MLPG) method [13], moving least square differential quadrature method [14], and so on.

Kansa [15] and Franke [16] developed the concept of solving PDEs using radial basis functions (RBFs). Many researchers find this method very attractive due to absence of mesh generation. Fedoseyev et al. [17] improved the accuracy of solution by placing the interior knots. Ferreira applied the radial basis functions for the analysis of thick laminated composite beams [18] using first-order shear deformation theory, and layerwise shear deformation theory for the analysis of laminated composite and sandwich plates [19].

Chen et al. [20] studied the free vibration analysis of circular and rectangular plates by employing the RBF in the

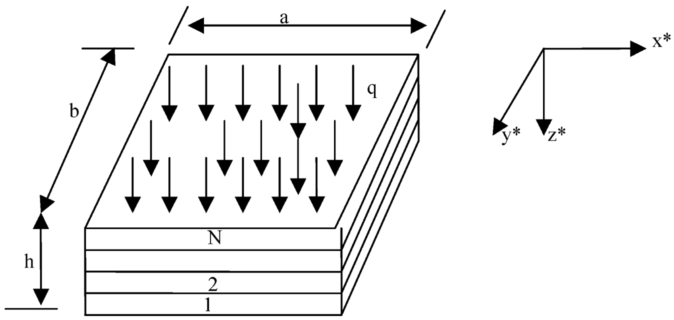

FIG. 1. Cross-ply laminate (0°/90°/90°/0°).

imaginary-part fundamental solution. Misra et al. [21] applied the collocation method based multi quadric radial basis function (MQRBF) method for static and dynamic analysis of anisotropic plates. The collocation-based algorithm has an advantage in the treatment of nonlinear terms and general boundary conditions. Analysis of banana fibers reinforced low-density polyethylene/polycaprolacton composites has been reported recently by Kumar and Misra [22]. The present paper employs MQRBF for static and dynamic analysis of laminates. Due to simplicity, the classical laminated plate theory [23] is used for the analysis of laminated plates, which is the extension of classical plate theory. The ill conditioning due to collocation method is efficiently handled by using multiple linear regression analysis.

BASIC EQUATIONS OF THE LAMINATED PLATES

Consider a rectangular plate consisting of N layers of orthotropic plies as shown in Figure 1. The undeformed mid-plane of the plate is assumed as the reference plane $z^* = 0$. From the classical laminated-plate theory [23], displacements u^*, v^* and w^*, in the direction of x^*, y^* and z^* can be expressed as:

$$u^* = -z^* \frac{\partial w^*}{\partial x^*} \tag{1}$$

$$v^* = -z^* \frac{\partial w^*}{\partial y^*} \tag{2}$$

The strain-displacement can be expressed as:

$$\begin{Bmatrix} \varepsilon_{x^*} \\ \varepsilon_{y^*} \\ \gamma_{x^*y^*} \end{Bmatrix} = -z^* \mathbf{L}\, w^* \tag{3}$$

$$\mathbf{L} = \left\{ \frac{\partial^2}{\partial x^{*2}}, \frac{\partial^2}{\partial y^{*2}}, 2\frac{\partial^2}{\partial x^* \partial y^*} \right\}^T \tag{4}$$

where $\mathbf{L}$ is a linear differential operator. Superscript T denotes the transpose of the matrix. Stresses at the Nth layer are

$$\begin{Bmatrix} \sigma_{x^*} \\ \sigma_{y^*} \\ \tau_{x^*y^*} \end{Bmatrix}^N = \begin{bmatrix} c_{11}^* & c_{12}^* & c_{13}^* \\ c_{21}^* & c_{22}^* & c_{23}^* \\ c_{31}^* & c_{32}^* & c_{33}^* \end{bmatrix}^N \begin{Bmatrix} \varepsilon_{x^*} \\ \varepsilon_{y^*} \\ \gamma_{x^*y^*} \end{Bmatrix} = c^{*N} \begin{Bmatrix} \varepsilon_{x^*} \\ \varepsilon_{y^*} \\ \gamma_{x^*y^*} \end{Bmatrix} \tag{5}$$

where c^N is the transformed stiffness matrix. It is given by

$$c^{*N} = T Q^N T^T \tag{6}$$

where T is transform matrix

$$T = \begin{bmatrix} l^2 & m^2 & lm \\ m^2 & l^2 & -lm \\ -2lm & 2lm & l^2 - m^2 \end{bmatrix} \tag{7}$$

and

$$\begin{aligned} l &= \cos\theta \\ m &= \sin\theta \end{aligned} \tag{8}$$

θ = Orientation of the fiber with respect to x-axis. The component of Q^N, stiffness matrix of the Nth layer, is:

$$\begin{aligned} Q_{11} &= \frac{E_!}{1-\nu_{12}\nu_{21}}, \quad Q_{12} = \frac{\nu_{12}E_2}{1-\nu_{12}\nu_{21}} = \frac{\nu_{21}E_1}{1-\nu_{12}\nu_{21}}, \\ Q_{22} &= \frac{E_2}{1-\nu_{12}\nu_{21}} \quad Q_{66} = G_{12}, \quad Q_{16} = Q_{26} = 0 \end{aligned} \tag{9}$$

where E_1 and E_2 are the Young's modulus in x* and y* direction, respectively. G_{12} and ν are shear modulus and Poisson ratio, respectively. Now bending moment

$$\begin{bmatrix} M_{x^*} \\ M_{y^*} \\ M_{x^*y^*} \end{bmatrix} = -D\mathbf{L}w^* \tag{10}$$

where D is the flexural rigidity matrix .It is given as

$$D = \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{21} & D_{22} & D_{26} \\ D_{61} & D_{62} & D_{66} \end{bmatrix} \tag{11}$$

$$D_{ij} = \frac{1}{3}\sum_{N=1}^{n} \left(c_{ij}^*\right)\left(Z_N^3 - Z_{N-1}^3\right) \tag{12}$$

The governing equation of the plate under vibration is written as [23]

$$\mathbf{L}^T D \mathbf{L} w + \rho h \ddot{w} = q \tag{13}$$

THE MULTIQUADRIC RADIAL BASIS FUNCTION METHOD

Consider a general differential equation

$$Aw = f(x, y) \quad \text{in} \quad \Omega \tag{14}$$

$$Bw = g(x, y) \quad \text{on} \quad \partial\Omega \tag{15}$$

In eigenvalue problem, equations (14) and (15) reduce to

$$Aw = \lambda^4 w \tag{16}$$
$$Bw = 0 \tag{17}$$

where A is a linear differential operator, B is a linear boundary operator imposed on boundary conditions and λ is eigen value. Let $\{P_i = (x_i, y_i)\}_{i=1}^{N}$ be N collocation points in domain Ω of which $\{(x_i, y_i)\}_{i=1}^{N_I}$ are interior points; $\{(x_i, y_i)\}_{i=N_I+1}^{N}$ are boundary points. In MQRBF method, the approximate solution for differential equation (14) and boundary conditions (15) can be expressed as:

$$w(x, y) = \sum_{j=1}^{N} w_j \varphi_j(x, y) \tag{18}$$

and multiquadric radial basis function as:

$$\varphi_{\mathrm{j}} = \sqrt{(x - x_j)^2 + (y - y_j)^2 + c^2} = \sqrt{r_j^2 + c^2} \tag{19}$$

where $\{w_j\}_{j=1}^{N}$ are the unknown coefficients to be determined, $\varphi_j(x_j, y_j)$ is a basis function. Here $r = \|P - P_j\|$ is the Euclidean norm between points $P = (x, y)$ and $P_j = (x_j, y_j)$. The Euclidian distance r is real and non-negative and c is a shape parameter, a positive constant. Shape parameter plays important role for getting the numerical solutions of composite laminates. Small shape parameter is considered for the interior nodes, but large value of shape parameter is considered on the boundary. In two-dimensional problems, shape parameter for interior points $c_{\mathrm{interior}} = 2/\sqrt{}$ (total number of points), and shape parameter for boundary points $c_{\mathrm{boundary}} = (100 \text{ to } 200) \times c_{\mathrm{interior}}$. Ling and Kansa [24] have discussed in detail about the shape parameter. Other widely used radial basis functions are:

$\varphi(r) = r^3$	Cubic
$\varphi(r) = r^2 \log(r)$	Thin plate splines
$\varphi(r) = (1 - r)_+^m p(r)$	Wendland functions
$\varphi(r) = e^{-(cr)^2}$	Gaussian
$\varphi(r) = (c^2 + r^2)^{-1/2}$	Inverse multiquadrics

CALCULATION OF EIGEN VALUES

Let N_B be the total points on the boundary $\partial\Omega$, N_I be the total points inside the domain Ω, and $N = N_I + N_B$. Applying the RBF in eqn (16),

$$\sum_{j=1}^{N} w_j A\varphi\|P_i - P_j\| = \lambda^4 \sum_{j=1}^{N} w_j \varphi\|P_i - P_j\| \tag{20}$$

Define,

$$\mathbf{L} = [A\phi(\|P_i - P_j\|)]_{N_I x N} \tag{21}$$
$$\mathbf{M} = [\phi\|P_i - P_j\|)]_{N_I x N} \tag{22}$$

Applying MQRBF in Eqn. (17),

$$\sum_{j=1}^{N} w_j B\phi(\|P_i - P_j\|) = 0 \tag{23}$$

where $i = N_{I+1}, N_{I+2} \ldots\ldots N$

$$\mathbf{K} = [B\phi(\|P_i - P_j\|)]_{N_B x N} \tag{24}$$
$$\mathbf{w} = [\mathrm{w}_1\ \mathrm{w}_2\ \mathrm{w}_3 \ldots \mathrm{w}_\mathrm{N}]^T \tag{25}$$

Equations (20) and (25) can be written as:

$$\mathbf{Lw} = \lambda^4 \mathbf{Mw} \tag{26}$$
$$\mathbf{Kw} = 0 \tag{27}$$

General eigenvalue problem in the matrix form becomes

$$\begin{bmatrix} \mathbf{L} \\ \mathbf{K} \end{bmatrix} \mathbf{w} = \lambda^4 \begin{bmatrix} \mathbf{M} \\ 0 \end{bmatrix} \mathbf{w} \tag{28}$$

The following algorithm [25] of the standard eigenvalue problem has been used in the present analysis.

$$\mathbf{L}^1 \mathbf{w^2} = \lambda^4 \mathbf{w^2} \tag{29}$$

where

$$\mathbf{L}^1 = \mathbf{LD}^{-1} \begin{bmatrix} \mathbf{I_{N_I x N_I}} \\ \mathbf{0_{N_B x N_I}} \end{bmatrix} \tag{30}$$
$$\mathbf{w^2} = [\mathrm{w}_1\ \mathrm{w}_2\ \mathrm{w}_3 \ldots \mathrm{w}_\mathrm{N}]^T \tag{31}$$
$$\mathbf{D} = \begin{bmatrix} \mathbf{M} \\ \mathbf{K} \end{bmatrix} \tag{32}$$

NUMERICAL RESULTS

Using multi-quadric radial basis function, static and dynamic response of cross ply laminates has been studied and natural frequencies are obtained. All layers of the laminates are assumed to have same thickness and density. Parameters of the composite materials are: $E_1/E_2 = 10, 20, 30$ and 40; $G_{12}/E_2 = 0.6$; $\nu_{12} = 0.25$. The subscripts 1 and 2 denote the normal x* and transverse directions y* of the fiber in the ply. ν_{12} is the Poisson's ratio and G_{12} is the shear modulus.

Case Study 1: Static and Dynamic Response of Laminates

Figure 1 shows the geometry, coordinate system and loading in cross-ply laminate. Neglecting the transverse shear and rotary inertia, equation of symmetric cross-ply laminate is expressed in non-dimensional form as [26]:

$$(w_{xxxx} + 2R^2\eta w_{xxyy} + R^4\psi w_{yyyy} + 4\mu w_{xxxy} + 4\chi w_{xyyy}) + w_{tt} + c_v w_t - Q(x, y, t) = 0 \tag{33}$$

where the subscript denotes the partial derivative with respect to the following suffix. The non-dimensional quantities are defined by

$$\begin{aligned} &w = w^*/h, x = x^*/a, y = y^*/b, R = a/b, \\ &t = t^*\sqrt{D_{11}/(\rho a^4 h)} \\ &Q = qa^4/(D_{11}h), C_v = (C_v^*/M)\sqrt{(\rho a^4 h)/D_{11}}, M = \rho abh \\ &\eta = (D_{12} + 2 * D_{66})/D_{11}, \psi = D_{22}/D_{11}, \\ &\mu = D_{16}/D_{11}, \chi = D_{26}/D_{11} \end{aligned}$$

Simply supported boundary conditions are

$$w = 0, \quad \frac{\partial^2 w}{\partial n^2} = 0 \tag{34}$$

Clamped boundary conditions are

$$w = 0, \quad \frac{\partial w}{\partial n} = 0 \tag{35}$$

n can be x or y, but it depends on the boundary conditions.

Free edge boundary conditions are

$$\left(Q_n + \frac{\partial M_{xy}}{\partial m}\right), M_n = 0 \tag{36}$$

Either $n = x, m = y$ or $n = y, m = x$.

Multiquadric radial basis function method is applied in the governing equations and boundary conditions. It generates more algebraic equations than the unknown coefficients. To solve this incompatibility, the multiple linear regression analysis (Appendix A) based on least-square error norms is employed. A computer program based on the finite difference method (FDM) proposed by Chadrashekara [27] is also developed. The results by the two methods are compared.

The results of the present method for simply supported cross-ply laminates (0°/90°/90°/0°) are shown in Table 1. Similar results of clamped immovable cross-ply laminates (0°/90°/90°/0°) are presented in Table 2. It is observed that for both simply supported and clamped boundary conditions the present results are in good agreement with those obtained by Ashton and Whitney [26]. The non-dimensional central deflection of cross-ply laminate (0°/90°/90°/0°) with parameters $E_1/E_2 = 10$, 20, 30 and 40, $G_{12}/E_2 = 0.6$ and $\nu_{12} = 0.25$ for simple supported and clamped edge boundary conditions are shown in Tables 3 and 4, respectively. The results obtained for various aspect ratios by using present method is in close agreement with those obtained by finite difference method (FDM). In both cases, the effect of E_1/E_2 in non-dimensional deflection is insignificant. However, there is significant change in α if aspect ratio changes from 1 to 2. In the case of simply supported boundary conditions, with the increase in aspect ratio it converges to 0.0139 whereas in the case of clamped boundary conditions it converges to 0.0044.

TABLE 1

The effect of aspect ratio (b/a) on $\alpha = \frac{D_{11} w^*_{max}}{qa^4}$ for simply supported cross-ply laminate ($E_1/E_2 = 14$, $G_{12}/E_2 = 0.5$, $\nu_{12} = 0.3$)

b/a	α Ashton and Whitney [26]	α Present method
1.0	0.0117	0.0118
2.0	0.0157	0.0156
3.0	0.0163	0.0164
4.0	0.0165	0.0165
5.0	0.0165	0.0165
6.0	0.0166	0.0167
7.0	0.0166	0.0167
8.0	0.0166	0.0167
9.0	0.0166	0.0167

A five-layer laminated square plate ($0^0/45^0/90^0/45^0/0^0$) is now considered. The material properties are equal for all layers: $E_1 = 30$ GPa, $E_2 = 5$ GPa, $G_{12} = 10$ GPa, $\mu_{12} = 0.25$. The dimensions of the plate are $a = b = 350$ mm, and the thickness of the layers are $h_1 = h_5 = 1.0$ mm, $h_2 = h_4 = 1.5$ mm, $h_3 = 2.0$ mm. Square plate is subjected to uniform lateral load q = 0.02 N/mm^2. Deflections at the center of the laminate for different boundary conditions are presented in Table 5. It can be observed that the maximum difference between the present results and [28] is less than 1%.

Dynamic analyses of cross-ply (0°/90°/90°/0°) laminate with parameters $E_1/E_2 = 10$, 20, 30 $G_{12}/E_2 = 0.6$, and $\nu_{12} = 0.25$ are performed by MQRBF. The damped response of present and finite difference methods have been compared and shown in Figure 2 for $C_v = 1.25$, $Q = 100$ and $a/b = 1$. There is good agreement in results. Damped dynamic response of cross-ply laminate with parameters $E_1/E_2 = 10$, $G_{12}/E_2 = 0.6$, $Q = 100$, $a/b = 1$ and $\nu_{12} = 0.25$ at various damping factor

TABLE 2

The effect of plate ratio (b/a) on $\alpha = \frac{D_{11} w^*_{max}}{qa^4}$ for clamped edge cross-ply laminate ($E1/E2 = 14$, $G12/E2 = 0.5$, $\nu\ 12 = 0.3$)

b/a	α Ashton and Whitney [26]	α Present Method
1.0	0.00269	0.0027
2.0	0.00320	0.0033
3.0	0.00339	0.0034
4.0	0.00340	0.0034
5.0	0.00341	0.0034
6.0	0.00341	0.0034
7.0	0.00342	0.0034
8.0	0.00342	0.0034
9.0	0.00342	0.0034

TABLE 3
Central deflection $\alpha = \frac{D_{22}\cdot w^*_{\max}}{qb^4}$ of simply supported cross-ply laminate

	$E_1/E_2 = 10$		$E_1/E_2 = 20$		$E_1/E_2 = 30$		$E_1/E_2 = 40$	
a/b	FDM [27] α	Present α	FDM [27] α	Present α	FDM [27] α	Present α	FDM [27] α	Present α
1	0.0024	0.0025	0.0023	0.0022	0.0022	0.0021	0.0021	0.0020
2	0.0100	0.0106	0.0104	0.0106	0.0107	0.0106	0.0108	0.0106
3	0.0129	0.0132	0.0134	0.0134	0.0137	0.0134	0.0139	0.0135
4	0.0136	0.0138	0.0139	0.0139	0.0141	0.0140	0.0142	0.0140
5	0.0137	0.0139	0.0139	0.0139	0.0140	0.0139	0.0141	0.0140
6	0.0137	0.0139	0.0138	0.0139	0.0139	0.0139	0.0139	0.0139
7	0.0137	0.0139	0.0138	0.0139	0.0138	0.0139	0.0138	0.0138
8	0.0137	0.0139	0.0137	0.0139	0.0138	0.0139	0.0138	0.0138
∞	0.0137	0.0139	0.0137	0.0139	0.0137	0.0139	0.0137	0.0138

TABLE 4
Central deflection $\alpha = \frac{D_{22}\cdot w^*_{\max}}{qb^4}$ of clamped laminate

	$E_1/E_2 = 10$		$E_1/E_2 = 20$		$E_1/E_2 = 30$		$E_1/E_2 = 40$	
a/b	FDM [27] $\alpha \times 10^{-4}$	Present $\alpha \times 10^{-4}$	FDM [27] $\alpha \times 10^{-4}$	Present $\alpha \times 10^{-4}$	FDM [27] $\alpha \times 10^{-4}$	Present $\alpha \times 10^{-4}$	FDM [27] $\alpha \times 10^{-4}$	Present $\alpha \times 10^{-4}$
1	8.2145	8.047	7.029	7.017	6.5875	6.627	6.3565	6.422
2	34	33	33	33	33	33	33	33
3	39	41	40	41	40	42	40	42
4	39	43	40	44	40	42	40	44
5	39	44	39	44	39	44	40	45
6	39	44	39	44	39	45	39	45
∞	39	44	39	44	39	45	39	45

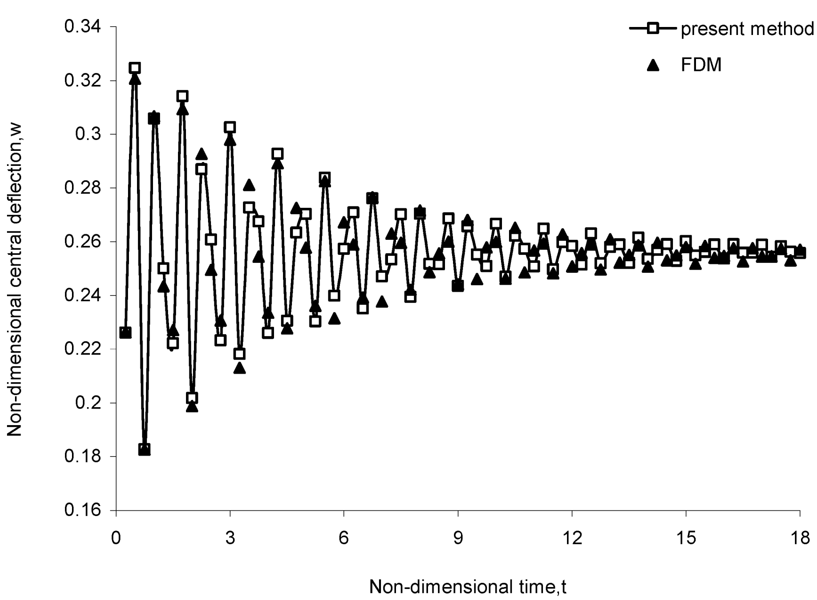

FIG. 2. Damped response of a simple supported cross-ply laminate.

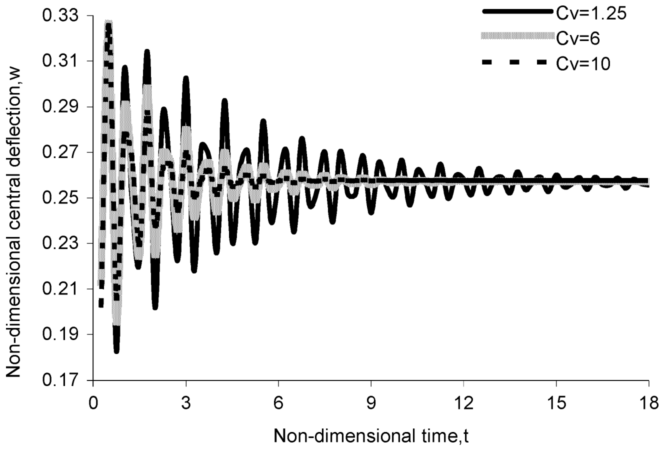

FIG. 3. Damped dynamic response of a simple supported cross-ply laminate at aspect ratio one.

for simply supported boundary conditions has been shown in Figure 3. It is observed that the maximum non-dimensional deflection is 0.3262. Static response can be noticed after t > 11.5 for Cv = 10, $t > 12.75$ for Cv = 6 and it takes greater time for Cv = 1.25. Similarly it can be observed in the case of clamped cross-ply laminate with parameters $E_1/E_2 = 30, G_{12}/E_2 = 0.6$, $Q = 100$, $a/b = 1$ and $\nu_{12} = 0.25$ in Figure 4 that maximum non-dimensional deflection is 0.0556. When damping factor increases maximum non-dimensional deflection decreases.

Damped dynamic response of a simple supported cross-ply laminate at various parameters of the composite materials $E_1/E_2 = 10$, 20, 30, $G_{12}/E_2 = 0.6$, Q = 100, Cv = 1.25, aspect ratio 1 and 1.5 is shown in Figures 5 and 6. At aspect ratio equal to unity, maximum non-dimensional deflections observed at $t = 0.5$ are 0.3247, 0.2979, 0.2823 respectively. The non-dimensional deflection decreases with increase of E_1/E_2 value and non-dimensional deflection increases with increase in aspect ratio. Similarly, the variation of non-dimensional central deflection with non-dimensional time t in case of clamped cross-ply laminate for the composite materials $E_1/E_2 = 10$, 20, 30, $G_{12}/E_2 = 0.6$, $Q = 100$, Cv = 1.25 and at aspect ratio 1 is shown in Figure 7. At aspect ratio 1 maximum non-dimensional deflection at t = 1 is 0.0688.

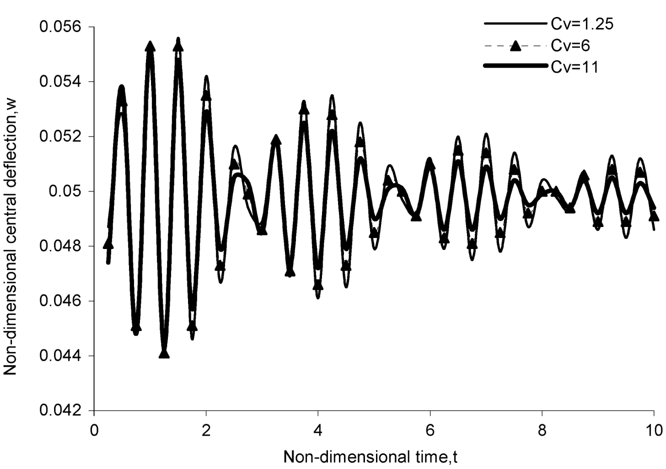

FIG. 4. Damped dynamic response of a clamped cross-ply laminate at aspect ratio one.

TABLE 5
Displacements on the center of the plate (unit: mm)

Boundary conditions	Rayleigh-Ritz [28]	SEM [28]	Present method
S-S-S-S	2.2905	2.3049	2.2985
C-C-C-C	0.7181	0.7200	0.7186
S-S-C-C	1.1960	1.2029	1.1978
C-S-C-S	0.8949	0.8986	0.8960

Case Study 2: Natural Frequency of Cross-Ply Laminates

The governing equation of free vibration of a thin rectangular plate in non-dimensional form is expressed as:

$$\frac{1}{a^4}[(D_{11}/D_{22})w_{xxxx} + 2R^2(H/D_{22})w_{xxyy} + R^4 w_{yyyy} + 4R(D_{16}/D_{22})w_{xxxy} + 4R^3(D_{26}/D_{22})w_{xyyy}] = \lambda^4 w(x, y) \quad (37)$$

where $\lambda^4 = \omega^2 \rho_0 h/D_{22}$ and $H = (D_{12} + 2 * D_{66})$

The natural frequency of simply supported symmetric cross-ply laminate is obtained by Ashton and Whitney [26] as:

$$\omega_{mn} = \frac{\pi^2}{R^2 b^2}\frac{1}{\sqrt{\rho_0}} \times \sqrt{D_{11}m^4 + 2(D_{12} + 2D_{66})m^2 n^2 R^2 + D_{22}n^4 R^4} \quad (38)$$

TABLE 6
The normalized fundamental frequency $\omega = k\pi^2/b^2\sqrt{D_{22}/\rho_0}$ of a simple supported cross-ply laminated

a/b	Mode	m	n	k Ashton and Whitney [26]	k (Present)
1	1st	1	1	2.5562	2.5598
	2nd	1	2	5.0603	5.0988
	3rd	1	3	9.8689	9.8705
	4th	2	1	8.5589	8.5611
2	1st	1	1	1.2651	1.2671
	2nd	1	2	4.1975	4.1940
	3rd	1	3	9.1824	9.1850
	4th	2	1	2.5562	2.5647
3	1st	1	1	1.0965	1.0969
	2nd	1	2	4.0810	4.0981
	3rd	1	3	9.0779	9.0816
	4th	2	1	1.5583	1.5569

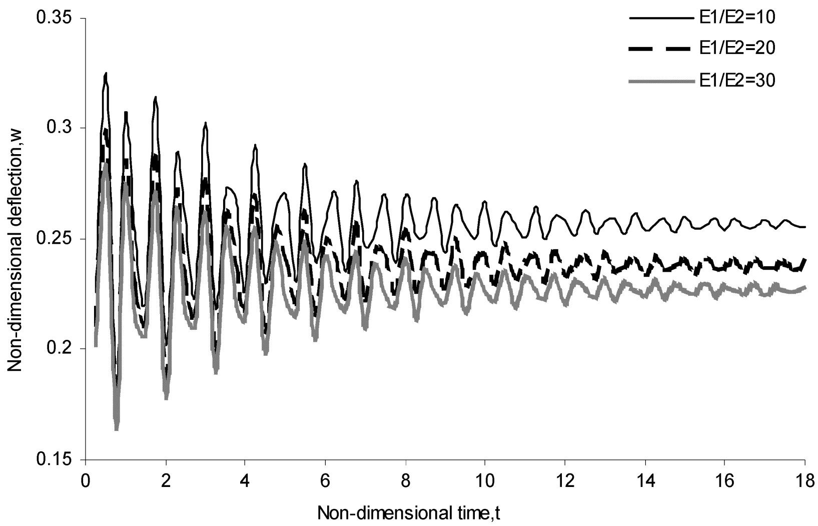

FIG. 5. Damped response of a simple supported cross-ply laminate at aspect ratio 1.

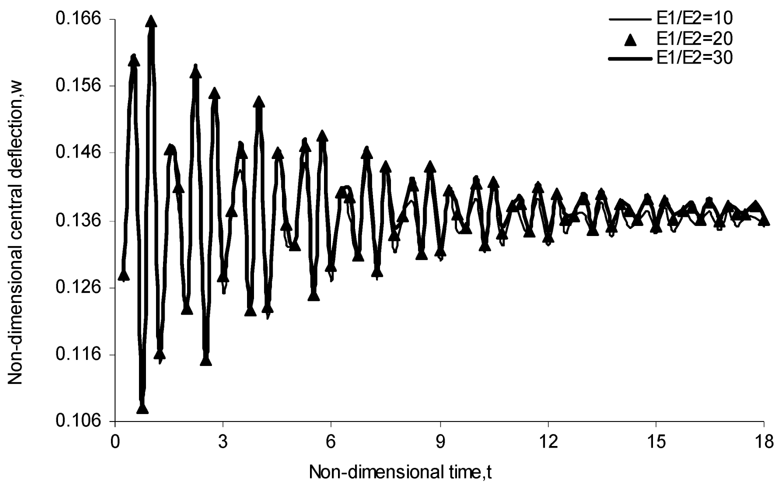

FIG. 6. Damped response of a simple supported cross-ply laminate at aspect ratio 1.5.

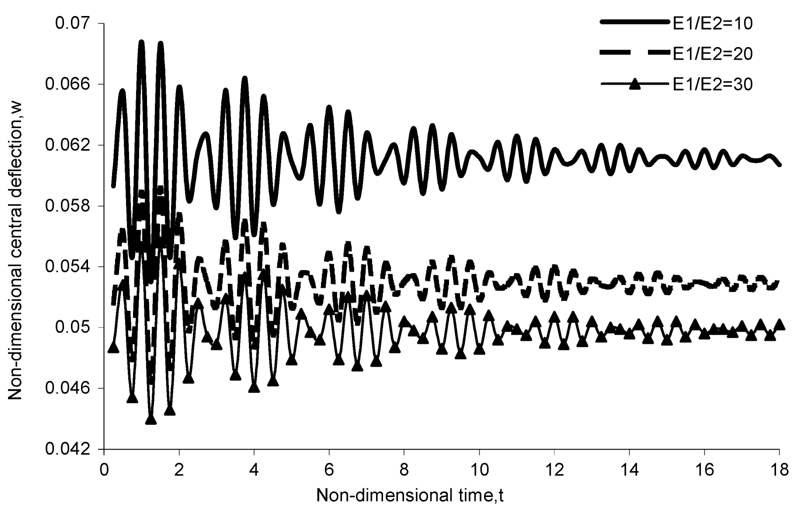

FIG. 7. Damped response of a clamped edge cross-ply laminate at aspect ratio 1.

TABLE 7
Non-dimensional natural frequencies, $\Omega = \omega a^2 \frac{\sqrt{\rho_0}}{D_{11}}$ obtained using simple supported square antisymmetric cross-ply laminated composite plates ($E_{11} = 130$ GPa, $E_{22} = 9$ GPa, $G_{12} = 4.8$ GPa, $\nu_{12} = 0.28$)

	Gin Boay Chai [29]				Present Method			
NL[a]	Mode 1	Mode 2	Mode 3	Mode 4	Mode 1	Mode 2	Mode 3	Mode 4
2	7.7985	21.1006	21.1006	31.1942	7.8001	21.1020	21.1023	31.2045
4	10.3384	28.8017	28.8017	41.3536	10.3403	28.8032	28.8036	41.3602
6	10.7431	30.0118	30.0118	42.9722	10.7488	30.0211	30.0211	42.9831
8	10.8811	30.4239	30.4239	43.5245	10.8834	30.4332	30.4332	43.5288
10	10.9445	30.6128	30.6128	43.7778	10.9455	30.6317	30.6320	43.7691
12	10.9787	30.7149	30.7149	43.9148	10.9881	30.7221	30.7221	43.9343
14	10.9993	30.7763	30.7763	43.9971	11.0115	30.7801	30.7804	44.0145
16	11.0126	30.8161	30.8161	44.0505	11.0165	30.8203	30.8203	44.0518
18	11.0218	30.8434	30.8434	44.0871	11.0278	30.8462	30.8465	44.0791
20	11.0283	30.8628	30.8628	44.1132	11.0288	30.8820	30.8821	44.1347

NL[a] (The number of layers in the laminated composite plate).

TABLE 8
Non-dimensional natural frequencies, $\Omega = \omega a^2 \frac{\sqrt{\rho_0}}{D_{11}}$ obtained using simple supported symmetric cross-ply laminated composite plates ($E_{11} = 130$ GPa, $E_{22} = 9$ GPa, $G_{12} = 4.8$ GPa, $\nu_{12} = 0.28$)

	Gin Boay Chai [29]				Present Method			
NL[a]	Mode 1	Mode 2	Mode 3	Mode 4	Mode 1	Mode 2	Mode 3	Mode 4
2	—	—	—	—	—	—	—	—
4	11.0561	21.1576	38.3097	41.3892	11.0566	21.1578	38.3102	41.3794
6	10.7431	30.0118	30.0118	42.9722	10.7543	30.0214	30.0314	42.9743
8	11.0561	26.5074	34.8229	44.2245	11.0566	26.5077	34.8190	44.2278
10	10.9445	30.6128	30.6128	43.7778	10.9541	30.6322	30.6127	43.7961
12	11.0561	28.0649	33.5803	44.2245	11.0569	28.0651	33.5905	44.2279
14	10.9993	30.7763	30.7763	43.9971	11.0214	30.7818	30.7764	43.8951
16	11.0561	28.8122	32.9414	44.2245	11.0569	28.8301	32.9381	44.2279
18	11.0218	30.8434	30.8434	44.0871	11.0219	30.8441	30.8452	44.0862
20	11.0561	29.2513	32.5521	44.2245	11.0569	29.2514	32.5601	44.2277

NL[a] (the number of layers in the laminated composite plate).

TABLE 9
Fundamental frequencies $\omega a^2 \sqrt{\frac{\rho_0}{E_{22} h^3}}$ of two-ply clamped edge square anti-symmetric cross-ply laminated plates ($\frac{E_{11}}{E_{22}} = 40$, $\frac{G_{12}}{E_{22}} = 0.5$, $\nu_{12} = 0.25$)

					Present Grid			
Aspect ratio	Asymptotic [32]	Fourier [34]	Polynom [33]	Gin Boay Chai [29]	7 × 7	9 × 9	11 × 11	13 × 13
1	23.638	24.527	24.033	24.037	24.169	24.088	24.065	24.042
2	17.111	21.171	17.297	17.299	17.509	17.403	17.358	17.326
3	16.640	21.171	16.782	16.784	16.986	16.872	16.823	16.798
4	16.543	17.200	16.673	16.675	16.871	16.736	16.698	16.680
5	16.509	16.974	16.636	16.638	16.857	16.714	16.680	16.658

TABLE 10
The fundamental frequency of the 3-layer [$0^0/90^0/0^0$] laminated square plate with various boundary conditions and span to thickness ratios ($\bar{\omega} = (\omega a^2/h)\sqrt{\rho/E_2}$, $E_1/E_2 = 40$)

a/h	Boundary Conditions	Exact [23]	Present Grid 7 × 7	9 × 9	11 × 11	13 × 13
100	SS	18.891	18.987	18.939	18.915	18.897
	SC	28.501	28.602	28.562	28.526	28.509
	CC	40.743	40.832	40.781	40.764	40.750
	FF	4.457	4.554	4.509	4.483	4.461
	FS	5.076	5.162	5.121	5.097	5.081
	FC	8.269	8.356	8.312	8.289	8.274

where m and n are integers. Different natural frequencies are obtained at different combinations of m and n. The normalized natural frequencies of cross-ply laminates (0°/90°/90°/0°) with parameters $E_1/E_2 = 10$, $G_{12}/E_2 = 0.6$, and $\nu_{12} = 0.25$ are obtained by using present method and compared with those obtained by Ashton and Whitney [26]. It can be observed in Table 6 that there is a very close agreement between the two results of simply supported boundary conditions. The non-dimensional natural frequencies for antisymmetric and symmetric cross-ply laminate at simple supported boundary conditions are compared with those obtained by Chai [29] (similar to Reddy [23], Jones [30] and Whitney [31]) and shown in Tables 7 and 8. Table 9 shows the fundamental frequencies of two-ply clamped edge square anti-symmetric cross-ply laminated plates and compared with Chai [29] and [32–34] ($\frac{E_{11}}{E_{22}} = 40, \frac{G_{12}}{E_{22}} = 0.5, \nu_{12} = 0.25$). It is found that the results are in very close agreement with other established results. Table 10 shows the dimensionless fundamental frequencies and influence of the mixed boundary conditions. In this case, plate is simple supported along the edges parallel to the x-axis and remaining two edges are simple supported (S) or clamped edge (C) or free (F). Here SS, SC, CC, FF, FS, and FC refer to the boundary conditions of the two edges parallel to the y-axis only. These results are very important from a design point of view. It is observed that non-dimensional frequency is minimum at FF boundary conditions and maximum at CC boundary conditions. Comparisons of the present results with exact results [23] have also been shown.

CONCLUSIONS

In this paper, MQRBF is successfully applied to analyze static, dynamic and free vibration response of laminates for various boundary conditions. The discretization procedure of the cross-ply laminates using MQRBF method possesses an advantage because it does not require generation of mesh. In this collocation method, more equations are generated compared to the unknowns. Multiple regression analysis is used to overcome this incompatibility. The present results are compared with those obtained by the finite difference method as well as the results available in literature. This method is found very effective in the study of static and dynamic response, and estimation of natural frequencies of laminates.

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APPENDIX

A Multiple regression analysis

$$A.a = p$$

where A is $(l * k)$coefficient matrix, a is (k $*$ 1) vector, p is $(l * 1)$ load vector. Approximating the solution by introducing the error vector e,

$$p = Aa + e$$

where e is $(l * 1)$ vector. To minimize the error norm, define a function S as

$$S(a) = e^T e = (p - Aa)^T (p - Aa)$$

The least-square norm must satisfy

$$(\partial S/\partial a)_a = -2A^T p + 2A^T Aa = 0$$

This can be expressed as

$$a = (A^T A)^{-1} A^T P$$

or

$$a = B \cdot P$$

The matrix B is evaluated once and stored for subsequent usages.