International Journal of Fracture (2005) 133:389–405
DOI 10.1007/s10704-005-5993-1

Numerical analysis of doubly periodic array of cracks/rigid-line inclusions in an infinite isotropic medium using the boundary integral equation method

C.Y. DONG[1,*] and KANG YONG LEE[2]
[1]*Department of Applied Mechanics, Beijing Institute of Technology, Beijing 100081, China*
[2]*School of Mechanical Engineering, Yonsei University, Seoul, 120-749, Korea*
[*]*Author for correspondence (E-mail: chunyingdong@yahoo.com)*

Received 22 December 2004; accepted in revised form 18 April 2005

Abstract. In this paper, the boundary integral equation approaches are used to study the doubly periodic array of cracks/rigid-line inclusions in an infinite isotropic plane medium. For the doubly periodic rigid-line inclusion problems, the special integral equation containing the axial and shear forces within the rigid-line inclusion is used. The doubly periodic crack problems are dealt with using the displacement discontinuous integral equation approach. Stress intensity factors, effective elastic properties for doubly periodic array of cracks/rigid-line inclusions are calculated and compared with the available numerical solutions.

Key words: Boundary integral equation approach, cracks, doubly periodic problem, effective elastic properties, rigid-line inclusions.

1. Introduction

As shown by Wang et al. (2000), "the significance of studying the elastic behavior and fracture characteristics of a media containing multiple cracks or microvoids hardly needs stressing. In fiber-reinforced brittle matrices, such as ceramics and cement, the fiber bridging over multiple cracks is an important mechanism for increasing the toughness of the composites". Multiple crack problems have been widely studied and a lot of references are available, e.g. Li et al. (2003), Wang and Feng (2001), Helsing (1999) and Nishimura et al. (1999). For actual media containing multiple cracks, it is very difficult to obtain the corresponding solutions, even using numerical methods (e.g. finite element methods and boundary element methods). Therefore, the simplified model, e.g. the regular distribution of multiple cracks, have to be assumed. Thus, the solution of the problems is easily obtained using analytical/numerical methods. The monograph by Chen et al. (2003) contains some singular integral equation methods which have been used to solve multiple crack problems. Doubly period array of cracks in an infinite isotropic plane medium can be considered as a special one of general multiple crack problems (Chen and Lee, 2002). The weight functions for an infinite row of periodic cracks in an infinite isotropic medium (Karihaloo and Wang, 1997) were used to calculate the stress intensity factor at the tip of a crack. The adopted kernel decays exponentially for the field points over the remote cracks. Therefore, an infinite series in the obtained equations (see their Equation (1)) can be replaced by a finite number of terms based on the desired accuracy. Similar method has been adopted to obtain the effective Young's modulus of doubly periodic array of cracks in an infinite isotropic medium (Wang et al., 2000). Unfortunately, other effective elastic properties are not available in their paper. A simply and robust method has been proposed to investigate doubly periodic array of cracks in an infinite plate (Chen and Lee, 2002). Based on the property of doubly periodic array of crack medium subjected to remote in-plane tension and shear loadings, the corresponding cracked cells with the known boundary conditions are cut from an infinite medium. Following this, the eigenfunction expansion variational method (Chen, 1983) was used to solve the cracked cells. Stress intensity factors at the crack tip and effective elastic properties of the medium were obtained.

Compared to multiple crack problems, a relatively few publications for multiple rigid line inclusions can be found in the literature. As well known, stress analysis of composites must consider the interaction of the fibers embedded in the matrix. Since the fibers in composite materials might be straight or curved, short or long, aligned or oriented arbitrarily, and distributed uniformly or randomly (Liu et al., 2005), all these properties make the simulation of fiber-reinforced composites become very difficult. Therefore, the simplified modeling has been assumed to investigate the interaction of multiple rigid line inclusions. This assumption is reasonable when the fiber is very hard compared to its surrounding matrix. Recently, Nishimula and Liu (2004) used the boundary integral equation approach and a rigid-line inclusion model to carry out thermal analysis of carbon-nanotube composites. Chen et al. (2003) adopted the complex potentials (Muskhelishvili, 1953) to derive some singular integral equations for rigid-line inclusions. The boundary integral equation formulation for the interaction of cracks and rigid-line inclusions was proposed by Dong et al. (2003). Some numerical results were compared with those available and showed that the presented integral equation can produce the convincible results.

In this paper, the boundary integral equation approaches are used to study the doubly periodic array of cracks/rigid-line inclusions in an infinite isotropic plane medium under remote in-plane loadings. Following the work by Chen and Lee (2002), a rectangular cell is cut from the infinite plane medium. Thus, the boundary conditions of the cell are available for two loading systems, i.e. tension and shear. Therefore, the solution of the simplified problems can be easily obtained using the corresponding boundary integral equation approaches for crack/rigid-line inclusion problems. Based on the results from the boundary integral equation approaches, the elastic properties of the equivalent orthotropic medium can be obtained. We need to stress that the novelty in the paper is that the coupling of existing methods, i.e. the boundary integral equation methods and the simple computational model proposed by Chen and Lee (2002), is used to solve the doubly periodic crack/rigid-line inclusion problems. Especially for doubly periodic rigid-line inclusion problems, to the best of authors' knowledge, there is no result available in the literature. The authors think that the obtained results can be considered as the benchmark results for future research.

2. Computational models

Figure 1a shows doubly periodic array of cracks/rigid-line inclusions in an infinite isotropic plane medium subjected to the remote in-plane tension and shear

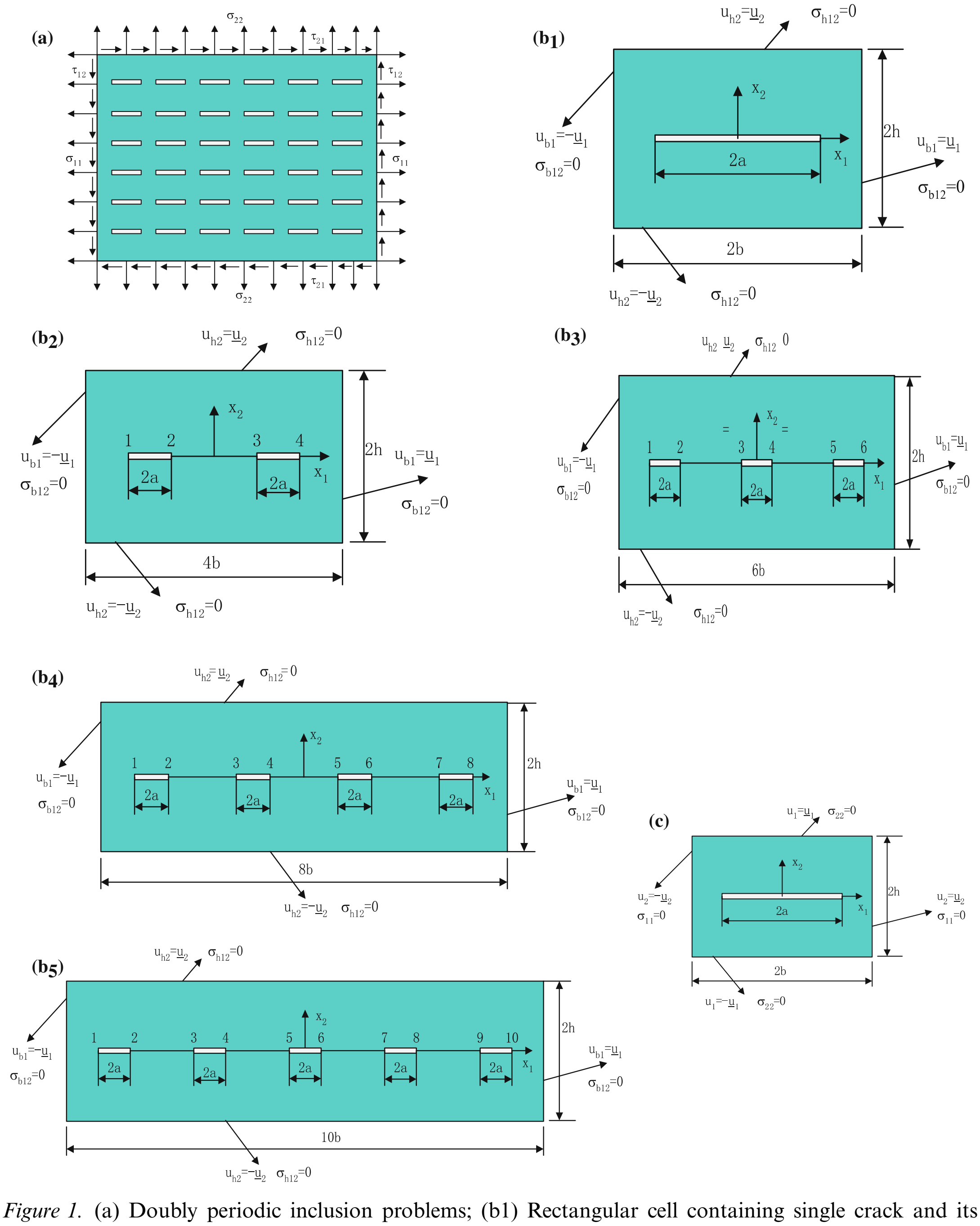

Figure 1. (a) Doubly periodic inclusion problems; (b1) Rectangular cell containing single crack and its boundary condition for tension loading; (b2) Rectangular cell containing two cracks and its boundary condition for tension loading; (b3) Rectangular cell containing three cracks and its boundary condition for tension loading; (b4) Rectangular cell containing four cracks and its boundary condition for tension loading; (b5) Rectangular cell containing five cracks and its boundary condition for tension loading; (c) Rectangular cell containing single crack and its boundary condition for shear loading.

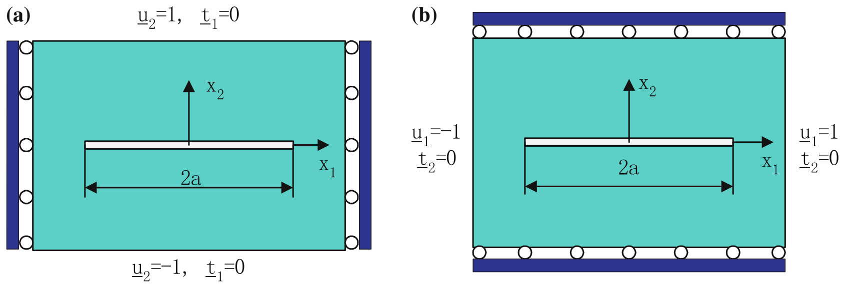

Figure 2. Sub-problems from the boundary value problem as shown in Figure 1b1. The boundary condition of sub-problems b2–b5 is the same as sub-problem (b1).

loadings. Two rectangular cells corresponding to the remote in-plane tension and shear loadings are, respectively, shown in Figure 1b1 and c in which $2b$ and $2h$ are respectively the edge lengths of the rectangular cells along x_1 and x_2 directions. The parameter $2a$ is the crack/rigid-line inclusion length. For remote tension and shear loadings, the boundary conditions along the cell boundaries are also shown in Figure 1b1 and c, respectively.

Following Chen and Lee (2002), the model as shown in Figure 1b1 can be decomposed into two sub-models as shown in Figure 2(a) and b which can be solved using the boundary integral equation method presented in the next section. The equilibrium relationship of the cell of doubly periodic medium subjected to a remote loading $\sigma_{22} = \sigma$ along x_2 direction can be written as

$$\begin{cases} u_{h2} \int_{-b}^{b} t_{2(2(a))}(x_1, h)\,\mathrm{d}x_1 + u_{b1} \int_{-b}^{b} t_{2(2(b))}(x_1, h)\,\mathrm{d}x_1 = 2b\sigma, \\ u_{h2} \int_{-h}^{h} t_{1(2(a))}(b, x_2)\,\mathrm{d}x_2 + u_{b1} \int_{-h}^{h} t_{1(2(b))}(b, x_2)\,\mathrm{d}x_2 = 0, \end{cases} \tag{1}$$

where u_{b1} and u_{h2} are the displacements on the boundary $x_1 = b$ along x_1 direction and on the boundary $x_2 = h$ along x_2 direction, respectively. $t_{1(2(a))}$ and $t_{1(2(b))}$ are the traction components on the boundary $x_1 = b$ along x_1 direction produced by sub-problems 2(a) and (b), respectively. $t_{2(2(a))}$ and $t_{2(2(b))}$ are the traction components on the boundary $x_2 = h$ along x_2 direction produced by sub-problems 2(a) and (b), respectively. The integrals in Equation (1) and the following Equations (2) and (3) can easily be calculated using the trapezoidal rule of numerical integration (Scheid, 1988).

From Equation (1), one can obtain u_{b1} and u_{h2}. Thus the effective elastic properties E_2 and ν_{21} of doubly periodic array of cracks/rigid-line inclusions in an infinite isotropic plane medium can be calculated using Equation (14) presented in Section 4.

Similarly, the equilibrium relationship of the cell of doubly periodic medium subjected to a remote loading $\sigma_{11} = \sigma$ along x_1 direction is as follows

$$\begin{cases} u_{h2} \int_{-b}^{b} t_{2(2(a))}(x_1, h)\,\mathrm{d}x_1 + u_{b1} \int_{-b}^{b} t_{2(2(b))}(x_1, h)\,\mathrm{d}x_1 = 0, \\ u_{h2} \int_{-h}^{h} t_{1(2(a))}(b, x_2)\,\mathrm{d}x_2 + u_{b1} \int_{-h}^{h} t_{1(2(b))}(b, x_2)\,\mathrm{d}x_2 = 2h\sigma. \end{cases} \tag{2}$$

Once u_{b1} and u_{h2} are solved by Equation (2), the effective elastic modulus E_1 and ν_{12} can be achieved using Equation (16) presented in Section 4.

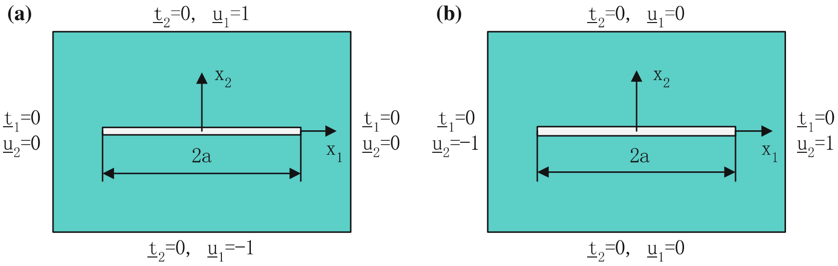

Figure 3. Sub-problems from the boundary value problem as shown in Figure 1c.

For the model as shown in Figure 1c which can be decomposed into two sub-problems as shown in Figure 3a and b, the equilibrium relationship of the cell of doubly periodic medium subjected to a remote shear loading $\sigma_{12}=\sigma$ is as follows

$$\begin{cases} u_{h1}\int_{-b}^{b} t_{1(3(a))}(x_1,h)\,\mathrm{d}x_1 + u_{b2}\int_{-b}^{b} t_{1(3(b))}(x_1,h)\,\mathrm{d}x_1 = 2b\sigma, \\ u_{h1}\int_{-h}^{h} t_{2(3(a))}(b,x_2)\,\mathrm{d}x_2 + u_{b2}\int_{-h}^{h} t_{2(3(b))}(b,x_2)\,\mathrm{d}x_2 = 2h\sigma, \end{cases} \tag{3}$$

where u_{b2} and u_{h1} are the displacements on the boundary $x_1=b$ along x_2 and on the boundary $x_2=h$ along x_1 directions, respectively. $t_{1(3(a))}$ and $t_{1(3(b))}$ are the traction components on the boundary $x_2=h$ along x_1 direction produced by sub-problems 3(a) and (b), $t_{2(3(a))}$ and $t_{2(3(b))}$ are the traction components on the boundary $x_1=b$ along x_2 direction produced by sub-problems 3(a) and (b), respectively.

Once u_{b2} and u_{h1} are available, the effective shear modulus of doubly periodic inclusion problems can be obtained using Equation (18) presented in Section 4.

Similar to the rectangular cell with single crack/rigid-line inclusion, the rectangular cell with two to five cracks/rigid-line inclusions (see Figure 1b2 – b5 can be also calculated).

3. Basic formulation

In order to study the doubly periodic array of cracks/rigid-line inclusions in an infinite isotropic plane medium, some related boundary integral equations are shown below for easy of reference. More details can be found in the mentioned references.

Crack problems have been widely investigated using the boundary integral equation approach. A lot of references are available, e.g. Pan (1997), Portela et al. (1992), Cruse (1988), Hong and Chen (1988). The interaction between cracks and rigid-line inclusions has been studied by Dong et al. (2003) using an integral equation method.

For the point p being on the outside boundary of the cell, the following formulations are adopted for crack problems:

$$c_{ij}(p)\,u_j(p) = \int_{\Gamma} U_{ij}(p,q)\,t_j(q)\,\mathrm{d}\Gamma - \int_{\Gamma} T_{ij}(p,q)\,u_j(q)\,\mathrm{d}\Gamma + \int_{\Gamma_c} T_{ij}(p,q)\,D_j(q)\,\mathrm{d}\Gamma \tag{4}$$

and for rigid-line inclusion problems:

$$c_{ij}(p)\,u_j(p) = \int_{\Gamma} U_{ij}(p,q)\,t_j(q)\,\mathrm{d}\Gamma - \int_{\Gamma} T_{ij}(p,q)\,u_j(q)\,\mathrm{d}\Gamma + \int_{\Gamma_r} \overline{U}_{ij}(p,q)\,Q_j(q)\mathrm{d}\Gamma, \tag{5}$$

where $c_{ij}(p)$ can be indirectly determined using rigid body displacements. Γ, Γ_c and Γ_r are respectively the outside boundary of the cell, crack surface and rigid-line inclusion. u_j and t_j are the displacement and traction components over the cell outside boundary. $D_j(q) = u_j(q^-) - u_j(q^+)$ is the discontinuous displacement component between the point q^- (being on the lower surface) and the corresponding point q^+ (being on the upper surface) over the crack, and $q = q^-$ for brevity. Q_j is the j-th component of the internal force along the rigid-line inclusion. U_{ij}, T_{ij} and $\overline{U}_{ij}$ are the fundamental solutions of 2-D elastic problems (Brebbia and Dominguez, 1992; Dong et al., 2003) which are shown in appendix.

For the point p being on the crack surface, the stress integral equation can be written as

$$\sigma_{ij}(p) = \int_{\Gamma} U_{ijk}(p,q)\, t_k(q)\, \mathrm{d}\Gamma - \int_{\Gamma} T_{ijk}(p,q)\, u_k(q)\, \mathrm{d}\Gamma + \int_{\Gamma_c} T_{ijk}(p,q)\, D_k(q)\, \mathrm{d}\Gamma \quad (6)$$

and for the point p being on the rigid-line inclusion, the corresponding integral equation is given as (Dong et al., 2003)

$$u_{i,j}(p) = \int_{\Gamma} U_{ik,j}(p,q)\, t_k(q)\, \mathrm{d}\Gamma - \int_{\Gamma} T_{ik,j}(p,q)\, u_k(q)\, \mathrm{d}\Gamma + \int_{\Gamma_r} \overline{U}_{ik,j}(p,q)\, Q_k(q)\, \mathrm{d}\Gamma, \quad (7)$$

where U_{ijk}, T_{ijk} and $\overline{U}_{ik,j}$ are available in appendix, the derivation ",j" in $u_{i,j}$, $U_{ik,j}$, $T_{ik,j}$ and $\overline{U}_{ik,j}$ is with respect to the source point p. The hypersingular integral over the crack surface/rigid-line-inclusion can be calculated using the well-known methods, e.g. Aliabadi (2002), Hui and Shia (1999) and Kutt (1975).

After discretization of Equations (4) and (6) or (5) and (7), one can obtain the system of equations as follows

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \begin{Bmatrix} \mathbf{X} \\ \mathbf{Y} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \end{Bmatrix}, \quad (8)$$

where $\mathbf{A}_{ij}$ $(i, j = 1, 2)$ are the associated coefficient matrices, $\mathbf{X}$ is the vector of the unknowns of the outside boundary of the cell, $\mathbf{Y}$ is the vector of the discontinuous displacements ($\mathbf{Y} = \mathbf{D}$)/the internal force along the rigid-line inclusion ($\mathbf{Y} = \mathbf{Q}$), $\mathbf{f}_1$ and $\mathbf{f}_2$ are the associated known values.

Once $\mathbf{Y}$ is obtained, stress intensity factors at the tips of cracks/rigid-line-inclusions can be calculated using the following formulations (Dong and de Pater, 2002; Dong et al., 2003)
for crack problems:

$$K_{\mathrm{I}} = \frac{G D_{\mathrm{n}}(r)}{4(1-\nu)} \sqrt{\frac{2\pi}{r}}, \quad (9a)$$

$$K_{\mathrm{II}} = \frac{G D_{\mathrm{s}}(r)}{4(1-\nu)} \sqrt{\frac{2\pi}{r}}, \quad (9b)$$

and for rigid-line-inclusion problems:

$$K_{\mathrm{I}} = \frac{(3-4\nu)\, Q_{\mathrm{a}}(r)}{4(1-\nu)} \sqrt{\frac{\pi}{2r}}, \quad (10a)$$

$$K_{\mathrm{II}} = \frac{(3-4\nu)\,Q_{\mathrm{s}}(r)}{4(1-\nu)}\sqrt{\frac{\pi}{2r}}, \tag{10b}$$

where D_{n} and D_{s} are the normal and tangential components of the crack opening displacements, Q_{a} and Q_{s} are the axial and shear forces of a field point at distance r to the end of the rigid-line inclusion.

Having **X** from Equation (8) and based on the methods from Sections 2 and 4, the effective elastic properties of doubly periodic array of crack/rigid-line-inclusion problems can be obtained.

4. Effective elastic properties of the medium

Following Chen and Lee (2002) for doubly periodic crack problem, doubly periodic crack/rigid-line-inclusion medium can be considered as an equivalent homogeneous orthotropic problem, the constitutive relation of which is as follows (Lekhnitsky, 1963)

$$\varepsilon_{11} = \frac{1}{E_1}\sigma_{11} - \frac{\nu_{21}}{E_2}\sigma_{22}, \tag{11a}$$

$$\varepsilon_{22} = \frac{1}{E_2}\sigma_{22} - \frac{\nu_{12}}{E_1}\sigma_{11}, \tag{11b}$$

$$\gamma_{12} = \frac{1}{G_{12}}\sigma_{12}. \tag{11c}$$

In order to obtain the effective elastic properties E_2 and γ_{21}, the remote loading can be assumed as

$$\sigma_{11}^{0} = 0, \quad \sigma_{22}^{0} = \sigma, \quad \sigma_{12}^{0} = 0. \tag{12}$$

For the equivalent homogeneous orthotropic medium, its stress and strain states are as follows

$$\bar{\sigma}_{11} = 0, \quad \bar{\sigma}_{22} = \sigma, \quad \bar{\sigma}_{12} = 0, \quad \bar{\varepsilon}_{11} = \frac{u_{b1}}{b} \quad \text{and} \quad \bar{\varepsilon}_{22} = \frac{u_{h2}}{h}, \tag{13}$$

where the bar over the above given symbols denotes the average value in a rectangular cell. u_{b1} and u_{h2} are obtained from the associated equations given in Section 2.

Substituting Equation (13) into (11), one can obtain

$$E_2 = \frac{\sigma h}{u_{h2}} \quad \text{and} \quad \gamma_{21} = -\frac{h u_{b1}}{b u_{h2}}. \tag{14}$$

Similarly, when the remote loading is given as follows

$$\sigma_{11}^{0} = \sigma, \quad \sigma_{22}^{0} = 0, \quad \sigma_{12}^{0} = 0 \tag{15}$$

one can have

$$E_1 = \frac{\sigma b}{u_{b1}} \quad \text{and} \quad \gamma_{12} = -\frac{b u_{h2}}{h u_{b1}}. \tag{16}$$

To find the effective shear modulus, the remote shear loading should be applied, i.e.

$$\sigma_{11}^0 = 0, \quad \sigma_{22}^0 = 0, \quad \sigma_{12}^0 = \sigma. \tag{17}$$

Substituting Equation (17) to Equation (11c), one can obtain

$$G_{12} = \frac{\sigma}{\left(\frac{u_{b2}}{h} + \frac{u_{h1}}{b}\right)} \tag{18}$$

For doubly periodic array of cracks/rigid-line inclusions in an infinite isotropic plane medium, the corresponding effective elastic properties can be obtained using Equations (14), (16) and (18) based on the associated boundary integral equation methods presented in Section 3.

5. Numerical results

5.1. Doubly periodic array of cracks

In this example, the doubly periodic array of cracks in an infinite isotropic plane medium is considered. The elastic modulus and Poisson's ratio of the medium are assumed as $E_0 = 1$ and $\nu_0 = 0.3$, respectively. The computational model as shown in Figure $1b_1$ is first used to check the numerical accuracy of the present method.

In numerical implementation, the crack is discretized into 20 quadratic discontinuous elements, whilst each boundary of the cell is meshed into 20 quadratic elements. For the remote loading σ in x_2 direction, stress intensity factor at crack tip is shown in Table 1 together with other results. One can find that the present result is in good agreement with those from different methods. For different rectangular cells (e.g. $h/b = 1.5$ and $a/b = 1.0$) as shown in Figure 1b2–b5 subjected to remote loading σ in x_2 direction, each crack is discretized into 20 quadratic discontinuous elements,

Table 1. Normalized stress intensity factor $K_I/\left(\sigma\sqrt{\pi a}\right)$.

h/b	a/b				Method
	0.05	0.10	0.15	0.20	
0.5	0.984	0.948	0.922	–	Karihaloo et al. (1996)
	0.983	0.948	0.922	–	Isida et al. (1981)
	0.982	0.928	0.833	–	Horii and Sahasakmontri (1990)
	0.983	0.948	0.921	0.922	Present
1.0	1.003	1.012	1.031	–	Karihaloo et al. (1996)
	1.003	1.012	1.031	–	Isida et al. (1981)
	1.003	1.011	1.025	–	Horii and Sahasakmontri (1990)
	1.002	1.011	1.030	1.061	Present
1.5	1.004	1.017	1.039	1.074	Karihaloo et al. (1996)
	1.004	1.017	1.039	1.074	Isida et al. (1981)
	1.004	1.016	1.036	1.064	Horii and Sahasakmontri (1990)
	1.003	1.016	1.038	1.073	Present

whilst each cell boundary is meshed into 20 quadratic elements. Stress intensity factor at each crack tip is the same for four decimal places, i.e. $K_I/\left(\sigma\sqrt{\pi a}\right) = 1.073$. The results show that the model containing single crack can produce enough exact numerical results. Therefore, the model containing more cracks is not obligatory for doubly periodic array of crack problems. In the analysis below, only the model containing single crack is adopted.

For different values of h/b and a/b, stress intensity factors at crack tip are shown in Figures 4 and 5 together with those from Chen and Lee (2002). One can observe that the present results and those from Chen and Lee (2002) are in good agreement. Figures 6 and 7 show the effective elastic properties E_2 and G_{12} of the medium which are also in agreement with those from Chen and Lee (2002) (see their Figures 2 and 3). Besides, the effective Poisson's ratio ν_{21}, which is not available in the reference (Chen and Lee, 2002), is shown in Figure 8. Effective properties E_1 and ν_{12} remains unchanged for doubly periodic array of cracks in an infinite isotropic plane medium. Note that for various rectangular cells containing different cracks as shown in Figure 1b2–b5 the corresponding results are not shown in Figures 4–8 since they are almost the same as those from the model with single crack and difficult to see their difference from these figures.

5.2. Doubly periodic array of rigid-line inclusions

Doubly periodic array of rigid-line inclusions in an infinite isotropic plane medium are studied in this example. Similar to the example above, the elastic parameters of

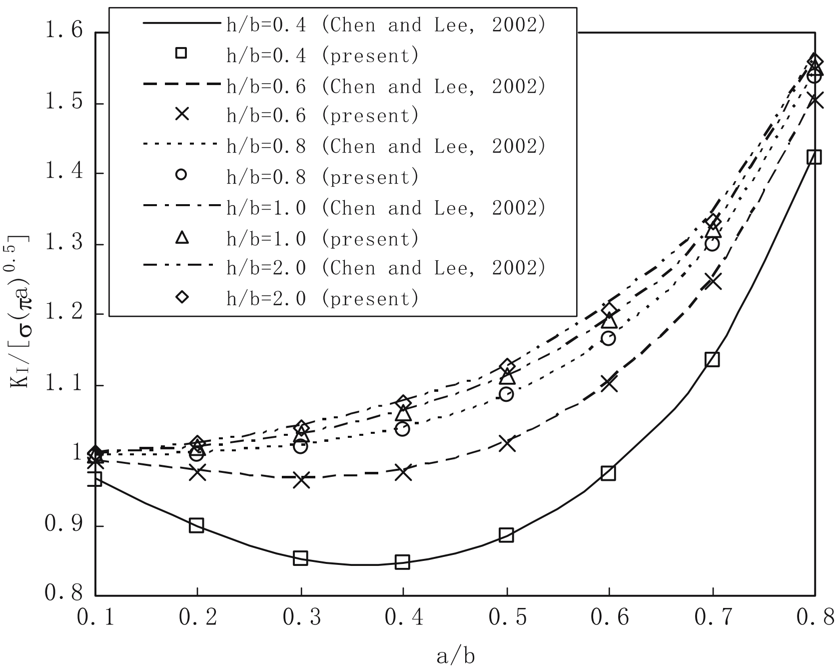

Figure 4. Normalized stress intensity factor $K_I/\left[\sigma\sqrt{\pi a}\right]$ for an infinite domain containing doubly periodic array of cracks subjected to a remote tension loading σ along x_2-direction.

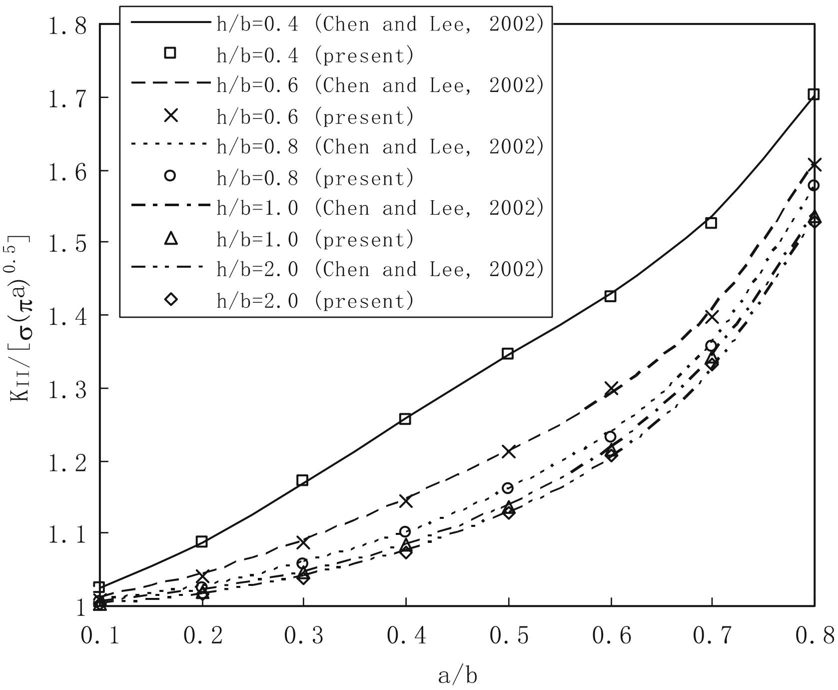

Figure 5. Normalized stress intensity factor $K_{\mathrm{II}}/\left[\sigma\sqrt{\pi a}\right]$ for an infinite domain containing doubly periodic array of cracks subjected to a remote shear loading σ.

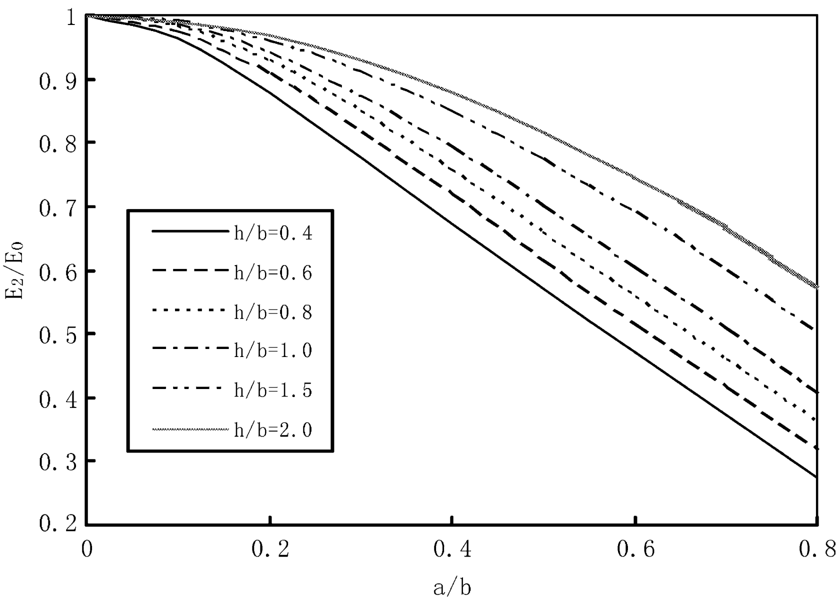

Figure 6. Normalized effective elastic modulus E_2/E_0 for an infinite domain containing doubly periodic array of cracks.

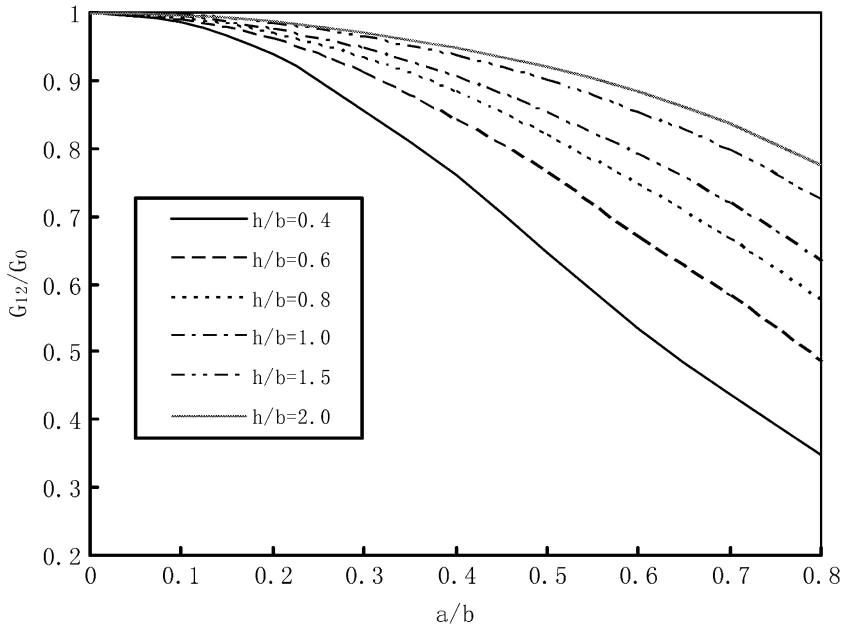

Figure 7. Normalized effective shear modulus G_{12}/G_0 for an infinite domain containing doubly periodic array of cracks.

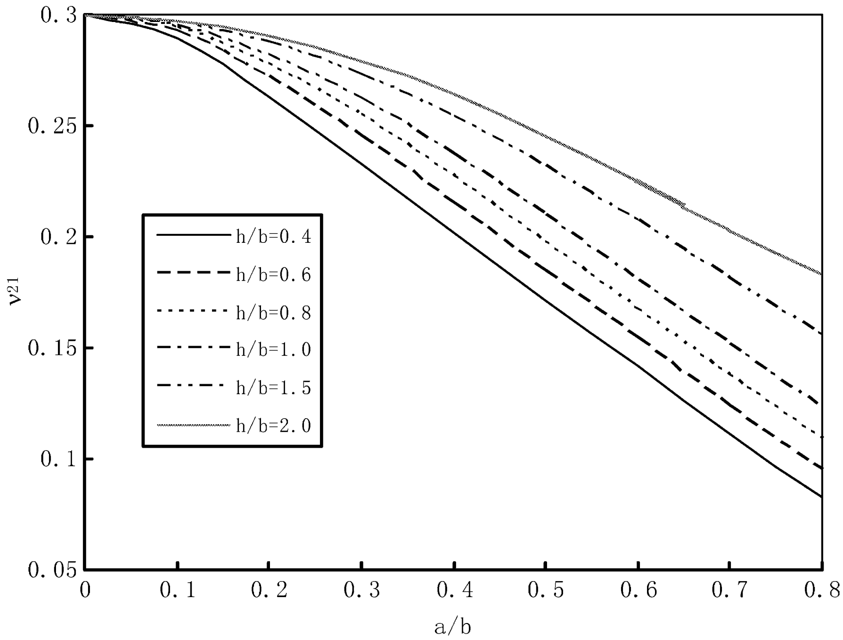

Figure 8. Effective Poisson's ratio ν_{21} for an infinite domain containing doubly periodic array of cracks.

the medium are given as $E_0=1$ and $\nu_0=0.3$. The computational model as shown in Figure 1b1 is first used to check the numerical accuracy of the present method.

The rigid-line inclusion is discretized into 20 quadratic discontinuous elements, whilst the discretization of the cell boundary is the same as the above example. For single rigid-line inclusion in an infinite isotropic plane medium, the computational model as shown in Figure 1b1 is taken as a rectangular cell with the boundary lengths $h/a=b/a=20$, and the boundaries $x_2=h$ and $x_2=-h$ are subjected to tension loading $\sigma_{22}=\sigma$, respectively. The obtained stress intensity factor $K_I/[\sigma\sqrt{\pi a}]=-0.2996$ is very close to analytical solution $K_I/[\sigma\sqrt{\pi a}]=-0.3$ (Brussat and Westmann, 1975). For various models as shown in Figure 1b2–b5, stress intensity factors at each crack tip are the same for four decimal places, i.e. $K_I/[\sigma\sqrt{\pi a}]=-0.2996$. Following this, and based on the model containing single crack, various cases are studied for doubly periodic array of rigid-line inclusions in an infinite isotropic plane medium. Stress intensity factors K_I at rigid-line-inclusion tip are shown in Figures 9 and 10 for two remote loading cases, whilst the corresponding stress intensity factors K_{II} at rigid-line-inclusion tip are zero. Figures 11–13 show the obtained effective elastic properties of doubly periodic array of rigid-line inclusions in an infinite isotropic plane medium. Note that shear modulus G_{12} and Poisson's ratio ν_{12} keep unchanged. Since the solution of the similar problem is not available in the published literatures, we think that the obtained results can be considered as a reference for future research.

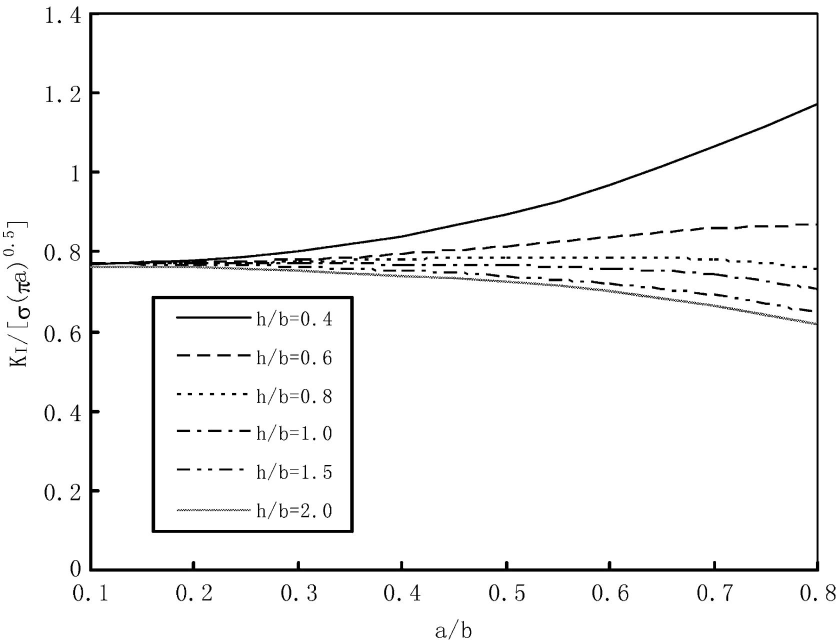

Figure 9. Normalized stress intensity factor $K_I/[\sigma\sqrt{\pi a}]$ for an infinite domain containing doubly periodic array of rigid-line inclusions subjected to a remote tension loading σ along x_1-direction.

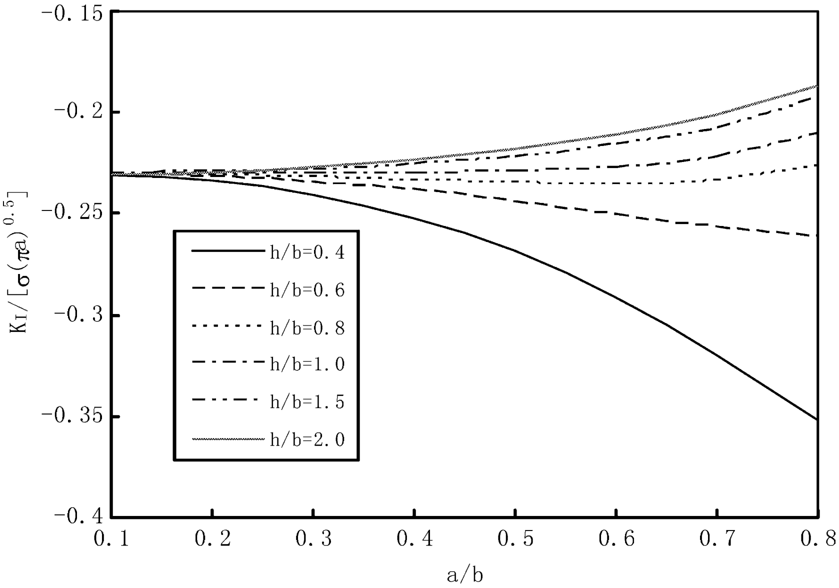

Figure 10. Normalized stress intensity factor $K_I/[\sigma\sqrt{\pi a}]$ for an infinite domain containing doubly periodic array of rigid-line inclusions subjected to a remote tension loading σ along x_2-direction.

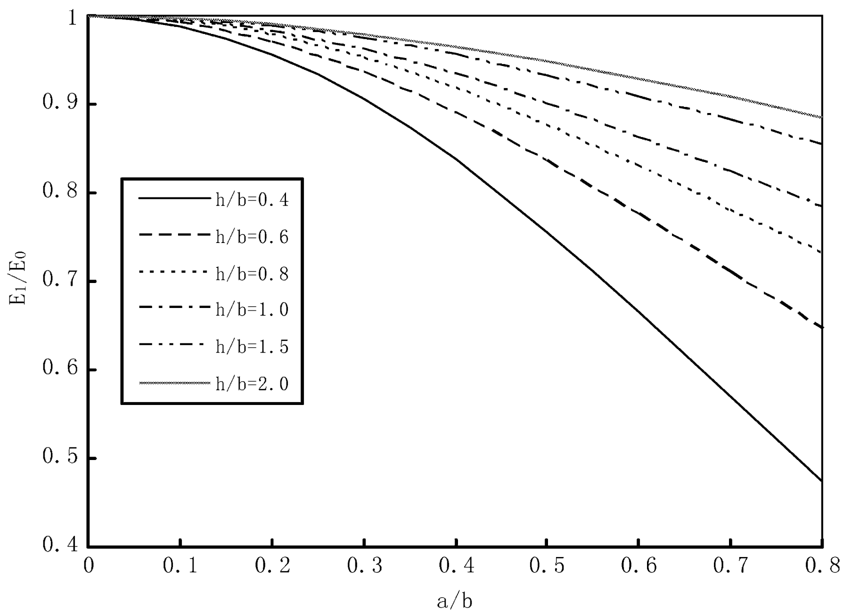

Figure 11. Normalized effective elastic modulus E_1/E_0 for an infinite domain containing doubly periodic array of rigid-line inclusions.

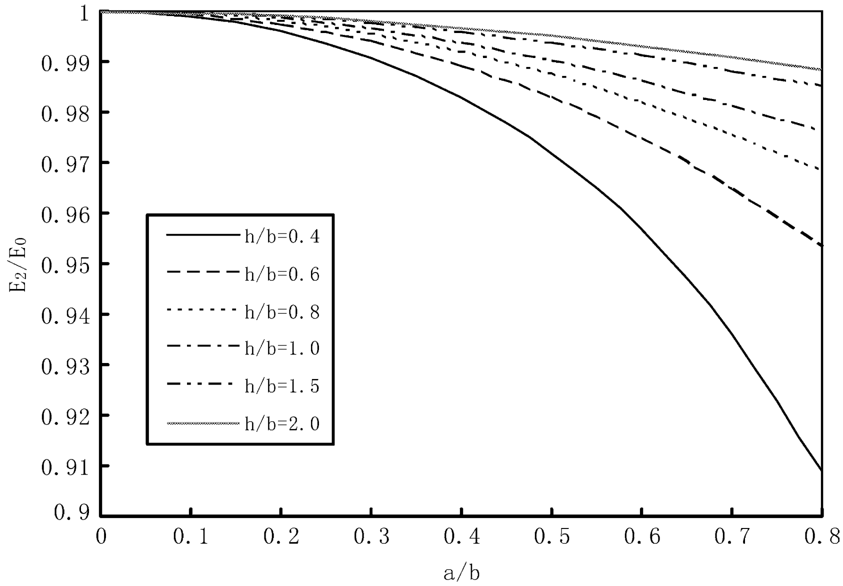

Figure 12. Normalized effective elastic modulus E_2/E_0 for an infinite domain containing doubly periodic array of rigid-line inclusions.

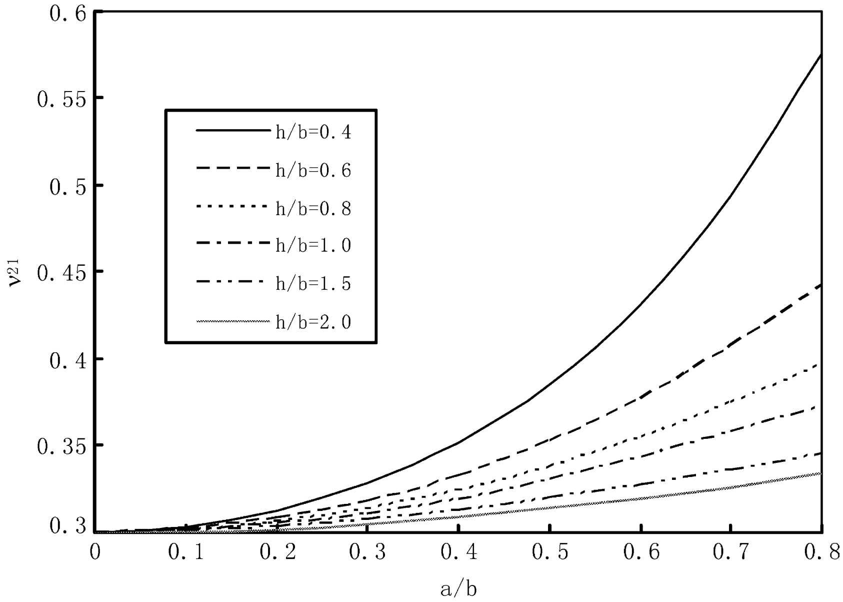

Figure 13. Effective Poisson's ratio ν_{21} for an infinite domain containing doubly periodic array of rigid-line inclusions.

6. Conclusions

The boundary integral equation approaches have been used to obtain the effective elastic properties of doubly periodic array of cracks/rigid-line inclusions in an infinite isotropic plane medium. For the studied crack problems, the present results are in good agreement with the results available. To the best of the authors' knowledge, the related results for doubly period array of rigid-line inclusions in an infinite isotropic plane medium have not yet been published in the literature. Therefore, the present results can be considered as benchmark results for further researches.

Acknowledgement

CY Dong is grateful for the support provided by a Brain Pool grant from the Korea Science and Engineering Foundation (KOSEF).

Appendix

Some formulations are as follows (Brebbia and Dominguez, 1992; Dong et al., 2003)

$$U_{ij} = \frac{1}{8\pi G(1-\nu)}\left[(3-4\nu)\ln\frac{1}{r}\delta_{ij} + r_{,i}r_{,j}\right], \tag{a1}$$

$$T_{ij} = -\frac{1}{4\pi(1-\nu)r}\left\{\frac{\partial r}{\partial n}\left[(1-2\nu)\delta_{ij} + 2r_{,i}r_{,j}\right] - (1-2\nu)\left(r_{,i}n_j - r_{,j}n_i\right)\right\}, \tag{a2}$$

$$U_{ijk} = \frac{1}{4\pi(1-\nu)r}\left[(1-2\nu)\left(r_{,k}\delta_{ij} + r_{,j}\delta_{ik} - r_{,i}\delta_{jk}\right) + 2r_{,i}r_{,j}r_{,k}\right], \tag{a3}$$

$$\begin{aligned} T_{ijk} = \frac{G}{2\pi(1-2\nu)r^2}\Bigg\{ & 2\frac{\partial r}{\partial n}\left[(1-2\nu)\delta_{ij}r_{,k} + \nu\left(r_{,j}\delta_{ik} + r_{,i}\delta_{jk}\right) - 4r_{,i}r_{,j}r_{,k}\right] \\ & + 2\nu\left(n_i r_{,j}r_{,k} + n_j r_{,i}r_{,k}\right) + (1-2\nu)\left(2n_k r_{,i}r_{,j} + n_j\delta_{ik} + n_i\delta_{jk}\right) - (1-4\nu)n_k\delta_{ij}\Bigg\}, \end{aligned} \tag{a4}$$

$$\overline{U}_{ij} = \frac{1}{8\pi G(1-\nu)r}\left\{-(3-4\nu)\delta_{ij}r_{,k} + r_{,i}\delta_{jk} + r_{,j}\delta_{ik} - 2r_{,i}r_{,j}r_{,k}\right\}e_{km}n_m, \tag{a5}$$

$$U_{ik,j} = \frac{1}{8\pi G(1-\nu)r}\left[(3-4\nu)r_{,j}\delta_{ik} - r_{,i}\delta_{kj} - r_{,k}\delta_{ij} + 2r_{,i}r_{,j}r_{,k}\right], \tag{a6}$$

$$\begin{aligned} T_{ik,j} = \frac{1}{4\pi(1-\nu)r^2}\Bigg\{ & r_{,j}\frac{\partial r}{\partial n}\left[(1-2\nu)\delta_{ik} + 2r_{,i}r_{,k}\right] - (1-2\nu)r_{,j}\left(r_{,i}n_k - r_{,k}n_i\right) \\ & + \left(r_{,m}r_{,j} - \delta_{mj}\right)n_m\left[(1-2\nu)\delta_{ik} + 2r_{,i}r_{,k}\right] + 2\frac{\partial r}{\partial n}\left[\left(r_{,i}r_{,j} - \delta_{ij}\right)r_{,k} + \left(r_{,k}r_{,j} - \delta_{kj}\right)r_{,i}\right] \\ & - (1-2\nu)\left[\left(r_{,i}r_{,j} - \delta_{ij}\right)n_k - \left(r_{,k}r_{,j} - \delta_{kj}\right)n_i\right]\Bigg\} \end{aligned} \tag{a7}$$

and

$$\begin{aligned} \overline{U}_{ik,j} = \frac{e_{mn}n_n}{8\pi G(1-\nu)r^2}\Big\{ & -(3-4\nu)\delta_{ik}\left(2r_{,m}r_{,j} - \delta_{mj}\right) + \left(2r_{,i}r_{,j} - \delta_{ij}\right)\delta_{km} \\ & + \left(2r_{,k}r_{,j} - \delta_{kj}\right)\delta_{im} + 2\delta_{mj}r_{,i}r_{,k} + 2\delta_{ij}r_{,m}r_{,k} + 2\delta_{kj}r_{,m}r_{,i} - 8r_{,i}r_{,j}r_{,k}r_{,m}\Big\} \end{aligned} \tag{a8}$$

where $e_{11}=0$, $e_{12}=1$, $e_{21}=-1$, $e_{22}=0$. G and ν are the shear modulus and Poisson's ratio of the medium, respectively. δ_{ij} is the Kronecker delta. $r_{,i}=\frac{\partial r(p,q)}{\partial x_i(q)}$ in which r is the distance between the field q and the source point p. n_i is the directional cosine of the normal at the boundary point q with respect to x_i. $\frac{\partial r}{\partial n}=r_{,i}n_i$.

References

Aliabadi, M.H. (2002). *The Boundary Element Method – Applications in Solids and Structures*. John Wiley & Sons, Ltd.

Brebbia, C.A. and Dominguez, J. (1992). *Boundary Elements – An Introduction Course*. Computational Mechanics Publications, Southampton.

Brussat, T.R. and Westmann, R.A. (1975). A Westergaard-type stress function for line inclusion problems. *International Journal of Solids and Structures* **11**, 665–677.

Chen, Y.Z. (1983). An investigation of the stress intensity factors for a finite internally cracked plate by using variational method. *Engineering Fracture Mechanics* **17**, 387–394.

Chen, Y.Z., Hasebe and Lee, K.Y. (2003). *Multiple Crack Problems in Elasticity*. WIT Press, Southampton.

Chen, Y.Z. and Lee, K.Y. (2002). An infinite plate weakened by periodic cracks. *ASME Journal of Applied Mechanics* **69**, 552–555.

Cruse, T.A. (1988). *Boundary Element Analysis in Computational Fracture Mechanics.* Kluwer, Dordrecht.

Dong, C.Y., Lo, S.H. and Cheung, Y.K. (2003). Interaction between cracks and rigid-line inclusions by an integral equation approach. *Computational Mechanics* **31**, 238–252.

Dong, C.Y. and de Pater, C.J. (2001). Numerical implementation of displacement discontinuous method and its application in hydraulic fracturing. *Computer Methods in Applied Mechanics Engineering* **19**, 745–760.

Helsing, J. (1999). Fast and accurate numerical solution to an elastostatic problem involving ten thousand randomly oriented cracks. *International Journal of Fracture* **100**, 321–327.

Horii, H. and Sahasakmontri, K. (1990). Mechanical properties of cracked solids: validity of the selfconsistent method. In: *Micromechanics and Inhomogeneity* (edited by G.J. Weng, M. Taya and H. Abe). Springer-Verlag, New York, 137–137.

Hong, H. and Chen, J. (1988). Derivations of integral equations of elasticity. *ASCE Journal of Engineering Mechanics* **114**, 1028–1044.

Hui, C.Y. and Shia, D. (1999). Evaluations of hypersingular integrals using Gaussian quadrature. *International Journal of Numerical Methods in Engineering* **44**, 205–214.

Isida, M., Ushijima, N. and Kishine, N. (1981). Rectangular plates, strips and wide plates containing internal cracks under various boundary conditions. *Transactions of the Japan Society of Mechanical Engineering Series A* **47**, 27–35.

Karihaloo, B.L. and Wang, J. (1997). On the solution of doubly array of cracks. *Mechanics of Materials.* **26**, 209–212.

Karihaloo, B.L., Wang, J. and Grzybowski, M. (1996). Doubly periodic arrays of bridged cracks and short fibre-reinforced cementitious composites. *Journal of the Mechanics and Physics of Solids* **44**, 1565–1586.

Kutt, H.R. (1975). Quadrature formulae for finite-part integrals. CSIR Special report, *National Research Institute for Mathematical Sciences*, WISK 178.

Li, Y.P., Tham, L.G., Wang, Y.H. and Tsui, Y. (2003). A modified Kachanov method for analysis of solids with multiple cracks. *Engineering Fracture Mechanics* **70**, 1115–1129.

Liu, Y.J., Nishimura, N., Otani, Y., Takahashi, T., Chen, X.L. and Munakata, H. (2005). A fast boundary element method for the analysis of fiber-reinforced composites based on a rigid-inclusion model. *ASME Journal of Applied Mechanics* **72**, 1–14.

Muskhelishvili, N.I. (1953). *Some Basic Problems of the Mathematical Theory of Elasticity*. Noordhoff, Netherlands.

Nishimula, N. and Liu, Y.J. (2004). Thermal analysis of carbon-nanotube composites using a rigid-line inclusion model by the boundary integral equation method. *Computational Mechanics* **35**, 1–10.

Nishimura, N., Yoshida, K. and Kobayashi, S., (1999). A fast multipole boundary integral equation method for crack problems in 3D. *Engineering Analysis with Boundary Elements* **23**, 97–105.

Pan, E. (1997). A general boundary element analysis of 2-D linear elastic fracture mechanics. *International Journal of Fracture* **88**, 41–59.

Portela, A., Aliabadi, M.H. and Rooke, D.P. (1992). Dual boundary element method: effective implementation for crack problems. *International Journal for Numerical Methods in Engineering* **33**, 1269–1287.

Wang, G.S. and Feng, X.T. (2001). The interaction of multiple rows of periodic cracks. *International Journal of Fracture* **110**, 73–100.

Wang, J., Fang, J. and Karihaloo, B.L. (2000). Asymptotic of multiple crack interactions and prediction of effective modulus. *International Journal of Solids and Structures* **37**, 4261–4273.