An Application of New Error Estimation Technique to the Boundary Element Method

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(NSC Project Number: NSC-98-2221-E-197-013)

ABSTRACT

In this study, we develop a novel estimation technique to obtain the optimal number of elements in the boundary element method (BEM) without having analytical solution. By using the complete Trefftz set as the analytical solution, namely quasi-analytical solution, the new error estimator is presented in the paper. The error curve versus different number of elements can be derived in the proposed techniques by comparing numerical solution with the quasi-analytical solution. By observing the error curve, we can obtain the optimal number of elements in BEM. One numerical example is taken to demonstrate the accuracy and efficiency of the proposed estimation technique.

Keywords: Trefftz complete set, boundary element method, estimation technique, quasi-analytical solution.

1. INTRODUCTION

Discretization of the boundary integral equation is an important stage of the Boundary Element Method (BEM) in solving engineering problems [1, 2, 4, 6, 7, 8, 11, 12], the discretization process, which transforms a continuous system into a discrete system with finite number of degrees of freedom, results in errors. Because of the fact that the reliability of the boundary element approximation is directly related to the discrete boundary element model, in which a proper mesh should be used to represent accurately the original problem both in its geometry and condition. In general, the discretization error is generated from the difference between the exact solution and the numerical result of the governing equation, but the exact solution of engineering problems is difficult to find from mathematical formulation. Furthermore, in the boundary element analysis, number of degrees of freedom depends solely on an analyst's experience and his/her intuition. Sometime we can get the accurate numerical solution, and sometimes we can get the poor results without having exact solution when we choose the different number of elements. Obviously, the choice of number of elements is a very objective and time-consuming process, and there is no guarantee that the final solution is sufficiently accurate. Obtaining a reliable error estimator is very important in order to guarantee a certain level of accuracy of the numerical result, and is a important ingredient of the stability analysis in numerical methods. Thus, estimation of the discretization error in the Boundary Element Method (BEM) is worthy of study.

Different integral equations can be used to find the residual of discretization [1, 4]. A large number of studies applied the hypersingular equation to find the residual as error estimator [1, 4]. Both the singular integral equation UT and hypersingular integral equation LM in the dual BEM can independently determine the unknown boundary data for the problems without a degenerate boundary [1]. The residuals obtained from these two equations can be used as indexes of error estimation. This provides a guide for remeshing without the problem of mismatch of the collocation points on the boundary in the sample point error method. However, it cannot be compared in the total error quantity in different number of mesh since it is pointwise error which depends on the number of elements. In this paper, we want to find a way of objective criterion to compare the error quantities in different number of mesh.

Therefore, we develop the novel error estimator to obtain the optimal number of elements of the BEM without having analytical solution. The convergent numerical solutions of the BEM can be obtained after adopting the optimal number of elements in unavailable analytic solution condition. This study has presented a way of calculating the total error quantity as an asymptotically exact error estimator by implementing the new estimator in BEM based on complete Trefftz set [3, 5, 8, 9, 10, 13] in solving potential problem. A quasi-analytical solution is simulated to substitute for real analytical solution by employing the aid of the Trefftz set. The convergence analysis of BEM versus different number of elements can be derived in the proposed techniques by comparing with the quasi-analytical solution. By observing the error curve versus different number of mesh, we can obtain the optimal number of elements in BEM. We develop a systematic error estimation scheme to search for the optimal number of elements.

2. PROBLEM STATEMENT AND METHOD OF SOLUTION

2.1 Problem statement

We consider the behavior of the medium governed by the Laplace equation with the mixed-type boundary conditions as:

$$\nabla^2 u(x) = 0, x \in D \tag{1}$$

$$u(x) = \bar{u}(x), x \in B_1 \tag{2}$$

$$t(x) = \frac{\partial u(x)}{\partial n_x} = \bar{t}(x), x \in B_2 \tag{3}$$

where ∇^2 is the operator with problem, $u(x)$ is the potential, D is the computational domain of the problem, $B = B_1 \cup B_2 = \partial D$ denotes the whole boundary of the domain D, in which of B_1 is the essential boundary (Dirichlet boundary) in which the potential is prescribed, B_2 is the natural boundary (Neumann boundary) where the normal derivative of the potential in the n_x direction is specified.

2.2 BEM formulation

The boundary integral equation for the domain point can be derived from Green's second identity as:

$$\begin{aligned} 2\pi u(x) &= \int T(s,x)u(s)dB(s) \\ &\quad - \int U(s,x)t(s)dB(s), x \in D \end{aligned} \tag{4}$$

where $U(s,x)$ is the fundamental solution which satisfies:

$$\nabla^2 U(s,x) = \delta(x-s) \tag{5}$$

in which $U(s,x) = \ln r$ and $\delta(x-s)$ is the Dirac-delta function, and $T(x,s)$ is defined by

$$T(s,x) = \frac{\partial U(s,x)}{\partial n_s} \tag{6}$$

in which n_s is the out-normal direction at the boundary point s. Discretizing the boundary B into N boundary elements in Eq.(4) as follows:

$$\begin{aligned} \pi u(x) &= CPV \int_B T(s,x)u(s)dB(s) \\ &\quad - RPV \int_B U(s,x)t(s)dB(s), x \in B \end{aligned} \tag{7}$$

where CPV is the Cauchy principal value and RPV is the Riemann principal value. The boundary integral equation is discretized by using N number of constant boundary elements, then the resulting algebraic system (UT method: conventional BEM of singular integral formulation) can be obtained as:

$$[U]\{t\} = [T]\{u\}. \tag{8}$$

For the problem with mixed-type boundary conditions, Eq.(8) can be decomposed into

$$[U_L \vdots U_R]_{N\times N} \begin{Bmatrix} t \\ \bar{t} \end{Bmatrix}_{N\times 1} = [T_L \vdots T_R]_{N\times N} \begin{Bmatrix} \bar{u} \\ u \end{Bmatrix}_{N\times 1}. \tag{9}$$

By collecting the given and unknown sets, we rearrange the influence matrices into

$$[U_L \vdots -T_R]_{N\times N}\begin{Bmatrix} t \\ u \end{Bmatrix}_{N\times 1} = [T_L \vdots -U_R]_{N\times N}\begin{Bmatrix} \bar{u} \\ \bar{t} \end{Bmatrix} \quad (10)$$

Eq.(8) can be simplified to

$$[A]_{N\times N}\, \underset{\sim}{x}_{N\times 1} = \underset{\sim}{b}_{N\times 1} \quad (11)$$

in which

$$[A] = [U_L \vdots -T_R],\ \underset{\sim}{x} = \begin{Bmatrix} \mathrm{t} \\ \mathrm{u} \end{Bmatrix} \quad (12)$$

$$\underset{\sim}{b} = [T_L \vdots -U_R]_{N\times N}\begin{Bmatrix} \bar{u} \\ \bar{t} \end{Bmatrix}. \quad (13)$$

We can derive the unknown vector, x, by utilizing the linear algebra solver. Next, we can collocate the points in the interested domain and calculate the potential by using the Eq.(8).

3. Novel error estimation technique

The derivation in formulating the analytical solution in the realistic engineering problem is not obtained easily. To overcome the drawback, an alternative problem is defined to be solved by implementing BEM. The domain shape and boundary condition type in the new problem are the same with the original problem. Furthermore, the alternative analytical solution in the new problem is similar with the real analytical solution in the original problem, namely quasi-analytical solution. After solving the new problem by BEM and comparing with the quasi-analytical solution, we develop a novel error estimation technique in this study. The derivation in the novel error estimation technique is presented as:

3.1 Definition of new problem

1. Quasi-analytical solution

In this study, a new boundary-value problem are derived based on the Trefftz concept, this geometry contour and boundary condition type in the new problem is the same with original problem, and also satisfies the same differential equation (DE) operator. The potential, $u(x)$, in the new problem at arbitrary point x in the domain is the known by function, the linear combination of the T-complete set functions as follows:

$$u^q(x) = \sum_{j=1}^{M} v_j \phi_j(x), x \in D \quad (14)$$

$\phi(x)$ can be chose by the T-complete set functions which satisfies the governing Eq.(1), M is the total number of the T-complete functions and v_j denotes the undetermined coefficient. Each of the functions of T-complete set functions satisfies the governing Laplace equation in Eq.(1) as:

$$\begin{aligned}&\nabla^2[\phi_{(1)}(x)] = 0, \nabla^2[\phi_{(2)}(x)] = 0, \cdots \\ &, \nabla^2[\phi_{(M-1)}(x)] = 0, \nabla^2[\phi_{(M)}(x)] = 0\end{aligned} \quad (15)$$

Because of the linear property of differential equation operator in G.E., the potential, $u^q(x)$, satisfies the G.E. as:

$$\begin{aligned}&\nabla^2[u^q(x)] = v_1\nabla^2[\phi_{(1)}(x)] + v_2\nabla^2[\phi_{(2)}(x)] + \\ &\cdots + v_{M-1}\nabla^2[\phi_{(M-1)}(x)] + v_M\nabla^2[\phi_{(M)}(x)] = 0\end{aligned} \quad (16)$$

By collocating M number of collocation points to match original B.C. in original problem, the undetermined coefficient, v_j, can be determined. Therefore, the quasi-analytical solution is similar to real analytical solution. The two problems have the same boundary contour and boundary condition type, and the boundary conditions of the new problem are given as:

$$\bar{\bar{u}}(x) = \sum_{j=1}^{M} v_j \phi_j(x), x \in B_1 \quad (17)$$

and its derivative in the normal direction (flux) as follows:

$$\begin{aligned}\bar{\bar{t}}(x) &= \frac{\partial \bar{\bar{u}}(x)}{\partial n_x} = \sum_{j=1}^{M} v_j \frac{\partial \phi_j(x)}{\partial n_x} \\ &= \sum_{j=1}^{M} v_j \omega_j(x), x \in B_2\end{aligned} \quad (18)$$

where $\bar{\bar{u}}(x)$ and $\bar{\bar{t}}(x)$ are the known potential and its derivative in the normal direction (flux). The error analysis between the new defined problem and the original problem are formulated on the next section as:

2. Error analysis between the new defined problem and the original problem

The relationship between the real analytical solution and quassi-analytical solution is shown as:

$$u^e(x) = u^q(x) + R_n(x) \tag{19}$$

where $R_n(x) = \sum_{j=M+1}^{\infty} v_j \phi_j(x)$

The remainder function $R_N(x)$ satisfies the G. E. and it is exponential convergence as

$$\|R_N(x)\| = O(r^n), 0 < r < 1$$

Therefore, the difference in the two solver of space is derived as:

$$\left\|u^e(\underset{\sim}{x}) - u^q(\underset{\sim}{x})\right\| = \|R_N(x)\| \le C(r^{-N}),$$

where C is bounded constant.

3. R.M.S comparing with quasi-analytical solution

The error quantity of numerical solution adopts the root mean squared (r.m.s) error by solving the new problem and comparing with quasi-analytical solution, which is defined as follows:

$$rel = \sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}(\tilde{u}-u)^2} \Bigg/ \sqrt{\frac{1}{N_t}\sum_{i=1}^{N_t}u^2} \tag{20}$$

where N_t is the number of field points, $\tilde{u}(x)$ is the numerical solution of the new problem by the BEM.

The flowchart of the formulation in implementing the novel errormator is shown in Fig.1.

4. NUMERICAL EXAMPLE

To show the performance of the error estimation scheme, we consider a rather standard problem, subjected to the mixed-type boundary condition as shown in Fig.2 (a) which have been solved by CHEN etc [5]. Our error estimation scheme can be applied to more general case with loss of generality through the case. The analytical solution of the radial temperature distribution is given by

$$u(r) = u(R_1) + u_0 \ln(r/R_1) \tag{21}$$

in which

$$u_0 = [u(R_2) - u(R_1)]/\ln(R_1/R_2) \tag{22}$$

where R_1 and R_2 denote the inner and outer radii, respectively. The nodes distribution is plotted in Fig.2 (b), the field solution in Eq. (21) is depicted in Fig.3 (a). The RMS results with different number of terms of Trefftz complete set function comparing with quasi analytical solution and real analytical solution, respectively, are shown in Fig.4. By observing the error curves, the optimal number of elements is 400. The potential is plotted by using the 400 number of elements (optimal elements) in Fig.3 (b) and 80 elements (less optimal elements) in Fig.3 (c), respectively. The field solution along the radius r=2.43 by using the different elements are show in Fig.5, and the potential at the point u(x=0.248, y=0.391) by using the different elements are shown in Fig.6.

5. CONCLUSION

In this paper, a new estimation technique is developed to obtain the optimal number of elements for the BEM, we successfully applied the systematic error estimation scheme to solve 2-D potential problems without having analytical solution. The numerical examination verifies the validity of the error estimator technique. The technique plays a role in determining the optimal number of elements which can be seen as a objective way to obtain the relative errors in different number of elements without having analytical solution, and we can obtain the numerical solution efficiently. The perplexing number of elements in the BEM can get. The convergent result is found from the convergent study in the case. Numerical results agreed very well with the analytical solutions.

6. ACKNOWLEDGEMENT

Financial support from the National Science Council under Grant no.NSC-98-2221-E-197-013 to the first author for Nation Ilan University is gratefully acknowledged.

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應用在邊界元素法的新誤差評估技術

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摘要

本研究發展出新的誤差評估方法，在不參照解析解的情況下，獲得邊界元素法的最佳元素數目。藉由使用 Trefftz 完全集合函數來創造新的解析解，並比較此新的解析解，我們可以得到邊界元素法在不同元素下的收斂行為分析。我們發展一套系統化的誤差評估技術來搜尋邊界元素法的最佳元素數目，最後提供數值案例證明在無需解析解的情況下，所提出的系統化誤差評估技術的有效性和準確性。

關鍵詞：元素離散，Trefftz 完全集合函數，收斂行為分析，系統化的誤差評估技術。

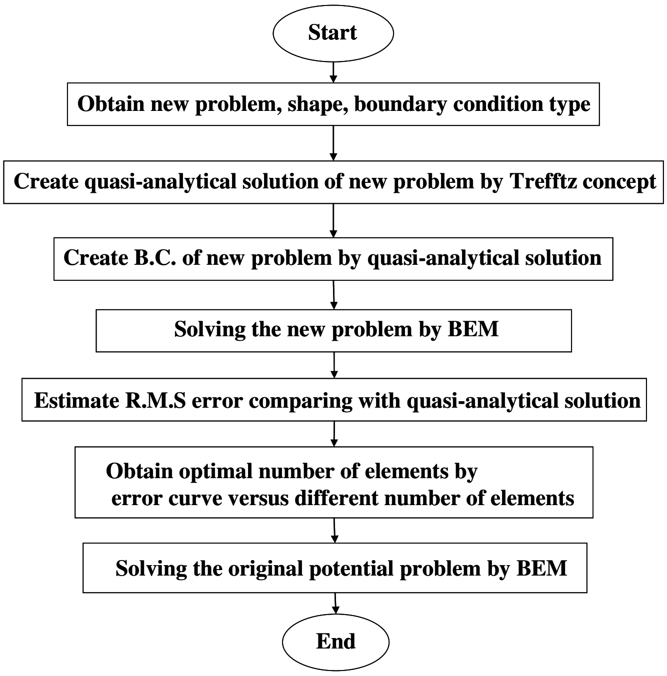
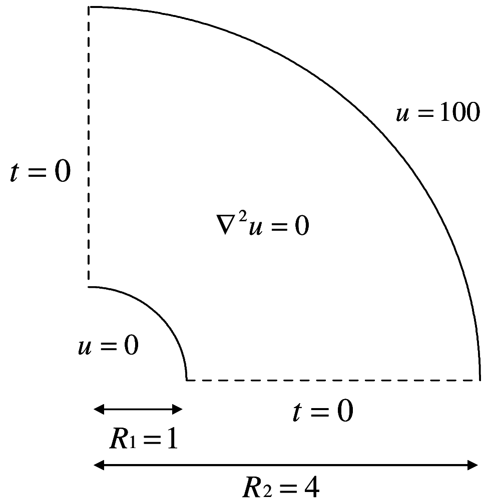
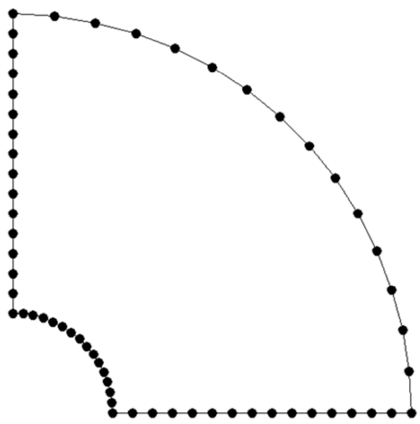


Fig. 1. Flowchart of the systematic error estimation scheme.

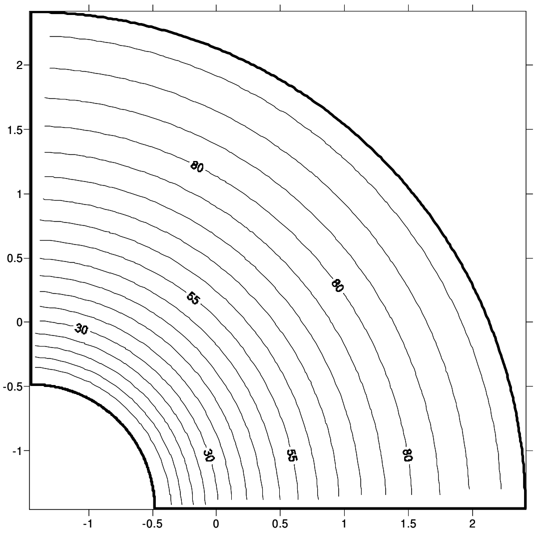


(a). problem sketch

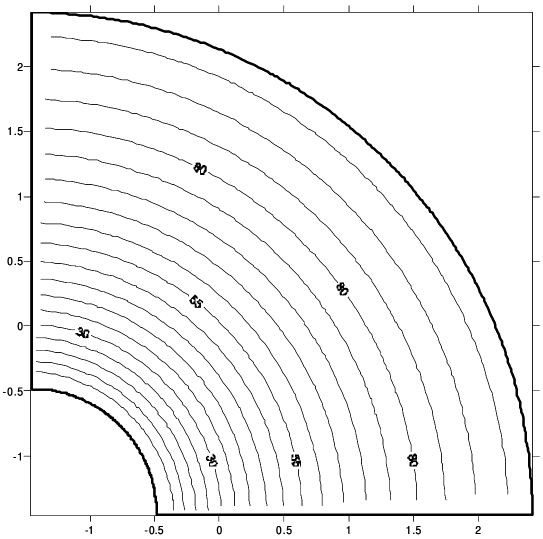


(b). element mesh

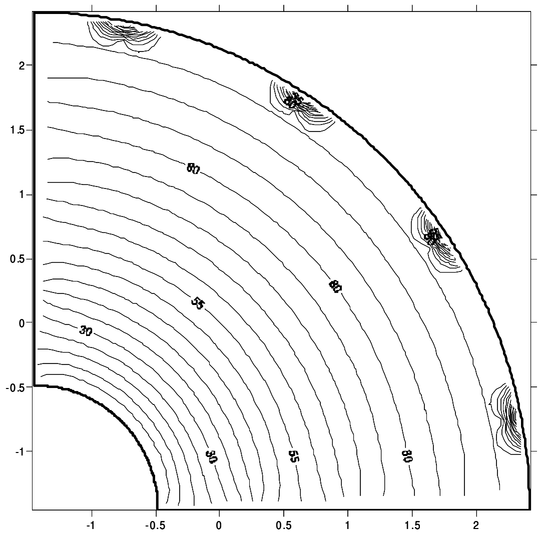
Fig. 2.(a) problem sketch, (b) element mesh..



(a). analytical solution



(b). 400 elements



(c). 80 elements

Fig. 3. Field solutions (a) analytical solution,

(b) 400 elements (optimal), (c) 80 elements.

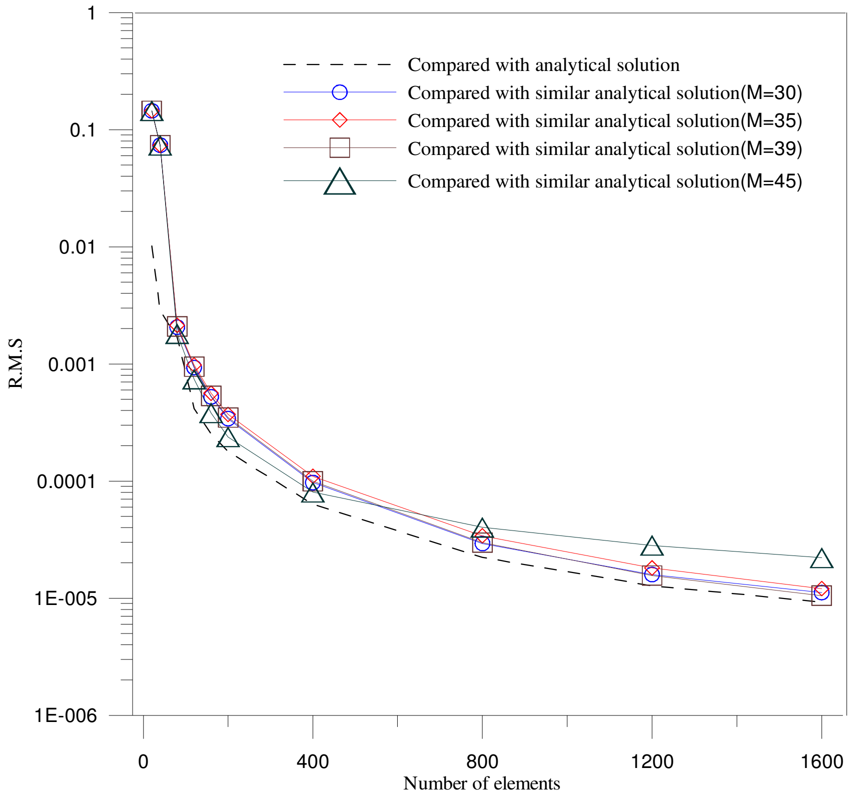


Fig. 4. The error analysis for the field solution with the different terms of Trefftz basis.

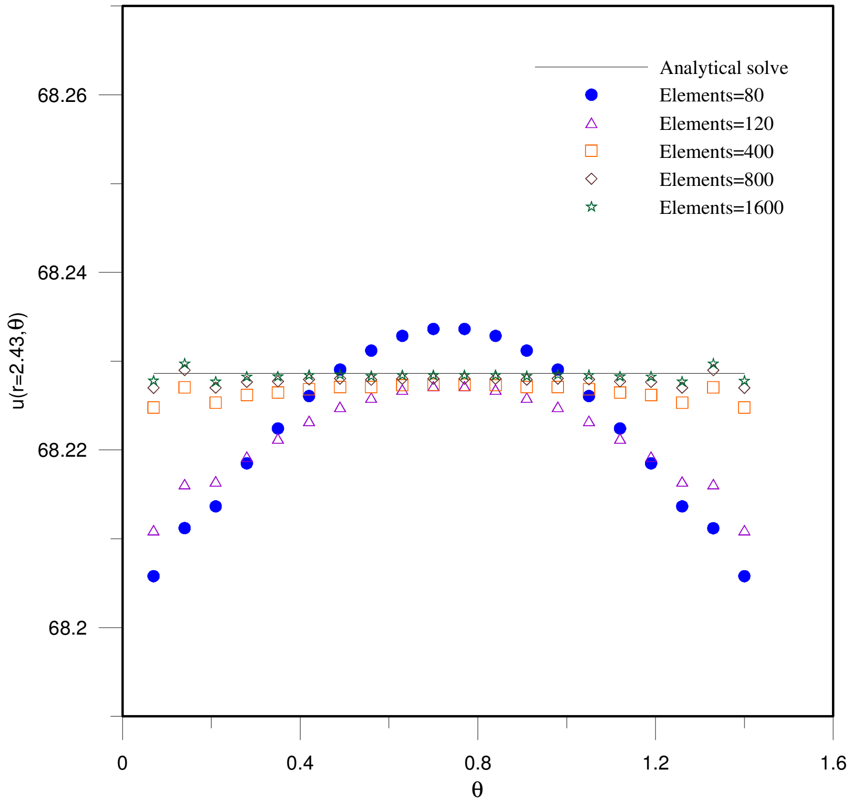


Fig. 5. The error analysis for the field solution along the radius r=2.43 with the different elements.

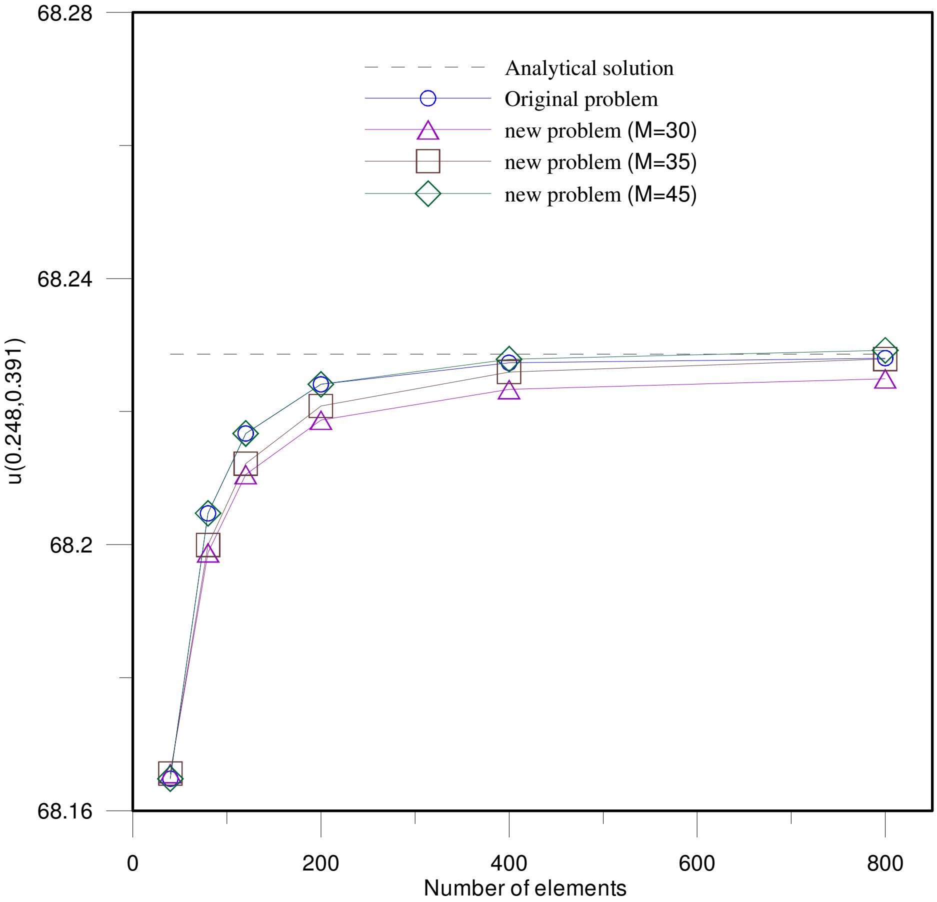


Fig. 6. u (0.248, 0.391) versus number of elements and with the different terms of Trefftz basis.