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ON SINGULAR INTEGRAL EQUATIONS AND FUNDAMENTAL SOLUTIONS OF POROELASTICITY

A. H.-D. CHENG†
Department of Civil and Environmental Engineering, University of Delaware, Newark, Delaware 19716, U.S.A.

E. DETOURNAY
Department of Civil Engineering, University of Minnesota, Minneapolis, Minnesota 55455, U.S.A.

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Abstract—This paper presents a unified formulation of the various singular integral equations used in the boundary element methods (BEM) for the solution of linear, quasi-static, anisotropic poroelasticity. In particular, a derivation is provided that connects the "direct method" with the "indirect methods". The presentation begins with an alternative derivative of the time and space dependent reciprocal integral. The Somigliana-type integral equations for the direct BEM are first constructed. By summing integral equations representing an interior and an exterior domain problem, the Somigliana (displacement discontinuity) and Volterra (stress discontinuity) type dislocation equations for indirect BEM are obtained. An extension to the edge dislocation method is discussed. These stress and displacement discontinuity equations are then combined to construct a symmetric Galerkin integral equation system. Through such construction, many intriguing connections among Green's functions of fluid source, dipole, dilatation, fluid body force, total body force, and displacement discontinuity are revealed. Finally, a complete compilation of fundamental solutions for the isotropic case is provided. © 1998 Elsevier Science Ltd. All rights reserved.

1. INTRODUCTION

In the boundary element method (BEM) literature, two types of integral equation representations are referred to: a "direct method" and an "indirect method" (Jaswon and Symm, 1977; Banerjee and Butterfield, 1981; Brebbia *et al.*, 1984). In a direct method, the integral equations are derived from a Green's second identity, or a reciprocity of work principle. The parameters in the integral expressions are "physical" quantities, such as potential, flux, displacement, stress. In contrast, an "indirect method" is based on distributing singular solutions at "fictitious" densities. After solving for the densities, a second application of the integral equations restores the desirable physical quantities. In yet another approach, integral equations are assembled following physical arguments; for example, singularities that simulate the opening of a fracture in an elastic solid (Crouch and Starfield, 1983) or fluid extraction from a fracture in a porous medium (Gringarten *et al.*, 1974) are distributed at magnitudes that represent crack opening displacement or fluid extraction rate.

The theoretical links between the direct and the indirect boundary integral equations for the Laplace (Jaswon and Symm, 1977; Brebbia and Butterfield, 1978), diffusion (Banerjee *et al.*, 1981) and elasticity operators (Jaswon and Symm, 1977; Altiero and Gavazza, 1980) are well known. The so-called fictitious densities are associated with jumps between the solutions of an exterior and an interior domain problem under the same set of boundary conditions. A theoretical connection also exists among the dislocation, the displacement discontinuity, and other types of integral equation methods (Mura, 1982; Hong and Chen, 1988).

The present paper focuses on the coupled theory of linear, quasi-static poroelasticity (Biot, 1941, 1955; Rice and Cleary, 1976; Detournay and Cheng, 1993; Coussy, 1995).

† Author to whom correspondence should be addressed. Tel.: 001 302 831 6787. Fax: 001 302 831 3640. E-mail: cheng@chaos.ce.udel.edu

Boundary element methods have been formulated for the direct method (Cheng and Liggett, 1984a; Cheng and Predeleanu, 1987; Nishimura and Kobayashi, 1989; Dargush and Banerjee, 1989, 1991), the displacement discontinuity method (Detournay and Cheng, 1987), and the stress discontinuity method (Carvalho, 1990). A symmetric Galerkin integral equation system has also recently been constructed by Pan and Maier (1997). Following the links that have been established in elasticity and potential theory, the connections between some of the integral equations have been demonstrated (Cheng *et al.*, 1990; Pan, 1991).

In this paper, a comprehensive presentation that unifies all the integral equation formulations is provided. We begin with an alternative derivation that leads to the reciprocity integral equations without invoking an adjoint system of equations as needed in the conventional derivation. Somigliana-type integral equations are created by using point force and point fluid dilatation Green's functions. These equations form the foundation of the direct BEM. Indirect integral equations are then obtained by adding the direct integral equations for the interior and the complementary exterior domain problem. Depending on the assumed jump and continuity conditions at the boundary between the interior and the exterior domain, two kinds of dislocation equations are derived: a Somigliana-type and a Volterra-type. In engineering terminology, they are, respectively, referred to as displacement and stress discontinuity method. Also, the equivalent of the dislocation method widely used to solve elastic fracture mechanics problems is derived. By combining the stress and displacement discontinuity integral equations, a symmetric Galerkin integral equation system is then obtained. The relations among the integral equations shed light into the various connections among Green's functions. Finally, a complete listing of the Green's functions appearing in the integral equations is provided in closed-form in an Appendix, for the isotropic case.

2. GOVERNING DIFFERENTIAL EQUATIONS

The governing equations of linear, quasi-static poroelasticity can be expressed as follows (Biot, 1955; Coussy, 1995; Cheng, 1997):

constitutive relations

$$\sigma_{ij} = M_{ijkl}e_{kl} - \alpha_{ij}p \tag{1}$$

$$p = M(\zeta - \alpha_{ij}e_{ij}) \tag{2}$$

equilibrium equations

$$\sigma_{ij,j} = -F_i \tag{3}$$

Darcy's law

$$q_i = -\kappa_{ij}(p_{,j} - f_j) \tag{4}$$

continuity equation

$$\frac{\partial \zeta}{\partial t} + q_{i,i} = \gamma \tag{5}$$

In the above, σ_{ij} is the total stress tensor, p the pore pressure, e_{ij} the strain tensor defined as $e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$, with u_i the solid displacement vector, ζ the variation of fluid content per unit volume of porous material, q_i the specific discharge vector, M_{ijkl} the drained elastic modulus tensor, α_{ij} the Biot stress coefficients tensor, M the Biot modulus, κ_{ij} the permeability tensor, F_i the total or bulk body force, f_i the fluid body force, and γ the fluid

source. We note the symmetry of the following material coefficients which is essential for deriving some of the relations below:

$$M_{ijkl} = M_{jikl} = M_{ijlk} = M_{klij} \tag{6}$$

$$\alpha_{ij} = \alpha_{ji} \tag{7}$$

$$\kappa_{ij} = \kappa_{ji} \tag{8}$$

For later use, we also define the following two quantities: a fluid relative displacement vector,

$$v_i = \int_0^t q_i \, \mathrm{d}t \tag{9}$$

and the volume of injected fluid due to a source γ,

$$Q = \int_0^t \gamma \, \mathrm{d}t \tag{10}$$

3. RECIPROCITY RELATION

Using the constitutive equations, (1) and (2), and the symmetry of the material coefficients, (6) and (7), it can be shown that the following reciprocity of work principle exists:

$$\sigma_{ij}^{(1)} e_{ij}^{(2)} + p^{(1)} \zeta^{(2)} = \sigma_{ij}^{(2)} e_{ij}^{(1)} + p^{(2)} \zeta^{(1)} \tag{11}$$

where superscripts (1) and (2) denote quantities under two independent stress and strain states at different spatial and time coordinates. For the present purpose, the following form is taken:

$$\begin{aligned} &\sigma_{ij}^{(1)}(\chi, \tau) e_{ij}^{(2)}(\chi - \mathbf{x}, t - \tau) + p^{(1)}(\chi, \tau) \zeta^{(2)}(\chi - \mathbf{x}, t - \tau) \\ &\qquad = \sigma_{ij}^{(2)}(\chi - \mathbf{x}, t - \tau) e_{ij}^{(1)}(\chi, \tau) + p^{(2)}(\chi - \mathbf{x}, t - \tau) \zeta^{(1)}(\chi, \tau) \end{aligned} \tag{12}$$

This special form allows an alternative derivation of the reciprocal integral equation (see Appendix A), without invoking an adjoint system of equation as in the traditional procedure. Integrating (12) over the solution domain and time, and applying the divergence theorem, yields the reciprocal integral equation (see Appendix A)

$$\begin{aligned} &\int_0^t \int_\Gamma (\sigma_{ij}^{(1)} n_j u_i^{(2)} - \sigma_{ij}^{(2)} n_j u_i^{(1)}) \, \mathrm{d}\chi \, \mathrm{d}\tau - \int_0^t \int_\Gamma (p^{(1)} v_i^{(2)} n_i - p^{(2)} v_i^{(1)} n_i) \, \mathrm{d}\chi \, \mathrm{d}\tau \\ &\quad + \int_0^t \int_\Omega (F_i^{(1)} u_i^{(2)} - F_i^{(2)} u_i^{(1)}) \, \mathrm{d}\chi \, \mathrm{d}\tau + \int_0^t \int_\Omega (f_i^{(1)} v_i^{(2)} - f_i^{(2)} v_i^{(1)}) \, \mathrm{d}\chi \, \mathrm{d}\tau \\ &\quad - \int_0^t \int_\Omega (Q^{(1)} p^{(2)} - Q^{(2)} p^{(1)}) \, \mathrm{d}\chi \, \mathrm{d}\tau + \int_0^t \int_\Omega (E_{ij}^{(1)} \sigma_{ij}^{(2)} - E_{ij}^{(2)} \sigma_{ij}^{(1)}) \, \mathrm{d}\chi \, \mathrm{d}\tau \\ &\quad + \int_0^t \int_\Omega (p^{(1)} \zeta_0^{(2)} - p^{(2)} \zeta_0^{(1)}) \, \mathrm{d}\chi \, \mathrm{d}\tau = 0 \end{aligned} \tag{13}$$

in which Γ is the bounding surface of the domain Ω, and n_i the component of the unit

outward normal to Γ. A new quantity E_{ij}, a nucleus of strain, has been introduced to keep track of the displacement discontinuity solution (see Appendix A). Integral equations similar to (13) were initially derived by Predeleanu (1968) and Cleary (1977). However, the above form which explicitly keeps track of the total and fluid forces, and fluid volume injection, was first presented by Cheng and Predeleanu (1987). The reciprocal integral eqn (13) has further incorporated the nucleus of strain E_{ij} whose usefulness will become evident later. These terms provide the pathway for the creation of the various integral equation representations as demonstrated below.

4. DIRECT METHOD

To obtain singular integral equations of Somigliana type, the following substitutions, respectively, corresponding to an instantaneous point total force in the x_k-direction, an instantaneous point fluid force in the x_k-direction, an instantaneous fluid volume dilatation, and an instantaneous displacement discontinuity, are made

$$F_{ik}^{(2)} = \delta_{ik}\delta(\chi-\mathbf{x})\delta(t-\tau) \tag{14}$$

$$f_{ik}^{(2)} = \delta_{ik}\delta(\chi-\mathbf{x})\delta(t-\tau) \tag{15}$$

$$Q^{(2)} = \delta(\chi-\mathbf{x})\delta(t-\tau) \tag{16}$$

$$E_{ijkl}^{(2)} = -\tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{jk}\delta_{il})\delta(\chi-\mathbf{x})\delta(t-\tau) \tag{17}$$

where δ_{ik} is the Kronecker delta, and $\delta()$ the Dirac delta function with singularity located at point $\mathbf{x}$ and time τ, respectively. The expanded indices for the body forces F_i and f_i, and nucleus of strain E_{ij}, means that multiple substitutions, $k, l = 1, 2$ for 2-D and $1, 2, 3$ for 3-D, are made, such that multiple integral equations are generated. The above substitutions yield

$$\begin{aligned}\beta u_k(\mathbf{x},t) = &\int_0^t\int_\Gamma [u_{ik}^{Fi}(\chi-\mathbf{x},t-\tau)\sigma_{ij}(\chi,\tau)n_j(\chi)-\sigma_{ijk}^{Fi}(\chi-\mathbf{x},t-\tau)n_j(\chi)u_i(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau\\ &-\int_0^t\int_\Gamma [v_{ik}^{Fi}(\chi-\mathbf{x},t-\tau)p(\chi,\tau)n_i(\chi)-p_k^{Fi}(\chi-\mathbf{x},t-\tau)v_i(\chi,\tau)n_i(\chi)]\,\mathrm{d}\chi\,\mathrm{d}\tau\\ &+\int_0^t\int_\Omega [u_{ik}^{Fi}(\chi-\mathbf{x},t-\tau)F_i(\chi,\tau)+v_{ik}^{Fi}(\chi-\mathbf{x},t-\tau)f_i(\chi,\tau)\\ &-p_k^{Fi}(\chi-\mathbf{x},t-\tau)Q(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \qquad +\int_0^t\int_\Omega p_k^{Fi}(\chi-\mathbf{x},t-\tau)\zeta(\chi,0)\,\mathrm{d}\chi\,\mathrm{d}\tau\end{aligned} \tag{18}$$

$$\begin{aligned}\beta v_k(\mathbf{x},t) = &\int_0^t\int_\Gamma [u_{ik}^{fi}(\chi-\mathbf{x},t-\tau)\sigma_{ij}(\chi,\tau)n_j(\chi)-\sigma_{ijk}^{fi}(\chi-\mathbf{x},t-\tau)n_j(\chi)u_i(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau\\ &-\int_0^t\int_\Gamma [v_{ik}^{fi}(\chi-\mathbf{x},t-\tau)p(\chi,\tau)n_i(\chi)-p_k^{fi}(\chi-\mathbf{x},t-\tau)v_i(\chi,\tau)n_i(\chi)]\,\mathrm{d}\chi\,\mathrm{d}\tau\\ &+\int_0^t\int_\Omega [u_{ik}^{fi}(\chi-\mathbf{x},t-\tau)F_i(\chi,\tau)+v_{ik}^{fi}(\chi-\mathbf{x},t-\tau)f_i(\chi,\tau)\\ &-p_k^{fi}(\chi-\mathbf{x},t-\tau)Q(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \qquad +\int_0^t\int_\Omega p_k^{fi}(\chi-\mathbf{x},t-\tau)\zeta(\chi,0)\,\mathrm{d}\chi\,\mathrm{d}\tau\end{aligned} \tag{19}$$

$$-\beta p(\mathbf{x},t) = \int_0^t \int_\Gamma [u_i^{li}(\chi-\mathbf{x},t-\tau)\sigma_{ij}(\chi,\tau)n_j(\chi) - \sigma_{ij}^{li}(\chi-\mathbf{x},t-\tau)n_j(\chi)u_i(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$-\int_0^t \int_\Gamma [v_i^{li}(\chi-\mathbf{x},t-\tau)n_i(\chi)p(\chi,\tau) - p^{li}(\chi-\mathbf{x},t-\tau)v_i(\chi,\tau)n_i(\chi)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+\int_0^t \int_\Omega [u_i^{li}(\chi-\mathbf{x},t-\tau)F_i(\chi,\tau) + v_i^{li}(\chi-\mathbf{x},t-\tau)f_i(\chi,\tau)$$
$$-p^{li}(\chi-\mathbf{x},t-\tau)Q(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \qquad + \int_0^t \int_\Omega p^{li}(\chi-\mathbf{x},t-\tau)\zeta(\chi,0)\,\mathrm{d}\chi\,\mathrm{d}\tau \tag{20}$$

$$-\beta\sigma_{kl}(\mathbf{x},t) = \int_0^t \int_\Gamma [u_{ikl}^{di}(\chi-\mathbf{x},t-\tau)\sigma_{ij}(\chi,\tau)n_j(\chi) - \sigma_{ijkl}^{di}(\chi-\mathbf{x},t-\tau)n_j(\chi)u_i(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$-\int_0^t \int_\Gamma [v_{ikl}^{di}(\chi-\mathbf{x},t-\tau)p(\chi,\tau)n_i(\chi) - p_{kl}^{di}(\chi-\mathbf{x},t-\tau)v_i(\chi,\tau)n_i(\chi)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+\int_0^t \int_\Omega [u_{ikl}^{di}(\chi-\mathbf{x},t-\tau)F_i(\chi,\tau) + v_{ikl}^{di}(\chi-\mathbf{x},t-\tau)f_i(\chi,\tau)$$
$$-p_{kl}^{di}(\chi-\mathbf{x},t-\tau)Q(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \qquad + \int_0^t \int_\Omega p_{kl}^{di}(\chi-\mathbf{x},t-\tau)\zeta(\chi,0)\,\mathrm{d}\chi\,\mathrm{d}\tau \tag{21}$$

in which we have set $E_{ij}^{(1)} = 0$ by not considering its physical presence, yet have retained total and fluid body forces, and fluid source. In the above equations, β is a constant of geometry determined by a Cauchy principal value integration. β is equal to 0, 1 and 1/2, respectively, when the "base point" $\mathbf{x}$ is located outside, inside the domain Ω, and on a smooth part of the boundary Γ. On a corner, β is proportional to the interior angle (2-D) or interior solid angle (3-D). The quantities denoted by superscripts are free-space Green's functions governed by (A2), (A3), (3)–(5), (9) and (10), with the respective substitution of the forcing terms shown in (14)–(17). The conventions for the superscripts used to denote the various singular solutions (force, source, continuous, instantaneous, etc.) are given in Table 1. Figure 1 gives a graphic illustration of the sign convention of some of these singularities. For the case of isotropy, a complete listing of these solutions in closed form is compiled and given in Appendix D.

Equations (18)–(20) are not yet in a form suitable for a BEM implementation. Since the physical conditions are hardly given in terms of fluid displacement v_i and fluid injection volume Q, an integration by parts is performed to convert them into fluid discharge q_i, and source intensity γ, respectively. We can also carry out the time integration for terms containing the initial distribution $\zeta(\chi,0)$. The following integration equations are obtained

Table 1. Convention for naming singular solutions

Singularity type	Instantaneous	Continuous
Total force (F_i)	*Fi*	*Fc*
Fluid force (f_i)	*fi*	*fc*
Fluid source (γ)	*si*	*sc*
Fluid dipole (γ_i)	*pi*	*pc*
Fluid dilatation (Q)	*li*	*lc*
Displacement discontinuity	*di*	*dc*
Edge dislocation	*ei*	*ec*

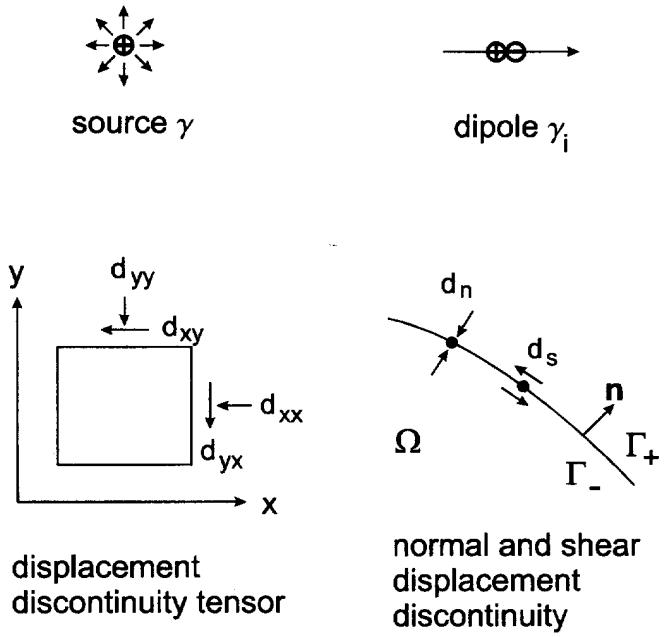

Fig. 1. Convention of singularities.

$$\beta u_k(\mathbf{x},t) = \int_0^t \int_\Gamma [u_{ik}^{Fi}(\chi-\mathbf{x},t-\tau)\sigma_{ij}(\chi,\tau)n_j(\chi) - \sigma_{ijk}^{Fi}(\chi-\mathbf{x},t-\tau)n_j(\chi)u_i(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$-\int_0^t \int_\Gamma [q_{ik}^{Fc}(\chi-\mathbf{x},t-\tau)n_i(\chi)p(\chi,\tau) - p_k^{Fc}(\chi-\mathbf{x},t-\tau)q_i(\chi,\tau)n_i(\chi)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+\int_0^t \int_\Omega [u_{ik}^{Fi}(\chi-\mathbf{x},t-\tau)F_i(\chi,\tau) + v_{ik}^{Fi}(\chi-\mathbf{x},t-\tau)f_i(\chi,\tau)$$
$$-p_k^{Fc}(\chi-\mathbf{x},t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \qquad -\int_\Omega p_k^{Fc}(\chi-\mathbf{x},t)\zeta(\chi,0)\,\mathrm{d}\chi \tag{22}$$

$$\beta v_k(\mathbf{x},t) = \int_0^t \int_\Gamma [u_{ik}^{fi}(\chi-\mathbf{x},t-\tau)\sigma_{ij}(\chi,\tau)n_j(\chi) - \sigma_{ijk}^{fi}(\chi-\mathbf{x},t-\tau)n_j(\chi)u_i(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$-\int_0^t \int_\Gamma [q_{ik}^{fc}(\chi-\mathbf{x},t-\tau)n_i(\chi)p(\chi,\tau) - p_k^{fc}(\chi-\mathbf{x},t-\tau)q_i(\chi,\tau)n_i(\chi)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+\int_0^t \int_\Omega [u_{ik}^{fi}(\chi-\mathbf{x},t-\tau)F_i(\chi,\tau) + v_{ik}^{fi}(\chi-\mathbf{x},t-\tau)f_i(\chi,\tau)$$
$$-p_k^{fc}(\chi-\mathbf{x},t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \qquad -\int_\Omega p_k^{fc}(\chi-\mathbf{x},t)\zeta(\chi,0)\,\mathrm{d}\chi \tag{23}$$

$$-\beta p(\mathbf{x},t) = \int_0^t \int_\Gamma [u_i^{li}(\chi-\mathbf{x},t-\tau)\sigma_{ij}(\chi,\tau)n_j(\chi) - \sigma_{ij}^{li}(\chi-\mathbf{x},t-\tau)n_j(\chi)u_i(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$-\int_0^t \int_\Gamma [q_i^{lc}(\chi-\mathbf{x},t-\tau)n_i(\chi)p(\chi,\tau) - p^{lc}(\chi-\mathbf{x},t-\tau)q_i(\chi,\tau)n_i(\chi)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$

$$+\int_0^t\int_\Omega [u_i^{li}(\chi-\mathbf{x},t-\tau)F_i(\chi,\tau)+v_i^{li}(\chi-\mathbf{x},t-\tau)f_i(\chi,\tau)$$

$$-p^{lc}(\chi-\mathbf{x},t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \qquad -\int_\Omega p^{lc}(\chi-\mathbf{x},t)\zeta(\chi,0)\,\mathrm{d}\chi \tag{24}$$

$$-\beta\sigma_{kl}(\mathbf{x},t) = \int_0^t\int_\Gamma [u_{ikl}^{di}(\chi-\mathbf{x},t-\tau)\sigma_{ij}(\chi,\tau)n_j(\chi)-\sigma_{ijkl}^{di}(\chi-\mathbf{x},t-\tau)n_j(\chi)u_i(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$

$$-\int_0^t\int_\Gamma [q_{ikl}^{dc}(\chi-\mathbf{x},t-\tau)n_i(\chi)p(\chi,\tau)-p_{kl}^{dc}(\chi-\mathbf{x},t-\tau)q_i(\chi,\tau)n_i(\chi)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$

$$+\int_0^t\int_\Omega [u_{ikl}^{di}(\chi-\mathbf{x},t-\tau)F_i(\chi,\tau)+v_{ikl}^{di}(\chi-\mathbf{x},t-\tau)f_i(\chi,\tau)$$

$$-p_{kl}^{dc}(\chi-\mathbf{x},t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \qquad -\int_\Omega p_{kl}^{dc}(\chi-\mathbf{x},t)\zeta(\chi,0)\,\mathrm{d}\chi \tag{25}$$

in which we have replaced some of the Green's functions by equivalent quantities using the following general formulae:

$$\frac{\partial(\sim)_{(\cdots)}^{(*)c}}{\partial t} = (\sim)_{(\cdots)}^{(*)i} \tag{26}$$

$$v_{i(\cdots)}^{(*)i} = q_{i(\cdots)}^{(*)c} \tag{27}$$

Although not directly used here, we also introduce several relations for future use:

$$\frac{\partial(\sim)_{(\cdots)}^{s(*)}}{\partial t} = (\sim)_{(\cdots)}^{l(*)} \tag{28}$$

$$(\sim)_{(\cdots)}^{lc} = (\sim)_{(\cdots)}^{si} \tag{29}$$

$$v_i^{li} = q_i^{si} \tag{30}$$

In the above, $(\sim)$ is to be replaced by any of the Green's function entities, such as u_{ij}, σ_{ijk}, etc. The superscript (*) corresponds to a forcing function designation: F, f, s, etc. for the first superscript, and i or c for the second, and the ellipses $(\cdots)$ in the subscript mean any number of indices as needed.

The origin of (26) is obvious as an instantaneous forcing function is the time derivative of a continuous one. Equation (28) is based on (10). However, rather than differentiate the dilation influence functions, we differentiate the source expressions. This is because that fluid dilatation influence functions, introduced by substituting Q by $\delta(\chi-\mathbf{x})\delta(t-\tau)$, are the time derivatives of fluid source influence functions, associated with $\gamma = \delta(\chi-\mathbf{x})\delta(t-\tau)$. Equation (29) is a consequence of both (26) and (28). The conversion between fluid displacement and the specific flux terms as shown in (27) is based on the definition (9). The last relation (30) is also evident from the various arguments above.

Although four integral equations are presented, only (22) and (24) are needed in a BEM implementation for the solution of an initial/boundary problem. In a typical problem for poroelasticity,

- either boundary tractions $t_i = \sigma_{ij}n_{ij}$ or displacements u_i, and
- either fluid pressure p or normal flux $q = q_i n_i$

are prescribed on a given part of the boundary. Equations (22) and (24) are enforced at a set of boundary nodes, leading to a collocation procedure. Due to the transient nature of the integral equations, the discretization takes place both in time and in space. Through a time-stepping or convolutional integral process, the missing boundary data in terms of the physical parameters of traction, displacement, pressure or flux are solved. We shall refrain from discussing further details of numerical issues. Refer to Cheng and Detournay (1988), Vandamme *et al.* (1989), Dargush and Banerjee (1989, 1991) for typical numerical implementations.

To avoid the numerical handling of time integration, integral transforms are sometimes utilized. For example, applying Laplace transformation to (22) and (24) and utilizing the convolutional theorem, yields the following equations from which the time integrals have been eliminated (Cheng and Liggett, 1984a; Cheng and Detournay, 1988; Badmus *et al.*, 1993)

$$\begin{aligned}\beta\tilde{u}_k(\mathbf{x},s) = &\int_\Gamma [\tilde{u}_{ik}^{Fi}(\chi-\mathbf{x},s)\tilde{\sigma}_{ij}(\chi,s)n_j(\chi) - \tilde{\sigma}_{ijk}^{Fi}(\chi-\mathbf{x},s)n_j(\chi)\tilde{u}_i(\chi,s)]\,\mathrm{d}\chi \\ &- \int_\Gamma \frac{1}{s}[\tilde{q}_{ik}^{Fi}(\chi-\mathbf{x},s)n_i(\chi)\tilde{p}(\chi,s) - \tilde{p}_k^{Fi}(\chi-\mathbf{x},s)\tilde{q}_i(\chi,s)n_i(\chi)]\,\mathrm{d}\chi \\ &+ \int_\Omega \left[\tilde{u}_{ik}^{Fi}(\chi-\mathbf{x},s)\tilde{F}_i(\chi,s) + \tilde{v}_{ik}^{Fi}(\chi-\mathbf{x},s)\tilde{f}_i(\chi,s) - \frac{1}{s}\tilde{p}_k^{Fi}(\chi-\mathbf{x},s)\tilde{\gamma}(\chi,s)\right]\mathrm{d}\chi \\ &- \int_\Omega \frac{1}{s}\tilde{p}_k^{Fi}(\chi-\mathbf{x},s)\zeta(\chi,0)\,\mathrm{d}\chi \end{aligned} \tag{31}$$

$$\begin{aligned}-\beta\tilde{p}(\mathbf{x},s) = &\int_\Gamma s[\tilde{u}_i^{si}(\chi-\mathbf{x},s)\tilde{\sigma}_{ij}(\chi,s)n_j(\chi) - \tilde{\sigma}_{ij}^{si}(\chi-\mathbf{x},s)n_j(\chi)\tilde{u}_i(\chi,s)]\,\mathrm{d}\chi \\ &- \int_\Gamma [\tilde{q}_i^{si}(\chi-\mathbf{x},s)n_i(\chi)\tilde{p}(\chi,s) - \tilde{p}^{si}(\chi-\mathbf{x},s)\tilde{q}_i(\chi,s)n_i(\chi)]\,\mathrm{d}\chi \\ &+ \int_\Omega [s\tilde{u}_i^{si}(\chi-\mathbf{x},s)\tilde{F}_i(\chi,s) + s\tilde{v}_i^{si}(\chi-\mathbf{x},s)\tilde{f}_i(\chi,s) - \tilde{p}^{si}(\chi-\mathbf{x},s)\tilde{\gamma}(\chi,s)]\,\mathrm{d}\chi \\ &- \int_\Omega \tilde{p}^{si}(\chi-\mathbf{x},s)\zeta(\chi,0)\,\mathrm{d}\chi \end{aligned} \tag{32}$$

where the tilde indicates a Laplace transform, and s is the transform parameter. We note that the Green's functions are unified in such a way that only two types of singularities appear: an instantaneous point force (Fi) and an instantaneous point source (si). In doing so, the following relations based on (26) and (28) are invoked:

$$(\sim)_{(\cdots)i}^{Fi} = s(\sim)_{(\cdots)i}^{Fc} \tag{33}$$

$$(\sim)_{(\cdots)}^{li} = s(\sim)_{(\cdots)}^{si} \tag{34}$$

We also notice the utilization of (29). Integral equations of similar nature can be derived in the frequency domain or via Fourier transform (Cheng and Liu, 1986), and in terms of a steadily moving coordinate system (Cheng and Liggett, 1984b).

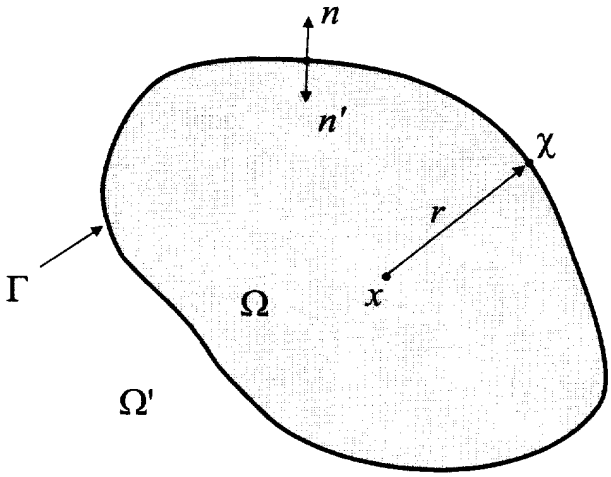

Fig. 2. Complementary regions bounded by a common surface.

5. INDIRECT METHOD

Consider a region Ω bounded by Γ. Its complementary region is denoted as Ω' as depicted in Fig. 2. The unit outward normal $\mathbf{n}$ is associated with Ω, $\mathbf{n}'$ with Ω'. Equations (22)–(25) are now written for Ω', which is free from body forces and fluid sources

$$0 = \int_0^t \int_\Gamma (u_{ik}^{Fi}\sigma'_{ij}n'_j - \sigma_{ijk}^{Fi}n'_j u'_i)\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_0^t \int_\Gamma (q_{ik}^{Fc}n'_i p' - p_k^{Fc}q'_i n'_i)\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_{\Omega'} p_k^{Fc}\zeta'_0\,\mathrm{d}\chi \tag{35}$$

$$0 = \int_0^t \int_\Gamma (u_{ik}^{fi}\sigma'_{ij}n'_j - \sigma_{ijk}^{fi}n'_j u'_i)\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_0^t \int_\Gamma (q_{ik}^{fc}n'_i p' - p_k^{fc}q'_i n'_i)\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_{\Omega'} p_k^{fc}\zeta'_0\,\mathrm{d}\chi \tag{36}$$

$$0 = \int_0^t \int_\Gamma (u_i^{li}\sigma'_{ij}n'_j - \sigma_{ij}^{li}n'_j u'_i)\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_0^t \int_\Gamma (q_i^{lc}n'_i p' - p^{lc}q'_i n'_i)\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_{\Omega'} p^{lc}\zeta'_0\,\mathrm{d}\chi \tag{37}$$

$$0 = \int_0^t \int_\Gamma (u_{ikl}^{di}\sigma'_{ij}n'_j - \sigma_{ijkl}^{di}n'_j u'_i)\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_0^t \int_\Gamma (q_{ikl}^{dc}n'_i p' - p_{kl}^{dc}q'_i n'_i)\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_{\Omega'} p_{kl}^{dc}\zeta'_0\,\mathrm{d}\chi \tag{38}$$

where the prime is used to denote quantities associated with Ω'. We have retained the initial condition ζ'_0 in the above, for a reason that will become evident later. We also note that $n'_i = -n_i$ as the outward normals of Ω and Ω' oppose each other. The left-hand sides of (35)–(37) are zero because the base point is located in Ω. Summing (35)–(37) with (22)–(24), the following expressions are obtained

$$\beta u_k = \int_0^t \int_\Gamma [u_{ik}^{Fi}(\sigma_{ij}-\sigma'_{ij})n_j - \sigma_{ijk}^{Fi}n_j(u_i-u'_i)]\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_0^t \int_\Gamma [q_{ik}^{Fc}n_i(p-p') - p_k^{Fc}(q_i-q'_i)n_i]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+ \int_0^t \int_\Omega (u_{ik}^{Fi}F_i + v_{ik}^{Fi}f_i - p_k^{Fc}\gamma)\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_{\Omega+\Omega'} p_k^{Fc}\zeta_0\,\mathrm{d}\chi \tag{39}$$

$$\beta v_k = \int_0^t \int_\Gamma [u_{ik}^{fi}(\sigma_{ij}-\sigma'_{ij})n_j - \sigma_{ijk}^{fi}n_j(u_i-u'_i)]\,\mathrm{d}\chi\,\mathrm{d}\tau - \int_0^t \int_\Gamma [q_{ik}^{fc}n_i(p-p') - p_k^{fc}(q_i-q'_i)n_i]\,\mathrm{d}\chi\,\mathrm{d}\tau$$

$$+\int_0^t\int_\Omega (u_{ik}^{fi}F_i+v_{ik}^{fi}f_i-p_k^{fc}\gamma)\,\mathrm{d}\chi\,\mathrm{d}\tau-\int_{\Omega+\Omega'} p_k^{fc}\zeta_0\,\mathrm{d}\chi \quad (40)$$

$$-\beta p=\int_0^t\int_\Gamma [u_i^{li}(\sigma_{ij}-\sigma'_{ij})n_j-\sigma_{ij}^{li}n_j(u_i-u'_i)]\,\mathrm{d}\chi\,\mathrm{d}\tau-\int_0^t\int_\Gamma [q_i^{lc}n_i(p-p')-p^{lc}(q_i-q'_i)n_i]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+\int_0^t\int_\Omega (u_i^{li}F_i+v_i^{li}f_i-p^{lc}\gamma)\,\mathrm{d}\chi\,\mathrm{d}\tau-\int_{\Omega+\Omega'} p^{lc}\zeta_0\,\mathrm{d}\chi \quad (41)$$

$$-\beta\sigma_{kl}=\int_0^t\int_\Gamma [u_{ikl}^{di}(\sigma_{ij}-\sigma'_{ij})n_j-\sigma_{ijkl}^{di}n_j(u_i-u'_i)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$-\int_0^t\int_\Gamma [q_{ikl}^{dc}n_i(p-p')-p_{kl}^{dc}(q_i-q'_i)n_i]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+\int_0^t\int_\Omega (u_{ikl}^{di}F_i+v_{ikl}^{di}f_i-p_{kl}^{dc}\gamma)\,\mathrm{d}\chi\,\mathrm{d}\tau-\int_{\Omega+\Omega'} p_{kl}^{dc}\zeta_0\,\mathrm{d}\chi \quad (42)$$

These expressions form the basis of the integral equations presented below.

6. STRESS DISCONTINUITY METHOD

For a problem defined in Ω, we can impose a complementary problem in Ω' in which the solid displacement and fluid normal flux along the boundary are identical to that of the primary problem ($u_i = u'_i, q_i n_i = q'_i n_i$). Equations (39)–(42) therefore, reduce to

$$\beta u_k(\mathbf{x},t)=\int_0^t\int_\Gamma [u_{ki}^{Fi}(\mathbf{x}-\chi,t-\tau)s_i(\chi,\tau)+q_{ki}^{Fc}(\mathbf{x}-\chi,t-\tau)n_i(\chi)s(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+\int_0^t\int_\Omega [u_{ki}^{Fi}(\mathbf{x}-\chi,t-\tau)F_i(\chi,\tau)+v_{ki}^{Fi}(\mathbf{x}-\chi,t-\tau)f_i(\chi,\tau)$$
$$+p_k^{Fc}(\mathbf{x}-\chi,t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau+\int_{\Omega+\Omega'} p_k^{Fc}(\mathbf{x}-\chi,t)\zeta(\chi,0)\,\mathrm{d}\chi \quad (43)$$

$$\beta v_k(\mathbf{x},t)=\int_0^t\int_\Gamma [u_{ki}^{fi}(\mathbf{x}-\chi,t-\tau)s_i(\chi,\tau)+q_{ki}^{fc}(\mathbf{x}-\chi,t-\tau)n_i(\chi)s(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+\int_0^t\int_\Omega [u_{ki}^{fi}(\mathbf{x}-\chi,t-\tau)F_i(\chi,\tau)+v_{ki}^{fi}(\mathbf{x}-\chi,t-\tau)f_i(\chi,\tau)$$
$$+p_k^{fc}(\mathbf{x}-\chi,t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau+\int_{\Omega+\Omega'} p_k^{fc}(\mathbf{x}-\chi,t)\zeta(\chi,0)\,\mathrm{d}\chi \quad (44)$$

$$\beta p(\mathbf{x},t)=\int_0^t\int_\Gamma [u_i^{li}(\mathbf{x}-\chi,t-\tau)s_i(\chi,\tau)+q_i^{lc}(\mathbf{x}-\chi,t-\tau)n_i(\chi)s(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$
$$+\int_0^t\int_\Omega [u_i^{li}(\mathbf{x}-\chi,t-\tau)F_i(\chi,\tau)+v_i^{li}(\mathbf{x}-\chi,t-\tau)f_i(\chi,\tau)$$
$$+p^{lc}(\mathbf{x}-\chi,t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau+\int_{\Omega+\Omega'} p^{lc}(\mathbf{x}-\chi,t)\zeta(\chi,0)\,\mathrm{d}\chi \quad (45)$$

$$\beta\sigma_{kl}(\mathbf{x},t) = \int_0^t\int_\Gamma [u_{ikl}^{di}(\mathbf{x}-\boldsymbol{\chi},t-\tau)s_i(\boldsymbol{\chi},\tau)+q_{ikl}^{dc}(\mathbf{x}-\boldsymbol{\chi},t-\tau)n_i(\boldsymbol{\chi})s(\boldsymbol{\chi},\tau)]\,\mathrm{d}\boldsymbol{\chi}\,\mathrm{d}\tau$$
$$+\int_0^t\int_\Omega [u_{ikl}^{di}(\mathbf{x}-\boldsymbol{\chi},t-\tau)F_i(\boldsymbol{\chi},\tau)+v_{ikl}^{di}(\mathbf{x}-\boldsymbol{\chi},t-\tau)f_i(\boldsymbol{\chi},\tau)$$
$$+p_{kl}^{dc}(\mathbf{x}-\boldsymbol{\chi},t-\tau)\gamma(\boldsymbol{\chi},\tau)]\,\mathrm{d}\boldsymbol{\chi}\,\mathrm{d}\tau+\int_{\Omega+\Omega'} p_{kl}^{dc}(\mathbf{x}-\boldsymbol{\chi},t)\zeta(\boldsymbol{\chi},0)\,\mathrm{d}\boldsymbol{\chi} \quad (46)$$

where s_i and s represent the traction and the pressure jumps across the boundary Γ, respectively,

$$s_i = s_{ij}n_j = (\sigma_{ij}-\sigma'_{ij})n_j \quad (47)$$

$$s = -(p-p') \quad (48)$$

and s_{ij} is the stress discontinuity tensor. It may be argued that s_i and s do not have a physical meaning as the complementary problem is fictitious. Hence these quantities may be referred to as "fictitious densities".

Following the spirit of an indirect method, the roles of $\mathbf{x}$ and $\boldsymbol{\chi}$ in the Green's functions have been reversed in writing (43)–(46), as compared to (22)–(25). The singularities for the indirect method are distributed along the integration contour $\boldsymbol{\chi}\in\Gamma$, while that for the direct method is at a fixed point $\mathbf{x}$. Consequently, the indices u_{ik}^{Fi} have also been reversed to that the integral representation of the k-component displacement u_k is associated with the distribution of k-component Green's functions u_{ki}^{Fi} (since the second index of u_{ki}^{Fi} denotes the vector component of the forcing function). These adjustments are performed under the following rules:

$$\begin{aligned}
u_i^{li}(\boldsymbol{\chi}-\mathbf{x},t-\tau) &= -u_i^{li}(\mathbf{x}-\boldsymbol{\chi},t-\tau); \quad q_i^{lc}(\boldsymbol{\chi}-\mathbf{x},t-\tau) = -q_i^{lc}(\mathbf{x}-\boldsymbol{\chi},t-\tau)\\
v_i^{li}(\boldsymbol{\chi}-\mathbf{x},t-\tau) &= -v_i^{li}(\mathbf{x}-\boldsymbol{\chi},t-\tau); \quad p^{li}(\boldsymbol{\chi}-\mathbf{x},t-\tau) = p^{li}(\mathbf{x}-\boldsymbol{\chi},t-\tau)\\
u_{ik}^{(*)i}(\boldsymbol{\chi}-\mathbf{x},t-\tau) &= u_{ki}^{(*)i}(\mathbf{x}-\boldsymbol{\chi},t-\tau); \quad q_{ik}^{(*)c}(\boldsymbol{\chi}-\mathbf{x},t-\tau) = q_{ki}^{(*)c}(\mathbf{x}-\boldsymbol{\chi},t-\tau)\\
v_{ik}^{(*)i}(\boldsymbol{\chi}-\mathbf{x},t-\tau) &= v_{ki}^{(*)i}(\mathbf{x}-\boldsymbol{\chi},t-\tau); \quad p_k^{(*)c}(\boldsymbol{\chi}-\mathbf{x},t-\tau) = -p_k^{(*)c}(\mathbf{x}-\boldsymbol{\chi},t-\tau)\\
u_{ikl}^{di}(\boldsymbol{\chi}-\mathbf{x},t-\tau) &= -u_{ikl}^{di}(\mathbf{x}-\boldsymbol{\chi},t-\tau); \quad q_{ikl}^{dc}(\boldsymbol{\chi}-\mathbf{x},t-\tau) = -q_{ikl}^{dc}(\mathbf{x}-\boldsymbol{\chi},t-\tau)\\
v_{ikl}^{di}(\boldsymbol{\chi}-\mathbf{x},t-\tau) &= -v_{ikl}^{di}(\mathbf{x}-\boldsymbol{\chi},t-\tau); \quad p_{kl}^{dc}(\boldsymbol{\chi}-\mathbf{x},t-\tau) = p_{kl}^{dc}(\mathbf{x}-\boldsymbol{\chi},t-\tau)
\end{aligned} \quad (49)$$

in which the superscripts (*) are replaced by F or f. The sign change in the above expressions is determined by the type of singularity and the order of spatial differentiation involved in obtaining these quantities. We note that the indices for three of the displacement discontinuity quantities, u_{ikl}^{di}, q_{ikl}^{dc} and v_{ikl}^{di}, are not switched, as symmetry does not exist between the first and the third index.

Although (43)–(46) are mathematically correct, they are not yet in the form that is most appealing to physical intuition. Indeed, to obtain displacement u_k at a point $\mathbf{x}$ and a time t using (43), various influence function quantities, such as solid and fluid displacements, pressure, and flux, are distributed. A more intuitive approach is to distribute only displacement influence functions but created by various singularities, such as total and fluid force, source, etc. A similar statement can be made for (44)–(46). Interestingly, as proven in (26)–(30) and Appendix C, the following relations among Green's functions exist

$$q_{ij}^{Fc} = u_{ij}^{fi}; \quad v_{ij}^{Fi} = u_{ij}^{fi}; \quad p_i^{Fc} = u_i^{si}; \quad q_{ij}^{fc} = v_{ij}^{fi},$$
$$p_i^{fc} = v_i^{si}; \quad u_i^{li} = p_i^{Fi}; \quad q_i^{lc} = p_i^{fi}; \quad v_i^{li} = p_i^{fi},$$

$$p^{lc} = p^{si}; \quad u_{ijk}^{di} = \sigma_{kji}^{Fi}; \quad q_{ijk}^{dc} = \sigma_{kji}^{fi}; \quad v_{ijk}^{di} = \sigma_{kji}^{fi};$$
$$p_{ij}^{dc} = \sigma_{ij}^{si} \tag{50}$$

Hence (43)–(46) can be expressed as

$$\begin{aligned}\beta u_k(\mathbf{x}, t) = &\int_0^t \int_\Gamma [u_{ki}^{Fi}(\mathbf{x}-\chi, t-\tau)s_i(\chi,\tau) + u_{ki}^{fi}(\mathbf{x}-\chi, t-\tau)n_i(\chi)s(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \\ &+ \int_0^t \int_\Omega [u_{ki}^{Fi}(\mathbf{x}-\chi, t-\tau)F_i(\chi,\tau) + u_{ki}^{fi}(\mathbf{x}-\chi, t-\tau)f_i(\chi,\tau) \\ &+ u_k^{si}(\mathbf{x}-\chi, t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau + \int_{\Omega+\Omega'} u_k^{si}(\mathbf{x}-\chi, t)\zeta(\chi, 0)\,\mathrm{d}\chi\end{aligned} \tag{51}$$

$$\begin{aligned}\beta v_k(\mathbf{x}, t) = &\int_0^t \int_\Gamma [v_{ki}^{Fi}(\mathbf{x}-\chi, t-\tau)s_i(\chi,\tau) + v_{ki}^{fi}(\mathbf{x}-\chi, t-\tau)n_i(\chi)s(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \\ &+ \int_0^t \int_\Omega [v_{ki}^{Fi}(\mathbf{x}-\chi, t-\tau)F_i(\chi,\tau) + v_{ki}^{fi}(\mathbf{x}-\chi, t-\tau)f_i(\chi,\tau) \\ &+ v_k^{si}(\mathbf{x}-\chi, t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau + \int_{\Omega+\Omega'} v_k^{si}(\mathbf{x}-\chi, t)\zeta(\chi, 0)\,\mathrm{d}\chi\end{aligned} \tag{52}$$

$$\begin{aligned}\beta p(\mathbf{x}, t) = &\int_0^t \int_\Gamma [p_i^{Fi}(\mathbf{x}-\chi, t-\tau)s_i(\chi,\tau) + p_i^{fi}(\mathbf{x}-\chi, t-\tau)n_i(\chi)s(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \\ &+ \int_0^t \int_\Omega [p_i^{Fi}(\mathbf{x}-\chi, t-\tau)F_i(\chi,\tau) + p_i^{fi}(\mathbf{x}-\chi, t-\tau)f_i(\chi,\tau) \\ &+ p^{si}(\mathbf{x}-\chi, t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau + \int_{\Omega+\Omega'} p^{si}(\mathbf{x}-\chi, t)\zeta(\chi, 0)\,\mathrm{d}\chi\end{aligned} \tag{53}$$

$$\begin{aligned}\beta \sigma_{kl}(\mathbf{x}, t) = &\int_0^t \int_\Gamma [\sigma_{kli}^{Fi}(\mathbf{x}-\chi, t-\tau)s_i(\chi,\tau) + \sigma_{kli}^{fi}(\mathbf{x}-\chi, t-\tau)n_i(\chi)s(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \\ &+ \int_0^t \int_\Omega [\sigma_{kli}^{Fi}(\mathbf{x}-\chi, t-\tau)F_i(\chi,\tau) + \sigma_{kli}^{fi}(\mathbf{x}-\chi, t-\tau)f_i(\chi,\tau) \\ &+ \sigma_{kl}^{si}(\mathbf{x}-\chi, t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau + \int_{\Omega+\Omega'} \sigma_{kl}^{si}(\mathbf{x}-\chi, t)\zeta(\chi, 0)\,\mathrm{d}\chi\end{aligned} \tag{54}$$

The two equations, (51) and (53), are weakly singular, and are equivalent to a "single-layer method" of the potential theory (Jaswon and Symm, 1977) in which weak singularities of order $\ln r$ for 2-D and $1/r$ for 3-D problems are distributed. We also note that the domain integration of the initial condition $\zeta(\chi, 0)$ is performed for both Ω and Ω'. As pointed out by Sharp (1983), the need for integrating in the complementary domain is dependent on the numerical algorithm adopted. When the time integration in (51)–(54) is carried out in the convolutional sense (i.e. tracing back to the time origin for solution at any given time), it can simply be assumed that $\zeta(\chi, 0) = 0$ for $\chi \in \Omega'$; this approach eliminates the need of integrating in the complementary domain. If a time stepping scheme is adopted instead, advancing the solution to the next time level is based on information at the present time, not at the time origin. In that case, it is necessary to keep track of the evolution of ζ in both

Ω and Ω' for the domain integral, which is a serious drawback in terms of computational effort.

In order to solve mixed boundary value problems, we also need integral expressions for the boundary $t_i = \sigma_{ij}n_j$ and the normal fluid flux $q = q_i n_i$. While (54) comes handy to form such a quantity, (52) must be differentiated according to the kinematic condition (9). This yields the following strong (Cauchy) singular integral equations

$$\beta t_i(\mathbf{x}, t) = \int_0^t \int_\Gamma [\sigma_{ijk}^{Fi}(\mathbf{x}-\chi, t-\tau)s_k(\chi, \tau) + \sigma_{ijk}^{fi}(\mathbf{x}-\chi, t-\tau)n_k(\chi)s(\chi, \tau)]n_j(\mathbf{x})\, \mathrm{d}\chi\, \mathrm{d}\tau$$
$$+ \int_0^t \int_\Omega [\sigma_{ijk}^{Fi}(\mathbf{x}-\chi, t-\tau)F_k(\chi, \tau) + \sigma_{ijk}^{fi}(\mathbf{x}-\chi, t-\tau)f_k(\chi, \tau)$$
$$+ \sigma_{ij}^{si}(\mathbf{x}-\chi, t-\tau)\gamma(\chi, \tau)]n_j(\mathbf{x})\, \mathrm{d}\chi\, \mathrm{d}\tau + \int_{\Omega+\Omega'} \sigma_{ij}^{si}(\mathbf{x}-\chi, t)\zeta(\chi, 0)n_j(\mathbf{x})\, \mathrm{d}\chi \quad (55)$$

$$\beta q(\mathbf{x}, t) = \int_0^t \int_\Gamma [q_{ij}^{Fi}(\mathbf{x}-\chi, t-\tau)s_j(\chi, \tau) + q_{ij}^{fi}(\mathbf{x}-\chi, t-\tau)n_j(\chi)s(\chi, \tau)]n_i(\mathbf{x})\, \mathrm{d}\chi\, \mathrm{d}\tau$$
$$+ \int_0^t \int_\Omega [q_{ij}^{Fi}(\mathbf{x}-\chi, t-\tau)F_j(\chi, \tau) + q_{ij}^{fi}(\mathbf{x}-\chi, t-\tau)f_j(\chi, \tau)$$
$$+ q_i^{si}(\mathbf{x}-\chi, t-\tau)\gamma(\chi, \tau)]n_i(\mathbf{x})\, \mathrm{d}\chi\, \mathrm{d}\tau + \int_{\Omega+\Omega'} q_i^{si}(\mathbf{x}-\chi, t)\zeta(\chi, 0)n_i(\mathbf{x})\, \mathrm{d}\chi \quad (56)$$

The four eqns (51), (53), (55) and (56) can now be exploited in a boundary collocation procedure to determine the unknown distribution densities s_i and s.

7. DISPLACEMENT DISCONTINUITY METHOD

In contrast to the stress discontinuity method, we now consider the case where the boundary traction and the pore pressure for the interior and exterior domain problems are set equal. The following set of integral equations are then deduced from (39)–(42). (Note that relations similar to (49) have been used for the adjustment of signs.)

$$\beta u_k(\mathbf{x}, t) = \int_0^t \int_\Gamma [\sigma_{ijk}^{Fi}(\mathbf{x}-\chi, t-\tau)n_j(\chi)d_i(\chi, \tau) + p_k^{Fc}(\mathbf{x}-\chi, t-\tau)d(\chi, \tau)]\, \mathrm{d}\chi\, \mathrm{d}\tau$$
$$+ \int_0^t \int_\Omega [u_{ki}^{Fi}(\mathbf{x}-\chi, t-\tau)F_i(\chi, \tau) + v_{ki}^{Fi}(\mathbf{x}-\chi, t-\tau)f_i(\chi, \tau)$$
$$+ p_k^{Fc}(\mathbf{x}-\chi, t-\tau)\gamma(\chi, \tau)]\, \mathrm{d}\chi\, \mathrm{d}\tau + \int_{\Omega+\Omega'} p_k^{Fc}(\mathbf{x}-\chi, t)\zeta(\chi, 0)\, \mathrm{d}\chi \quad (57)$$

$$\beta v_k(\mathbf{x}, t) = \int_0^t \int_\Gamma [\sigma_{ijk}^{fi}(\mathbf{x}-\chi, t-\tau)n_j(\chi)d_i(\chi, \tau) + p_k^{fc}(\mathbf{x}-\chi, t-\tau)d(\chi, \tau)]\, \mathrm{d}\chi\, \mathrm{d}\tau$$
$$+ \int_0^t \int_\Omega [u_{ki}^{fi}(\mathbf{x}-\chi, t-\tau)F_i(\chi, \tau) + v_{ki}^{fi}(\mathbf{x}-\chi, t-\tau)f_i(\chi, \tau)$$
$$+ p_k^{fc}(\mathbf{x}-\chi, t-\tau)\gamma(\chi, \tau)]\, \mathrm{d}\chi\, \mathrm{d}\tau + \int_{\Omega+\Omega'} p_k^{fc}(\mathbf{x}-\chi, t)\zeta(\chi, 0)\, \mathrm{d}\chi \quad (58)$$

$$\beta p(\mathbf{x}, t) = \int_0^t \int_\Gamma [\sigma_{ij}^{li}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) n_j(\boldsymbol{\chi}) d_i(\boldsymbol{\chi}, \tau) + p^{lc}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) d(\boldsymbol{\chi}, \tau)] \, \mathrm{d}\boldsymbol{\chi} \, \mathrm{d}\tau$$
$$+ \int_0^t \int_\Omega [u_i^{li}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) F_i(\boldsymbol{\chi}, \tau) + v_i^{li}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) f_i(\boldsymbol{\chi}, \tau)$$
$$+ p^{lc}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) \gamma(\boldsymbol{\chi}, \tau)] \, \mathrm{d}\boldsymbol{\chi} \, \mathrm{d}\tau + \int_{\Omega+\Omega'} p^{lc}(\mathbf{x}-\boldsymbol{\chi}, t) \zeta(\boldsymbol{\chi}, 0) \, \mathrm{d}\boldsymbol{\chi} \quad (59)$$

$$\beta \sigma_{kl}(\mathbf{x}, t) = \int_0^t \int_\Gamma [\sigma_{klij}^{di}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) n_j(\boldsymbol{\chi}) d_i(\boldsymbol{\chi}, \tau) + p_{kl}^{dc}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) d(\boldsymbol{\chi}, \tau)] \, \mathrm{d}\boldsymbol{\chi} \, \mathrm{d}\tau$$
$$+ \int_0^t \int_\Omega [u_{ikl}^{di}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) F_i(\boldsymbol{\chi}, \tau) + v_{ikl}^{di}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) f_i(\boldsymbol{\chi}, \tau)$$
$$+ p_{kl}^{dc}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) \gamma(\boldsymbol{\chi}, \tau)] \, \mathrm{d}\boldsymbol{\chi} \, \mathrm{d}\tau + \int_{\Omega+\Omega'} p_{kl}^{dc}(\mathbf{x}-\boldsymbol{\chi}, t) \zeta(\boldsymbol{\chi}, 0) \, \mathrm{d}\boldsymbol{\chi} \quad (60)$$

where

$$d_i = d_{ij} n_j = u_i - u_i' \quad (61)$$

$$d = -(q_i - q_i') n_i \quad (62)$$

In the above d is the fluid normal flux discontinuity associated with the surface, d_{ij} is the displacement discontinuity tensor, and with the presence of a surface, d_{ij} contracts to d_i, the surface displacement discontinuity. The sign convention for the components of the displacement discontinuity tensor is shown in Fig. 1.

The integral equations can again be cast into a physically more appealing form in which influence functions of the same kind are distributed:

$$\beta u_k(\mathbf{x}, t) = \int_0^t \int_\Gamma [u_{kji}^{di}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) n_j(\boldsymbol{\chi}) d_i(\boldsymbol{\chi}, \tau) + u_k^{si}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) d(\boldsymbol{\chi}, \tau)] \, \mathrm{d}\boldsymbol{\chi} \, \mathrm{d}\tau$$
$$+ \int_0^t \int_\Omega [u_{ki}^{Fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) F_i(\boldsymbol{\chi}, \tau) + u_{ki}^{fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) f_i(\boldsymbol{\chi}, \tau)$$
$$+ u_k^{si}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) \gamma(\boldsymbol{\chi}, \tau)] \, \mathrm{d}\boldsymbol{\chi} \, \mathrm{d}\tau + \int_{\Omega+\Omega'} u_k^{si}(\mathbf{x}-\boldsymbol{\chi}, t) \zeta(\boldsymbol{\chi}, 0) \, \mathrm{d}\boldsymbol{\chi} \quad (63)$$

$$\beta v_k(\mathbf{x}, t) = \int_0^t \int_\Gamma [v_{kji}^{di}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) n_j(\boldsymbol{\chi}) d_i(\boldsymbol{\chi}, \tau) + v_k^{si}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) d(\boldsymbol{\chi}, \tau)] \, \mathrm{d}\boldsymbol{\chi} \, \mathrm{d}\tau$$
$$+ \int_0^t \int_\Omega [v_{ki}^{Fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) F_i(\boldsymbol{\chi}, \tau) + v_{ki}^{fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) f_i(\boldsymbol{\chi}, \tau)$$
$$+ v_k^{si}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) \gamma(\boldsymbol{\chi}, \tau)] \, \mathrm{d}\boldsymbol{\chi} \, \mathrm{d}\tau + \int_{\Omega+\Omega'} v_k^{si}(\mathbf{x}-\boldsymbol{\chi}, t) \zeta(\boldsymbol{\chi}, 0) \, \mathrm{d}\boldsymbol{\chi} \quad (64)$$

$$\beta p(\mathbf{x}, t) = \int_0^t \int_\Gamma [p_{ji}^{di}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) n_j(\boldsymbol{\chi}) d_i(\boldsymbol{\chi}, \tau) + p^{si}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) d(\boldsymbol{\chi}, \tau)] \, \mathrm{d}\boldsymbol{\chi} \, \mathrm{d}\tau$$

$$+\int_0^t\int_\Omega [p_i^{Fi}(\mathbf{x}-\chi, t-\tau)F_i(\chi,\tau)+p_i^{fi}(\mathbf{x}-\chi, t-\tau)f_i(\chi,\tau)$$

$$+p^{si}(\mathbf{x}-\chi, t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau+\int_{\Omega+\Omega'} p^{si}(\mathbf{x}-\chi, t)\zeta(\chi,0)\,\mathrm{d}\chi \quad (65)$$

$$\beta\sigma_{kl}(\mathbf{x},t) = \int_0^t\int_\Gamma [\sigma_{klij}^{di}(\mathbf{x}-\chi, t-\tau)n_j(\chi)d_i(\chi,\tau)+\sigma_{kl}^{si}(\mathbf{x}-\chi, t-\tau)d(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau$$

$$+\int_0^t\int_\Omega [\sigma_{kli}^{Fi}(\mathbf{x}-\chi, t-\tau)F_i(\chi,\tau)+\sigma_{kli}^{fi}(\mathbf{x}-\chi, t-\tau)f_i(\chi,\tau)$$

$$+\sigma_{kl}^{si}(\mathbf{x}-\chi, t-\tau)\gamma(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau+\int_{\Omega+\Omega'} \sigma_{kl}^{si}(\mathbf{x}-\chi, t)\zeta(\chi,0)\,\mathrm{d}\chi \quad (66)$$

We notice that all the conversions of Green's functions have been earlier defined in (50), except for one:

$$\sigma_{ij}^{li} = p_{ji}^{di} \quad (67)$$

This is also proven in Appendix C.

Equations (63) and (65) hence define another indirect method in which solid displacement discontinuity and fluid source singularities are distributed. These equations are Cauchy singular. This indirect method may be viewed as an equivalent of the "double-layer method" in potential theory.

In order to solve mixed boundary value problems, (64) and (66) are utilized to obtain the following hypersingular (Hadamard) equations:

$$\beta t_i(\mathbf{x},t) = \int_0^t\int_\Gamma [\sigma_{ijkl}^{di}(\mathbf{x}-\chi, t-\tau)n_k(\chi)d_l(\chi,\tau)+\sigma_{ij}^{si}(\mathbf{x}-\chi, t-\tau)d(\chi,\tau)]n_j(\mathbf{x})\,\mathrm{d}\chi\,\mathrm{d}\tau$$

$$+\int_0^t\int_\Omega [\sigma_{ijk}^{Fi}(\mathbf{x}-\chi, t-\tau)F_k(\chi,\tau)+\sigma_{ijk}^{fi}(\mathbf{x}-\chi, t-\tau)f_k(\chi,\tau)$$

$$+\sigma_{ij}^{si}(\mathbf{x}-\chi, t-\tau)\gamma(\chi,\tau)]n_j(\mathbf{x})\,\mathrm{d}\chi\,\mathrm{d}\tau+\int_{\Omega+\Omega'} \sigma_{ij}^{si}(\mathbf{x}-\chi, t)\zeta(\chi,0)n_j(\mathbf{x})\,\mathrm{d}\chi \quad (68)$$

$$\beta q(\mathbf{x},t) = \int_0^t\int_\Gamma [q_{ijk}^{di}(\mathbf{x}-\chi, t-\tau)n_j(\chi)d_k(\chi,\tau)+q_i^{si}(\mathbf{x}-\chi, t-\tau)d(\chi,\tau)]n_i(\mathbf{x})\,\mathrm{d}\chi\,\mathrm{d}\tau$$

$$+\int_0^t\int_\Omega [q_{ij}^{Fi}(\mathbf{x}-\chi, t-\tau)F_j(\chi,\tau)+q_{ij}^{fi}(\mathbf{x}-\chi, t-\tau)f_j(\chi,\tau)$$

$$+q_i^{si}(\mathbf{x}-\chi, t-\tau)\gamma(\chi,\tau)]n_i(\mathbf{x})\,\mathrm{d}\chi\,\mathrm{d}\tau+\int_{\Omega+\Omega'} q_i^{si}(\mathbf{x}-\chi, t)\zeta(\chi,0)n_i(\mathbf{x})\,\mathrm{d}\chi \quad (69)$$

By this derivation we can loosely interpret that the constant β takes the same value as it is in (63)–(66). More rigorously, however, the Hadamard finite-part argument needs to be invoked to obtain its limiting value. In the numerical implementation, certain regularization process is needed to evaluate this kind of hypersingular equation (Krishnasamy *et al.*, 1992).

Equations (63), (65), (68) and (69) now form the basis of a poroelastic displacement discontinuity method.

As a final remark, we present an interesting observation. The stress expression of an instantaneous displacement discontinuity can be obtained from the constitutive equation,

$$\sigma_{ijkl}^{di} = G(u_{ikl,j}^{di} + u_{jkl,i}^{di}) + \frac{2G\nu}{1-2\nu}\delta_{ij}u_{mkl,m}^{di} - \alpha\delta_{ij}p_{kl}^{di}$$

$$= G(\sigma_{lki,j}^{Fi} + \sigma_{lkj,i}^{Fi}) + \frac{2G\nu}{1-2\nu}\delta_{ij}\sigma_{lkm,m}^{Fi} - \alpha\delta_{ij}\sigma_{lk}^{li} \tag{70}$$

where isotropy is assumed. In the above equation, the second line is substituted by relations in (50) and (67). For a continuous displacement discontinuity, (70) can be integrated with respect to time to give

$$\sigma_{ijkl}^{dc} = G(\sigma_{lki,j}^{Fc} + \sigma_{lkj,i}^{Fc}) + \frac{2G\nu}{1-2\nu}\delta_{ij}\sigma_{lkm,m}^{Fc} - \alpha\delta_{ij}\sigma_{lk}^{si} \tag{71}$$

where we have utilized (29). Equation (71) shows that the continuous displacement discontinuity is a combination of a number of singular solutions, including a continuous force dipole (also known as double force, which is the spatial derivative of a point force), a quadrapole for 2-D or an hexapole for 3-D (respectively, two and three pairs of mutually orthogonal double forces), and also an instantaneous fluid source. Such a superposition of singular solutions, based on physical arguments and the requirement that the final combination contain all the necessary properties, is one of the approaches that has been used to derive some of the displacement discontinuity solutions (Curran and Carvalho, 1987; Carvalho, 1990; Carvalho and Curran, 1998). Here its theoretical connection is formally established.

8. DISLOCATION METHOD

This dislocation method is often used to model linear fracture problems and is a reduced version of the displacement discontinuity method. In this case, Γ represents the fracture locus, d_i is the actual displacement jump due to the opening and sliding of fracture walls, and d the actual fluid flux jump as the result of fluid injection. For example, for the geometry of a linear fracture under plane strain conditions in an infinite domain (Fig. 3), the integral equations can be written as

$$\sigma_n(x,t) = \int_0^t \int_{-L}^{L} [\sigma_{2222}^{di}(x-\chi, t-\tau)d_n(\chi,\tau) + \sigma_{2221}^{di}(x-\chi, t-\tau)d_s(\chi,\tau) + \sigma_{22}^{si}(x-\chi, t-\tau)d(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \tag{72}$$

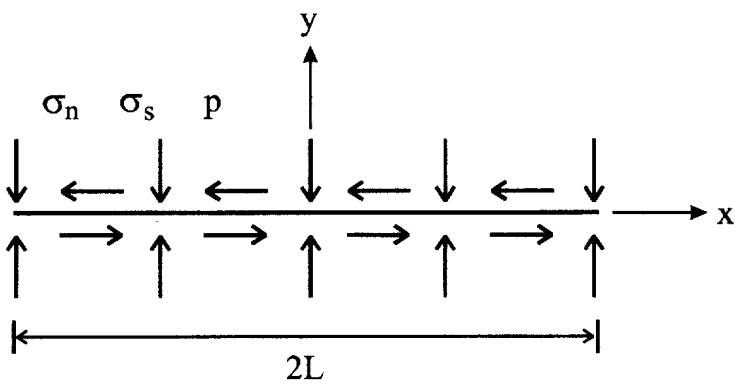

Fig. 3. A linear fracture in poroelastic medium.

$$\sigma_s(x,t) = \int_0^t \int_{-L}^{L} [\sigma_{1222}^{di}(x-\chi, t-\tau)d_n(\chi,\tau) + \sigma_{1221}^{di}(x-\chi, t-\tau)d_s(\chi,\tau) + \sigma_{12}^{si}(x-\chi, t-\tau)d(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \quad (73)$$

$$p(x,t) = \int_0^t \int_{-L}^{L} [p_{22}^{di}(x-\chi, t-\tau)d_n(\chi,\tau) + p_{21}^{di}(x-\chi, t-\tau)d_s(\chi,\tau) + p^{si}(x-\chi, t-\tau)d(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \quad (74)$$

in which σ_n and σ_s are, respectively, the normal and shear stress on the fracture surface, d_n and d_s are the normal (mode 1) and shear (mode 2) displacement jumps across the fracture. These equations are apparently a reduced version of (65) and (68). They have been utilized for the numerical solution of hydraulic fracture embedded in porous formation, opened by fluid pressurization (Detournay and Cheng, 1987; Vandamme *et al.*, 1989; Renshaw and Harvey, 1994).

An edge dislocation method, similar to that used in elasticity (Bilby and Eshelby, 1968), can also be devised. For example, performing an integration by parts on (72) produces the edge dislocation formula

$$\sigma_n(x,t) = \int_0^t \int_{-L}^{L} [\sigma_{2222}^{ei}(x-\chi, t-\tau)d_n'(\chi,\tau) + \sigma_{2221}^{ei}(x-\chi, t-\tau)d_s'(\chi,\tau) + \sigma_{22}^{si}(x-\chi, t-\tau)d(\chi,\tau)]\,\mathrm{d}\chi\,\mathrm{d}\tau \quad (75)$$

In the above, d_n' and d_s' are the derivatives of d_n and d_s with respect to χ. The influence functions denoted by the superscript e are edge dislocation solutions obtained as

$$\sigma_{2222}^{ei}(x-\chi, t-\tau) = \int_{-\infty}^{\chi} \sigma_{2222}^{di}(x-\chi', t-\tau)\,\mathrm{d}\chi' \quad (76)$$

and so forth. Equation (75) has a singularity of $1/r$ as compared with $1/r^2$ in (72), which is of some numerical advantage. In addition, the stress intensity factor can directly be solved in the edge dislocation method, which should be more accurate than that determined from the slope of displacement in the point dislocation method. The poroelastic edge dislocation method has been numerically implemented (Cheng *et al.*, 1988; Detournay and Cheng, 1991).

For curved fracture, an orthonormal coordinate system can be used. A 2-D representation, in which Γ is a curve with a + and a − side, is illustrated in Fig. 1, where u_i^- corresponds to u_i, and u_i^+ to u_i'. The normal d_n and shear d_s component of the displacement discontinuity vector are then given by

$$d_n = d_{ij}n_i n_j \quad (77)$$

$$d_s = d_{jk}n_i n_k \varepsilon_{ji} \quad (78)$$

where ε_{ij} is the 2-D Levi–Cevita permutation symbol.

9. SYMMETRIC GALERKIN INTEGRAL EQUATIONS

A shortcoming common to the boundary integral equation methods presented so far is that the coefficient matrix of the discretized, linear solution system is non-symmetric in contrast to the symmetric coefficient matrices of finite element methods. Since there are considerable computational advantages to be gained from a symmetric coefficient matrix,

symmetric, "Galerkin formulations", mimicking the Galerkin weighted residual formulations of finite elements, have been proposed for boundary integral equations. For elasticity, the Galerkin BEM has been well expounded (Kane, 1994; Bonnet, 1995). Recently, Pan and Maier (1997) derived the symmetric Galerkin BEM for poroelasticity. Based on the Galerkin formulation viewpoint, the Green's functions in the integral equations are weighing functions which are chosen, among many other possible weighing functions. In this section, we investigate the theoretical foundation of the symmetric Galerkin integral equation system and unify it with the present family of integral equations.

Equation (51), which is a single layer potential representation of the displacement vector, is repeated below

$$\beta u_i(\mathbf{x}, t) = \int_0^t \int_\Gamma [u_{ik}^{Fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau)s_k(\boldsymbol{\chi}, \tau) + u_{ik}^{fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau)n_k(\boldsymbol{\chi})s(\boldsymbol{\chi}, \tau)]\, \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\tau \tag{79}$$

where we have dropped the domain integrals associated with body forces, sources, etc., for simplicity. Similarly, based on (63), the double layer representation is given as

$$\beta u_i(\mathbf{x}, t) = \int_0^t \int_\Gamma [u_{ikl}^{di}(\mathbf{x}-\boldsymbol{\chi}, t-\tau)n_k(\boldsymbol{\chi})d_l(\boldsymbol{\chi}, \tau) + u_i^{li}(\mathbf{x}-\boldsymbol{\chi}, t-\tau)D(\boldsymbol{\chi}, \tau)]\, \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\tau \tag{80}$$

Here we note that we have performed an integration by parts with respect to time on the second term in the integrand such that

$$u_i^{li} = \frac{\partial u_i^{si}}{\partial t} \tag{81}$$

and following (62),

$$D = \int_0^t d\, \mathrm{d}t = -(v_i - v_i')n_i \tag{82}$$

is the fluid relative displacement discontinuity. The above two equations, (79) and (80), can be added to give a mixed-type integral equation

$$2\beta u_i = \int_0^t \int_\Gamma [u_{ik}^{Fi}s_k + u_{ik}^{fi}n_k s + u_{ikl}^{di}n_k d_l + u_i^{li}D]\, \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\tau \tag{83}$$

Following the same strategy, and utilizing the single and double layer integral equations of Sections 6 and 7, three more mixed type integral equations can be constructed

$$2\beta t_i = \int_0^t \int_\Gamma [\sigma_{ijk}^{Fi}s_k + \sigma_{ijk}^{fi}n_k s + \sigma_{ijkl}^{di}n_k d_l + \sigma_{ij}^{li}D]n_j\, \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\tau \tag{84}$$

$$2\beta v = \int_0^t \int_\Gamma [v_{ik}^{Fi}s_k + v_{ik}^{fi}n_k s + v_{ikl}^{di}n_k d_l + v_i^{li}D]n_i\, \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\tau \tag{85}$$

$$2\beta p = \int_0^t \int_\Gamma [p_k^{Fi}s_k + p_k^{fi}n_k s + p_{kl}^{di}n_k d_l + p^{li}D]\, \mathrm{d}\boldsymbol{\chi}\, \mathrm{d}\tau \tag{86}$$

In the above, $v = v_i n_i$, is the normal fluid relative displacement. These four eqns (83)–(86) can be put into a matrix form

$$\int_0^t \int_\Gamma \begin{bmatrix} \sigma_{ijkl}^{di} n_k n_j & \sigma_{ijk}^{Fi} n_j & \sigma_{ijk}^{fi} n_k n_j & \sigma_{ij}^{li} n_j \\ u_{ikl}^{di} n_k & u_{ik}^{Fi} & u_{ik}^{fi} n_k & u_i^{li} \\ v_{ikl}^{di} n_k n_i & v_{ik}^{Fi} n_i & v_{ik}^{fi} n_k n_i & v_i^{li} n_i \\ p_{kl}^{di} n_k & p_k^{Fi} & p_k^{fi} n_k & p^{li} \end{bmatrix} \begin{Bmatrix} d_l \\ s_k \\ s \\ D \end{Bmatrix} \mathrm{d}\chi \, \mathrm{d}\tau = 2\beta \begin{Bmatrix} t_i \\ u_i \\ v \\ p \end{Bmatrix} \tag{87}$$

We notice that the coefficient matrix is symmetric because of the following relations $\sigma_{ijk}^{Fi} = u_{kji}^{di}$, $\sigma_{ijk}^{fi} = v_{kji}^{di}$, $\sigma_{ij}^{li} = p_{ji}^{di}$, $u_{ij}^{fi} = v_{ji}^{Fi}$, $u_i^{li} = p_i^{Fi}$, and $v_i^{li} = p_i^{fi}$, all of which have been proven in (50) and (67). Similar to the stress and the displacement discontinuity methods, only half of the above equations are needed at each collocation node, depending on the type of boundary conditions (traction vs displacement, and pressure vs flux).

10. SUMMARY AND CONCLUSION

In this paper we have unified the direct and indirect boundary integral equations for the theory of linear quasi-static anisotropic poroelasticity. The direct integral equations were derived based on the reciprocity relation. The indirect integral equations were obtained by summing up the integral representation of the direct method for an interior and an exterior domain problem. The choices of the boundary conditions led to two methods, a stress and a displacement discontinuity method. Each of the two methods were expressed in two forms. One was based on distributing various influence functions of stresses, displacements, etc., created by the same singularity. The other uses the same kind of influence function as the quantity represented by the integral, yet associated with various kind of singularities. Through these relations, various connections among Green's functions were observed and proved under general anisotropy condition. Two variations of the displacement discontinuity method, a point and an edge dislocation method, were introduced to model fracture problems. By combining the stress and the displacement discontinuity equations, an integral equation system whose coefficient matrix is symmetric is formed. This is similar to the Galerkin formulation in FEM. Finally, a complete listing of all isotropic Green's functions was provided as an Appendix.

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APPENDIX A: DERIVATION OF RECIPROCAL INTEGRAL EQUATIONS

The governing eqns (1)–(5) are not self-adjoint due to the presence of a time derivative term in (5). Following the derivation of the reciprocal integral equation for the diffusion equation (Morse and Feshbach, 1953), the second system in (11) is represented by the adjoint system of equations, which involves switching the sign of the time derivative term in (5). Since the second system is replaced by Green's functions, a reversal of the field and source parameters is necessary to restore the physical nature of the Green's functions. These procedures, adopted

for the derivation of reciprocal integral equations (Predeleanu, 1968; Cheng and Predeleanu, 1987), are often a source of confusion. In the derivation below, it is demonstrated that such procedures are not necessary.

Another innovative feature of this Appendix is the explicit use of a nucleus of strain which naturally leads to the introduction of the displacement discontinuity solution. To achieve this, we recognize that the strain field e_{ij}^* for a displacement discontinuity consists of two terms, an elastic strain e_{ij} and a singularity E_{ij}:

$$e_{ij}^* = e_{ij} + E_{ij} \tag{A1}$$

The singularity E_{ij} will be referred to as a "nucleus of strain", borrowing an expression coined by Love (1944). The nucleus of strain introduced as (17) is related to but different from the nuclei of strain introduced by Love (1944) in elasticity and also from the poroelastic point dilation and point slip derived by Cleary (1977). Its counterpart in elasticity was presented by Nedelec (1986) and Becache *et al.* (1993).

The elastic strain e_{ij} that satisfies the constitutive eqns (1) and (2) can be expressed in terms of $e_{ij}^* = \frac{1}{2}(u_{i,j}^* + u_{j,i}^*)$ and E_{ij} using the decomposition (A1); hence

$$\sigma_{ij}^* = M_{ijkl}e_{kl}^* - \alpha_{ij}p^* - M_{ijkl}E_{kl} \tag{A2}$$

$$p^* = M(\zeta^* - \alpha_{ij}e_{ij}^*) + M\alpha_{ij}E_{ij} \tag{A3}$$

The asterisk superscripts are used to denote the displacement discontinuity solution. Note that when E_{ij} is zero, the asterisks drop out. When a proper substitution of singularity is made, the asterisks are replaced by the displacement discontinuity notation, *di* or *dc*. As no confusion will result, the asterisk superscripts will henceforth be dropped.

The reciprocity of work principle (11) can now be rewritten as

$$\sigma_{ij}^{(1)}e_{ij}^{(2)} + p^{(1)}\zeta^{(2)} - \sigma_{ij}^{(1)}E_{ij}^{(2)} = \sigma_{ij}^{(2)}e_{ij}^{(1)} + p^{(2)}\zeta^{(1)} - \sigma_{ij}^{(2)}E_{ij}^{(1)} \tag{A4}$$

to accommodate the introduction of nuclei of strain. First, note that

$$\sigma_{ij}^{(1)}e_{ij}^{(2)} = (\sigma_{ij}^{(1)}u_i^{(2)})_{,j} + u_i^{(2)}F_i^{(1)} \tag{A5}$$

where the equilibrium eqns (3) have been utilized. Integrating the continuity eqn (5) we obtain

$$\zeta = \zeta_0 - v_{i,i} + Q \tag{A6}$$

in which ζ_0 is the initial value of ζ.

It can then be shown

$$p^{(1)}\zeta^{(2)} = -(p^{(1)}v_i^{(2)})_{,i} - b_{ij}q_i^{(1)}v_j^{(2)} + v_i^{(2)}f_i^{(1)} + p^{(1)}Q^{(2)} + p^{(1)}\zeta_0^{(2)} \tag{A7}$$

where we have utilized Darcy's law (4), and $b_{ij} = [\kappa_{ij}]^{-1}$ is the resistivity tensor, or the matrix inverse of the permeability tensor.

Substituting (A5), (A7) and their counterparts into (A4), and integrating over the problem domain Ω and with respect to time yields:

$$\begin{aligned}\int_0^t \int_\Omega & [(\sigma_{ij}^{(1)}u_i^{(2)})_{,j} - (\sigma_{ij}^{(2)}u_i^{(1)})_{,j}] - [(p^{(1)}v_i^{(2)})_{,i} - (p^{(2)}v_i^{(1)})_{,i}] \\ & + [F_i^{(1)}u_i^{(2)} - F_i^{(2)}u_i^{(1)}] + [f_i^{(1)}v_i^{(2)} - f_i^{(2)}v_i^{(1)}] - [Q^{(1)}p^{(2)} - Q^{(2)}p^{(1)}] \\ & + [E_{ij}^{(1)}\sigma_{ij}^{(2)} - E_{ij}^{(2)}\sigma_{ij}^{(1)}] + [p^{(1)}\zeta_0^{(2)} - p^{(2)}\zeta_0^{(1)}] - [b_{ij}q_i^{(1)}v_j^{(2)} - b_{ij}q_i^{(2)}v_j^{(1)}]\,\mathrm{d}\chi\,\mathrm{d}\tau = 0\end{aligned} \tag{A8}$$

We note that the last group of the integrand can be transformed as follows:

$$\begin{aligned} b_{ij}(q_i^{(1)}v_j^{(2)} - q_i^{(2)}v_j^{(1)}) = b_{ij}(q_i^{(1)}v_j^{(2)} - q_j^{(2)}v_i^{(1)}) &= b_{ij}\left[\frac{\partial v_i^{(1)}(\tau)}{\partial \tau}v_j^{(2)}(t-\tau) - \frac{\partial v_j^{(2)}(t-\tau)}{\partial(t-\tau)}v_i^{(1)}(\tau)\right] \\ &= b_{ij}\frac{\partial v_i^{(1)}(\tau)v_j^{(2)}(t-\tau)}{\partial \tau}\end{aligned} \tag{A9}$$

in which we have utilized the symmetry of b_{ij}. Integrating the above we find:

$$\int_0^t b_{ij}\frac{\partial v_i^{(1)}(\tau)v_j^{(2)}(t-\tau)}{\partial \tau}\mathrm{d}\tau = b_{ij}v_i^{(1)}(\tau)v_j^{(2)}(t-\tau)\Big|_{\tau=0}^{\tau=t} = 0 \tag{A10}$$

where we note $v_i(0) = 0$, following the definition (9). Hence the last part of the integrand of (A8) vanishes. Finally, (13) is obtained by applying the divergence theorem to the first two parts of (A8).

APPENDIX B: DERIVATION OF GREEN'S FUNCTIONS FOR ISOTROPIC POROELASTICITY

In this Appendix, we outline a methodology to derive the solutions of the various singularities presented in Appendix D. This approach is based on a particular decomposition of the displacement field originally proposed by Biot (1956). It was first used by Cheng and Liggett (1984a) to derive fundamental solutions (see also Cheng and Predeleanu (1987), Detournay and Cheng (1987)), and is a more systematic approach than other methods that have been described in the literature (see, for example, Cleary (1977), Rudnicki (1981, 1987)).

Consistent with the scope of Appendix D, the following discussion is limited to the case of material isotropy. The governing eqns (A2), (A3) and (4) reduce to:

$$\sigma_{ij} = 2Ge_{ij} + \frac{2G\nu}{1-2\nu}\delta_{ij}e - \alpha\delta_{ij}p - 2GE_{ij} - \frac{2G\nu}{1-2\nu}\delta_{ij}E_{kk} \tag{B1}$$

$$p = M(\zeta - \alpha e) + M\alpha E_{ii} \tag{B2}$$

$$q_i = \kappa(p_{,i} - f_i) \tag{B3}$$

Here for convenience we have dropped the asterisk superscripts as shown in (A2) and (A3). The material constants are the shear modulus G, the Poisson ratio ν, the mobility or permeability κ, and the Biot modulus M, and the Biot effective stress coefficient α.

B.1. Biot's decomposition

We shall first ignore the presence of E_{ij}, thus the displacement discontinuity solution. It will be separately discussed below. In that case, the governing equations, (3), (5), (B1)–(B3) can be combined to give the following field equations in terms of the solid displacement u_i and the variation of fluid content ζ (see, for example, Detournay and Cheng (1993))

$$Gu_{i,jj} + \frac{G}{(1-2\nu_u)}u_{j,ji} = \alpha M\zeta_{,i} - F_i \tag{B4}$$

$$\frac{\partial\zeta}{\partial t} - c\nabla^2\zeta = \frac{\eta c}{G}F_{i,i} + \gamma - \kappa f_{i,i} \tag{B5}$$

In the above ν_u is the undrained Poisson ratio, and the diffusivity (generalized consolidation coefficient) c can be expressed as

$$c = \frac{2\kappa G(\nu_u - \nu)(1-\nu)}{\alpha^2(1-\nu_u)(1-2\nu)^2} \tag{B6}$$

Note that ν_u is not an independent material coefficient as it is related to the other coefficients according to

$$M = \frac{2G(\nu_u - \nu)}{\alpha^2(1-2\nu_u)(1-2\nu)} \tag{B7}$$

We also introduce here the dimensionless coefficient η defined as

$$\eta = \frac{\alpha(1-2\nu)}{2(1-\nu)} \tag{B8}$$

The displacement decomposition proposed by Biot (1956) consists in writing

$$u_i = u_i^0 + \frac{\eta c}{\kappa G}\phi_{,i} \tag{B9}$$

where u_i^0 satisfies the undrained Navier equation of elasticity

$$Gu_{i,jj}^0 + \frac{G}{(1-2\nu_u)}u_{j,ji}^0 = -F_i \tag{B10}$$

and the scalar potential ϕ defined as

$$\nabla^2\phi = \zeta \tag{B11}$$

is governed by an inhomogeneous diffusion equation

$$\frac{\partial\phi}{\partial t} - c\nabla^2\phi = \frac{\eta c}{G}g_1 - \kappa g_2 + g_3 \tag{B12}$$

The functions g_1, g_2 and g_3 are such that

$$\nabla^2 g_1 = F_{i,i} \tag{B13}$$

$$\nabla^2 g_2 = \kappa f_{i,i} \tag{B14}$$

$$\nabla^2 g_3 = \gamma \tag{B15}$$

Here we note that ϕ is determined only to within an arbitrary harmonic function. We also notice these useful formulae:

$$p = \frac{1}{\kappa}\frac{\partial \phi}{\partial t} + g_2 - \frac{1}{\kappa} g_3 \tag{B16}$$

$$v_i = \left(-\phi + \int_0^t g_3 \, \mathrm{d}t\right)_{,i} \tag{B17}$$

$$\zeta = \frac{\kappa}{c} p - \frac{\eta}{G} g_1 \tag{B18}$$

The functions g_1, g_2 and g_3 provide the entry point for various singularities.

B.2. Irrotational singular solutions

All the singular solutions associated with the fluid (i.e. solutions with a singular term in the continuity equation or in Darcy's law) are characterized by $F_i = 0$. Equations (B9) and (B10) show that $u_i^0 = 0$ and the displacement u_i can be expressed as the gradient of a potential ϕ, hence the reference to the term "irrotational". Solutions that fall into this category include instantaneous and continuous fluid source, dipole, fluid body force, and dilation.

The solution procedure involves assigning proper Dirac delta and Heaviside step functions to f_i or γ. Through (B14) or (B15), the corresponding g function is found. Equation (B12) is used to find ϕ and (B9) is used for u_i. Further use of (B16), (B17) and other constitutive eqns (B1)–(B3) leads to the complete set of solutions of displacement, stress, pressure, flux, etc. We give a quick illustration below.

Take, for example, the continuous source, which corresponds to $\gamma = \delta(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau)$. From (B15) we find that

$$\begin{aligned} g_3 &= \frac{\ln r}{2\pi}\mathrm{H}(t-\tau) \quad \text{for 2-D} \\ &= -\frac{1}{4\pi r}\mathrm{H}(t-\tau) \quad \text{for 3-D} \end{aligned} \tag{B19}$$

Substituting the above into the right-hand-side of (B12), we find the scalar potential ϕ^{sc} for a unit continuous source as:

$$\begin{aligned} \phi^{sc} &= \frac{r^2}{16\pi c}[(1+\xi^{-2})\mathrm{E}_1(\xi^2) + 2\xi^{-2}\ln r - \xi^{-2} e^{-\xi^2})] \quad \text{for 2-D} \\ &= \frac{r}{16\pi c}[(2+\xi^{-2})\,\mathrm{erfc}(\xi^2) - \frac{2}{\sqrt{\pi}}\xi^{-1} e^{-\xi^2} - \xi^{-2}] \quad \text{for 3-D} \end{aligned} \tag{B20}$$

where

$$\xi = \frac{|\mathbf{x}-\boldsymbol{\chi}|}{\sqrt{4c(t-\tau)}} \tag{B21}$$

We can clearly derive u_i^{sc} from

$$u_i^{sc} = \frac{\eta c}{\kappa G}\phi_{,i}^{sc} \tag{B22}$$

and p^{sc} from

$$p^{sc} = \frac{1}{\kappa}\frac{\partial \phi^{sc}}{\partial t} - \frac{1}{\kappa} g_3 \tag{B23}$$

The rest of the solution then follows from the constitutive equations.

Once the continuous source solution is obtained, other irrotational singular solutions can be found as follows:

- All instantaneous solutions can be found as the time derivative of the continuous solutions [see (26)].
- The fluid dipole solution is by definition the spatial derivative of the source solution (with a minus sign)

$$(\sim)^{p(*)}_{(\cdots)i} = -\frac{\partial(\sim)^{s(*)}_{(\cdots)}}{\partial x_i} \tag{B24}$$

• The fluid dilation solution is the time derivative of the source solution [see (28)].

By observing the relations shown above, we also detect the following connections among influence functions. In view of the way that f_i and γ are combined in the field eqns (B4), and the delta functions substituted, the solution for a unit fluid body force in terms of displacement, stress, pressure, etc., is obtained by multiplying by κ the solution for a dipole of the same orientation

$$(\sim)^{f(*)}_{(\cdots)i} = \kappa(\sim)^{p(*)}_{(\cdots)i} \tag{B25}$$

This however is not true for two expressions, q_{ij} and v_{ij}, as evident from (B3) and (B17).

Observing (B9) and (B17), we find that as long as the singularity is not a total force, a fluid source, or derived ones such as dipole and dilation, u_i and v_i are related according to

$$v^{f(*)}_{ij} = -\frac{\kappa G}{\eta c} u^{f(*)}_{ij} \tag{B26}$$

From (B18), we find that for all irrotational solutions (source, dipole, dilation, fluid body force, instantaneous and continuous)

$$\zeta^{(*)}_{(\cdots)} = \frac{\kappa}{p} p^{(*)}_{(\cdots)} \tag{B27}$$

Executing the above procedures, all singular "fluid" solutions can be found. They are presented in Appendix D.

B.3. Total force solution

The continuous point force solution is achieved by the introduction of $F_{ij} = \delta_{ij}\delta(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau)$ to the right-hand-side of (B10). Now the solution in (B9) contains a rotational part corresponding to the non-trivial u^0_i. Equation (B10) shows that u^0_i is exactly the classical elastic point force solution multiplied by a Heaviside function, but with the drained Poisson ratio ν replaced by the undrained one, ν_u. The "time-dependent" component of the displacement is contained in the potential as

$$u^{Fc}_{ij}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) = u^0_{ij}(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau) + \frac{\eta c}{\kappa G}\phi^{Fc}_{j,i}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) \tag{B28}$$

An inspection of the definition (B13) and (B14), and the diffusion eqn (B12), reveals that the potential of the total force ϕ^{Fc}_i is related to the potential of the fluid force ϕ^{fc}_i:

$$\phi^{Fc}_i = -\frac{\eta c}{G\kappa}\phi^{fc}_i \tag{B29}$$

Or, from (B25), it is related to the fluid dipole as:

$$\phi^{Fc}_i = -\frac{\eta c}{G}\phi^{pc}_i \tag{B30}$$

From what we already know about the fluid singular solutions, (B28) can be easily assembled.

Also, based on the definition of ζ in (B11), we can easily deduce from (B29) and (B30) that

$$\zeta^{F(*)}_i = -\frac{\eta c}{G}\zeta^{p(*)}_i = -\frac{\eta c}{\kappa G}\zeta^{f(*)}_i \tag{B31}$$

B.4. Displacement discontinuity solution

The governing equations can be assembled into field equations in terms of u_i and p, instead of u_i and ζ as in (B4) and (B5) (Detournay and Cheng, 1993),

$$Gu_{i,jj} + \frac{G}{(1-2\nu)}u_{j,ji} - \alpha p_{,i} = -F_i + 2GE_{ij,j} + \frac{2G\nu}{(1-2\nu)}E_{jj,i} \tag{B32}$$

$$\frac{\partial p}{\partial t} - \kappa M\nabla^2 p + \alpha M\frac{\partial e}{\partial t} = M\gamma - \kappa M f_{i,i} + \alpha M\frac{\partial E_{ii}}{\partial t} \tag{B33}$$

Here we have retained the presence of E_{ij}. For a continuous displacement discontinuity [cf (17)]

$$E_{ijkl} = -\tfrac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau) \tag{B34}$$

we find on the right hand side of (B32) and (B33) such singularities

$$Gu_{i,jj}+\frac{G}{(1-2\nu)}u_{j,ji}-\alpha p_{,i} = -G[\delta_{ik}\delta_{,l}(\mathbf{x}-\boldsymbol{\chi})+\delta_{il}\delta_{,k}(\mathbf{x}-\boldsymbol{\chi})]\mathrm{H}(t-\tau)-\frac{2G\nu}{(1-2\nu)}\delta_{kl}\delta_{,i}(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau) \quad \text{(B35)}$$

$$\frac{\partial p}{\partial t}-\kappa M\nabla^2 p+\alpha M\frac{\partial e}{\partial t} = -\alpha M\delta_{kl}\delta(\mathbf{x}-\boldsymbol{\chi})\delta(t-\tau) \quad \text{(B36)}$$

We observe through the F_i term that the introduction of a singularity $\delta_{ik}\delta(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau)$ for F_i in (B35) creates a solution of u_{ik}^{Fc}. If we introduce the spatial derivative of a delta function $\delta_{ik}\delta_{,l}(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau)$, a solution $u_{ik,l}^{Fc}$ results. Similar argument can be made in (B36) through the source term γ such that a u_k^{fi} solution is produced. We then realize that the displacement discontinuity solution is given by the following combination of solutions:

$$u_{ikl}^{dc} = G(u_{ik,l}^{Fc}+u_{il,k}^{Fc})+\frac{2G\nu}{1-2\nu}\delta_{kl}u_{mi,m}^{Fc}-\alpha\delta_{kl}u_i^{fi} \quad \text{(B37)}$$

Through the substitution of known relations, we also find

$$u_{lki}^{Fc} = G(u_{ki,l}^{Fc}+u_{li,k}^{Fc})+\frac{2G\nu}{1-2\nu}\delta_{kl}u_{mi,m}^{Fc}-\alpha\delta_{kl}p_i^{Fc} \quad \text{(B38)}$$

We realize that this is exactly the constitutive relation. Hence it provides a side proof of the displacement discontinuity solution (B37). We also notice that (B37) is consistent with (71).

APPENDIX C: RELATIONS AMONG GREEN'S FUNCTIONS

In this Appendix, it is proven that the following relations among Green's functions, as associated with (50) and (67), exist:

$$u_i^{li} = p_i^{Fi} \quad \text{(C1)}$$

$$v_{ij}^{Fi} = u_{ij}^{fi} \quad \text{(C2)}$$

$$v_i^{li} = p_i^{fi} \quad \text{(C3)}$$

$$u_{ijk}^{di} = \sigma_{kji}^{Fi} \quad \text{(C4)}$$

$$v_{ijk}^{di} = \sigma_{kji}^{fi} \quad \text{(C5)}$$

$$p_{ij}^{di} = \sigma_{ji}^{li} \quad \text{(C6)}$$

Although these relations can be directly observed from the isotropic solutions as displayed in Appendix D, they need to be proven for the general anisotropic case. A proof based on the reciprocity principle is given here. First consider the reciprocal work principle (11), where the two states (1) and (2) are now taken to be contemporary (i.e. they are occurring at the same time)

$$\sigma_{ij}^{(1)}(\boldsymbol{\chi}-\mathbf{x}_1,t-\tau)e_{ij}^{(2)}(\boldsymbol{\chi}-\mathbf{x}_2,t-\tau)+p^{(1)}(\boldsymbol{\chi}-\mathbf{x}_1,t-\tau)\zeta^{(2)}(\boldsymbol{\chi}-\mathbf{x}_2,t-\tau)$$
$$= \sigma_{ij}^{(2)}(\boldsymbol{\chi}-\mathbf{x}_2,t-\tau)e_{ij}^{(1)}(\boldsymbol{\chi}-\mathbf{x}_1,t-\tau)+p^{(2)}(\boldsymbol{\chi}-\mathbf{x}_2,t-\tau)\zeta^{(1)}(\boldsymbol{\chi}-\mathbf{x}_1,t-\tau) \quad \text{(C7)}$$

Following the similar procedure that produced (13), but without the integration with time, we obtain

$$\int_\Gamma(\sigma_{ij}^{(1)}n_ju_i^{(2)}-\sigma_{ij}^{(2)}n_ju_i^{(1)})\,\mathrm{d}\boldsymbol{\chi}-\int_\Gamma(p^{(1)}v_i^{(2)}n_i-p^{(2)}v_i^{(1)}n_i)\,\mathrm{d}\boldsymbol{\chi}+\int_\Omega(F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)})\,\mathrm{d}\boldsymbol{\chi}+\int_\Omega(f_i^{(1)}v_i^{(2)}-f_i^{(2)}v_i^{(1)})\,\mathrm{d}\boldsymbol{\chi}$$
$$-\int_\Omega(Q^{(1)}p^{(2)}-Q^{(2)}p^{(1)})\,\mathrm{d}\boldsymbol{\chi}+\int_\Omega(E_{ij}^{(1)}\sigma_{ij}^{(2)}-E_{ij}^{(2)}\sigma_{ij}^{(1)})\,\mathrm{d}\boldsymbol{\chi}=0 \quad \text{(C8)}$$

where we have ignored the initial condition. Consider Ω as a circular (2-D) or spherical (3-D) domain with a radius R. We shall take limit of $R\to\infty$. We note that the first two integrals are performed over the boundary. If the products in the integrand, $\sigma_{ij}^{(1)}u_i^{(2)}$, $p^{(1)}v_i^{(2)}$, etc., decay fast enough, say, of order $R^{-\alpha}$, where $\alpha>1$ for 2-D and $\alpha>2$ for 3-D cases, the boundary integrals vanish. Hence

$$\int_{\Omega_\infty}(F_i^{(1)}u_i^{(2)}-F_i^{(2)}u_i^{(1)})\,\mathrm{d}\boldsymbol{\chi}+\int_{\Omega_\infty}(f_i^{(1)}v_i^{(2)}-f_i^{(2)}v_i^{(1)})\,\mathrm{d}\boldsymbol{\chi}$$
$$-\int_{\Omega_\infty}(Q^{(1)}p^{(2)}-Q^{(2)}p^{(1)})\,\mathrm{d}\boldsymbol{\chi}+\int_{\Omega_\infty}(E_{ij}^{(1)}\sigma_{ij}^{(2)}-E_{ij}^{(2)}\sigma_{ij}^{(1)})\,\mathrm{d}\boldsymbol{\chi}=0 \quad \text{(C9)}$$

To prove (C1), we substitute in the following singularities and their corresponding influence functions:

$$F_i^{(1)} = \delta_{ij}\delta(\boldsymbol{\chi}-\mathbf{x}_1)\delta(t-\tau) \tag{C10}$$

$$Q^{(2)} = \delta(\boldsymbol{\chi}-\mathbf{x}_2)\delta(t-\tau) \tag{C11}$$

$$F_i^{(2)} = Q^{(1)} = f_i^{(1)} = f_i^{(2)} = E_{ij}^{(1)} = E_{ij}^{(2)} = 0 \tag{C12}$$

Equation (C9) reduces to

$$u_i^{fi}(\mathbf{x}_1-\mathbf{x}_2, t-\tau)+p_i^{Fi}(\mathbf{x}_2-\mathbf{x}_1, t-\tau) = 0 \tag{C13}$$

Since $\mathbf{x}_1$ and $\mathbf{x}_2$ are arbitrary, we can rewrite the above as

$$u_i^{fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) = -p_i^{Fi}(\boldsymbol{\chi}-\mathbf{x}, t-\tau) = p_i^{Fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) \tag{C14}$$

Relation (C1) is thus proven. To demonstrate (C2), we adopt

$$F_i^{(1)} = \delta_{ij}\delta(\boldsymbol{\chi}-\mathbf{x}_1)\delta(t-\tau) \tag{C15}$$

$$f_i^{(2)} = \delta_{ij}\delta(\boldsymbol{\chi}-\mathbf{x}_2)\delta(t-\tau) \tag{C16}$$

$$F_i^{(2)} = Q^{(1)} = Q^{(2)} = f_i^{(1)} = E_{ij}^{(1)} = E_{ij}^{(2)} = 0 \tag{C17}$$

from which we deduce that

$$u_{ij}^{fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) = v_{ij}^{Fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) \tag{C18}$$

Using in (C9)

$$f_i^{(1)} = \delta_{ij}\delta(\boldsymbol{\chi}-\mathbf{x}_1)\delta(t-\tau) \tag{C19}$$

$$Q^{(2)} = \delta(\boldsymbol{\chi}-\mathbf{x}_2)\delta(t-\tau) \tag{C20}$$

$$F_i^{(1)} = F_i^{(2)} = Q^{(1)} = f_i^{(2)} = E_{ij}^{(1)} = E_{ij}^{(2)} = 0 \tag{C21}$$

(C3) can readily be proven

$$v_i^{fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) = p_i^{fi}(\mathbf{x}-\boldsymbol{\chi}, t-\tau) \tag{C22}$$

Also, selecting

$$E_{ij}^{(1)} = -\tfrac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\delta(\boldsymbol{\chi}-\mathbf{x}_1)\delta(t-\tau) \tag{C23}$$

and alternately assigning the delta function to $F_i^{(2)}$, $f_i^{(2)}$ and $Q^{(2)}$, the remaining three relations in (C4)–(C6) can be demonstrated.

Equations (C1)–(C6) are six fundamental relations that cannot be proven by casual observation. Other relations contained in (50) and (67) can be obtained from (C1)–(C6) by manipulating the conversion rules shown as (26)–(30).

APPENDIX D: TABLE OF GREEN'S FUNCTIONS

A comprehensive list of Green's functions for quasi-static isotropic poroelasticity is given below. We utilize the following notations

$$r = |\mathbf{x}-\boldsymbol{\chi}| \tag{D1}$$

$$r_{,i} = \frac{\partial r}{\partial x_i} = \frac{x_i-\chi_i}{r} \tag{D2}$$

$$\xi = \frac{r}{\sqrt{4c(t-\tau)}} \tag{D3}$$

and use erf and erfc, respectively, to denote error function and complementary error function, while E_1 is the exponential integral.

D.1. Continuous source
$\gamma = \delta(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau)$

2-D

$$u_i^{sc} = \frac{\eta}{8\pi G\kappa} r r_{,i}\left[\mathrm{E}_1(\xi^2) + \frac{1-e^{-\xi^2}}{\xi^2}\right]$$

$$\sigma_{ij}^{sc} = \frac{\eta}{4\pi\kappa}\left[(\delta_{ij}-2r_{,i}r_{,j})\frac{1-e^{-\xi^2}}{\xi^2} - \delta_{ij}\mathrm{E}_1(\xi^2)\right]$$

$$p^{sc} = \frac{1}{4\pi\kappa}\mathrm{E}_1(\xi^2) = \frac{c}{\kappa}\zeta^{sc}$$

$$v_i^{sc} = \frac{1}{8\pi c} r r_{,i}\left[\frac{e^{-\xi^2}}{\xi^2} - \mathrm{E}_1(\xi^2)\right]$$

$$q_i^{sc} = \frac{1}{2\pi}\frac{r_{,i}}{r}e^{-\xi^2} = v_i^{si} = \kappa p_i^{pc} = c\zeta_i^{rpc} = v_i^{lc} = p_i^{fc} = \frac{c}{\kappa}\zeta_i^{fc} = -\frac{G}{\eta}\zeta_i^{Fc}$$

$$\zeta^{sc} = \frac{\kappa}{c}p^{sc}$$

3-D

$$u_i^{sc} = \frac{\eta}{16\pi G\kappa} r_{,i}\left[2\,\mathrm{erfc}(\xi) + \frac{\mathrm{erf}(\xi)}{\xi^2} - \frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi}\right]$$

$$\sigma_{ij}^{sc} = \frac{\eta}{8\pi\kappa}\frac{1}{r}\left\{(\delta_{ij}-3r_{,i}r_{,j})\left[\frac{\mathrm{erf}(\xi)}{\xi^2}\frac{2}{\sqrt{\pi}} - \frac{e^{-\xi^2}}{\xi}\right] - 2(\delta_{ij}+r_{,i}r_{,j})\,\mathrm{erfc}(\xi)\right\}$$

$$p^{sc} = \frac{1}{4\pi\kappa}\frac{\mathrm{erfc}(\xi)}{r} = \frac{c}{\kappa}\zeta^{sc}$$

$$v_i^{sc} = \frac{1}{16\pi c} r_{,i}\left[\frac{\mathrm{erfc}(\xi)}{\xi^2} - 2\,\mathrm{erfc}(\xi) + \frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi}\right]$$

$$q_i^{sc} = \frac{1}{4\pi}\frac{r_{,i}}{r^2}\left[\frac{2}{\sqrt{\pi}}\xi e^{-\xi^2} + \mathrm{erfc}(\xi)\right] = v_i^{si} = \kappa p_i^{pc} = c\zeta_i^{rpc} = v_i^{lc} = p_i^{fc} = \frac{c}{\kappa}\zeta_i^{fc} = -\frac{G}{\eta}\zeta_i^{Fc}$$

$$\zeta^{sc} = \frac{\kappa}{c}p^{sc}$$

D.2. Instantaneous source
$\gamma = \delta(\mathbf{x}-\boldsymbol{\chi})\delta(t-\tau)$

2-D

$$u_i^{si} = \frac{\eta c}{2\pi G\kappa}\frac{r_{,i}}{r}(1-e^{-\xi^2}) = p_i^{Fc} = u_i^{lc}$$

$$\sigma_{ij}^{si} = \frac{\eta c}{\pi\kappa}\frac{1}{r^2}[(\delta_{ij}-2r_{,i}r_{,j})(1-e^{-\xi^2}) - 2(\delta_{ij}-r_{,i}r_{,j})\xi^2 e^{-\xi^2}]$$

$$= \sigma_{ij}^{lc} = p_{ij}^{dc}$$

$$p^{si} = \frac{c}{\pi\kappa}\frac{1}{r^2}\xi^2 e^{-\xi^2} = \frac{c}{\kappa}\zeta^{si} = p^{lc} = \frac{c}{\kappa}\zeta^{lc}$$

$$v_i^{si} = q_i^{sc} = \kappa p_i^{pc} = c\zeta_i^{rpc} = v_i^{lc} = p_i^{fc} = \frac{c}{\kappa}\zeta_i^{fc}$$

$$q_i^{si} = \frac{2c}{\pi}\frac{r_{,i}}{r^3}\xi^4 e^{-\xi^2} = \kappa p_i^{pi} = c\zeta_i^{pi} = q_i^{lc} = v_i^{li} = p_i^{fi} = \frac{c}{\kappa}\zeta_i^{fi} = -\frac{G}{\eta}\zeta_i^{Fi}$$

$$\zeta^{si} = \frac{\kappa}{c}p^{si} = \frac{\kappa}{c}p^{lc} = \zeta^{lc}$$

3-D

$$u_i^{si} = \frac{\eta c}{4\pi G\kappa}\frac{r_{,i}}{r^2}\left[\operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}}\xi\, e^{-\xi^2}\right] = p_i^{Fc} = u_i^{lc}$$

$$\sigma_{ij}^{si} = \frac{\eta c}{2\pi\kappa}\frac{1}{r^3}\left\{(\delta_{ij} - 3r_{,i}r_{,j})\left[\operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}}\xi\, e^{-\xi^2}\right] - \frac{4}{\sqrt{\pi}}(\delta_{ij} - r_{,i}r_{,j})\xi^3\, e^{-\xi^2}\right\}$$

$$= \sigma_{ij}^{lc} = p_{ij}^{dc}$$

$$p^{si} = \frac{c}{\pi^{3/2}\kappa}\frac{1}{r^3}\xi^3\, e^{-\xi^2} = \frac{c}{\kappa}\zeta^{si} = p^{lc} = \frac{c}{\kappa}\zeta^{lc}$$

$$v_i^{si} = q_i^{sc} = \kappa p_i^{pc} = c\zeta_i^{pc} = v_i^{lc} = p_i^{fc} = \frac{c}{\kappa}\zeta_i^{fc}$$

$$q_i^{si} = \frac{2c}{\pi^{3/2}}\frac{r_{,i}}{r^4}\xi^5\, e^{-\xi^2} = \kappa p_i^{pi} = c\zeta_i^{pi} = q_i^{lc} = v_i^{li} = p_i^{fi} = \frac{c}{\kappa}\zeta_i^{fi} = -\frac{G}{\eta}\zeta_i^{Fi}$$

$$\zeta^{si} = \frac{\kappa}{c}p^{si} = \frac{\kappa}{c}p^{lc} = \zeta^{lc}$$

D.3. Continuous dipole
$\gamma_k = -\delta(\mathbf{x}-\boldsymbol{\chi})_{,k}\mathrm{H}(t-\tau)$

2-D

$$u_{ik}^{pc} = \frac{\eta}{8\pi G\kappa}\left[(2r_{,i}r_{,k} - \delta_{ik})\frac{1-e^{-\xi^2}}{\xi^2} - \delta_{ik}\mathrm{E}_1(\xi^2)\right]$$

$$= \frac{1}{\kappa}u_{ik}^{fc} = -\frac{\eta c}{\kappa^2 G}v_{ik}^{fc} = \frac{1}{\kappa}v_{ik}^{Fc}$$

$$\sigma_{ijk}^{pc} = \frac{\eta}{2\pi\kappa}\frac{1}{r}\left[(\delta_{ik}r_{,j} + \delta_{jk}r_{,i} + \delta_{ij}r_{,k} - 4r_{,i}r_{,j}r_{,k})\frac{1-e^{-\xi^2}}{\xi^2} + 2(r_{,i}r_{,j}r_{,k} - \delta_{ij}r_{,k})\, e^{-\xi^2}\right]$$

$$= \frac{1}{\kappa}\sigma_{ijk}^{fc} = \frac{1}{\kappa}v_{kji}^{dc}$$

$$p_k^{pc} = \frac{1}{\kappa}q_k^{sc} = \frac{1}{\kappa}v_k^{si} = \frac{c}{k}\zeta_k^{pc} = \frac{1}{\kappa}v_k^{lc} = \frac{1}{\kappa}p_k^{fc} = \frac{c}{\kappa^2}\zeta_k^{fc} = -\frac{G}{\eta\kappa}\zeta_k^{Fc}$$

$$v_{ik}^{pc} = \frac{1}{8\pi c}\left[(2r_{,i}r_{,k} - \delta_{ik})\frac{e^{-\xi^2}}{\xi^2} + \delta_{ik}\mathrm{E}_1(\xi^2)\right]$$

$$q_{ik}^{pc} = \frac{1}{2\pi}\frac{1}{r^2}[2r_{,i}r_{,k}(1+\xi^2) - \delta_{ik}]\, e^{-\xi^2} = v_{ik}^{pi}$$

$$\zeta_k^{pc} = \frac{1}{c}q_k^{sc} = \frac{1}{c}v_k^{si} = \frac{\kappa}{c}p_k^{pc} = \frac{1}{c}v_k^{lc} = \frac{1}{c}p_k^{fc} = \frac{1}{\kappa}\zeta_k^{fc} = -\frac{G}{\eta c}\zeta_k^{Fc}$$

3-D

$$u_{ik}^{pc} = \frac{\eta}{16\pi G\kappa}\frac{1}{r}\left\{(3r_{,i}r_{,k} - \delta_{ik})\left[\frac{\operatorname{erf}(\xi)}{\xi^2} - \frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi}\right] + 2(r_{,i}r_{,k} - \delta_{ik})\operatorname{erfc}(\xi)\right\}$$

$$= \frac{1}{\kappa}u_{ik}^{fc} - \frac{\eta c}{\kappa^2 G}v_{ik}^{fc} = \frac{1}{\kappa}v_{ik}^{Fc}$$

$$\sigma_{ijk}^{pc} = \frac{\eta}{8\pi\kappa}\frac{1}{r^2}\left\{(\delta_{ij}r_{,k} + \delta_{ik}r_{,j} + \delta_{jk}r_{,i} - 5r_{,i}r_{,j}r_{,k})\left[\frac{3\operatorname{erf}(\xi)}{\xi^2} - \frac{6}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi} + 2\operatorname{erfc}(\xi)\right]\right.$$

$$\left. + 4(r_{,i}r_{,j}r_{,k} - \delta_{ij}r_{,k})\left[\operatorname{erfc}(\xi) + \frac{2}{\sqrt{\pi}}\xi\, e^{-\xi^2}\right]\right\}$$

$$= \frac{1}{\kappa}\sigma_{ijk}^{fc} = \frac{1}{\kappa}v_{kji}^{dc}$$

$$p_k^{pc} = \frac{1}{\kappa} q_k^{sc} = \frac{1}{\kappa} v_k^{si} = \frac{c}{k} \zeta_k^{pc} = \frac{1}{\kappa} v_k^{fc} = \frac{1}{\kappa} p_k^{fc} = \frac{c}{\kappa^2} \zeta_k^{fc} = -\frac{G}{\eta\kappa} \zeta_k^{Fc}$$

$$v_{ik}^{pc} = \frac{1}{16\pi c} \frac{1}{r} \left\{ (3r_{,i}r_{,k} - \delta_{ik}) \left[\frac{\operatorname{erfc}(\xi)}{\xi^2} + \frac{2}{\sqrt{\pi}} \frac{e^{-\xi^2}}{\xi} \right] + 2(\delta_{ik} - r_{,i}r_{,k}) \operatorname{erfc}(\xi) \right\}$$

$$q_{ik}^{pc} = \frac{1}{4\pi} \frac{1}{r^3} \left\{ (3r_{,i}r_{,k} - \delta_{ik}) \left[\operatorname{erfc}(\xi) + \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} \right] + \frac{4}{\sqrt{\pi}} r_{,i}r_{,k} \xi^3 e^{-\xi^2} \right\} = v_{ik}^{pi}$$

$$\zeta_k^{pc} = \frac{1}{c} q_k^{sc} = \frac{1}{c} v_k^{si} = \frac{\kappa}{c} p_k^{pc} = \frac{1}{c} v_k^{fc} = \frac{1}{c} p_k^{fc} = \frac{1}{\kappa} \zeta_k^{fc} = -\frac{G}{\eta c} \zeta_k^{Fc}$$

D.4. Instantaneous dipole
$\gamma_k = \delta(\mathbf{x} - \boldsymbol{\chi})_{,k} \delta(t - \tau)$

2-D

$$u_{ik}^{pi} = \frac{\eta c}{2\pi G\kappa} \frac{1}{r^2} [(2r_{,i}r_{,k} - \delta_{ik})(1 - e^{-\xi^2}) - 2r_{,i}r_{,k}\xi^2 e^{-\xi^2}]$$

$$= -\frac{\eta c}{\kappa^2 G} q_{ik}^{fc} = \frac{1}{\kappa} u_{ik}^{fi} = -\frac{\eta c}{\kappa^2 G} v_{ik}^{fi} = \frac{1}{\kappa} q_{ik}^{Fc} = \frac{1}{\kappa} v_{ik}^{Fi}$$

$$\sigma_{ijk}^{pi} = \frac{2\eta c}{\pi\kappa} \frac{1}{r^3} [(\delta_{ik}r_{,j} + \delta_{jk}r_{,i} + \delta_{ij}r_{,k} - 4r_{,i}r_{,j}r_{,k})(1 - e^{-\xi^2} - \xi^2 e^{-\xi^2}) + 2(r_{,i}r_{,j}r_{,k} - \delta_{ij}r_{,k})\xi^4 e^{-\xi^2}]$$

$$= \frac{1}{\kappa} \sigma_{ijk}^{fi} = \frac{1}{\kappa} q_{kji}^{dc} = \frac{1}{\kappa} v_{kji}^{di}$$

$$p_k^{pi} = \frac{1}{k} q_k^{si} = \frac{c}{\kappa} \zeta_k^{pi} = \frac{1}{k} q_k^{fc} = \frac{1}{k} v_k^{fi} = \frac{1}{k} p_k^{fi} = \frac{c}{\kappa^2} \zeta_k^{fi} = -\frac{G}{\eta\kappa} \zeta_k^{Fi}$$

$$v_{ik}^{pi} = q_{ik}^{pc}$$

$$q_{ik}^{pi} = \frac{2c}{\pi} \frac{1}{r^4} (2r_{,i}r_{,k}\xi^2 - \delta_{ik}) \xi^4 e^{-\xi^2}$$

$$\zeta_k^{pi} = \frac{1}{c} q_k^{si} = \frac{\kappa}{c} p_k^{pi} = \frac{1}{c} q_k^{fc} = \frac{1}{c} v_k^{fi} = \frac{1}{c} p_k^{fi} = \frac{1}{\kappa} \zeta_k^{fi} = -\frac{G}{\eta c} \zeta_k^{Fi}$$

3-D

$$u_{ik}^{pi} = \frac{\eta c}{4\pi G\kappa} \frac{1}{r^3} \left\{ (3r_{,i}r_{,k} - \delta_{ik}) \left[\operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} \right] - \frac{4}{\sqrt{\pi}} r_{,i}r_{,k} \xi^3 e^{-\xi^2} \right\}$$

$$= -\frac{\eta c}{\kappa^2 G} q_{ik}^{fc} = \frac{1}{\kappa} u_{ik}^{fi} = -\frac{\eta c}{\kappa^2 G} v_{ik}^{fi} = \frac{1}{\kappa} q_{ik}^{Fc} = \frac{1}{\kappa} v_{ik}^{Fi}$$

$$\sigma_{ijk}^{pi} = \frac{\eta c}{2\pi\kappa} \frac{1}{r^4} \left\{ \frac{8}{\sqrt{\pi}} (r_{,i}r_{,j}r_{,k} - \delta_{ij}r_{,k}) \xi^5 e^{-\xi^2} \right.$$

$$\left. + (\delta_{ij}r_{,k} + \delta_{ik}r_{,j} + \delta_{jk}r_{,i} - 5r_{,i}r_{,j}r_{,k}) \left[3 \operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}} (3 + 2\xi^2) \xi e^{-\xi^2} \right] \right\}$$

$$= \frac{1}{\kappa} \sigma_{ijk}^{fi}$$

$$p_k^{pi} = \frac{1}{k} q_k^{si} = \frac{c}{\kappa} \zeta_k^{pi} = \frac{1}{k} q_k^{fc} = \frac{1}{k} v_k^{fi} = \frac{1}{k} p_k^{fi} = \frac{c}{\kappa^2} \zeta_k^{fi} = -\frac{G}{\eta\kappa} \zeta_k^{Fi}$$

$$v_{ik}^{pi} = q_{ik}^{pc}$$

$$q_{ik}^{pi} = \frac{2c}{\pi^{3/2}} \frac{1}{r^5} (2r_{,i}r_{,k}\xi^2 - \delta_{ik}) \xi^5 e^{-\xi^2}$$

$$\zeta_k^{pi} = \frac{1}{c} q_k^{si} = \frac{\kappa}{c} p_k^{pi} = \frac{1}{c} q_k^{lc} = \frac{1}{c} v_k^{li} = \frac{1}{c} p_k^{fi} = \frac{1}{\kappa} \zeta_k^{fi} = -\frac{G}{\eta c} \zeta_k^{Fi}$$

D.5. Continuous fluid dilatation
$Q = \sigma(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau)$

2-D, 3-D

$$(u_i^{lc}, \sigma_{ij}^{lc}, p^{lc}, v_i^{lc}, q_i^{lc}, \zeta^{lc}) = (u_i^{si}, \sigma_{ij}^{si}, p^{si}, v_i^{si}, q_i^{si}, \zeta^{si})$$

D.6. Instantaneous fluid dilatation
$Q = \delta(\mathbf{x}-\boldsymbol{\chi})\delta(t-\tau)$

2-D

$$u_i^{li} = \frac{\eta c}{2\pi G\kappa} \frac{r_{,i}}{r} \left[\delta(t-\tau) - 4c \frac{1}{r^2} \xi^4 e^{-\xi^2} \right] = p_i^{Fi}$$

$$\sigma_{ij}^{li} = \frac{\eta c}{\pi\kappa} \frac{1}{r^2} \left\{ (\delta_{ij} - 2r_{,i}r_{,j})\delta(t-\tau) + 4c \frac{1}{r^2} [\delta_{ij}(1-2\xi^2) + 2r_{,i}r_{,j}\xi^2] \xi^4 e^{-\xi^2} \right\}$$

$$= p_{ij}^{di}$$

$$p^{li} = \frac{4c^2}{\pi\kappa} \frac{1}{r^4} (\xi^2 - 1)\xi^4 e^{-\xi^2} = \frac{c}{\kappa} \zeta^{li}$$

$$v_i^{li} = q_i^{si} = \kappa p_i^{pi} = c\zeta_i^{pi} = q_i^{lc} = p_i^{fi} = \frac{c}{\kappa} \zeta_i^{fi} = -\frac{G}{\eta} \zeta_i^{Fi}$$

$$q_i^{li} = \frac{8c^2}{\pi} \frac{r_{,i}}{r^5} (\xi^2 - 2)\xi^6 e^{-\xi^2}$$

$$\zeta^{li} = \frac{\kappa}{c} p^{li}$$

3-D

$$u_i^{li} = \frac{\eta c}{4\pi G\kappa} \frac{r_{,i}}{r^2} \left[\delta(t-\tau) - \frac{8c}{\sqrt{\pi}} \frac{1}{r^2} \xi^5 e^{-\xi^2} \right] = p_i^{Fi}$$

$$\sigma_{ij}^{li} = \frac{\eta c}{2\pi\kappa} \frac{1}{r^3} \left\{ (\delta_{ij} - 3r_{,i}r_{,j})\delta(t-\tau) + \frac{16c}{\sqrt{\pi}} \frac{1}{r^2} [\delta_{ij}(1-\xi^2) + r_{,i}r_{,j}\xi^2] \xi^5 e^{-\xi^2} \right\}$$

$$= p_{ij}^{di}$$

$$p^{li} = \frac{2c^2}{\pi^{3/2}\kappa} \frac{1}{r^5} (2\xi^2 - 3)\xi^5 e^{-\xi^2} = \frac{c}{\kappa} \zeta^{li}$$

$$v_i^{li} = q_i^{si} = \kappa p_i^{pi} = c\zeta_i^{pi} = q_i^{lc} = p_i^{fi} = \frac{c}{\kappa} \zeta_i^{fi} = -\frac{G}{\eta} \zeta_i^{Fi}$$

$$q_i^{li} = \frac{4c^2}{\pi^{3/2}} \frac{r_{,i}}{r^6} (2\xi^2 - 5)\xi^7 e^{-\xi^2}$$

$$\zeta^{li} = \frac{\kappa}{c} p^{li}$$

D.7. Continuous fluid force
$f_{ik} = \delta_{ik}\delta(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau)$

2-D, 3-D

$$u_{ik}^{fc} = \kappa u_{ik}^{pc} = -\frac{\eta c}{\kappa G} v_{ik}^{fc} = v_{ik}^{Fc}$$

$$\sigma_{ijk}^{fc} = \kappa \sigma_{ijk}^{pc} = v_{kji}^{dc}$$

$$p_k^{fc} = q_k^{sc} = v_k^{si} = \kappa p_k^{pc} = c\zeta_k^{pc} = v_k^{lc} = \frac{c}{\kappa} \zeta_k^{fc} = -\frac{G}{\eta} \zeta_k^{Fc}$$

$$v_{ik}^{fc} = -\frac{\kappa^2 G}{\eta c} u_{ik}^{pc} = -\frac{\kappa G}{\eta c} u_{ik}^{fc} = -\frac{\kappa G}{\eta c} v_{ik}^{Fc}$$

$$q_{ik}^{fc} = -\frac{\kappa^2 G}{\eta c} u_{ik}^{pi} = -\frac{\kappa G}{\eta c} u_{ik}^{fi} = v_{ik}^{fi} = -\frac{\kappa G}{\eta c} q_{ik}^{Fc} = -\frac{\kappa G}{\eta c} v_{ik}^{Fi}$$

$$\zeta_k^{fc} = \frac{\kappa}{c} q_k^{sc} = \frac{\kappa}{c} v_k^{si} = \frac{\kappa^2}{c} p_k^{pc} = \kappa \zeta_k^{pc} = \frac{\kappa}{c} v_k^{lc} = \frac{\kappa}{c} p_k^{fc} = -\frac{\kappa G}{\eta c} \zeta_k^{Fc}$$

D.8. Instantaneous fluid force
$f_{ik} = \delta_{ik}\delta(\mathbf{x}-\boldsymbol{\chi})\delta(t-\tau)$

2-D

$$u_{ik}^{fi} = \kappa u_{ik}^{pi} = -\frac{\eta c}{\kappa G} q_{ik}^{fc} = -\frac{\eta c}{\kappa G} v_{ik}^{fi} = q_{ik}^{Fc} = v_{ik}^{Fi}$$

$$\sigma_{ijk}^{fi} = \kappa \sigma_{ijk}^{pi} = q_{kji}^{dc} = v_{kji}^{di}$$

$$p_k^{fi} = q_k^{si} = \kappa p_k^{pi} = c\zeta_k^{pi} = q_k^{lc} = v_k^{li} = \frac{c}{\kappa}\zeta_k^{fi} = -\frac{G}{\eta}\zeta_k^{Fi}$$

$$v_{ik}^{fi} = -\frac{\kappa^2 G}{\eta c} u_{ik}^{pi} = q_{ik}^{fc} = -\frac{\kappa G}{\eta c} u_{ik}^{fi} = -\frac{\kappa G}{\eta c} q_{ik}^{Fc} = -\frac{\kappa G}{\eta c} v_{ik}^{Fi}$$

$$q_{ik}^{fi} = -\frac{\kappa}{2\pi}\frac{1}{r^2}\left[(2r_{,i}r_{,k}-\delta_{ik})\delta(t-\tau) + 4c\frac{1}{r^2}(\delta_{ik}-2r_{,i}r_{,k}\xi^2)\xi^4 e^{-\xi^2}\right]$$

$$\zeta_k^{fi} = \frac{\kappa}{c} q_k^{si} = \frac{\kappa^2}{c} p_k^{pi} = \kappa\zeta_k^{pi} = \frac{\kappa}{c} q_k^{lc} = \frac{\kappa}{c} v_k^{li} = \frac{\kappa}{c} p_k^{fi} = -\frac{\kappa G}{\eta c}\zeta_k^{Fi}$$

3-D

$$u_{ik}^{fi} = \kappa u_{ik}^{pi} = -\frac{\eta c}{\kappa G} q_{ik}^{fc} = -\frac{\eta c}{\kappa G} v_{ik}^{fi} = q_{ik}^{Fc} = v_{ik}^{Fi}$$

$$\sigma_{ijk}^{fi} = \kappa \sigma_{ijk}^{pi} = q_{kji}^{dc} = v_{kji}^{di}$$

$$p_k^{fi} = q_k^{si} = \kappa p_k^{pi} = c\zeta_k^{pi} = q_k^{lc} = v_k^{li} = \frac{c}{\kappa}\zeta_k^{fi} = -\frac{G}{\eta}\zeta_k^{Fi}$$

$$v_{ik}^{fi} = -\frac{\kappa^2 G}{\eta c} u_{ik}^{pi} = q_{ik}^{fc} = -\frac{\kappa G}{\eta c} u_{ik}^{fi} = -\frac{\kappa G}{\eta c} q_{ik}^{Fc} = -\frac{\kappa G}{\eta c} v_{ik}^{Fi}$$

$$q_{ik}^{fi} = -\frac{\kappa}{4\pi}\frac{1}{r^3}\left[(3r_{,i}r_{,k}-\delta_{ik})\delta(t-\tau) + \frac{8c}{\sqrt{\pi}}\frac{1}{r^2}(\delta_{ik}-2r_{,i}r_{,k}\xi^2)\xi^5 e^{-\xi^2}\right]$$

$$\zeta_k^{fi} = \frac{\kappa}{c} q_k^{si} = \frac{\kappa^2}{c} p_k^{pi} = \kappa\zeta_k^{pi} = \frac{\kappa}{c} q_k^{lc} = \frac{\kappa}{c} v_k^{li} = \frac{\kappa}{c} p_k^{fi} = -\frac{\kappa G}{\eta c}\zeta_k^{Fi}$$

D.9. Continuous total force
$F_{ik} = \delta_{ik}\delta(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau)$

2-D

$$u_{ik}^{Fc} = \frac{1}{8\pi G(1-\nu_u)}[r_{,i}r_{,k} - (3-4\nu_u)\delta_{ik}\ln r]\mathrm{H}(t-\tau)$$

$$+ \frac{\eta^2 c}{8\pi G^2\kappa}\left[(\delta_{ik}-2r_{,i}r_{,k})\frac{1-e^{-\xi^2}}{\xi^2} + \delta_{ik}\mathrm{E}_1(\xi^2)\right]$$

$$\sigma_{ijk}^{Fc} = \frac{1}{4\pi(1-\nu_u)}\frac{1}{r}[(1-2\nu_u)(\delta_{ij}r_{,k}-\delta_{jk}r_{,i}-\delta_{ik}r_{,j}) - 2r_{,i}r_{,j}r_{,k}]\mathrm{H}(t-\tau)$$

$$+ \frac{\eta^2 c}{2\pi G\kappa}\frac{1}{r}\left[(4r_{,i}r_{,j}r_{,k}-\delta_{ik}r_{,j}-\delta_{jk}r_{,i}-\delta_{ij}r_{,k})\frac{1-e^{-\xi^2}}{\xi^2} + 2(\delta_{ij}r_{,k}-r_{,i}r_{,j}r_{,k})e^{-\xi^2}\right]$$

$$= u_{kji}^{dc}$$

$$p_k^{Fc} = u_k^{si} = u_k^{lc}$$

$$v_{ik}^{Fc} = \kappa u_{ik}^{pc} = u_{ik}^{fc} = -\frac{\eta c}{\kappa G} v_{ik}^{fc}$$

$$q_{ik}^{Fc} = \kappa u_{ik}^{pi} = -\frac{\eta c}{\kappa G} q_{ik}^{fc} = u_{ik}^{fi} = -\frac{\eta c}{\kappa G} v_{ik}^{fi} = v_{ik}^{Fi}$$

$$\zeta_k^{Fc} = -\frac{\eta}{G} q_k^{sc} = -\frac{\eta}{G} v_k^{si} = -\frac{\eta\kappa}{G} p_k^{pc} = -\frac{\eta c}{G} \zeta_k^{pc} = -\frac{\eta}{G} v_k^{lc} = -\frac{\eta}{G} p_k^{fc} = -\frac{\eta c}{\kappa G} \zeta_k^{fc}$$

3-D

$$u_{ik}^{Fc} = \frac{1}{16\pi G(1-\nu_u)} \frac{1}{r} [r_{,i} r_{,k} + (3-4\nu_u)\delta_{ik}] \mathrm{H}(t-\tau)$$

$$+ \frac{\eta^2 c}{16\pi G^2 \kappa} \frac{1}{r} \left\{ (\delta_{ik} - 3r_{,i} r_{,k}) \left[\frac{\mathrm{erf}(\xi)}{\xi^2} - \frac{2}{\sqrt{\pi}} \frac{e^{-\xi^2}}{\xi} \right] + 2(\delta_{ik} - r_{,i} r_{,k}) \,\mathrm{erfc}(\xi) \right\}$$

$$\sigma_{ijk}^{Fc} = \frac{1}{8\pi(1-\nu_u)} \frac{1}{r^2} [(1-2\nu_u)(\delta_{ij} r_{,k} - \delta_{jk} r_{,i} - \delta_{ik} r_{,j}) - 3r_{,i} r_{,j} r_{,k}] \mathrm{H}(t-\tau)$$

$$+ \frac{\eta^2 c}{8\pi G\kappa} \frac{1}{r^2} \left\{ 4(\delta_{ij} r_{,k} - r_{,i} r_{,j} r_{,k}) \left[\mathrm{erfc}(\xi) + \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} \right] \right.$$

$$\left. + (5r_{,i} r_{,j} r_{,k} - \delta_{ij} r_{,k} - \delta_{ik} r_{,j} - \delta_{jk} r_{,i}) \left[\frac{3\,\mathrm{erf}(\xi)}{\xi^2} - \frac{6}{\sqrt{\pi}} \frac{e^{-\xi^2}}{\xi} + 2\,\mathrm{erfc}(\xi) \right] \right\}$$

$$= u_{kji}^{dc}$$

$$p_k^{Fc} = u_k^{si} = u_k^{lc}$$

$$v_{ik}^{Fc} = \kappa u_{ik}^{pc} = u_{ik}^{fc} = -\frac{\eta c}{\kappa G} v_{ik}^{fc}$$

$$q_{ik}^{Fc} = \kappa u_{ik}^{pi} = -\frac{\eta c}{\kappa G} q_{ik}^{fc} = u_{ik}^{fi} = -\frac{\eta c}{\kappa G} v_{ik}^{fi} = v_{ik}^{Fi}$$

$$\zeta_k^{Fc} = -\frac{\eta}{G} q_k^{sc} = -\frac{\eta}{G} v_k^{si} = -\frac{\eta\kappa}{G} p_k^{pc} = -\frac{\eta c}{G} \zeta_k^{pc} = -\frac{\eta}{G} v_k^{lc} = -\frac{\eta}{G} p_k^{fc} = -\frac{\eta c}{\kappa G} \zeta_k^{fc}$$

D.10. *Instantaneous total force*
$F_{ik} = \delta_{ik} \delta(\mathbf{x} - \boldsymbol{\chi}) \delta(t-\tau)$

2-D

$$u_{ik}^{Fi} = \frac{1}{8\pi G(1-\nu_u)} \delta(t-\tau) [r_{,i} r_{,k} - (3-4\nu_u)\delta_{ik} \ln r]$$

$$+ \frac{\eta^2 c^2}{2\pi G^2 \kappa} \frac{1}{r^2} [(\delta_{ik} - 2r_{,i} r_{,k})(1 - e^{-\xi^2}) + 2r_{,i} r_{,k} \xi^2 e^{-\xi^2}]$$

$$\sigma_{ijk}^{Fi} = \frac{1}{4\pi(1-\nu_u)} \delta(t-\tau) \frac{1}{r} [(1-2\nu_u)(\delta_{ij} r_{,k} - \delta_{jk} r_{,i} - \delta_{ik} r_{,j}) - 2r_{,i} r_{,j} r_{,k}]$$

$$+ \frac{2\eta^2 c^2}{\pi G\kappa} \frac{1}{r^3} [(4r_{,i} r_{,j} r_{,k} - \delta_{ik} r_{,j} - \delta_{jk} r_{,i} - \delta_{ij} r_{,k})(1 - e^{-\xi^2} - \xi^2 e^{-\xi^2})$$

$$- 2(r_{,i} r_{,j} r_{,k} - \delta_{ij} r_{,k}) \xi^4 e^{-\xi^2}]$$

$$= u_{kji}^{di}$$

$$p_k^{Fi} = u_k^{li}$$

$$v_{ik}^{Fi} = \kappa u_{ik}^{pi} = -\frac{\eta c}{\kappa G} q_{ik}^{fc} = u_{ik}^{fi} = -\frac{\eta c}{\kappa G} v_{ik}^{fi} = q_{ik}^{Fc}$$

$$q_{ik}^{Fi} = \frac{\eta c}{2\pi G} \frac{1}{r^2} \left[(2r_{,i} r_{,k} - \delta_{ik}) \delta(t-\tau) + 4c \frac{1}{r^2} (\delta_{ik} - 2r_{,i} r_{,k} \xi^2) \xi^4 e^{-\xi^2} \right]$$

$$\zeta_k^{Fi} = -\frac{\eta}{G} q_k^{si} = -\frac{\eta\kappa}{G} p_k^{pi} = -\frac{\eta c}{G} \zeta_k^{pi} = -\frac{\eta}{G} q_k^{lc} = -\frac{\eta}{G} v_k^{li} = -\frac{\eta}{G} p_k^{fi} = -\frac{\eta c}{\kappa G} \zeta_k^{fi}$$

3-D

$$u_{ik}^{Fi} = \frac{1}{16\pi G(1-\nu_u)} \delta(t-\tau) \frac{1}{r} [r_{,i} r_{,k} + (3-4\nu_u)\delta_{ik}]$$

$$+ \frac{\eta^2 c^2}{4\pi G^2 \kappa} \frac{1}{r^3} \left\{ (\delta_{ik} - 3r_{,i} r_{,k}) \left[\operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}} \xi e^{-\xi^2} \right] + \frac{4}{\sqrt{\pi}} r_{,i} r_{,k} \xi^3 e^{-\xi^2} \right\}$$

$$\sigma_{ijk}^{Fi} = \frac{1}{8\pi(1-\nu_u)} \delta(t-\tau) \frac{1}{r^2} [(1-2\nu_u)(\delta_{ij} r_{,k} - \delta_{jk} r_{,i} - \delta_{ik} r_{,j}) - 3r_{,i} r_{,j} r_{,k}]$$

$$+ \frac{\eta^2 c^2}{2\pi G \kappa} \frac{1}{r^4} \left\{ \frac{8}{\sqrt{\pi}} (\delta_{ij} r_{,k} - r_{,i} r_{,j} r_{,k}) \xi^5 e^{-\xi^2} \right.$$

$$\left. - (\delta_{ij} r_{,k} + \delta_{ik} r_{,j} + \delta_{jk} r_{,i} - 5r_{,i} r_{,j} r_{,k}) \left[3 \operatorname{erf}(\xi) - \frac{2}{\sqrt{\pi}} (3+2\xi^2) \xi e^{-\xi^2} \right] \right\}$$

$$= u_{kji}^{di}$$

$$p_k^{Fi} = u_k^{li}$$

$$v_{ik}^{Fi} = \kappa u_{ik}^{pi} = -\frac{\eta c}{\kappa G} q_{ik}^{fc} = u_{ik}^{fi} = -\frac{\eta c}{\kappa G} v_{ik}^{fi} = q_{ik}^{Fc}$$

$$q_{ik}^{Fi} = \frac{\eta c}{4\pi G} \frac{1}{r^3} \left[(3r_{,i} r_{,k} - \delta_{ik}) \delta(t-\tau) + \frac{8c}{\sqrt{\pi}} \frac{1}{r^2} (\delta_{ik} - 2r_{,i} r_{,k} \xi^2) \xi^5 e^{-\xi^2} \right]$$

$$\zeta_k^{Fi} = -\frac{\eta}{G} q_k^{si} = -\frac{\eta\kappa}{G} p_k^{pi} = -\frac{\eta c}{G} \zeta_k^{pi} = -\frac{\eta}{G} q_k^{lc} = -\frac{\eta}{G} v_k^{li} = -\frac{\eta}{G} p_k^{fi} = -\frac{\eta c}{\kappa G} \zeta_k^{fi}$$

D.11. *Continuous displacement discontinuity*
$E_{ijkl} = -\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})\delta(\mathbf{x}-\boldsymbol{\chi})\mathrm{H}(t-\tau)$

2-D

$$u_{ikl}^{dc} = \sigma_{lki}^{Fc}$$

$$\sigma_{ijkl}^{dc} = \frac{G}{2\pi(1-\nu_u)} \frac{1}{r^2} [8r_{,i} r_{,j} r_{,k} r_{,l}$$

$$- (1-2\nu_u)(\delta_{il}\delta_{jk} + \delta_{ik}\delta_{jl} - \delta_{ij}\delta_{kl} + 2\delta_{kl} r_{,i} r_{,j} + 2\delta_{ij} r_{,k} r_{,l})$$

$$- 2\nu_u(\delta_{ij}\delta_{kl} + \delta_{ij} r_{,l} r_{,k} + \delta_{il} r_{,j} r_{,k} + \delta_{jk} r_{,i} r_{,l} + \delta_{ik} r_{,j} r_{,l})] \mathrm{H}(t-\tau)$$

$$+ \frac{\eta^2 c}{\pi\kappa} \frac{1}{r^2} \left[(4\delta_{kj} r_{,i} r_{,l} + 4\delta_{kl} r_{,i} r_{,j} + 4\delta_{kl} r_{,i} r_{,j} + 4\delta_{ij} r_{,l} r_{,k} + 4\delta_{li} r_{,k} r_{,j} \right.$$

$$+ 4\delta_{ij} r_{,k} r_{,l} - \delta_{ij}\delta_{kl} - \delta_{kj}\delta_{il} - \delta_{ki}\delta_{jl} - 24 r_{,i} r_{,j} r_{,k} r_{,l}) \frac{1-e^{-\xi^2}}{\xi^2}$$

$$+ 2(2\delta_{ij}\delta_{kl} - 3\delta_{kl} r_{,i} r_{,j} - \delta_{ij} r_{,l} r_{,k} - \delta_{il} r_{,k} r_{,j} - \delta_{kj} r_{,i} r_{,l} - \delta_{ki} r_{,l} r_{,j}$$

$$- 3\delta_{ij} r_{,l} r_{,k} + 8 r_{,i} r_{,j} r_{,k} r_{,l}) e^{-\xi^2}$$

$$\left. + 4(\delta_{ij}\delta_{kl} - \delta_{kl} r_{,i} r_{,j} - \delta_{ij} r_{,l} r_{,k} + r_{,i} r_{,j} r_{,k} r_{,l}) \xi^2 e^{-\xi^2} \right]$$

$$p_{kl}^{dc} = \sigma_{kl}^{si} = \sigma_{kl}^{lc}$$

$$v_{ikl}^{dc} = \kappa \sigma_{lki}^{pc} = \sigma_{lki}^{fc}$$

$$q_{ikl}^{dc} = \kappa \sigma_{lki}^{pi} = \sigma_{lki}^{fi} = v_{ikl}^{di}$$

$$\zeta_{kl}^{dc} = \frac{\eta}{\pi} \frac{1}{r^2} [2r_{,k} r_{,l} - \delta_{kl} - 2(\delta_{kl} - r_{,k} r_{,l}) \xi^2] e^{-\xi^2}$$

3-D

$$u_{ikl}^{dc} = \sigma_{lki}^{Fc}$$

$$\begin{aligned}
\sigma_{ijkl}^{dc} = {} & \frac{G}{4\pi(1-\nu_u)}\frac{1}{r^3}[15r_{,i}r_{,j}r_{,k}r_{,l} \\
& -(1-2\nu_u)(\delta_{kj}\delta_{il}+\delta_{ki}\delta_{jl}-\delta_{ij}\delta_{kl}+3\delta_{kl}r_{,i}r_{,j}+3\delta_{ij}r_{,k}r_{,l}) \\
& -\nu_u(2\delta_{ij}\delta_{kl}+3\delta_{lj}r_{,i}r_{,k}+3\delta_{il}r_{,k}r_{,j}+3\delta_{kj}r_{,i}r_{,l}+3\delta_{ki}r_{,l}r_{,j})]\mathrm{H}(t-\tau) \\
& +\frac{\eta^2 c}{4\pi\kappa}\frac{1}{r^3}\left[\frac{8}{\sqrt{\pi}}(-2\delta_{ij}\delta_{kl}+4\delta_{kl}r_{,i}r_{,j}+4\delta_{ij}r_{,k}r_{,l}+\delta_{lj}r_{,i}r_{,k}+\delta_{il}r_{,k}r_{,j}\right. \\
& +\delta_{kj}r_{,i}r_{,l}+\delta_{ki}r_{,l}r_{,j}-10r_{,i}r_{,j}r_{,k}r_{,l})\xi\, e^{-\xi^2} \\
& +\frac{16}{\sqrt{\pi}}(-\delta_{ij}\delta_{kl}+\delta_{kl}r_{,i}r_{,j}+\delta_{ij}r_{,k}r_{,l}-r_{,i}r_{,j}r_{,k}r_{,l})\xi^3\, e^{-\xi^2} \\
& +3(\delta_{lj}\delta_{ki}+\delta_{kj}\delta_{il}+\delta_{ji}\delta_{kl}-5\delta_{lj}r_{,i}r_{,k}-5\delta_{il}r_{,k}r_{,j}-5\delta_{kj}r_{,i}r_{,l}-5\delta_{ki}r_{,l}r_{,j} \\
& -5\delta_{ij}r_{,k}r_{,l}-5\delta_{kl}r_{,i}r_{,j}+35r_{,i}r_{,j}r_{,k}r_{,l})\left(\frac{\mathrm{erf}(\xi)}{\xi^2}-\frac{2}{\sqrt{\pi}}\frac{e^{-\xi^2}}{\xi}\right) \\
& +2(\delta_{kj}\delta_{il}+\delta_{ki}\delta_{jl}-3\delta_{ij}\delta_{kl}+3\delta_{kl}r_{,i}r_{,j}+3\delta_{ij}r_{,k}r_{,l}-3\delta_{il}r_{,k}r_{,j}-3\delta_{kj}r_{,i}r_{,l} \\
& \left.-3\delta_{ki}r_{,l}r_{,j}-3\delta_{lj}r_{,i}r_{,k}+15r_{,i}r_{,j}r_{,k}r_{,l})\,\mathrm{erfc}(\xi)\right]
\end{aligned}$$

$$p_{kl}^{dc} = \sigma_{kl}^{si} = \sigma_{kl}^{lc}$$

$$v_{ikl}^{dc} = \kappa\sigma_{lki}^{pc} = \sigma_{lki}^{fc}$$

$$q_{ikl}^{dc} = \kappa\sigma_{lki}^{pi} = \sigma_{lki}^{fi} = v_{ikl}^{di}$$

$$\zeta_{kl}^{dc} = \frac{\eta}{2\pi}\frac{1}{r^3}\left\{(3r_{,k}r_{,l}-\delta_{kl})\left[\mathrm{erfc}(\xi)+\frac{2}{\sqrt{\pi}}\xi\, e^{-\xi^2}\right]-\frac{4}{\sqrt{\pi}}(\delta_{kl}-r_{,k}r_{,l})\xi^3\, e^{-\xi^2}\right\}$$

D.12. *Instantaneous displacement discontinuity*
$E_{ijkl} = -\frac{1}{2}(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\delta(\mathbf{x}-\boldsymbol{\chi})\delta(t-\tau)$

2-D

$$u_{ikl}^{di} = \sigma_{lki}^{Fi}$$

$$\begin{aligned}
\sigma_{ijkl}^{di} = {} & \frac{G}{2\pi(1-\nu_u)}\delta(t-\tau)\frac{1}{r^2}[8r_{,i}r_{,j}r_{,k}r_{,l} \\
& -(1-2\nu_u)(\delta_{il}\delta_{jk}+\delta_{ik}\delta_{jl}-\delta_{ij}\delta_{kl}+2\delta_{kl}r_{,i}r_{,j}+2\delta_{ij}r_{,k}r_{,l}) \\
& -2\nu_u(\delta_{ij}\delta_{kl}+\delta_{lj}r_{,i}r_{,k}+\delta_{il}r_{,j}r_{,k}+\delta_{jk}r_{,i}r_{,l}+\delta_{ik}r_{,j}r_{,l})] \\
& +\frac{4\eta^2 c^2}{\pi\kappa}\frac{1}{r^4}\{(4\delta_{kj}r_{,i}r_{,l}+4\delta_{ki}r_{,l}r_{,j}+4\delta_{kl}r_{,i}r_{,j}+4\delta_{lj}r_{,i}r_{,k}+4\delta_{li}r_{,k}r_{,j} \\
& +4\delta_{ij}r_{,k}r_{,l}-\delta_{ij}\delta_{kl}-\delta_{kj}\delta_{il}-\delta_{ki}\delta_{jl}-24r_{,i}r_{,j}r_{,k}r_{,l})\,[1-(1+\xi^2)\,e^{-\xi^2}] \\
& +2(6r_{,i}r_{,j}r_{,k}r_{,l}-\delta_{kl}r_{,i}r_{,j}-\delta_{lj}r_{,i}r_{,k}-\delta_{li}r_{,k}r_{,j}-\delta_{jk}r_{,i}r_{,l}-\delta_{ki}r_{,l}r_{,j}-\delta_{ij}r_{,k}r_{,l})\xi^4\, e^{-\xi^2} \\
& +4(r_{,i}r_{,j}r_{,k}r_{,l}+\delta_{ij}\delta_{kl}-\delta_{ij}r_{,k}r_{,l}-\delta_{kl}r_{,i}r_{,j})\xi^6\, e^{-\xi^2}\}
\end{aligned}$$

$$p_{kl}^{di} = \sigma_{kl}^{li}$$

$$v_{ikl}^{di} = \kappa\sigma_{lki}^{pi} = \sigma_{lki}^{fi} = q_{ikl}^{dc}$$

$$\begin{aligned}
q_{ikl}^{di} = {} & \frac{2\eta c}{\pi}\delta(t-\tau)\frac{1}{r^3}(\delta_{il}r_{,k}+\delta_{kl}r_{,i}+\delta_{ik}r_{,l}-4r_{,i}r_{,k}r_{,l}) \\
& -\frac{8\eta c^2}{\pi}\frac{1}{r^5}[(\delta_{il}r_{,k}+\delta_{ik}r_{,l}-3\delta_{kl}r_{,i})+2(\delta_{kl}r_{,i}-r_{,i}r_{,k}r_{,l})\xi^2]\xi^6\, e^{-\xi^2}
\end{aligned}$$

$$\zeta_{kl}^{di} = \frac{4\eta c}{\pi}\frac{1}{r^4}[\delta_{kl}(1-2\xi^2)+2r_{,k}r_{,l}\xi^2]\xi^4 e^{-\xi^2}$$

3-D

$$u_{ikl}^{di} = \sigma_{lki}^{Fi}$$

$$\begin{aligned}
\sigma_{ijkl}^{di} = {} & \frac{G}{4\pi(1-\nu_u)}\frac{1}{r^3}\delta(t-\tau)[15r_{,i}r_{,j}r_{,k}r_{,l} \\
& -(1-2\nu_u)(\delta_{kj}\delta_{il}+\delta_{ki}\delta_{jl}-\delta_{ij}\delta_{kl}+3\delta_{kl}r_{,i}r_{,j}+3\delta_{ij}r_{,k}r_{,l}) \\
& -2\nu_u(\delta_{ij}\delta_{kl}+\delta_{ij}r_{,l}r_{,k}+\delta_{il}r_{,j}r_{,k}+\delta_{jk}r_{,i}r_{,l}+\delta_{ik}r_{,j}r_{,l})] \\
& +\frac{\eta^2c^2}{\pi\kappa}\frac{1}{r^5}\Big\{8(7r_{,i}r_{,j}r_{,k}r_{,l}-\delta_{ij}\delta_{kl}-\delta_{kl}r_{,i}r_{,j}-\delta_{ij}r_{,k}r_{,l}-\delta_{ij}r_{,l}r_{,k} \\
& -\delta_{il}r_{,k}r_{,j}-\delta_{kj}r_{,i}r_{,l}-\delta_{ki}r_{,l}r_{,j})\xi^5 e^{-\xi^2} \\
& +16(r_{,i}r_{,j}r_{,k}r_{,l}+\delta_{ij}\delta_{kl}-\delta_{kl}r_{,i}r_{,j}-\delta_{ij}r_{,k}r_{,l})\xi^7 e^{-\xi^2} \\
& -(\delta_{ij}\delta_{kl}+\delta_{kj}\delta_{il}+\delta_{ji}\delta_{kl}-5\delta_{ij}r_{,l}r_{,k}-5\delta_{il}r_{,k}r_{,j}-5\delta_{kj}r_{,i}r_{,l}-5\delta_{kl}r_{,l}r_{,j} \\
& -5\delta_{ij}r_{,k}r_{,l}-5\delta_{kl}r_{,i}r_{,j}+35r_{,i}r_{,j}r_{,k}r_{,l})\left[3\,\mathrm{erf}(\xi)-\frac{2}{\sqrt{\pi}}(3+2\xi^2)\xi e^{-\xi^2}\right]\Big\}
\end{aligned}$$

$$p_{kl}^{pi} = \sigma_{kl}^{li}$$

$$v_{ikl}^{di} = \kappa\sigma_{lki}^{pi} = \sigma_{lki}^{fi} = q_{ikl}^{dc}$$

$$\begin{aligned}
q_{ikl}^{di} = {} & \frac{3\eta c}{2\pi}\delta(t-\tau)\frac{1}{r^4}(\delta_{il}r_{,k}+\delta_{kl}r_{,i}+\delta_{ik}r_{,l}-5r_{,i}r_{,k}r_{,l}) \\
& -\frac{8\eta c^2}{\pi^{3/2}}\frac{1}{r^6}[(\delta_{il}r_{,k}+\delta_{ik}r_{,l}-4\delta_{kl}r_{,i})+2(\delta_{kl}r_{,i}-r_{,i}r_{,k}r_{,l})\xi^2]\xi^7 e^{-\xi^2}
\end{aligned}$$

$$\zeta_{kl}^{di} = \frac{8\eta c}{\pi^{3/2}}\frac{1}{r^5}[\delta_{kl}(1-\xi^2)+r_{,k}r_{,l}\xi^2]\xi^5 e^{-\xi^2}$$