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THE TRACTION BOUNDARY CONTOUR METHOD FOR LINEAR ELASTICITY

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SUMMARY

This paper presents a further development of the boundary contour method. The boundary contour method is extended to cover the traction boundary integral equation. A traction boundary contour method is proposed for linear elastostatics. The formulation of traction boundary contour method is regular for points except the ends of the boundary element and corners. The present approach only requires line integrals for three-dimensional problems and function evaluations at the ends of boundary elements for two-dimensional cases. The implementation of the traction boundary contour method with quadratic boundary elements is presented for two-dimensional problems. Numerical results are given for some two-dimensional examples, and these are compared with analytical solutions. This method is shown to give excellent results for illustrative examples. Copyright © 1999 John Wiley & Sons, Ltd.

KEY WORDS: traction boundary contour method; boundary contour method; boundary element method; elasticity

1. INTRODUCTION

The Boundary Element Method (BEM) has become an efficient numerical method in solving practical engineering problems. The main advantage of the BEM is the reduction of the dimension of boundary value problems. The traditional BEM, however, requires numerical evaluation of line integrals for two-dimensional problems and surface integrals for three-dimensional ones. Moreover, since the boundary integrals involve singular kernels, the BEM encounters difficulty in the numerical evaluation of the singular integrals, and the overall accuracy of this method is largely dependent on the precision of which the various integrals, especially the singular integrals, are evaluated. Many papers have been devoted to such computational aspects of the BEM as the treatment of singular integrals in the Cauchy singular integral equations and the hypersingular integral equations [1–5].

Recently, a novel variant of the boundary element method, called the Boundary Contour Method (BCM), was first proposed by Nagarajan *et al.* [6, 7] for linear elasticity. The central idea

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of the new approach is the exploitation of the divergence free property of the BEM integrand. This property of the integrand demonstrates the path-independent integral in the boundary integral equation for two-dimensional problems. For three-dimensional problems, surface integrals on boundary elements are converted, through an application of Stokes' theorem, into line integrals on the boundary contours of these elements. Thus, the BCM requires simply the evaluation of potential functions at the end points of boundary elements in two-dimensional problems and only numerical evaluation of line integrals in three-dimensional cases. The BCM with quadratic boundary elements is presented for two-dimensional problems by Phan *et al.* [8] and for three-dimensional problems by Mukherjee *et al.* [9]. These works are all based on the conventional displacement boundary integral equation. However, this equation has non-unique solution for the traction in some two-dimensional Boundary Value Problem (BVP) [10, 11]. The authors give the equivalent BCM for two-dimensional problems in Reference 12, which starts from the equivalent displacement BIE. This approach always gives accurate results for two-dimensional linear elasticity.

During recent years, when using the integral equations in fracture mechanics, considerable attention has been given to the traction boundary integral equation. The use of the traction boundary integral equation was first reported by Hong and Chen [13], who employed both the displacement and traction boundary integral equations to solve crack problems. This method is called the Dual Boundary Element Method (DBEM). The DBEM has been widely applied to the analysis of various crack problems [14–16]. Recent papers have been devoted to the treatments of the hypersingular integral term that arises in the Somigliana stress identity [17–20]. A non-singular form of the traction BIE is obtained by Huang *et al.* [17], who used only the traction BIE for stress analysis. A detailed discussion of this type of equation is given by Cruse *et al.* [18]. In previous works, the traction boundary integral equation is numerically solved by the traditional boundary element method.

In this paper, the Somigliana stress identity is written in a vector form. The divergence of the integrand vector is proofed to be zero. Based on this property, the traction boundary contour method (TBCM) is proposed for linear elasticity. The formulations and implementation of the traction boundary contour method with quadratic boundary elements are addressed for two-dimensional problems. Finally, numerical results for some examples are presented and compared with the analytical solutions and those from the BCM based on the conventional displacement BIE.

2. SOMIGLIANA STRESS IDENTITY

We begin with the Somigliana stress identity at internal points for an elasticity problem without body forces [16]

$$\sigma_{ij}(p) = \int_{\partial B} [D_{ijk}(q, p)t_k(q) - S_{ijk}(q, p)u_k(q)]\, \mathrm{d}S(q) \tag{1}$$

where p is a source point, q a field point, u_k a displacement component and t_k a traction component. The integral kernels are as follows:

$$D_{ijk}(p, q) = C_{jkml} u^{*}_{im,L}(p, q) \tag{2}$$

$$S_{ijk}(p, q) = C_{jkml} t^{*}_{im,L}(p, q) = C_{jkml} \Sigma_{i\beta m,L}(p, q) n_{\beta} \tag{3}$$

in which the elastic constant

$$C_{jkml} = \lambda\delta_{jk}\delta_{ml} + \mu(\delta_{jm}\delta_{kl} + \delta_{jl}\delta_{km}) \tag{4}$$

and comma with the capital letter shows the derivation with relation to the source point p. The displacement fundamental solution u^*_{im} and the fundamental solution stress tensor $\Sigma_{i\beta m}$ for two-dimensional plane strain problems are given as

$$u^*_{im}(p, q) = \frac{-1}{8\pi(1-\nu)\mu}\left[(3-4\nu)\delta_{im}\ln\left(\frac{r}{a}\right) - r_{,i}r_{,m}\right] \tag{5}$$

$$\Sigma_{i\beta m}(p, q) = \frac{-1}{4\pi(1-\nu)r}[(1-2\nu)(\delta_{im}r_{,\beta} + \delta_{\beta m}r_{,i} - \delta_{i\beta}r_{,m}) + 2r_{,i}r_{,\beta}r_{,m})] \tag{6}$$

where r is the distance between p and q and a is a constant with the length dimension.

In the conventional BEM, the integral of equation (1) is singular as q is near p since the integral kernels D_{ijk} and S_{ijk} behave like r^{-2} and r^{-3} in three-dimensional problems, and r^{-1} and r^{-2} in two-dimensional cases, respectively. Numerical evaluation of these integrals is difficult. Equation (1) needs to be regularized before the appropriate integral can be evaluated. An inaccurate evaluation of principal value integrals will give rise to the incorrect results of stresses at internal points close to the boundary.

3. BCM FORMULATIONS FOR SOMIGLIANA STRESS IDENTITY

Noting equation (3), equation (1) can be expressed as

$$\sigma_{ij}(p) = \int_{\partial B}\{D_{ijk}(p, q)\sigma_{k\beta}(q) - C_{jkml}\Sigma_{i\beta m,L}(p, q)u_k(q)\}\mathbf{e}_\beta \cdot \mathrm{d}\mathbf{S}(q) \tag{7}$$

Rewriting the vector in the integrand of equation (7) as $\mathbf{F}_{ij}$, we get

$$\sigma_{ij}(p) = \int_{\partial B}\mathbf{F}_{ij}(p, q)\cdot \mathrm{d}\mathbf{S}(q) \tag{8}$$

The vector $\mathbf{F}_{ij}$ in the above equation is

$$\mathbf{F}_{ij} = F_{ij\beta}\mathbf{e}_\beta = \{D_{ijk}(p, q)\sigma_{k\beta}(q) - C_{jkml}\Sigma_{i\beta m,L}(p, q)u_k(q)\}\mathbf{e}_\beta \tag{9}$$

Now, substituting equation (2) into the above equation and taking the divergence of $\mathbf{F}_{ij}$ at a field point q, we derive the following result:

$$\nabla_q\cdot\mathbf{F}_{ij} = F_{ij\beta,\beta} = C_{jkml}\{\varepsilon^*_{i\beta m,L}\sigma_{k\beta} - \Sigma_{i\beta m,L}\varepsilon_{k\beta} + u^*_{im,L}\sigma_{k\beta,\beta} - \Sigma_{i\beta m,\beta L}u_k(q)\} \tag{10}$$

where the fundamental solution strain tensor $\varepsilon^*_{i\beta m}$ and the strain field $\varepsilon_{k\beta}$ are

$$\varepsilon^*_{i\beta m} = \tfrac{1}{2}(u^*_{im,\beta} + u^*_{\beta m,i}) \tag{11}$$

and

$$\varepsilon_{k\beta} = \tfrac{1}{2}(u_{k,\beta} + u_{\beta,k}) \tag{12}$$

There is the result that the divergence of $\mathbf{F}_{ij}$ is zero everywhere except at the source point, provided that u_k and $\sigma_{k\beta}$ correspond to a vanishing body force elastostatic state with the same elastic constants as the fundamental solution, that is

$$\nabla_q \cdot \mathbf{F}_{ij} = 0 \tag{13}$$

The above property demonstrates the existence of vector potential functions $\mathbf{V}_{ij}$, as shown in [6], such that

$$\mathbf{F}_{ij} = \nabla_q \times \mathbf{V}_{ij} \tag{14}$$

Thus, the Somigliana stress identity is in a form similar to the displacement boundary integral equation. When the whole boundary is discretized into N boundary elements, the BCM formulation is

$$\sigma_{ij}(p) = \sum_{n=1}^{N} \int_{\partial B_n} \nabla_q \times \mathbf{V}_{ij} \cdot \mathrm{d}\mathbf{S} = \sum_{n=1}^{N} \oint_{C_n} \mathbf{V}_{ij} \cdot \mathrm{d}\mathbf{r} \tag{15}$$

for three-dimensional problems. Here, C_n is the bounding contour of the boundary element ∂B_n.

In two-dimensional problems, since the integral of vector F_{ij} is path-independent, the result is

$$\sigma_{ij}(p) = \sum_{n=1}^{N} \int_{\partial B_n} \mathbf{F}_{ij} \cdot \mathrm{d}\mathbf{S} = \sum_{n=1}^{N} \{G_{ij}^{n}(Q_{n2}) - G_{ij}^{n}(Q_{n1})\} \tag{16}$$

where G_{ij} is a potential related to F_{ij} by

$$\mathbf{F}_{ij} = F_{ijk}\mathbf{e}_k = \frac{\partial G_{ij}}{\partial y}\mathbf{e}_1 - \frac{\partial G_{ij}}{\partial x}\mathbf{e}_2 \tag{17}$$

The above results show that for three-dimensional problems, surface integrals are converted into line integrals on the bounding contour of the boundary elements. For two-dimensional problems, there is no numerical integration and only evaluations of the potential functions at the end points of boundary elements.

4. NUMERICAL IMPLEMENTATION FOR 2-D PROBLEMS

4.1. Shape functions

In the numerical implementation, the BCM is different from the traditional BEM. The vector $\mathbf{F}_{ij}$ contains the unknown field u_k and $\sigma_{k\beta}$. Thus, the shape functions must first be chosen for these variables, then potential functions can be determined. In order for the property of equation (13) to be valid, the displacement and stress shape functions must satisfy the equation of elasticity. The element displacement shape functions in the global co-ordinate system (x, y) can be expressed as

$$\begin{Bmatrix} u_1(q) \\ u_2(q) \end{Bmatrix} = [U(x, y)]\{a\} \tag{18}$$

For quadratic boundary elements [8]

$$[U(x, y)] = \begin{bmatrix} 1 & x & y & 0 & 0 & 0 & x^2 & y^2 & k_1xy & k_2xy \\ 0 & 0 & 0 & 1 & x & y & k_2xy & k_1xy & x^2 & y^2 \end{bmatrix} \tag{19}$$

in which $k_1 = -2(1-2\nu)$, $k_2 = -4(1-\nu)$. In this case, 10 artificial variables $\{a\} = \{a_1, a_2, \ldots, a_{10}\}^{\mathrm{T}}$.

Substituting equation (18) into Hooke's law, we write the stress shape function as

$$\{\sigma\} = [\sigma(x, y)]\{a\} \tag{20}$$

Then, the expression for the traction on the boundary is

$$\{\mathbf{t}\} = [t(x, y)]\{a\} \tag{21}$$

In order to relate these 10 artificial variables $\{a\}$ to 10 physical variables, a quadratic boundary element is chosen. The configuration of the chosen quadratic boundary element here is the same as the BCM based on the displacement BIE, shown in Figure 1. This configuration of the element avoids locating the traction nodes at corners. For the quadratic element, 10 artificial variables can be expressed with 10 boundary physical variables of the nth element, which are four traction components at two traction nodes and six displacement components at three displacement nodes, as follows

$$\{a\}^n = [T^n(x, y)]^{-1}\{b\}^n \tag{22}$$

where

$$\{b\}^n = \{u_1^{2n-1}, u_2^{2n-1}, t_1^{2n-1}, t_2^{2n-1}, u_1^{2n}, u_2^{2n}, t_1^{2n}, t_2^{2n}, u_1^{2n+1}, u_2^{2n+1}\}^{\mathrm{T}}.$$

For the sake of convenience, a new co-ordinate system (ξ, η) centred at the source point is introduced, so that the shape functions in the new co-ordinate system is

$$\begin{Bmatrix} u_1(q) \\ u_2(q) \end{Bmatrix} = [U(\xi, \eta)]\{\hat{a}\}^n \tag{23}$$

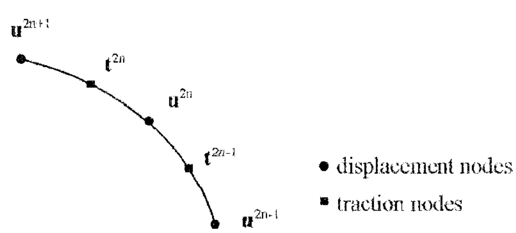

Figure 1. Quadratic boundary element

In the above equation

$$\{\hat{a}\}^n = [B_m]\{a\}^n \tag{24}$$

where $[B_m]$ is a transformation matrix that depends only on the co-ordinates of the mth source point.

4.2. Potential functions

The shape functions (23) can be considered to be a linear combination of the 10 basic shape functions in equation (19) with x and y replaced by ξ and η, respectively, with coefficient $\{\hat{a}\}^n$. Let $\mathbf{f}_{ijk}$ and g_{ijk} stand for the vector $\mathbf{F}_{ij}$ and potential function G_{ij}, respectively, which are determined by each one of 10 basic shape functions and its corresponding stress shape function. In this case, equation (17) has the form

$$\mathbf{f}_{ijk} = \frac{\partial g_{ijk}}{\partial \eta}\,\mathbf{e}_1 - \frac{\partial g_{ijk}}{\partial \xi}\,\mathbf{e}_2 \tag{25}$$

By substituting each basic shape function and its corresponding stress shape function into equation (9), we can obtain the basic vector $\mathbf{f}_{ijk}$. Then, equation (25) is solved to give the potential function g_{ijk}. These potential functions are listed in the appendix. Thus, the potential function G_{ij} can be expressed by the linear combination of g_{ijk} as

$$G_{ij} = g_{ijk}\hat{a}^n_k \tag{26}$$

in which $\hat{a}^n_k$ is the kth component of $\{\hat{a}\}^n$.

By using the potential function (26), equation (16) becomes

$$\begin{aligned}\sigma_{ij}(p) &= \sum_{n=1}^{N}\int_{\partial B_n}\mathbf{f}_{ijk}\hat{a}^n_k\cdot \mathrm{d}\mathbf{S} = \sum_{n=1}^{N}[g_{ijk}(Q_{n2}) - g_{ijk}(Q_{n1})]\hat{a}^n_k \\ &= \sum_{n=1}^{N}\Delta g^n_{ijk}\hat{a}^n_k = \sum_{n=1}^{N}\{\Delta g^n_{ij}\}[B_m][T^n(x,y)]^{-1}\{b\}^n \end{aligned}\tag{27}$$

The integrals in the Somigliana stress identity (27) are path-independent. Therefore, equation (27) remains valid when the source point is taken at a boundary point, provided it is not at the end of an element.

5. TBCM FOR 2-D PROBLEMS

In the BCM, traction nodes are located at the interior of elements, as shown in Figure 1, so that equation (27) is regular at traction nodes. Thus, the formulation of the traction boundary contour method has the result

$$\begin{aligned}t_i(p) &= n_j(p)\sum_{n=1}^{N}\int_{\partial B_n}\mathbf{f}_{ijk}\hat{a}^n_k\cdot \mathrm{d}\mathbf{S} = n_j(p)\sum_{n=1}^{N}[g_{ijk}(Q_{n2}) - g_{ijk}(Q_{n1})]\hat{a}^n_k \\ &= \sum_{n=1}^{N} n_j(p)\Delta g^n_{ijk}\hat{a}^n_k = \sum_{n=1}^{N}\Delta g^n_{ik}(p)\hat{a}^n_k \end{aligned}\tag{28}$$

where n_j denotes the jth component of the unit outward normal to the boundary, at the source point p.

For the mth node only corresponding to traction nodes, the discretized equation (28) can be expressed as the matrix form

$$\{t_m\} = \sum_{n=1}^{N} [\Delta g^n(p_m)][B_m][T^n(x, y)]^{-1}\{b\}^n = [\bar{M}_m]\{b\} \tag{29}$$

Here, $\{b\}$ contains node displacements and tractions on the whole boundary ∂B.

By combining free terms of the left-hand side of equation (29) with the right-hand side, we have the equation

$$[M_m]\{b\} = 0 \tag{30}$$

Equation (30) is reordered in accordance with node displacements and tractions to form

$$[H_m]\{u\} + [T_m]\{t\} = 0 \tag{31}$$

Equation (28) being forced to $2N$ source points only corresponding to $2N$ traction nodes, we get $2N$ relations of form (31). These relations are combined to form the final linear system of equations

$$[H]\{u\} + [T]\{t\} = 0 \tag{32}$$

According to the boundary conditions, equation (32) can be rearranged into a system of algebraic equations

$$[A]\{x\} = [B]\{y\} = \{z\} \tag{33}$$

where $\{x\}$ contains the boundary unknown u_i or t_i and $\{y\}$ contains the boundary known u_i or t_i.

In general, an over-determined matrix system with more equations than unknowns is often obtained when the BCM and the TBCM are independently used for solving some mixed boundary value problems. The over-determined system can be solved by the Householders' method to get the least-square solution. However, the determined system of equations can also be obtained for the mixed BVP if the formulations of the BCM and the TBCM are properly collocated at displacement and traction nodes, respectively.

6. NUMERICAL EXAMPLE

Example 1. A 2×2 square plate with material constants: $E = 1$, $v = 0{\cdot}3$ is subjected to a bending load as shown in Figure 2. The analytical solutions are

$$u_1 = -0{\cdot}195x^2 - 0{\cdot}455y^2, \quad u_2 = 0{\cdot}91xy, \quad \sigma_1 = 0, \quad \sigma_2 = x$$

The whole boundary is discretized by four quadratic elements coincided with natural boundaries. The boundary quantities obtained by the TBCM are very close to analytical solution. The numerical results of the interior points along the line k–k ($y = 0{\cdot}8$) shown in Table I are compared with analytical solutions. The results are exact to six significant digits in a single-precision implementation.

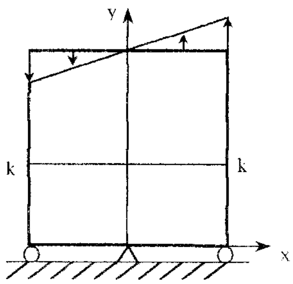

Figure 2. Square plate

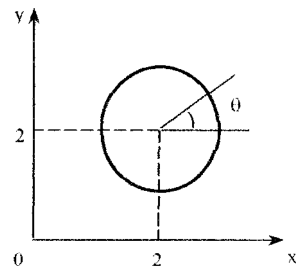

Figure 3. Circular body

Table I. Square plate subject to bending load

	TBCM results			Analytical solutions		
x	s_2	u_1	u_2	s_2	u_1	u_2
−1·0	−1·000001	−0·486202	−0·728001	−1·0	−0·4862	−0·728
−0·75	−0·750001	−0·400887	−0·546001	−0·75	−0·400888	−0·546
−0·5	−0·500000	−0·339951	−0·364001	−0·5	−0·33995	−0·364
−0·25	−0·250000	−0·303388	−0·182000	−0·25	−0·303388	−0·182
0·0	$0{\cdot}1\times10^{-7}$	−0·291202	$0{\cdot}4\times10^{-7}$	0·0	−0·2912	0·0
0·25	0·250000	−0·303388	0·182000	0·25	−0·303388	0·182
0·5	0·500000	−0·339951	0·364000	0·50	−0·33995	0·364
0·75	0·750001	−0·400889	0·546000	0·75	−0·400888	0·546
1·0	1·000001	−0·486201	0·728001	1·0	−0·4862	0·728

Example 2. Consider a circular body of unit radius centred at the point (2, 2) in the global (x, y) co-ordinate system as shown in Figure 3. The displacement field that is the exact solution of the elastic equation follows

$$u_1 = \frac{x}{x^2 + y^2}, \quad u_2 = \frac{y}{x^2 + y^2}$$

The material constants are Young's modulus $E = 2{\cdot}5$ and Poisson's ratio $\nu = 0{\cdot}3$. The above displacement fields are imposed at the displacement nodes on the boundary. This problem is solved by the linear and quadratic BCM in [6, 8]. The same problem is chosen here in order to test the accuracy of the TBCM for two-dimensional elasticity. The circular boundary is discretized by 10 quadratic elements spaced at equal increments. The numerical solutions of node tractions calculated by the TBCM are shown in Figure 4. These results are in very good agreement with the exact solutions. As $\theta = \pi/10$, physical quantities at the interior points are accurate to three decimal places. The results at the interior points located very close to the boundary shown in Table II are compared with analytical solution. It can be seen from Table II that displacement and traction values are sufficiently accurate, and the boundary layer effect does

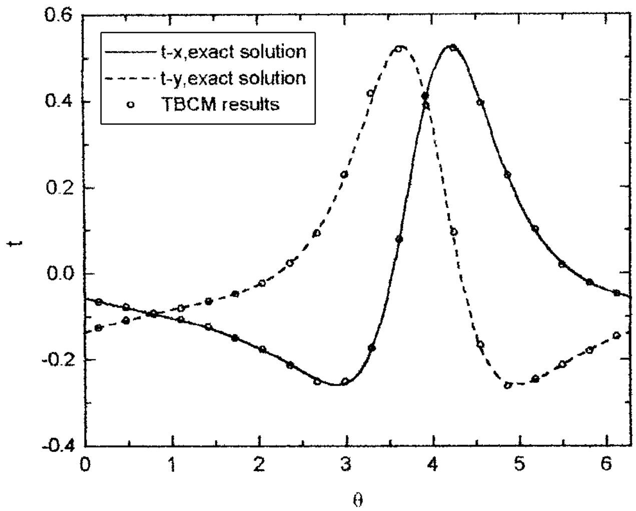

Figure 4. Traction components for circular body ($r = 1{\cdot}0$)

Table II. Displacement and stress components for circular body ($r = 1{\cdot}0$)

r	1·000	0·998	0·996	0·994	0·992	0·990
TBCM results						
u_1	0·21019	0·21026	0·21034	0·21041	0·21049	0·21056
u_2	0·16446	0·16458	0·16470	0·16482	0·16494	0·16506
σ_{11}	−0·03295	−0·03293	−0·03291	−0·03290	−0·03288	−0·03287
σ_{22}	0·03295	0·03293	0·03291	0·03290	0·03288	0·03287
σ_{12}	−0·13295	−0·13309	−0·13324	−0·13338	−0·13353	−0·13368
Analysis solutions						
u_1	0·21020	0·21028	0·21034	0·21042	0·21045	0·21057
u_2	0·16448	0·16460	0·16472	0·16482	0·16497	0·16508
σ_{11}	−0·03274	−0·03273	−0·03270	−0·03269	−0·03268	−0·03266
σ_{22}	0·03286	0·03284	0·03282	0·03280	0·03278	0·03276
σ_{12}	−0·13275	−0·13291	−0·13305	−0·13320	−0·13335	−0·13350

not exist in the BCM though the integral equation is singular. This is not surprising because the accurate values of singular integrals can be evaluated by potential functions.

Example 3. The conditions are the same as Example 2, but the radius of the circular body is taken according to the condition

$$\ln\left(\frac{r}{a}\right) = -\frac{1}{2(3-4\nu)}$$

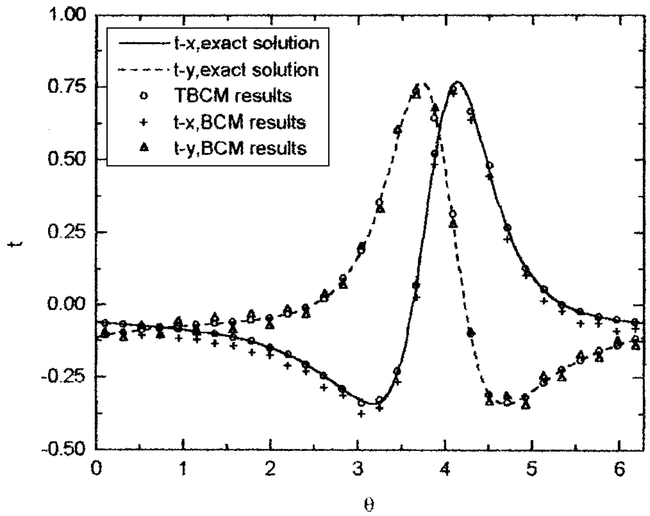

Figure 5. Traction components for circular body ($r = 1{\cdot}320193$)

We take $a = 1$, i.e. $r = 1{\cdot}320193$. The circular boundary is discretized by 15 quadratic elements spaced at equal increments. The numerical results shown in Figure 5 are compared with analytical solutions and those obtained by the BCM based on the conventional displacement BIE. The error of the traction obtained by the BCM is great in this example. The TBCM solution is agreement with the exact solution. In fact, the conventional displacement BIE does not have unique solution for the traction in the displacement BVP when the above condition is satisfied, as shown in [10].

7. CONCLUSION

In the present work, the idea of the BCM is extended to cover the traction BIE. A traction boundary contour method is proposed for linear elasticity. The formulations and implementation of the TBCM with quadratic boundary elements are presented for two-dimensional problems. The stresses at interior points and inside boundary elements are directly obtained from the BCM formulation of the Somigliana stress identity.

This method requires simply numerical evaluation of potential functions at the end of boundary elements for two-dimensional problems, even with curved boundary elements. For three-dimensional problems, surface integrals on the boundary elements can be converted, through an application of Stokes' theorem, into the line integrals over the bounding contours of these elements. Moreover, the formulation of TBCM is regular at the boundary points except the ends of boundary elements and corners. All integrals can be accurately calculated by potential functions in two-dimensional BCM so that the so-called boundary layer effect is eliminated. Besides those, the TBCM can give equivalent solutions to the boundary value problem of differential equations in two dimensions. The numerical analysis shows that the TBCM gives excellent results.

APPENDIX

Potential functions:

$$g_{111}(\xi,\eta) = k\left[\frac{\eta}{r^2} + \frac{2\xi^2\eta}{r^4}\right]$$

$$g_{112}(\xi,\eta) = k\left[\frac{2(1-v)^2}{(1-2v)}\,\mathrm{arctg}\,\frac{\eta}{\xi} + \frac{(2-3v)}{(1-2v)}\frac{\xi\eta}{r^2} + \frac{2\xi^3\eta}{r^4}\right]$$

$$g_{113}(\xi,\eta) = -k\left[\frac{(3-2v)}{2}\ln\left(\frac{r}{a}\right) + \frac{3(2v-1)}{2(1-2v)}\frac{\xi^2}{r^2} + \frac{2\xi^4}{r^4}\right]$$

$$g_{114}(\xi,\eta) = k\left[\frac{2\xi\eta^2}{r^4} - \frac{\xi}{r^2}\right]$$

$$g_{115}(\xi,\eta) = k\left[\frac{2\xi^2\eta^2}{r^4} - 0{\cdot}5(1-2v)\ln\left(\frac{r}{a}\right) - 1{\cdot}5\frac{\xi^2}{r^2}\right]$$

$$g_{116}(\xi,\eta) = k\left[\frac{2v(1-v)}{(1-2v)}\,\mathrm{arctg}\,\frac{\eta}{\xi} - \frac{(3v-2)}{(1-2v)}\frac{\xi\eta}{r^2} - \frac{2\xi^3\eta}{r^4}\right]$$

$$g_{117}(\xi,\eta) = k\left[(4v-3)\frac{\xi^2\eta}{r^2} + (10-8v)\frac{\xi^4\eta}{r^4} + 2(1-v)(1-2v)\eta\right]$$

$$g_{118}(\xi,\eta) = k\left[2(1-4v)\frac{\xi^4\eta}{r^4} - (1-4v)\frac{\xi^2\eta}{r^2} - (1+2v-4v^2)\eta\right]$$

$$g_{119}(\xi,\eta) = k\left[2(1-4v)\frac{\xi^5}{r^4} - (1-4v)\frac{\xi^3}{r^2} + (1-2v+4v^2)\xi\right]$$

$$g_{1110}(\xi,\eta) = k\left[(10-8v)\frac{\xi^5}{r^4} - (7-4v)\frac{\xi^3}{r^2} + (3-6v+4v^2)\xi\right]$$

$$g_{121}(\xi,\eta) = k\left[\frac{2\xi\eta^2}{r^4} - \frac{\xi}{r^2}\right]$$

$$g_{122}(\xi,\eta) = k\left[(1-v)\ln\left(\frac{r}{a}\right) - \frac{(2-3v)}{(1-2v)}\frac{\xi^2}{r^2} + \frac{2\xi^2\eta^2}{r^4}\right]$$

$$g_{123}(\xi,\eta) = k\left[(1-v)\,\mathrm{arctg}\,\frac{\eta}{\xi} + 1{\cdot}5\frac{\xi\eta}{r^2} - \frac{2\xi^3\eta}{r^4}\right]$$

$$g_{124}(\xi,\eta) = -g_{121}(\eta,\xi), \quad g_{125}(\xi,\eta) = -g_{123}(\eta,\xi)$$

$$g_{126}(\xi,\eta) = -g_{122}(\eta,\xi)$$

$$g_{127}(\xi,\eta) = k\left[(8v-10)\frac{\xi^5}{r^4} + (13-12v)\frac{\xi^3}{r^2} + 2(1-v)(1-2v)\xi\right]$$

$$g_{128}(\xi,\eta)=k\left[2(4v-1)\frac{\xi^5}{r^4}+3(1-4v)\frac{\xi^3}{r^2}-(1-2v+4v^2)\xi\right]$$

$$g_{129}(\xi,\eta)=-g_{128}(\eta,\xi),\quad g_{1210}(\xi,\eta)=-g_{127}(\eta,\xi)$$

$$g_{221}(\xi,\eta)=-g_{114}(\eta,\xi),\quad g_{222}(\xi,\eta)=-g_{116}(\eta,\xi)$$

$$g_{223}(\xi,\eta)=-g_{115}(\eta,\xi),\quad g_{224}(\xi,\eta)=-g_{111}(\eta,\xi)$$

$$g_{225}(\xi,\eta)=-g_{113}(\eta,\xi),\quad g_{226}(\xi,\eta)=-g_{112}(\eta,\xi)$$

$$g_{227}(\xi,\eta)=-g_{110}(\eta,\xi),\quad g_{228}(\xi,\eta)=-g_{119}(\eta,\xi)$$

$$g_{229}(\xi,\eta)=-g_{118}(\eta,\xi),\quad g_{2210}(\xi,\eta)=-g_{117}(\eta,\xi)$$

here $k=\mu/[2\pi(1-v)]$, μ is the shear modulus, $r^2=\xi^2+\eta^2$, and a is an artificial constant with the length dimension.

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