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A new boundary integral formulation for plane elastic bodies containing cracks and holes

K. T. Chau[a,*], Y. B. Wang[b]

[a] *Department of Civil and Structural Engineering, The Hong Kong Polytechnic University, Hung Hom, Kowloon, Hong Kong, SAR, China*

[b] *Department of Mechanics, Lanzhou University, Lanzhou 730000, People's Republic of China*

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Abstract

This paper presents a new boundary integral formulation for a plane elastic body containing an arbitrary number of cracks and holes. The body is assumed to be linear elastic and isotropic, but can be of either finite or infinite extend. The cracks inside the body can be either internal or edge crack, and either straight or curvilinear; and the holes can be of arbitrary number and shape. Starting from Somigliana formula, we obtain a system of boundary integral equations by applying integration by parts. In complex variables notation, the stress and displacement components can be expressed in terms of Muskhelishvili's analytic functions, which are in turn written as functions of boundary traction and displacement data in the form of Cauchy integral. The complex boundary integral equations for traction involve only singularity of order $1/r$, where r is the distance measured from the singular boundary points, and no hypersingular terms appear. This new boundary integral formulation provides an effective basis in solving problems both analytically and numerically. To illustrate the validity of our new integral formulation, a number of classical problems are re-examined analytically using the present formulation: (i) an infinite body containing a circular hole subject to far field biaxial stress, internal pressure, and a point force on the hole's boundary respectively; and (ii) an infinite body containing a circular-arc crack under remote uniaxial tension. To illustrate the applicability of the present formulation for boundary element method analysis, two numerical examples for the interactions between two collinear cracks are considered and the results agree well with the existing solutions by Chandra et al. (1995) for the case of finite rectangular plates and with Isida (cited in p. 195 of Murakami, 1987) for the case of infinite plates. © 1999 Elsevier Science Ltd. All rights reserved.

1. Introduction

The idea of using boundary integral formulation probably originates from its applications in potential theory (e.g. Kellogg, 1953). The first boundary integral formulation for three-dimensional

* Corresponding author. Tel.: 00 852 2766 6015; fax: 00 852 2334 6389; e-mail: cektchau@polyu.edu.hk

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elasticity was proposed by Kupradze (1953) (see also Kupradze, 1965) and was later derived independently by Kinoshita and Mura (1956) and Mikhlin (1965). This formulation is normally called boundary integral equation (BIE) method, and it is founded on the basis of Green's formula. In essence, it is a limiting case of the Somigliana's identity, which expresses the displacement field inside a body in terms of the surface traction and displacements when the point at which the displacement is evaluated approaches the boundary of the body (e.g. Brebbia et al., 1984).

Due to the rapid development of computers in the last few decades, numerical scheme (normally referred to as the boundary element method or BEM) has been developed in solving the boundary integral equations in elasticity (e.g. Rizzo, 1967; Cruse, 1969; Lacht and Watson, 1976; Brebbia et al., 1984). However, when the boundary element method is applied directly to crack problems, the geometrical overlapping of the upper and lower crack surfaces lead to an indeterminacy of the equations. Different approaches have been proposed to overcome this difficulty, such as: the Greens' function method, which requires the use of a Green's function for the particular crack problem (e.g. Snyder and Cruse, 1975; Ang and Clements, 1986, 1987; Ang, 1986, 1987, 1990); the displacement discontinuity method, which uses a point displacement Green's function instead of the point force Green's function (e.g. Crouch, 1976; Crouch and Starfield, 1983; Shou and Crouch, 1995); the subregional or multidomain method, which cuts the body into domains by introducing an artificial cut from the crack line to the external boundary (e.g. Blanford et al., 1991); the dual or hypersingular boundary element method (e.g. Portela et al., 1992; Saez et al., 1995; Chen and Chen, 1995); the body force method (e.g. Lee and Keer, 1986); and the dislocation density method, which reduces the singularity by one order using integration by parts and expressing the unknowns as dislocation densities (e.g. Bui, 1977; Weaver, 1977; Wang, 1990, 1993, 1995; Wang and Tang, 1988; Zhang and Gudmundson, 1988; Chang and Mear, 1995). The method of boundary integral equations and the boundary element method for crack problems has been and remains an area of active research (e.g. Aliabadi et al., 1989; Aliabadi and Brebbia, 1993; Ioakimidis, 1982, 1983, 1985; Lavit, 1994; Martin and Rizzo, 1989; Sladek and Sladek, 1982, 1990; Pan and Amadei, 1996; Stephan, 1986; Wang and Chen, 1993; Jiang et al., 1996; Wendland and Stephen, 1990; Hong and Chen, 1988; Zang, 1990; Takakuda et al., 1985; Nishimura and Kobayashi, 1988; Chen and Hasebe, 1996).

When a solid contains both cracks and holes, the interactions among them may be significant as the distance between a crack-tip and the neighbouring holes and cracks decreases. For problems with straight cracks and regularly-shaped holes, the complex variable techniques by Muskhelishvili (1975) can be used to estimate the stress intensity factor K_{I} at the crack-tip due to interactions. For examples, the effect of an elliptical and a square hole on the K_{I} of a straight crack have been evaluated using Muskhelishvili's (1975) method by Tang and Wang (1986) and by Wang and Tang (1988), respectively. However, when the cracks are curvilinear and/or the holes are irregular in shape, numerical method must inevitably be employed. To our best knowledge, boundary integral formulation for plane elastic solids containing both cracks and holes has not, however, been derived such that it can be applied to elastic bodies of either finite or infinite extend, containing either internal or edge cracks (these cracks can be either straight or curvilinear), and the holes inside the body can be of arbitrary shape.

Therefore, the main purpose of this study is to derive a system of boundary integral equations, which is useful in obtaining analytical solutions for simple problems and can also provide a base for accurate and efficient implementation of boundary element method. The formulation should also be applicable to a wide class of interaction problems between two-dimensional cracks and holes. In addition, we also expect that our boundary integral formulation bears close resemblance with the well-established analytical techniques, such as the complex variable method by Muskhelishvili (1975).

With all these requirements in mind, we first integrate the Somigliana formula by parts to yield a traction boundary integral equation in terms of boundary traction and displacement densities. The complex representation of stresses and displacements are then derived in terms of two analytic functions (Φ and Ψ), and both of them involve Cauchy integral of a boundary complex function $H(t)$, which is a complex combination of traction and displacement density on the boundary points t. Since the displacement density is involved in the boundary complex function, a single-valued condition is derived for $H(t)$. The physical meaning of $H(t)$ and the uniqueness of our boundary integral equation are also discussed. Our boundary integral equation is then specialized to cases of infinite body subject to far field stresses. To illustrate the correctness of our proposed formulation, the stress concentration at a circular opening in an infinite body subject to various kinds of loadings is considered. The problem of a circular-arc crack subject to far field uniaxial tension is re-examined; and, as expected, our solution is the same as those given by Tada et al. (1985) based upon Muskhelishvili (1975) method.

More importantly, our boundary integration formulation is motivated by its possible application to an efficient boundary element analysis. In this regard, an attractive feature for the present formulation is that our boundary integral formulation for traction is obtained by integration by parts on the Somigliana identity, and therefore, involves only singularity of order $1/r$, where r is the distance measured from crack tip. That is, no hypersingular term (i.e. $1/r^n$ terms with $n > 1$) appears in the boundary integral equation. As shown by Wang and Chau (1997), the present boundary integral formulation provides a firm base for the analysis of interaction between cracks and holes, in which cracks can, in general, be curvilinear and the hole can also be of arbitrary shape. Although the BEM formulation is out of the scope of the present paper, two numerical examples will be considered to illustrate the numerical applicability of the present boundary integral formulation to more complicated problems.

2. A new boundary integral formulation

In this section, we derive a new formulation for the boundary integral equations for two-dimensional bodies containing an arbitrary number of cracks and holes of arbitrary shape. As shown in Fig. 1, we consider a two-dimensional linear elastic body under plane condition containing m holes and n cracks. The body is multi-connected as its surface can be divided into two subsets: the ordinary boundary set $S = S_0 + S_1 + \cdots + S_m$; and the crack boundary set $\Gamma = \Gamma_1 + \Gamma_2 + \cdots + \Gamma_n$. The outer boundary of the body is denoted by S_0 which can either be of finite length (i.e. a finite body) or of infinite extend (i.e. an infinite body).

We first apply the following Somigliana formula, which expresses the displacement field at an interior point $\mathbf{x}$ in terms of the traction and displacement data on the boundary points $\mathbf{y}$ [i.e. $t_j(\mathbf{y}) = \sigma_{ij} n_i$ and $u_j(\mathbf{y})$], for a body containing both holes and cracks as shown in Fig. 1 (e.g. Cruse, 1988):

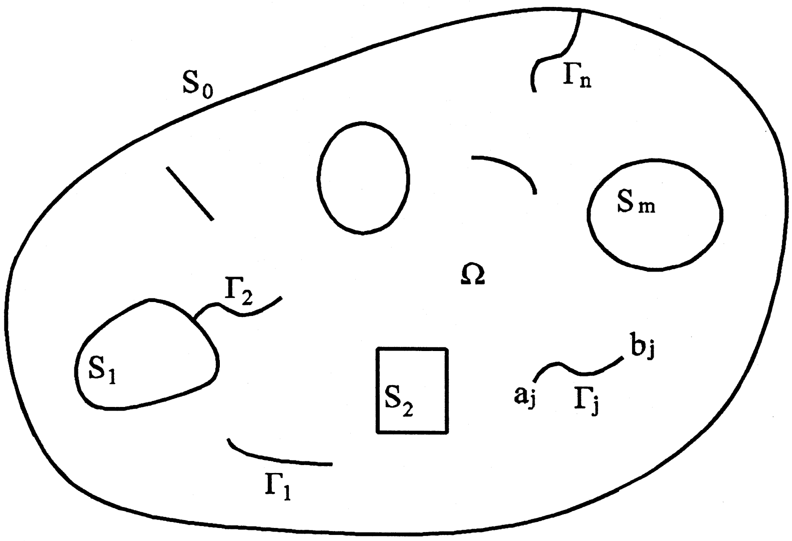

Fig. 1. A finite isotropic elastic domain containing n cracks and m holes with the outer boundary S_0.

$$u_i(\mathbf{x}) = \int_S U_{ij}(\mathbf{x},\mathbf{y})t_j(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) - \int_S T_{ij}(\mathbf{x},\mathbf{y})u_j(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) + \int_\Gamma U_{ij}(\mathbf{x},\mathbf{y})\Sigma t_j(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) - \int_\Gamma T_{ij}(\mathbf{x},\mathbf{y})\Delta u_j(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \quad (1)$$

where $s(\mathbf{y})$ is the arc length along the boundaries, either along S or Γ, and $i,j = 1,2$; $\Sigma t_j(y) = t_j(y^+)+t_j(y^-)$ is the sum of the tractions acting on the upper and lower crack surfaces; and $\Delta u_j(y) = u_j(y^+)-u_j(y^-)$ is the difference of the displacements between the upper and lower crack surfaces. The fundamental solutions $U_{ij}(\mathbf{x},\mathbf{y})$ and $T_{ij}(\mathbf{x},\mathbf{y})$ are the j-th displacement and traction at boundary point $\mathbf{y}$ caused by a unit point force along the i-th direction at a source point $\mathbf{x}$; and they can be expressed as follows (e.g. Brebbia 1984; Brebbia and Dominguez, 1992; Danson, 1983):

$$U_{ij}(\mathbf{x},\mathbf{y}) = \frac{1}{2\pi G(\kappa+1)}\left[\kappa\delta_{ij}\ln\left(\frac{1}{r}\right)+r_{,i}r_{,j}\right] \quad (2)$$

$$T_{ij} = -D_{ijk}(\mathbf{x},\mathbf{y})n_k(\mathbf{y}) \quad (3)$$

$$D_{ijk} = \frac{1}{\pi(\kappa+1)r}\left[\frac{1}{2}(\kappa-1)(\delta_{ij}r_{,k}+\delta_{ik}r_{,j}-\delta_{jk}r_{,i})+2r_{,i}r_{,j}r_{,k}\right] \quad (4)$$

where $r = |\mathbf{y}-\mathbf{x}| = [(y_2-x_2)^2+(y_1-x_1)^2]^{1/2}$ is the distance between points $\mathbf{x}$ and $\mathbf{y}$; G is the shear modulus of the body; the partial derivative $r_{,j}$ denotes $(\partial r)/(\partial y_j)$; $n_k(\mathbf{y})$ is the unit normal along the boundary on either S or Γ; κ equals $3-4v$ for plane strain and $(3-v)/(1+v)$ for plane stress, where v is the Poisson's ratio.

Since the displacement field given in (1) is valid at all internal points of the linear elastic body, thus, Hooke's law can be applied to obtain the stress field inside the body. In particular, we first differentiate u_i with respect to x_k to give

$$u_{i,k}(\mathbf{x}) = \int_S \frac{\partial U_{ij}(\mathbf{x},\mathbf{y})}{\partial x_k} t_j(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) - \int_S \frac{\partial T_{ij}(\mathbf{x},\mathbf{y})}{\partial x_k} u_j(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) + \int_\Gamma \frac{\partial U_{ij}(\mathbf{x},\mathbf{y})}{\partial x_k} \Sigma t_j(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) - \int_\Gamma \frac{\partial T_{ij}(\mathbf{x},\mathbf{y})}{\partial x_k} \Delta u_j(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \tag{5}$$

The displacement gradient terms in the kernels can be found by differentiating (2) to yield:

$$\frac{\partial U_{ij}}{\partial x_k} = -\frac{\partial U_{ij}}{\partial y_k} = -U_{ij,k} = -\frac{1}{2\pi G(\kappa+1)}\left[-\frac{\kappa}{r}\delta_{ij}r_{,k} + r_{,ik}r_{,j} + r_{,i}r_{,jk}\right] \tag{6}$$

The first of (6) can easily be shown by using the fact that $\partial r/\partial x_j = -\partial r/\partial y_j = -r_{,j}$. Unless otherwise stated, all subsequent $()_{,j}$ denotes the partial derivative with respect to y_j not x_j. Using the definition of r given after (4), it is straightforward to show that

$$r_{,i} = \frac{y_i - x_i}{r}, \quad r_{,ik} = \frac{1}{r}(\delta_{ik} - r_{,i}r_{,k}) \tag{7}$$

Substitution of (7) into (6) gives

$$\frac{\partial U_{ij}(\mathbf{x},\mathbf{y})}{\partial x_k} = \frac{1}{2\pi G(\kappa+1)}\left[\frac{\kappa}{r}\delta_{ij}r_{,k} + \frac{2}{r}r_{,i}r_{,j}r_{,k} - \frac{1}{r}(\delta_{ik}r_{,j} + \delta_{jk}r_{,i})\right] \tag{8}$$

For the traction gradient terms in the kernels of (5), we can differentiate T_{ij} given in (3) with respect to x_1 to give

$$\frac{\partial T_{ij}(\mathbf{x},\mathbf{y})}{\partial x_1} = -n_k(\mathbf{y})\frac{\partial D_{ijk}(\mathbf{x},\mathbf{y})}{\partial x_1} = n_k(\mathbf{y})D_{ijk,1}(\mathbf{x},\mathbf{y}) \tag{9}$$

where, as defined before, the $()_{,j}$ means $\partial()/\partial y_j$ which should not be confused with the partial derivative with respect to x_j. On the other hand, the fundamental solution D_{ijk} should satisfy the equilibrium equation, therefore, we have $D_{ijk,k} = 0$ or $D_{ij1,1} + D_{ij2,2} = 0$ for $\mathbf{y} \neq \mathbf{x}$. Using this information, (9) can be expressed as:

$$\frac{\partial T_{ij}(\mathbf{x},\mathbf{y})}{\partial x_1} = -\left[-n_2(\mathbf{y})\frac{\partial}{\partial y_1} + n_1(\mathbf{y})\frac{\partial}{\partial y_2}\right]D_{ij2}(\mathbf{x},\mathbf{y}) = -\frac{\partial D_{ij2}(\mathbf{x},\mathbf{y})}{\partial s(\mathbf{y})} \tag{10}$$

The second of (10) is resulted from the following identity: $\partial\mathbf{f}/\partial s = n_1(\partial\mathbf{f}/\partial y_2) - n_2(\partial\mathbf{f}/\partial y_1)$, which is obvious from the definition of n_1 and n_2 given in Fig. 2. Similarly, we can also show that

$$\frac{\partial T_{ij}(\mathbf{x},\mathbf{y})}{\partial x_2} = \frac{\partial D_{ij1}(\mathbf{x},\mathbf{y})}{\partial s(\mathbf{y})} \tag{11}$$

The results for (10)–(11) can be written in a more compact form as:

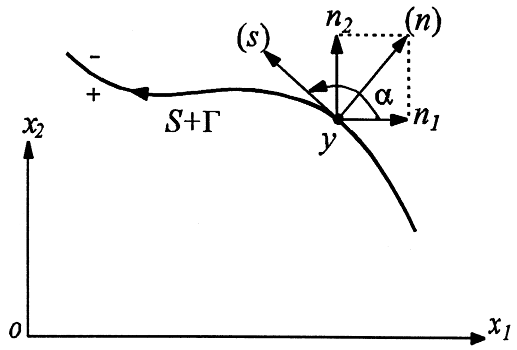

Fig. 2. A local orthogonal curvilinear system $n-s$ along the contour $S+\Gamma$. Any quantity on the left side of the contour line is taken as '+' and those on the right is taken as '−'. At any boundary point on $S+\Gamma$ with normal **n**, the angle α is taken as the angle of the s-direction measured from the x_1 axis.

$$\frac{\partial T_{ij}(\mathbf{x},\mathbf{y})}{\partial x_k} = -e_{k\beta}\frac{\partial D_{ij\beta}(\mathbf{x},\mathbf{y})}{\partial s(\mathbf{y})} \tag{12}$$

where $e_{ij} = \delta_{1i}\delta_{2j} - \delta_{2i}\delta_{1j}$ is the 2-D permutation tensor (i.e. $e_{11} = e_{22} = 0$, $e_{12} = -e_{21} = 1$).

Substitution of (8) and (12) into (5), then inserting the resulting expression into the following Hooke's law for 2-D isotropic solids

$$\sigma_{ij}(\mathbf{x}) = \lambda\delta_{ij}u_{m,m}(\mathbf{x}) + G[u_{i,j}(\mathbf{x}) + u_{j,i}(\mathbf{x})] \tag{13}$$

where $\lambda = G(3-\kappa)/(\kappa-1)$, gives an integral formulation for the stress:

$$\sigma_{ij}(\mathbf{x}) = \int_S D_{kij}(\mathbf{x},\mathbf{y})t_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) + \int_S \frac{\partial W_{kij}(\mathbf{x},\mathbf{y})}{\partial s(\mathbf{y})}u_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) + \int_\Gamma D_{kij}(\mathbf{x},\mathbf{y})\Sigma t_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) + \int_\Gamma \frac{\partial W_{kij}(\mathbf{x},\mathbf{y})}{\partial s(\mathbf{y})}\Delta u_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) \tag{14}$$

where

$$W_{kij} = \lambda\delta_{ij}e_{m\beta}D_{mk\beta}(\mathbf{x},\mathbf{y}) + G[e_{j\beta}D_{ik\beta}(\mathbf{x},\mathbf{y}) + e_{i\beta}D_{jk\beta}(\mathbf{x},\mathbf{y})] \tag{15}$$

Integrating (14) by parts and noting that $[W_{kij}u_k]$ vanishes around a closed boundary S_j (where $j = 1, 2, \ldots, m$) since it is a single-value function and $[W_{kij}\Delta u_k]$ vanishes on both tips of any crack Γ_j (where $j = 1, 2, \ldots, n$); thus, we finally obtain

$$\sigma_{ij}(\mathbf{x}) = \int_S D_{kij}(\mathbf{x},\mathbf{y})t_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) - \int_S W_{kij}(\mathbf{x},\mathbf{y})\frac{\partial u_k(\mathbf{y})}{\partial s(\mathbf{y})}\,\mathrm{d}s(\mathbf{y}) + \int_\Gamma D_{kij}(\mathbf{x},\mathbf{y})\Sigma t_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y}) - \int_\Gamma W_{kij}(\mathbf{x},\mathbf{y})\frac{\partial \Delta u_k(\mathbf{y})}{\partial s(\mathbf{y})}\,\mathrm{d}s(\mathbf{y}) \tag{16}$$

In addition, it is not difficult to show that this procedure of integration by parts also applies to

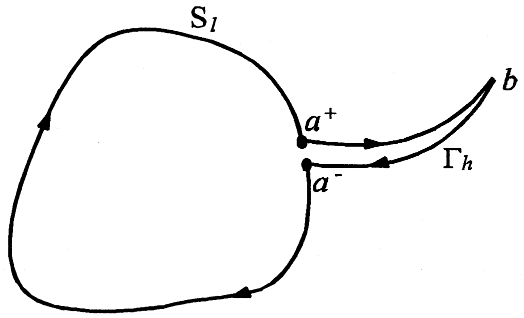

Fig. 3. A sketch for an edge crack Γ_j emanating from a hole S_i. The upper and lower points on the crack surfaces at the hole's boundary are denoted by a_j^+ and a_j^-, respectively, and b_j is the tip of the edge crack Γ_j.

edge cracks emanate from any hole or from the outer boundary S_0 (such as Γ_2 emanates from S_1 and Γ_n from S_0 shown in Fig. 1), and that (16) remains valid for bodies containing edge cracks. To see this, we consider an edge crack Γ_h emanates from an ordinary boundary S_l shown in Fig. 3. Applying integration by parts to (14) leads to the following quantities being evaluated on the boundary:

$$[W_{kij}(\mathbf{x},\mathbf{y})u_k(\mathbf{y})]_{S_l}+[W_{kij}(\mathbf{x},\mathbf{y})\Delta u_k(\mathbf{y})]_{\Gamma_h} = W_{kij}(\mathbf{x},\mathbf{a}^+)u_k(\mathbf{a}^+)-W_{kij}(\mathbf{x},\mathbf{a}^-)u_k(\mathbf{a}^-)$$
$$+W_{kij}(\mathbf{x},\mathbf{b})\Delta u_k(\mathbf{b})-W_{kij}(\mathbf{x},\mathbf{a})\Delta u_k(\mathbf{a}) = 0 \quad (17)$$

where $\mathbf{a}^+$ and $\mathbf{a}^-$ are the upper and lower points on crack surface at the hole's boundary, and $\mathbf{b}$ is the tip of the edge crack. The last of (17) is obtained by virtue of the fact that $\Delta u_k(\mathbf{b}) \equiv 0$, $\Delta u_k(\mathbf{a}) = u_k(\mathbf{a}^+)-u_k(\mathbf{a}^-)$ and W_{kij} is the same on both the lower and upper crack surfaces. Therefore, (16) remains valid for edge crack problems.

For any interior point $\mathbf{x}$, we have $r = |\mathbf{y}-\mathbf{x}| \neq 0$; thus, the integrals in (16) are regular (i.e. no singularity). Consider now the limit that $\mathbf{x}$ tends to a smooth boundary point $\mathbf{x}_0$ on either S or Γ. For such a limiting process, we decompose the ordinary boundary S or the crack boundary Γ into $S-S_\varepsilon$ and S_ε or $\Gamma-\Gamma_\varepsilon$ and Γ_ε, depending on where the boundary point $\mathbf{x}_0$ is, in which $S_\varepsilon \equiv \{\mathbf{y}|\mathbf{y}\in S, |\mathbf{y}-\mathbf{x}_0| < \varepsilon \text{ and } \mathbf{x}_0 \in S\}$ or $\Gamma_\varepsilon \equiv \{\mathbf{y}|\mathbf{y}\in\Gamma, |\mathbf{y}-\mathbf{x}_0| < \varepsilon \text{ and } \mathbf{x}_0\in\Gamma\}$. Note that this approach is different from the customary approach that the boundary (either S or Γ) around the boundary point $\mathbf{x}_0$ is deformed by incorporating a semicircular region with centre at $\mathbf{x}_0$. If the source point $\mathbf{x}$ approaches an ordinary boundary (or a crack boundary), the integral on the boundary $S-S_\varepsilon+\Gamma$ (or $S+\Gamma-\Gamma_\varepsilon$) should be interpreted in the Cauchy principal value sense. Whereas the integral on the boundary S_ε can be shown to be:

$$\lim_{\varepsilon\to 0}\lim_{\mathbf{x}\to\mathbf{x}_0}\left[n_j(\mathbf{x}_0)t_k(\mathbf{x}_0)\int_{S_\varepsilon} D_{kij}(\mathbf{x},\mathbf{y})\,\mathrm{d}s(\mathbf{y})-n_j(\mathbf{x}_0)\frac{\partial u_k(\mathbf{x}_0)}{\partial s(\mathbf{x}_0)}\int_{S_\varepsilon} W_{kij}(\mathbf{x},\mathbf{y})\,\mathrm{d}s(\mathbf{y})\right] = \frac{1}{2}t_i(\mathbf{x}_0) \quad (18)$$

where $n_j(\mathbf{x}_0)$ denotes the j-th component of the unit outward normal to the boundary S. Note that the mean value theorem for integrals have been applied such that t_k and $\partial u_k/\partial s$ are taken out of the integration. The full details of the integration involved in obtaining the right-hand side of (18) is quite straightforward, though tedious; thus, the details are omitted here. In short, we find that the first term on the left-hand side contributes to $\frac{1}{2}t_i(\mathbf{x}_0)$ appearing on the right-hand side and the second term on the left-hand side is identically zero. Similarly, if $\mathbf{x}$ approaches a crack boundary, the integral on Γ_ε can be shown to be

$$\lim_{\varepsilon\to 0}\lim_{\mathbf{x}\to\mathbf{x}_0^\pm}\left[n_j(\mathbf{x}_0^\pm)\Sigma t_k(\mathbf{x}_0)\int_{\Gamma_\varepsilon} D_{kij}(\mathbf{x},\mathbf{y})\,\mathrm{d}s(\mathbf{y})-n_j(\mathbf{x}_0^\pm)\frac{\partial\Delta u_k(\mathbf{x}_0)}{\partial s(\mathbf{x}_0)}\int_{\Gamma_\varepsilon} W_{kij}(\mathbf{x},\mathbf{y})\,\mathrm{d}s(\mathbf{y})\right]$$
$$=\frac{1}{2}\Sigma t_i(\mathbf{x}_0) \quad (19)$$

where $n_j(\mathbf{x}_0^+)$ and $n_j(\mathbf{x}_0^-)$ are the j-th component of the unit outward normal to the upper and lower crack surfaces, respectively.

In view of (18) and (19), the integral equation for the traction at an ordinary boundary point $\mathbf{x}_0$ on S becomes

$$\frac{1}{2}t_i(\mathbf{x}_0) = n_j(\mathbf{x}_0)\int_S D_{kij}(\mathbf{x}_0,\mathbf{y})t_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y})-n_j(\mathbf{x}_0)\int_S W_{kij}(\mathbf{x}_0,\mathbf{y})\frac{\partial u_k(\mathbf{y})}{\partial s(\mathbf{y})}\mathrm{d}s(\mathbf{y})$$
$$+n_j(\mathbf{x}_0)\int_\Gamma D_{kij}(\mathbf{x}_0,\mathbf{y})\Sigma t_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y})-n_j(\mathbf{x}_0)\int_\Gamma W_{kij}(\mathbf{x}_0,\mathbf{y})\frac{\partial\Delta u_k(\mathbf{y})}{\partial s(\mathbf{y})}\mathrm{d}s(\mathbf{y}) \quad (20)$$

where $i = 1, 2$; and, similarly, when $\mathbf{x}_0$ is on the crack surface Γ, we have

$$t_i(\mathbf{x}_0^\pm) = n_j(\mathbf{x}_0^\pm)\int_S D_{kij}(\mathbf{x}_0,\mathbf{y})t_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y})-n_j(\mathbf{x}_0^\pm)\int_S W_{kij}(\mathbf{x}_0,\mathbf{y})\frac{\partial u_k(\mathbf{y})}{\partial s(\mathbf{y})}\mathrm{d}s(\mathbf{y})$$
$$+n_j(\mathbf{x}_0^\pm)\int_\Gamma D_{kij}(\mathbf{x}_0,\mathbf{y})\Sigma t_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y})-n_j(\mathbf{x}_0^\pm)\int_\Gamma W_{kij}(\mathbf{x}_0,\mathbf{y})\frac{\partial\Delta u_k(\mathbf{y})}{\partial s(\mathbf{y})}\mathrm{d}s(\mathbf{y})+\frac{1}{2}\Sigma t_i(\mathbf{x}_0) \quad (21)$$

where, again, $i = 1, 2$. Although there seems four expressions in (21), only two of them are actually independent. In particular, we can set $n_j(\mathbf{x}_0) \equiv n_j(\mathbf{x}_0^+) = -n_j(\mathbf{x}_0^-)$ for any $\mathbf{x}_0$ on Γ such that the upper and lower crack tractions in (21) can be combined to yield:

$$\frac{1}{2}[t_i(\mathbf{x}_0^+)-t_i(\mathbf{x}_0^-)] = n_j(\mathbf{x}_0)\int_S D_{kij}(\mathbf{x}_0,\mathbf{y})t_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y})-n_j(\mathbf{x}_0)\int_S W_{kij}(\mathbf{x}_0,\mathbf{y})\frac{\partial u_k(\mathbf{y})}{\partial s(\mathbf{y})}\mathrm{d}s(\mathbf{y})$$
$$+n_j(\mathbf{x}_0)\int_\Gamma D_{kij}(\mathbf{x}_0,\mathbf{y})\Sigma t_k(\mathbf{y})\,\mathrm{d}s(\mathbf{y})-n_j(\mathbf{x}_0)\int_\Gamma W_{kij}(\mathbf{x}_0,\mathbf{y})\frac{\partial\Delta u_k(\mathbf{y})}{\partial s(\mathbf{y})}\mathrm{d}s(\mathbf{y}) \quad (22)$$

where $i = 1, 2$. Equations (20) and (22) provide a set of four integral equations for the tractions on both the hole and crack surface. Since for stress boundary value problems the traction should be prescribed on the boundary, only four of the eight independent variables in (20) and (22) are unknowns, i.e. $\partial u_k/\partial s$ are unknowns on hole boundary and $\partial\Delta u_k/\partial s$ are unknowns on crack surface (where $k = 1, 2$). For displacement boundary value problems or mixed boundary value problems, we will discuss them separately in our later publications.

An attractive feature of the present formulation is that all the crack boundary conditions (both upper and lower ones) have been incorporated into (22); thus, there is no need to discretize the upper and lower crack surfaces separately when boundary element method is applied. In addition, only stress singularity in the order of $1/r$ appears in the integrand and no hypersingularity (i.e. $1/r^2$) is involved.

To further simplify our boundary integral eqns (20) and (22), a complex representation will be introduced next.

3. Integral equations in complex variable notation

The integral formulation given in the previous section can significantly be simplified by using complex representation. More specifically, we introduce two complex variables $t = y_1 + iy_2$ and $z = x_1 + ix_2$ in replacement of $\mathbf{y}$ and $\mathbf{x}$, respectively, where $i = (-1)^{1/2}$ [recalling that $\mathbf{x} = (x_1, x_2)$ and $\mathbf{y} = (y_1, y_2)$]. It is obvious to show that $r^2 = (t-z)(\bar{t}-\bar{z})$ and

$$\frac{1}{r} r_{,1} = \frac{1}{2}\left[\frac{1}{t-z} + \frac{1}{\bar{t}-\bar{z}}\right], \quad \frac{1}{r} r_{,2} = \frac{1}{2i}\left[\frac{1}{\bar{t}-\bar{z}} - \frac{1}{t-z}\right] \tag{23}$$

To express the stress components given in (16) in terms of complex representation, we first note the following identities, which can be deduced directly from the definitions of D_{kij} and W_{kij} together with (23):

$$(D_{k11} + D_{k22})t_k = \frac{2}{\pi(\kappa+1)} \frac{r_{,k} t_k}{r} = \frac{1}{\pi(1+\kappa)}\left[\frac{p(t)}{(t-z)} + \frac{\bar{p}(t)}{(\bar{t}-\bar{z})}\right] \tag{24}$$

$$(W_{k11} + W_{k22})\frac{\partial u_k}{\partial s} = \frac{2Gi}{\pi(1+\kappa)}\left[\frac{U(t)}{t-z} - \frac{\bar{U}(t)}{\bar{t}-\bar{z}}\right] \tag{25}$$

$$(D_{k22} - D_{k11} + 2iD_{k12})t_k = \frac{-1}{\pi(\kappa+1)}\left[\frac{\kappa\bar{p}(t)}{t-z} + \frac{\bar{t}-\bar{z}}{(t-z)^2} p(t)\right] \tag{26}$$

$$(W_{k22} - W_{k11} + 2iW_{k12})\frac{\partial u_k}{\partial s} = -\frac{2Gi}{\pi(\kappa+1)}\left(\frac{\bar{t}-\bar{z}}{t-z}\right)\left(\frac{U(t)}{t-z} + \frac{\bar{U}(t)}{\bar{t}-\bar{z}}\right) \tag{27}$$

where $p(t) = t_1(t) + it_2(t)$ is the complex traction and $U(t) = \partial u_1/\partial s + i\,\partial u_2/\partial s$ is the complex displacement density on S. In addition, $\bar{U}(t)$ represents the complex conjugate of $U(t)$, and similar definitions are also applied to other complex functions. Expressions similar to (24)–(27) can also be obtained for quantities on Γ.

Applying the results (24)–(27) to (16), the stress components can be expressed as:

$$\sigma_{11} + \sigma_{22} = \mathrm{Re}\left[\frac{2}{\pi}\int_{S+\Gamma} \frac{F(t)\,\mathrm{d}s(t)}{t-z}\right] \tag{28}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = \frac{1}{\pi}\int_{S+\Gamma}\left[\frac{\bar{F}(t) - \bar{P}(t)}{t-z} - \frac{\bar{t}-\bar{z}}{(t-z)^2} F(t)\right]\mathrm{d}s(t) \tag{29}$$

where

$$F(t) = \frac{1}{\kappa+1}[P(t) - iw(t)] \tag{30}$$

In which, the following complex traction $P(t)$ and displacement density $w(t)$ are introduced:

$$w(t) = \begin{cases} 2G\dfrac{\partial}{\partial s(t)}[u_1(t) + iu_2(t)] & \text{for } t \text{ on } S \\ 2G\dfrac{\partial}{\partial s(t)}[\Delta u_1(t) + i\Delta u_2(t)] & \text{for } t \text{ on } \Gamma \end{cases} \tag{31}$$

$$P(t) = \begin{cases} t_1(t) + it_2(t) & \text{for } t \text{ on } S \\ \Sigma t_1(t) + i\Sigma t_2(t) & \text{for } t \text{ on } \Gamma \end{cases} \tag{32}$$

Alternatively, if we define the following analytic functions $\Phi(z)$ and $\Psi(z)$:

$$\Phi(z) = \frac{1}{2\pi}\int_{S+\Gamma} \frac{F(t)\,\mathrm{d}s(t)}{t-z} \tag{33}$$

$$\Psi(z) = \frac{1}{2\pi}\int_{S+\Gamma} \left\{\frac{1}{t-z}[\bar{F}(t) - \bar{P}(t)] - \frac{\bar{t}F(t)}{(t-z)^2}\right\}\mathrm{d}s(t) \tag{34}$$

where $F(t)$ and $P(t)$ are defined in (30) and (32), respectively, then the stress boundary integral formulation given in (28)–(29) can now be interpreted in terms of Muskhelishvili's (1975) formulation as:

$$\sigma_{11} + \sigma_{22} = 2[\Phi(z) + \bar{\Phi}(z)] \tag{35}$$

$$\sigma_{22} - \sigma_{11} + 2i\sigma_{12} = 2[\bar{z}\Phi'(z) + \Psi(z)] \tag{36}$$

To express the boundary integrals of (33) and (34) in terms of Cauchy integrals, we further introduce the following complex boundary function $H(t)$:

$$H(t) = iF(t)\,\mathrm{e}^{-i\alpha(t)} \tag{37}$$

where t is a boundary point on $S+\Gamma$. Consequently, (33) and (34) can be rewritten in terms of Cauchy integrals as:

$$\Phi(z) = \frac{1}{2\pi i}\int_{S+\Gamma} \frac{H(t)\,\mathrm{d}t}{t-z} \tag{38}$$

$$\Psi(z) = -\frac{1}{2\pi i}\int_{S+\Gamma} \left[\frac{\bar{H}(t) - \bar{q}(t)}{t-z}\mathrm{e}^{-2i\alpha(t)} + \frac{\bar{t}H(t)}{(t-z)^2}\right]\mathrm{d}t \tag{39}$$

where

$$q(t) = iP(t)\,\mathrm{e}^{-i\alpha(t)} \tag{40}$$

With the introduction of complex variable notation, the boundary integral equations for tractions given in (20) and (22) can now be simplified significantly. We first note that after a rather lengthy manipulation the following identities can be established:

$$n_j(D_{k1j}+iD_{k2j})t_k = \frac{i}{2\pi(\kappa+1)}\left\{\left[\kappa\frac{p(t)}{\bar{t}-\bar{z}}+\frac{\bar{p}(t)}{(\bar{t}-\bar{z})^2}(t-z)\right]e^{-i\alpha}-\left(\frac{p(t)}{t-z}+\frac{\bar{p}(t)}{\bar{t}-\bar{z}}\right)e^{i\alpha}\right\} \tag{41}$$

$$n_j(W_{k1j}+iW_{k2j})\frac{\partial u_k}{\partial s} = \frac{G}{\pi(\kappa+1)}\left\{\left[\frac{U(t)}{t-z}-\frac{\bar{U}(t)}{\bar{t}-\bar{z}}\right]e^{i\alpha}+\left[\frac{t-z}{(\bar{t}-\bar{z})^2}\bar{U}(t)+\frac{1}{\bar{t}-\bar{z}}U(t)\right]e^{-i\alpha}\right\} \tag{42}$$

where, similar to the derivation of (24)–(27), $p(t) = t_1(t)+it_2(t)$ is the complex traction and $U(t) = \partial u_1/\partial s+i\,\partial u_2/\partial s$ is the complex displacement density on S. Note that similar expressions can also be obtained for the integrands involving Σt_k and $\partial\Delta u_k/\partial s$.

Applying (41)–(42) to (20), the boundary integral equation for traction becomes

$$\frac{i}{2}P(t_0) = \frac{1}{2\pi}\int_{S+\Gamma}\left\{\left[\frac{F(t)}{t-t_0}+\frac{\bar{F}(t)}{\bar{t}-\bar{t}_0}\right]e^{i\alpha(t_0)}+\left[\frac{F(t)-P(t)}{\bar{t}-\bar{t}_0}-\frac{t-t_0}{(\bar{t}-\bar{t}_0)^2}\bar{F}(t)\right]e^{-i\alpha(t_0)}\right\}ds(t) \tag{43}$$

where $t_0 = x_1^0+ix_2^0$ is a boundary point on S and $P(t)$ is defined in (32). Similarly, (22) can be reduced to

$$\frac{i}{2}Q(t_0) = \frac{1}{2\pi}\int_{S+\Gamma}\left\{\left[\frac{F(t)}{t-t_0}+\frac{\bar{F}(t)}{\bar{t}-\bar{t}_0}\right]e^{i\alpha(t_0)}+\left[\frac{F(t)-P(t)}{\bar{t}-\bar{t}_0}-\frac{t-t_0}{(\bar{t}-\bar{t}_0)^2}\bar{F}(t)\right]e^{-i\alpha(t_0)}\right\}ds(t) \tag{44}$$

where t_0 is on Γ and $Q(t_0)$ is defined as:

$$Q(t_0) = [t_1(x_0^+)-t_1(x_0^-)]+i[t_2(x_0^+)-t_2(x_0^-)] \tag{45}$$

In view of the definition for $H(t)$, we can further simplify our boundary integral equations (43)–(44) for traction to the following unified form:

$$\pi i f(t_0) = \int_{S+\Gamma}\left\{\frac{H(t)}{t-t_0}-\frac{\bar{H}(t)}{\bar{t}-\bar{t}_0}e^{-2i\alpha(t)}+e^{-2i\alpha(t_0)}\left[\frac{H(t)-q(t)}{\bar{t}-\bar{t}_0}+\frac{t-t_0}{(\bar{t}-\bar{t}_0)^2}\bar{H}(t)\,e^{-2i\alpha(t)}\right]\right\}dt \tag{46}$$

where t_0 is on $S+\Gamma$ and $f(t_0)$ is defined as

$$f(t_0) = \begin{cases} q(t_0) = \sigma_n(x_0)+i\sigma_{ns}(x_0) & \text{for } t \text{ on } S \\ iQ(t_0)\,e^{-i\alpha(x_0)} = [\sigma_n(x_0^+)+\sigma_n(x_0^-)]+i[\sigma_{ns}(x_0^+)+\sigma_{ns}(x_0^-)] & \text{for } t \text{ on } \Gamma \end{cases} \tag{47}$$

This integral eqn (46) is much simpler than (20) and (22), and will be applied in later sections to obtain the solutions for some hole and crack problems; and the general numerical treatment of it by using boundary element method is given by Wang and Chau (1997).

4. Complex representation of displacement

Although the ordinary integral formulation for displacement given in (1) can, in general, be used to find the displacement field inside the body, we provide here a much simpler form of displacement in terms of complex representation. In particular, we first recall the following two-dimensional Hooke's law:

$$2G(\varepsilon_{11}+\varepsilon_{22}) = \tfrac{1}{2}(\kappa-1)(\sigma_{11}+\sigma_{22}), \quad 2G(\varepsilon_{22}-\varepsilon_{11}+2i\varepsilon_{12}) = \sigma_{22}-\sigma_{11}+2i\sigma_{12} \tag{48}$$

which can be specialized from (13), and where ε_{ij} $(i, j = 1, 2)$ are the components of the two-dimensional strain tensor. Substitution of (35)–(36) into (48) leads to

$$2G(\varepsilon_{11}+\varepsilon_{22}) = (\kappa-1)[\Phi(z)+\bar{\Phi}(z)], \quad 2G(\varepsilon_{22}-\varepsilon_{11}+2i\varepsilon_{12}) = 2[\bar{z}\Phi'(z)+\Psi(z)] \tag{49}$$

We first note that the rotation ω can be expressed in terms of the analytic function $\Phi(z)$ as:

$$\omega_3 = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1}-\frac{\partial u_1}{\partial x_2}\right) = \frac{\kappa+1}{4iG}[\Phi(z)-\bar{\Phi}(z)] \tag{50}$$

The second of (50) is obtained by substituting (1) into the first of (50), using (10)–(11) and (38)–(39) and applying integration by parts. Although the manipulation is somewhat lengthy, the procedure is quite straightforward and similar to that for (28)–(29).

To find the expression for u_j, we can integrate (49)–(50) directly. Our discussion here for obtaining the displacement field follows, however, a somewhat simpler approach. In particular, we consider the following partial derivative of the complex displacement field u_1+iu_2 at any point $z = x_1+ix_2$ with respect to s of an arbitrary set of local orthogonal curvilinear coordinate system (n, s), as shown in Fig. 2:

$$\frac{\partial}{\partial s}[2G(u_1+iu_2)] = 2G\left[\cos\alpha\left(\frac{\partial u_1}{\partial x_1}+i\frac{\partial u_2}{\partial x_1}\right)+\sin\alpha\left(\frac{\partial u_1}{\partial x_2}+i\frac{\partial u_2}{\partial x_2}\right)\right] \tag{51}$$

where α is the angle between the local coordinate s and the x_1 axis. With the following identities

$$\frac{\partial u_2}{\partial x_1} = \varepsilon_{12}+\omega_3, \quad \frac{\partial u_1}{\partial x_2} = \varepsilon_{12}-\omega_3 \tag{52}$$

(51) can be rewritten as

$$\frac{\partial}{\partial s}[2G(u_1+iu_2)] = G[(\varepsilon_{11}+\varepsilon_{22}+2i\omega_3)\,e^{i\alpha}-(\varepsilon_{22}-\varepsilon_{11}-2i\varepsilon_{12})\,e^{-i\alpha}] \tag{53}$$

Substitution of (49) and (50) into (53) gives

$$\frac{\partial}{\partial s}[2G(u_1+iu_2)] = [\kappa\Phi(z)-\bar{\Phi}(z)]\,e^{i\alpha}-[z\bar{\Phi}'(z)+\bar{\Psi}(z)]\,e^{-i\alpha} \tag{54}$$

We now introduce two new complex functions $\varphi(z)$ and $\psi(z)$ which satisfy

$$\varphi'(z) = \Phi(z), \quad \psi'(z) = \Psi(z), \tag{55}$$

Thus, substitution of (55) into (54) and simplification of the result yield

$$2G(u_1+iu_2) = \kappa\varphi(z)-z\bar{\varphi}'(z)-\bar{\psi}(z) \tag{56}$$

and, as expected, these expressions agree with the formulae given by Muskhelishvili (1975). In view of the definitions given in (55) and the expressions for $\Phi(z)$ and $\Psi(z)$ given in (38) and (39), we obtain

$$\varphi(z) = -\frac{1}{2\pi i}\int_{S+\Gamma} H(t)\ln(t-z)\,\mathrm{d}t \tag{57}$$

$$\psi(z) = \frac{1}{2\pi i}\int_{S+\Gamma}\left\{[\bar{H}(t)-\bar{q}(t)]\,\mathrm{e}^{-2i\alpha(t)}\ln(t-z)-\frac{\bar{t}H(t)}{t-z}\right\}\mathrm{d}t \tag{58}$$

Note that, in principle, the derivation of (56) can also be done directly from (1); however, the procedure is much more tedious than the one used here.

Although the appearance of both the stresses and displacements seems to be the same as those given by Muskhelishvili (1975), our complex functions $\Phi(z)$, $\Psi(z)$, $\varphi(z)$ and $\psi(z)$ are actually expressed in terms of boundary integrals for the surface $S+\Gamma$ of a multi-connected body. Such a form is especially suitable in considering bodies with multiple holes and cracks. Therefore, the present formulation actually provides a link between the boundary integral method and the Muskhelishvili's (1975) formalism for problems involving cracks and holes.

5. Physical meaning of the analytic function $\Phi(z)$ and $H(t)$

To see the physical meaning of $\Phi(z)$, we first rewrite (35)–(36) along the coordinate $n-s$ shown in Fig. 2 as

$$\sigma_n+\sigma_s = 2[\Phi(z)+\bar{\Phi}(z)] \tag{59}$$

$$\sigma_s-\sigma_n+i\sigma_{ns} = -2[\bar{z}\Phi'(z)+\Psi(z)]\,\mathrm{e}^{2i\alpha} \tag{60}$$

Equations (59) and (60) can be combined to yield

$$\sigma_n+i\sigma_{ns} = \Phi(z)+\bar{\Phi}(z)+[z\bar{\Phi}'(z)+\bar{\Psi}(z)]\,\mathrm{e}^{-2i\alpha} \tag{61}$$

The complex traction t_1+it_2 on a unit arc element at a point z with an outward normal $\mathbf{n} = (\sin\alpha$-, $-\cos\alpha)$ becomes

$$t_1+it_2 = -i(\sigma_n+i\sigma_{ns})\,\mathrm{e}^{i\alpha} = -i[\Phi(z)+\bar{\Phi}(z)]\,\mathrm{e}^{i\alpha}-i[z\bar{\Phi}'(z)+\bar{\Psi}(z)]\,\mathrm{e}^{-i\alpha} \tag{62}$$

where t_1 and t_2 are the traction components on $S+\Gamma$ along the x_1 and x_2 axes, respectively. Substitution of (62) into (54) gives the following physical interpretation of $\Phi(z)$:

$$\Phi(z) = i\,\mathrm{e}^{-i\alpha}\left[\frac{1}{\kappa+1}(t_1+it_2)-\frac{2iG}{\kappa+1}\frac{\partial}{\partial s}(u_1+iu_2)\right] \tag{63}$$

In particular, $\Phi(z)$ is a combination of complex traction and complex displacement gradient on an arc around the point z. When the complex function $\Phi(z)$ approaches the boundary, it can be related to the boundary complex function defined earlier as:

$$\begin{aligned} H(t) &= \Phi^+(t) && \text{for } t \text{ on } S \\ &= \Phi^+(t)-\Phi^-(t) && \text{for } t \text{ on } \Gamma \end{aligned} \tag{64}$$

where the superscript '+' indicates the limit of $\Phi(z)$ obtained from the left of $S+\Gamma$ as $z \to t$ while the superscript '−' indicates the limit obtained from the right. This definition for the limiting values of $\Phi(z)$ on the boundary bears a close resemblance with the notation used in Section 16 of Muskhelishvili (1953).

6. Single-valued condition of displacements

We have shown in (56) that the solutions for displacements can be expressed in terms of two complex functions $\varphi(z)$ and $\psi(z)$, which are in turn determined by boundary integrals (57)–(58). In general, both (57)–(58) can be multi-valued and to ensure the single-valued condition of displacements we consider the change of any complex quantity upon circulating the closed contours around these holes or cracks as shown in Fig. 4. The discussion employed here is very similar to those used by Section 35 of Muskhelishvili (1975).

In particular, when we consider an anti-clockwise loop Π_k around the hole boundary S_k, the terms on the right hand side of (56) undergo the following changes in value:

$$[\varphi(z)]_{\Pi_k} = -\int_{S_k} H(t)\,\mathrm{d}t, \quad [\bar{z}\varphi'(z)]_{\Pi_k} = 0 \tag{65}$$

$$[\psi(z)]_{\Pi_k} = \int_{S_k} [\bar{H}(t)-\bar{q}(t)]\,\mathrm{e}^{2i\alpha(t)}\,\mathrm{d}t \tag{66}$$

since the change in $[\ln(t-z)]$ upon circulating Π_k equals $2\pi i$, where the left-hand side of (65)–(66)

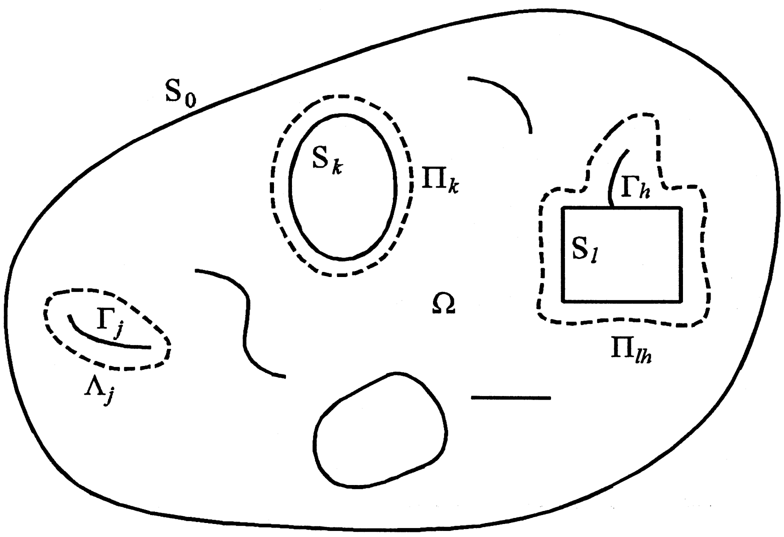

Fig. 4. A sketch showing the closed contours Λ_j, Π_k and Π_{1h} circulating around the crack Γ_j, the hole S_k, and the hole with an edge crack $S_1+\Gamma_h$, respectively.

indicates the increase of the term inside the bracket. Therefore, the changes on the left hand side of (56) becomes:

$$2G[u_1+iu_2]_{\Pi_k} = -(\kappa+1)\int_{S_k} H(t)\,\mathrm{d}t + i\int_{S_k} P(t)\,\mathrm{d}s(t) \tag{67}$$

The single-valued condition requires that (67) equals zero, or, we have

$$\int_{S_k} H(t)\,\mathrm{d}t = \frac{i}{\kappa+1}\int_{S_k} P(t)\,\mathrm{d}s(t) \tag{68}$$

The same procedure can also be applied to the loops Λ_j and Π_{1h} which circuit around an isolated crack Γ_j and the hole with edge crack $S_1+\Gamma_h$, as shown in Fig. 4. The corresponding condition for single-valuedness is again given by (68) except that S_k should be replaced by Γ_j and $S_1+\Gamma_h$, respectively. Physically, the integral on the right hand side is related to the resultant forces exerted on the corresponding boundary S_k. That is, we have

$$\frac{1}{\kappa+1}\int_{S_k} P(t)\,\mathrm{d}s(t) \equiv \frac{X_k+iY_k}{\kappa+1} \tag{69}$$

$$\frac{1}{\kappa+1}\int_{\Gamma_j} P(t)\,\mathrm{d}s(t) \equiv \frac{(X_j^+ + X_j^-)+i(Y_j^+ + Y_j^-)}{\kappa+1} \tag{70}$$

on the hole and crack boundaries, respectively. In these expressions, X_k+iY_k is the complex resultant force on the boundary S_k and $X_j^{\pm}+iY_j^{\pm}$ are the resultant forces on the crack surfaces $\Gamma_j^{\pm}$, where '+' and '−' denote the upper and lower crack surfaces, respectively.

7. Uniqueness of the solution *H*(*t*)

The issue on the uniqueness of $H(t)$ for our boundary integral formulation will be discussed briefly here since similar discussion can be found in Section 34 of Muskhelishvili (1975). In particular, we want to examine whether the solution for which $H(t)$ satisfies the compatibility eqn (68) is unique. As discussed by Muskhelishvili (1975), one can show that stress remains the same even though $\Phi(z)$, $\varphi(z)$ and $\psi(z)$ are replaced by

$$\Phi(z)+Ci, \quad \varphi(z)+Ciz+\gamma, \quad \psi(z)+\gamma' \tag{71}$$

respectively. Then, it follows from (64) that the same solution for the traction boundary integral equation (46) is obtained if $H(t)$ is replaced by $H(t)+iC$ on S_k. As shown by Muskhelishvili (1975), the constants C, γ, and γ' only relate to rigid body displacements; therefore, in case of displacement boundary value problem or mixed boundary value problem $H(t)$ is unique.

8. Formulae for multi-connected infinite domain

In this section, general formulae for the cases of infinite domain containing both cracks and holes are considered. Such formulae will be useful when we consider the interaction between the microcracks and micropores in brittle geomaterials, such as rock and concrete. In particular, we consider the special case that the outside boundary S_0 becomes unbounded and that stresses ($\sigma_1^\infty, \sigma_2^\infty$ and σ_{12}^∞) are applied at infinity, as shown in Fig. 5. That is, S_0 can be replaced by a circle such that $t = Re^{i\theta}$ on S_0 with $0 \leqslant \theta \leqslant 2\pi$ and $R \to \infty$. The far field complex traction can be expressed as:

$$q(t) = \sigma_\rho(R,\theta) + i\sigma_{\rho\theta}(R,\theta) = \tfrac{1}{2}[\sigma_1^\infty + \sigma_2^\infty - (\sigma_2^\infty - \sigma_1^\infty - 2i\sigma_{12}^\infty)\,\mathrm{e}^{-2i\theta}] \tag{72}$$

The complex functions $\Phi(z)$ and $\Psi(z)$ defined by (38)–(39) can first be decomposed into two parts:

$$\Phi(z) = \Phi_0(z) + \Phi_*(z), \quad \Psi(z) = \Psi_0(z) + \Psi_*(z), \tag{73}$$

where

$$\Phi_0(z) = \frac{1}{2\pi i}\int_{S_0} \frac{H(t)\,\mathrm{d}t}{t-z}, \quad \Psi_0(z) = \frac{R^2}{2\pi i}\int_{S_0}\left[\frac{\bar{H}(t)-\bar{q}(t)}{t^2(t-z)} - \frac{H(t)}{t(t-z)^2}\right]\mathrm{d}t \tag{74}$$

$$\Phi_*(z) = \frac{1}{2\pi i}\int_{S+\Gamma} \frac{H(t)\,\mathrm{d}t}{t-z}, \quad \Psi_*(z) = -\frac{1}{2\pi i}\int_{S+\Gamma}\left[\frac{\bar{H}(t)-\bar{q}(t)}{t-z}\mathrm{e}^{-2i\alpha(t)} + \frac{\bar{t}H(t)}{(t-z)^2}\right]\mathrm{d}t \tag{75}$$

where S denotes the union of holes only (i.e. excluding S_0). Applying (61) to the left of (72), we obtain

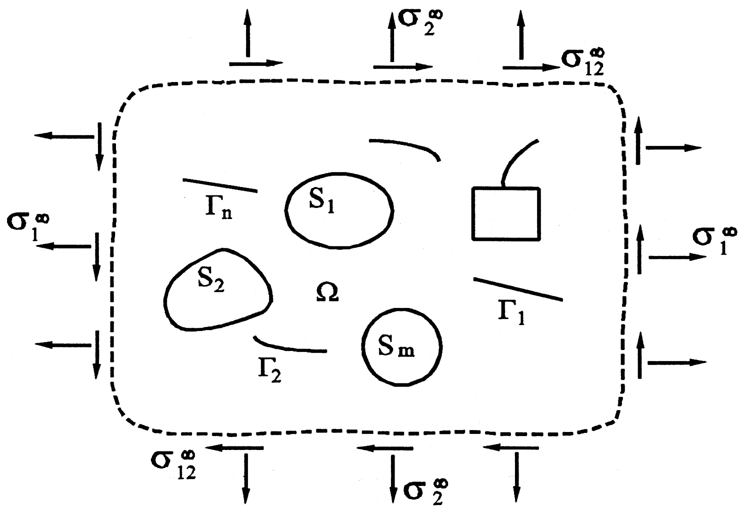

Fig. 5. An infinite elastic body containing n cracks and m holes subject to far field stresses σ_1^∞, σ_2^∞ and σ_{12}^∞.

$$q(t_0) = \Phi^+(t_0) + \bar{\Phi}^+(t_0) - \frac{R^2}{t_0^2}[t_0\bar{\Phi}'(t_0) + \bar{\Psi}(t_0)]^+ \tag{76}$$

where t_0 is on S_0. Substitution of (73)–(74) into (76) and using the following Plemelj formulae for $\Phi(t_0)$ (Muskhelishvili, 1953):

$$\Phi^+(t_0) = \frac{1}{2}H(t_0) + \frac{1}{2\pi i}\int_{S_0}\frac{H(t)\,\mathrm{d}t}{t-t_0} + \Phi_*(t_0) \tag{77}$$

for t_0 on S_0, we obtain the following integral equation for $H(t)$:

$$\frac{1}{\pi i}\int_{S_0}\frac{2H(t)-q(t)}{t-t_0}\mathrm{d}t = q(t_0) - 2A - 2\left\{\Phi_*(t_0) + \bar{\Phi}_*(t_0) - \frac{R^2}{t_0^2}[t_0\bar{\Phi}'_*(t_0) + \bar{\Psi}_*(t_0)]\right\} \tag{78}$$

where A is defined as

$$A = \frac{1}{2\pi i}\int_{S_0}\frac{\bar{H}(t)\,\mathrm{d}t}{t} = \bar{\Phi}_0(0) \tag{79}$$

Recalling (64) and (73), and applying the Plemelj formula (77), we obtain

$$\frac{1}{\pi i}\int_{S_0}\frac{H(t)\,\mathrm{d}t}{t-t_0} = H(t_0) - 2\Phi_*(t_0) \tag{80}$$

Finally, substitution of (80) into (78) yields a solution for $H(t_0)$:

$$H(t_0) = \frac{1}{2}q(t_0) + \frac{1}{2\pi i}\int_{S_0}\frac{q(t)\,\mathrm{d}t}{t-t_0} - A + \left\{\Phi_*(t_0) - \bar{\Phi}_*(t_0) + \frac{R^2}{t_0^2}[t_0\bar{\Phi}'_*(t_0) + \bar{\Psi}_*(t_0)]\right\} \tag{81}$$

Note that the first two terms on the right hand side can be simplified by using the Plemelj formula. More specifically, we set

$$F(z) = \frac{1}{2\pi i}\int_{S_0}\frac{q(t)\,\mathrm{d}t}{t-z} \tag{82}$$

Substitution of (72) into (82) and applying some elementary formulae for the resultant Cauchy integrals (e.g. Section 70 of Muskhelishvili, 1975) gives:

$$F^+(t_0) = \frac{1}{2}q(t_0) + \frac{1}{2\pi i}\int_{S_0}\frac{q(t)\,\mathrm{d}t}{t-t_0} = \frac{1}{2}(\sigma_1^\infty + \sigma_2^\infty) \tag{83}$$

Finally, by virtue of (83), (81) is simplified to

$$H(t_0) = \frac{1}{2}(\sigma_1^\infty + \sigma_2^\infty) - A + \left\{\Phi_*(t_0) - \bar{\Phi}_*(t_0) + \frac{R^2}{t_0^2}[t_0\bar{\Phi}'_*(t_0) + \bar{\Psi}_*(t_0)]\right\} \tag{84}$$

Substitution of (84) into (74) and noticing (72) and (75), then simplification of the result by applying the formulae for Cauchy integrals yields

$$\Phi_0(z) = \frac{1}{2}(\sigma_1^\infty + \sigma_2^\infty) - A - \frac{1}{2\pi i}\int_{S+\Gamma}\left[\frac{z}{R^2 - z\bar{t}} - \frac{R^2(t-z)}{(R^2 - z\bar{t})^2}\right]\bar{H}(t)\,\mathrm{d}\bar{t} - \frac{1}{2\pi i}\int_{S+\Gamma}\frac{\bar{t}[H(t)-q(t)]}{R^2 - z\bar{t}}\mathrm{d}t \tag{85}$$

$$\Psi_0(z) = \frac{1}{2}(\sigma_2^\infty - \sigma_1^\infty + 2i\sigma_{12}^\infty) + \frac{1}{2\pi i}\int_{S+\Gamma}\frac{\bar{t}\bar{H}(t)}{R^2 - z\bar{t}}\left[2 - \frac{\bar{t}(t-z)}{R^2 - z\bar{t}} + \frac{2R^2(R^2 - t\bar{t})}{(R^2 - z\bar{t})^2}\right]\mathrm{d}\bar{t} + \frac{1}{2\pi i}\int_{S+\Gamma}\frac{\bar{t}^3[H(t)-q(t)]}{(R^2 - z\bar{t})^2}\mathrm{d}t \tag{86}$$

The derivation of these expressions is quite tedious although the procedure is straightforward. Substitution of (85) into (79) yields the following form for A

$$A = \frac{1}{2}(\sigma_1^\infty + \sigma_2^\infty) - \bar{A} - \frac{1}{2\pi i R^2}\int_{S+\Gamma} t\bar{q}(t)\,\mathrm{d}\bar{t} - \frac{1}{2\pi i R^2}\left[\int_{S+\Gamma}\bar{t}H(t)\,\mathrm{d}t - \int_{S+\Gamma} t\bar{H}(t)\,\mathrm{d}\bar{t}\right] \tag{87}$$

Alternatively, we can write (87) as

$$A = \frac{1}{4}(\sigma_1^\infty + \sigma_2^\infty) - \frac{1}{4\pi i R^2}\int_{S+\Gamma} t\bar{q}(t)\,\mathrm{d}\bar{t} - \frac{1}{4\pi i R^2}\left[\int_{S+\Gamma}\bar{t}H(t)\,\mathrm{d}t - \int_{S+\Gamma} t\bar{H}(t)\,\mathrm{d}\bar{t}\right] - iC \tag{88}$$

where C, as remarked earlier, is a real constant only related to the rigid rotation of the body. Note that (87) is valid if and only if the imaginary part of the third term on the right hand side is zero. To see this, we consider the resultant moment M of the boundary traction about the origin of the coordinate system. It follows from (32) and (40) that

$$M = Re\left[i\int_{S_0+S+\Gamma} t\bar{p}(t)\,\mathrm{d}s(t)\right] = -Re\left[\int_{S_0+S+\Gamma} t\bar{q}(t)\,\mathrm{d}\bar{t}\right] = -Re\left[\int_{S+\Gamma} t\bar{q}(t)\,\mathrm{d}\bar{t}\right] = 0 \tag{89}$$

The last equality is due to the fact that no moment is applied at infinity,and this shows the validity of (87) or (88).

Now, considering the limit $R \to \infty$, we have (85), (86) and (88) being reduced to

$$\Phi_0(z) = \tfrac{1}{4}(\sigma_1^\infty + \sigma_2^\infty) + iC, \quad \Psi_0(z) = \tfrac{1}{2}(\sigma_2^\infty - \sigma_1^\infty + 2i\sigma_{12}^\infty), \quad A = \tfrac{1}{4}(\sigma_1^\infty + \sigma_2^\infty) - iC \tag{90}$$

Then, substitution of (90) and (75) into (73) yields

$$\Phi(z) = \frac{1}{4}(\sigma_1^\infty + \sigma_2^\infty) + iC + \frac{1}{2\pi i}\int_{S+\Gamma}\frac{H(t)\,\mathrm{d}t}{t-z}, \tag{91}$$

$$\Psi(z) = \frac{1}{2}(\sigma_2^\infty - \sigma_1^\infty + 2i\sigma_{12}^\infty) - \frac{1}{2\pi i}\int_{S+\Gamma}\left[\frac{\bar{H}(t)-\bar{q}(t)}{t-z}\mathrm{e}^{-2i\alpha(t)} + \frac{\bar{t}H(t)}{(t-z)^2}\right]\mathrm{d}t \tag{92}$$

where C can be related to the rotation at infinity. For infinite body with cracks and holes subject to far field loading, the stresses and displacements in the body can be determined from (35), (36) and (56) together with (91) and (92). By substituting (91) and (92) into (76), we can show that both force and moment equilibrium are satisfied.

For this case of infinite body subject to far field stress, the traction boundary integral equation similar to (46) becomes

$$\pi i[f(t_0)-g(t_0)] = \int_{S+\Gamma} \left\{ \frac{H(t)}{t-t_0} - \frac{\bar{H}(t)}{\bar{t}-\bar{t}_0} e^{-2i\alpha(t)} + e^{-2i\alpha(t_0)} \left[\frac{H(t)-q(t)}{\bar{t}-\bar{t}_0} + \frac{t-t_0}{(\bar{t}-\bar{t}_0)^2} \bar{H}(t) e^{-2i\alpha(t)} \right] \right\} dt \quad (93)$$

where $f(t_0)$ is given by (47) and

$$g(t_0) = \sigma_1^\infty + \sigma_2^\infty + (\sigma_2^\infty - \sigma_1^\infty - 2i\sigma_{12}^\infty) e^{-2i\alpha(t_0)} \quad (94)$$

The complex functions and the boundary integral equation given in (38)–(39) and (46) are now reduced to (91)–(93) for the problems of infinite body subject to far field stresses σ_1^∞, σ_2^∞ and σ_{12}^∞.

9. Some analytical solutions using the present approach

To illustrate the validity and the analytic power of the present method, we reconsider some classical problems of stress concentration at circular hole and stress intensity factor at a circular-arc crack.

9.1. Infinite body with a circular hole

Consider the stress boundary problem of an infinite elastic body with a circular hole with radius R. The domain Ω of the body is given by $z = x_1 + ix_2 = \rho e^{i\theta}$ with $R < \rho < \infty$ and the boundary $S \equiv S_1$ is defined by $t = y_1 + iy_2 = R e^{i\theta}$, with $0 \leqslant \theta \leqslant 2\pi$ for both z and t. Suppose that the boundary traction on the circular hole $\rho = R$ is given by

$$q(t) = \sigma_n(t) + i\sigma_{ns}(t) = N(\theta) + iT(\theta) \quad (95)$$

Since $t = R e^{i\theta}$, we have

$$\bar{t} = R e^{-i\theta} = R^2/t, \quad e^{-2i\alpha(t)} = -e^{-2i\theta} = -R^2/t^2 \quad (96)$$

Thus, the boundary integral eqn (93) and compatibility eqn (68) become

$$\frac{1}{\pi i}\int_S \frac{2H(t)-q(t)}{t-t_0} dt + \frac{1}{\pi i t_0}\int_S [H(t)-q(t)] dt = q(t_0) - g(t_0) - 2A, \quad (97)$$

$$\int_S H(t) dt = \frac{1}{\kappa+1}\int_S q(t) dt \quad (98)$$

where $t_0 = R\exp(i\theta_0)$ and

$$A = \frac{1}{2\pi i}\int_S \frac{\bar{H}(t)\,\mathrm{d}t}{t} \tag{99}$$

From (64) and (91), we obtain

$$\frac{1}{\pi i}\int_S \frac{H(t)\,\mathrm{d}t}{t-t_0} = H(t_0) - \frac{1}{2}(\sigma_1^\infty + \sigma_2^\infty) - 2iC \tag{100}$$

In view of (98) and (100), the exact solution for (97) is obtained as

$$H(t_0) = \frac{1}{2}q(t_0) + \frac{1}{2\pi i}\int_S \frac{q(t)\,\mathrm{d}t}{t-t_0} + \frac{\kappa}{2\pi i(\kappa+1)t_0}\int_S q(t)\,\mathrm{d}t - A + \frac{1}{2}(\sigma_1^\infty + \sigma_2^\infty) + 2iC - \frac{1}{2}g(t_0) \tag{101}$$

Substitution of (94) and (101) into (91) and (92), and simplification of the results give

$$\Phi(z) = \frac{1}{2\pi i}\int_S \frac{q(t)\,\mathrm{d}t}{t-z} + \frac{\kappa}{2\pi i(\kappa+1)z}\int_S q(t)\,\mathrm{d}t + \frac{1}{4}(\sigma_1^\infty + \sigma_2^\infty) + iC + \frac{1}{2}(\sigma_2^\infty - \sigma_1^\infty - 2i\sigma_{12}^\infty)\frac{R^2}{z^2} \tag{102}$$

$$\begin{aligned}\Psi(z) = &-\frac{R^2}{2\pi i}\int_S \left[\frac{\bar{q}(t)}{t^2(t-z)} + \frac{q(t)}{t(t-z)^2}\right]\mathrm{d}t + \frac{1}{2}(\sigma_2^\infty - \sigma_1^\infty + 2i\sigma_{12}^\infty) - \frac{R^2}{2\pi i(\kappa+1)z}\int_S \frac{\bar{q}(t)\,\mathrm{d}t}{t^2} \\ &+ \frac{R^2}{z^2}\left[\frac{1}{2\pi i}\int_S \frac{q(t)\,\mathrm{d}t}{t} - (A+\bar{A})\right] + \frac{\kappa R^2}{\pi i(\kappa+1)z^3}\int_S q(t)\,\mathrm{d}t + \frac{3}{2}(\sigma_2^\infty - \sigma_1^\infty - 2i\sigma_{12}^\infty)\frac{R^4}{z^4}\end{aligned} \tag{103}$$

where

$$\bar{A} = \frac{1}{2\pi i}\int_S \frac{H(t)\,\mathrm{d}t}{t} = \frac{1}{2\pi i}\int_S \Phi^+(t)\frac{\mathrm{d}t}{t} = -\Phi(\infty) = -\frac{1}{4}(\sigma_1^\infty + \sigma_2^\infty) - iC \tag{104}$$

The present solution, of course, agree with the classical solution, such as Muskhelishvili's (1975) results. To see illustrate this, we consider the following special cases of (102) and (103).

9.1.1. Biaxial compression

When the hole boundary is free of traction and the only loading is the biaxial compression applied at infinity, we have

$$\sigma_1^\infty = -\sigma_0, \quad \sigma_2^\infty = -\beta\sigma_0, \quad \sigma_{12}^\infty = 0, \quad C = 0, \quad q(t) = 0. \tag{105}$$

Substitution of (104) and (105) into (102) and (103) yields

$$\Phi(z) = \frac{\sigma_0}{4}\left[-(1+\beta) + 2(1-\beta)\frac{R^2}{z^2}\right], \quad \Psi(z) = \frac{\sigma_0}{2}\left[(1-\beta) - (1+\beta)\frac{R^2}{z^2} + 3(1-\beta)\frac{R^4}{z^4}\right] \tag{106}$$

Recalling the following identities for coordinate transformation:

$$\sigma_\rho+\sigma_\theta=\sigma_{11}+\sigma_{22},\quad \sigma_\theta-\sigma_\rho+2i\sigma_{\rho\theta}=(\sigma_{22}-\sigma_{11}+2i\sigma_{12})\,\mathrm{e}^{2i\theta} \tag{107}$$

and $z=\rho\exp(i\theta)$, we have the following stress components, as expected,

$$\sigma_\rho=-\frac{\sigma_0}{2}(1+\beta)\left(1-\frac{R^2}{\rho^2}\right)-\frac{\sigma_0}{2}(1-\beta)\left(1-\frac{4R^2}{\rho^2}+\frac{3R^4}{\rho^4}\right)\cos 2\theta \tag{108}$$

$$\sigma_\theta=-\frac{\sigma_0}{2}(1+\beta)\left(1+\frac{R^2}{\rho^2}\right)+\frac{\sigma_0}{2}(1-\beta)\left(1+\frac{3R^4}{\rho^4}\right)\cos 2\theta \tag{109}$$

$$\sigma_{\rho\theta}=\frac{\sigma_0}{2}(1-\beta)\left(1+\frac{2R^2}{\rho^2}-\frac{3R^4}{\rho^4}\right)\sin 2\theta \tag{110}$$

where $R\leqslant\rho$ and $0\leqslant\theta\leqslant 2\pi$. These expressions agree with those given in Section 56a of Muskhelishvili (1975).

9.1.2. Uniform internal pressure

If a uniform pressure of intensity p applied on the boundary of the circular hole, we have

$$q(t)=-p,\quad \sigma_1^\infty=\sigma_2^\infty=\sigma_{12}^\infty=0,\quad C=0 \tag{111}$$

Then the complex stress functions become

$$\Phi(z)=0,\quad \Psi(z)=\frac{pR^2}{z^2} \tag{112}$$

and the corresponding stress components are

$$\sigma_\rho=-\frac{pR^2}{\rho^2},\quad \sigma_\theta=\frac{pR^2}{\rho^2},\quad \sigma_{\rho\theta}=0 \tag{113}$$

which, of course, agree with the classical solutions (e.g. Timoshenko and Goodier, 1951).

9.1.3. Concentrated force

Let the concentrated point force P is applied on the boundary of a circular hole in an infinite solid with zero far field stress, as shown in Fig. 6. For this case, we can set

$$q(t)=-P\delta(R\theta),\quad \sigma_1^\infty=\sigma_2^\infty=\sigma_{12}^\infty=0,\quad C=0 \tag{114}$$

where $\delta(\xi)$ is the Dirac delta function. Complex functions $\Phi(z)$ and $\Psi(z)$ become

$$\Phi(z)=\frac{P}{2R\pi}\left(\frac{R}{R-z}+\frac{\kappa R}{(\kappa+1)z}\right),$$

$$\Psi(z)=-\frac{P}{2R\pi}\left[\frac{R}{R-z}+\frac{R^2}{(R-z)^2}+\frac{R}{(\kappa+1)z}-\frac{R^2}{z^2}-\frac{2\kappa R^3}{(\kappa+1)z^3}\right] \tag{115}$$

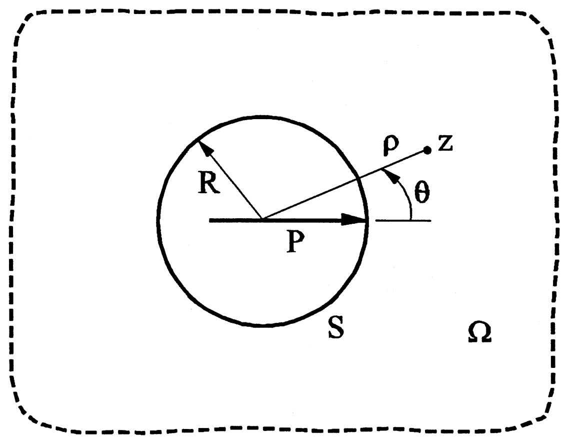

Fig. 6. An infinite elastic body containing a circular hole of radius R subject to a point force P on the hole's boundary.

where $R \leqslant \rho$ and $0 \leqslant \theta \leqslant 2\pi$. Substitution of (115) into (35)–(36) and (107) leads to the following hoop stress concentration on the boundary of the circular hole:

$$\sigma_\theta(R,\theta) = \frac{P}{2R\pi}\left(2+\frac{4\kappa}{\kappa+1}\cos\theta\right) \tag{116}$$

which, again, agrees with the result given by Timoshenko and Goodier (1951).

9.2. A circular-arc crack in an infinite body subject to uniaxial tension

This final example considers circular-arc crack in an infinite elastic body under remote uniaxial tension σ in the x_1 direction. As shown in Fig. 7, the crack surface Γ is defined as $t = y_1 + iy_2 = R\,e^{i\theta}$

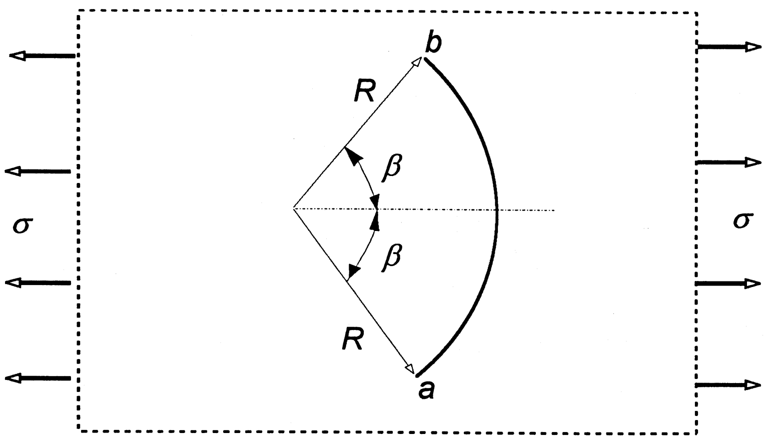

Fig. 7. A circular-arc crack sustaining an angle of 2β in an infinite body under far field tension σ.

with $-\beta < \theta < \beta$ and R being the radius of the circular crack. We have $\sigma_n^{\pm}$ and $\sigma_{ns}^{\pm}$ being zero on the crack surface Γ and $\sigma_{12}^{\infty} = \sigma_2^{\infty} = 0$ and $\sigma_1^{\infty} = \sigma$. By the definitions for (47) and (94), we obtain

$$q(t_0) = 0, \quad f(t_0) = 0, \quad g(t_0) = \sigma(1 - \mathrm{e}^{-2i\alpha(t_0)}) = \sigma(1 + R^2/t_0^2) \tag{117}$$

where $t_0 = R\exp(i\theta_0)$ and $-\beta < \theta_0 < \beta$. The complex functions $\Phi(z)$ and $\Psi(z)$ becomes

$$\Phi(z) = \frac{1}{2\pi i}\int_\Gamma \frac{H(t)\,\mathrm{d}t}{t-z} + \frac{\sigma}{4}, \tag{118}$$

$$\Psi(z) = \frac{R^2}{2\pi i}\int_\Gamma \left[\frac{\bar{H}(t)}{(t-z)t^2} - \frac{H(t)}{(t-z)^2 t}\right]\mathrm{d}t - \frac{\sigma}{2} \tag{119}$$

The boundary integral equation (93) for traction becomes

$$\frac{1}{\pi i}\int_\Gamma \frac{H(t)\,\mathrm{d}t}{t-t_0} = -\frac{1}{2}g(t_0) - A \tag{120}$$

with again t_0 being on Γ and A is defined by (99) with S being replaced by Γ. The compatibility condition (68) becomes

$$\int_\Gamma H(t)\,\mathrm{d}t = 0 \tag{121}$$

Apply the Plemelj formulae (Muskhelishvili, 1975) to (118), we obtain

$$\Phi^+(t_0) + \Phi^-(t_0) = \frac{1}{\pi i}\int_\Gamma \frac{H(t)\,\mathrm{d}t}{t-t_0} + \frac{\sigma}{2} \tag{122}$$

$$H(t_0) = \Phi^+(t_0) - \Phi^-(t_0) \tag{123}$$

Substitution of (117) and (122) into (120), the singular integral equation is reduced to the following non-homogeneous Hilbert problem (Muskhelishvili, 1975):

$$\Phi^+(t_0) + \Phi^-(t_0) = -A - \frac{\sigma R^2}{2t_0^2} \tag{124}$$

Following the procedure given by Muskhelishvili (1975), the following exact solution for (124) satisfying (118) is obtained as:

$$\Phi(z) = (C_0 + C_1 z)X(z) - \frac{X(z)}{2\pi i}\int_\Gamma \left(A + \frac{\sigma R^2}{2t^2}\right)\frac{\mathrm{d}t}{X^+(t)(t-z)} \tag{125}$$

where

$$X(z) = (z-b)^{-1/2}(z-a)^{-1/2}, \quad b = R\,\mathrm{e}^{i\beta}, \quad a = \bar{b} \tag{126}$$

The constants C_1 and C_0 are to be determined and $X^+(t)$ is the boundary value of $X(z)$ on the upper crack face (i.e. the concave side of the crack which is closer to the origin as shown in Fig.

7). As it is obvious that $X(z)$ is multi-valued, thus, we take $X(z)$ being the branch which for large $|z|$ has the following form:

$$X(z) = \frac{1}{z} + \frac{b+a}{2z^2} + O\left(\frac{1}{z^3}\right) = \frac{1}{z} + \frac{R\cos\beta}{z^2} + O\left(\frac{1}{z^3}\right) \tag{127}$$

$$\frac{1}{X(z)} = z - \frac{1}{2}(b+a) + O\left(\frac{1}{z}\right) = z - R\cos\beta + O\left(\frac{1}{z}\right) \tag{128}$$

To solve (125), we first note the following formulae

$$\frac{1}{2\pi i}\int_{\Lambda}\frac{\mathrm{d}\zeta}{X(\zeta)(\zeta-z)} = \frac{1}{X(z)} - z + R\cos\beta \tag{129}$$

$$\frac{1}{2\pi i}\int_{\Lambda}\frac{\mathrm{d}\zeta}{\zeta^2 X(\zeta)(\zeta-z)} = \frac{1}{z^2X(z)} - \frac{1}{z^2X(0)} + \frac{X'(0)}{z[X(0)]^2} \tag{130}$$

thus, we have

$$\frac{1}{\pi i}\int_{\Gamma}\left(A + \frac{\sigma R^2}{2t^2}\right)\frac{\mathrm{d}t}{X^+(t)(t-z)} = \frac{1}{2\pi i}\int_{\Lambda}\left(A + \frac{\sigma R^2}{2\zeta^2}\right)\frac{\mathrm{d}\zeta}{X(\zeta)(\zeta-z)}$$

$$= A\left[\frac{1}{X(z)} - z + R\cos\beta\right] + \frac{\sigma R^2}{2}\left[\frac{1}{z^2X(z)} - \frac{1}{z^2X(0)} + \frac{X'(0)}{z[X(0)]^2}\right] \tag{131}$$

The first of (131) relates to integral along Γ to that for the closed-loop Λ shown in Fig. 8(a). The

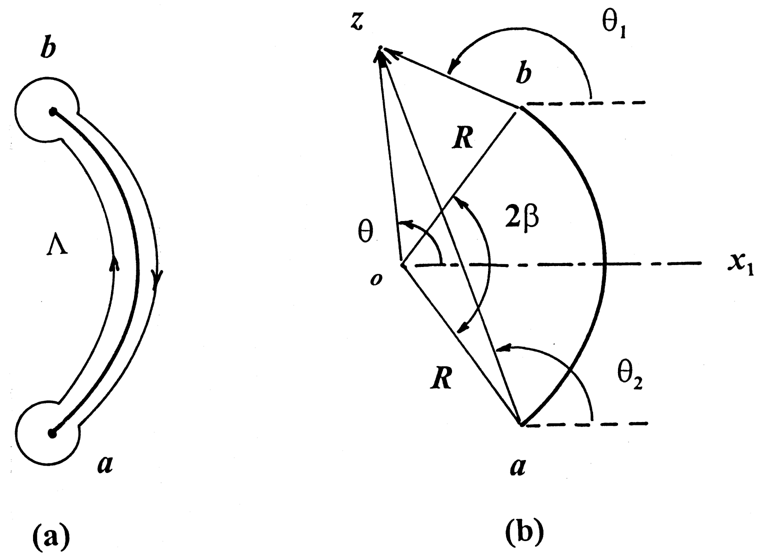

Fig. 8. (a) A closed contour Λ enclosing the arc crack with end points a and b; (b) the definitions of θ, θ_1 and θ_2 for the position point z are defined.

detailed argument being used is similar to that given in Section 110 of Muskhelishvili (1975), and will not be elaborated here. Finally, substitution of (131) into (125) gives

$$\Phi(z) = (C_0 + C_1 z)X(z) - \frac{A}{2}[1 - X(z)(z - R\cos\beta)] - \frac{\sigma R^2}{4}\left[\frac{1}{z^2} - \frac{X(z)}{X(0)}\left(\frac{1}{z^2} - \frac{X'(0)}{zX(0)}\right)\right] \tag{132}$$

By matching the values of $\Phi(z)$ at infinity by given (118) and (132), respectively, we get

$$C_1 = \frac{\sigma}{4} \tag{133}$$

To evaluate C_0, we note the following results

$$\int_\Lambda X(\zeta)\,\mathrm{d}\zeta = -2\pi i, \quad \int_\Lambda \zeta X(\zeta)\,\mathrm{d}\zeta = -2\pi i R\cos\beta, \quad \int_\Lambda \frac{X(\zeta)\,\mathrm{d}\zeta}{\zeta} = 2\pi i X(0),$$
$$\int_\Lambda \frac{X(\zeta)\,\mathrm{d}\zeta}{\zeta^2} = 2\pi i X'(0), \quad \int_\Lambda \frac{X(\zeta)\,\mathrm{d}\zeta}{\zeta^3} = \pi i X''(0) \tag{134}$$

Substitution of (123) and (132) into (121) leads to

$$\int_\Gamma H(t)\,\mathrm{d}t = \int_\Gamma [\Phi^+(t) - \Phi^-(t)]\,\mathrm{d}t = \int_\Lambda \Phi(\zeta)\,\mathrm{d}\zeta = -2\pi i(C_0 + C_1 R\cos\beta) = 0 \tag{135}$$

Therefore, this gives C_0 as

$$C_0 = -\frac{\sigma}{4}R\cos\beta \tag{136}$$

Utilizing the results in (132)–(136), we have the following expression for $\bar{A}$

$$\bar{A} = \frac{1}{2\pi i}\int_\Gamma \frac{H(t)\,\mathrm{d}t}{t} = \frac{1}{2\pi i}\int_\Gamma [\Phi^+(t) - \Phi^-(t)]\frac{\mathrm{d}t}{t} = \frac{1}{2\pi i}\int_\Lambda \Phi(\zeta)\frac{\mathrm{d}\zeta}{\zeta}$$
$$= C_0 X(0) - C_1 - \frac{A}{2}[1 + X(0)R\cos\beta] - \frac{\sigma R^2}{4}\left[\left(\frac{X'(0)}{X(0)}\right)^2 - \frac{X''(0)}{2X(0)}\right] \tag{137}$$

Using the branch for $X(z)$ defined by (127) and the notations given in Fig. 8(b), we obtain

$$X(0) = -\frac{1}{R}, \quad X'(0) = -\frac{\cos\beta}{R^2}, \quad X''(0) = -\frac{1}{R^3}(3\cos^2\beta - 1),$$
$$X^+(t) = -|t-b|^{-1/2}|t-a|^{-1/2}\,\mathrm{e}^{-i\theta/2} \tag{138}$$

With these results, (137) reduces further to

$$A = -\frac{\sigma(3+\cos\beta)(1-\cos\beta)}{4(3-\cos\beta)} \tag{139}$$

By using (133), (136) and (138), the complex function $\Phi(z)$ can also be simplified to

$$\Phi(z) = \frac{X(z)}{2}\left[\left(A+\frac{\sigma}{2}\right)(z-R\cos\beta)-\frac{\sigma R^3}{2z}\left(\frac{1}{z}-\frac{\cos\beta}{R}\right)\right]-\frac{A}{2}-\frac{\sigma R^2}{4z^2} \tag{140}$$

Consequently, substitution of (140) into (123) gives the following exact solution of the singular integral equation (120) satisfying the single-valued condition (121):

$$H(t) = X^+(t)\left[\left(A+\frac{\sigma}{2}\right)(t-R\cos\beta)-\frac{\sigma R^3}{2t}\left(\frac{1}{t}-\frac{\cos\beta}{R}\right)\right] \tag{141}$$

where $X^+(t)$ and A are given in (138) and (139), respectively. In obtaining (141), we have also used the fact that $X^+(t)+X^-(t) = 0$.

To obtain the stress intensity factors at the crack tips a and b shown in Fig. 7, we can first derive the following expressions for K_I and K_{II} in terms of $H(t)$ (Wang and Chau, 1997):

$$K_I(a)-iK_{II}(a) = -\lim_{t\to a}\sqrt{2\pi|t-a|}\cdot iH(t), \quad K_I(b)-iK_{II}(b) = \lim_{t\to b}\sqrt{2\pi|t-b|}\cdot iH(t) \tag{142}$$

The derivation of these formulae have been given in detail by Wang and Chau (1997) and, thus, will not be repeated here. Substitution of (141) into (142) then using the results in (138) and (139), we finally get

$$K_I(b) = K_I(a) = \frac{1}{2}\left[\cos\left(\frac{3\beta}{2}\right)+\frac{3+\cos^2\beta}{2(3-\cos\beta)}\cos\left(\frac{\beta}{2}\right)\right]\sigma\sqrt{\pi R\sin\beta} \tag{143}$$

$$K_{II}(b) = -K_{II}(a) = \frac{1}{2}\left[\sin\left(\frac{3\beta}{2}\right)+\frac{3+\cos^2\beta}{2(3-\cos\beta)}\sin\left(\frac{\beta}{2}\right)\right]\sigma\sqrt{\pi R\sin\beta} \tag{144}$$

which, as expected, agree with those given by formula 21.1 of Tada et al. (1985) when the applied stress σ is given along the x_1 axis.

Although the present formulation has only been applied to problems to which exact solutions exist, the present approach can also be applied to other problems of cracks and holes and provides firm basis for both analytic and numerical analyses. A numerical analysis for the present boundary integral formulation using a boundary element method is presented by Wang and Chau (1997).

10. Numerical results on two interacting collinear cracks

As discussed in the Introduction, the full discussion on the implementation of the boundary element method (BEM) for the present boundary integral formulation, either (46) for finite bodies or (93) for infinite bodies, is out of the scope of the present study. However, to illustrate the BEM based upon the present approach, we will summarize briefly the work by Wang and Chau (1997) and consider the interactions between two collinear cracks of equal size, in either a finite rectangular plate or in an infinite plate.

The numerical results to be presented here are obtained using the BEM discussed by Wang and Chau (1997). In particular, the boundary complex function $H(t)$ is the only unknown of the BEM formulation, which is governed by the integral equation, either (46) or (93), and the single-valued condition (68). The discretization of $H(t)$ uses the same linear interpolation for the boundary t, except that at the crack tips a singular shape function is used for the crack–tip-node. One nice feature of the proposed singular shape functions is that they allow the exact integration of the singular integrals at the element level, which are resulted from the overlapping of the source and field points (Wang and Chau, 1997). For each hole boundary, an addition constant is proposed in the interpolation such that the displacement compatibility condition (68) can be satisfied exactly. In addition, the stress intensity factor at the crack tips can be calculated directly from (142) in terms of $H(t)$ and no data interpolation is needed. The convergence of the BEM and the accuracy in calculating the crack–hole interactions have been demonstrated by considering the Griffith crack problem and the problem of a straight or kinked crack near to a circular hole, respectively (Wang and Chau, 1997).

For full details on the BEM formulation, the reader is referred to Wang and Chau (1997). In this section, we will, however, present two particular problems of crack–crack interactions shown in Figs 9 and 10.

10.1. Two interacting collinear cracks in a finite rectangular plate

Our first numerical example investigates the interaction between two collinear cracks of length $2a$ in a finite rectangular plate of dimension $2b \times 2h$, and the distance between the centres of these cracks is $2c$, as shown in Fig. 9. Uniform tension field σ is applied on the pair of boundaries of length $2b$. And, as shown in Fig. 9, the crack tip closer to the neighbouring crack or 'the inner

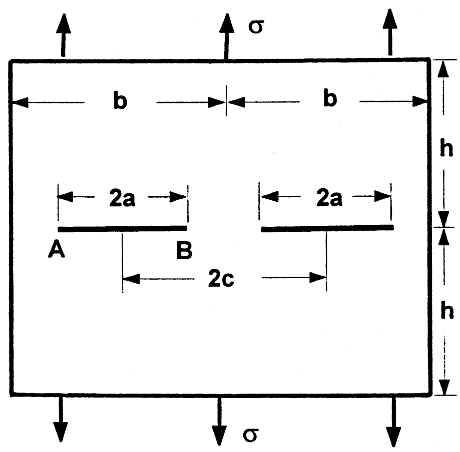

Fig. 9. A finite rectangular plate containing two equal-length collinear cracks under uniform tension σ along a direction perpendicular to the crack face.

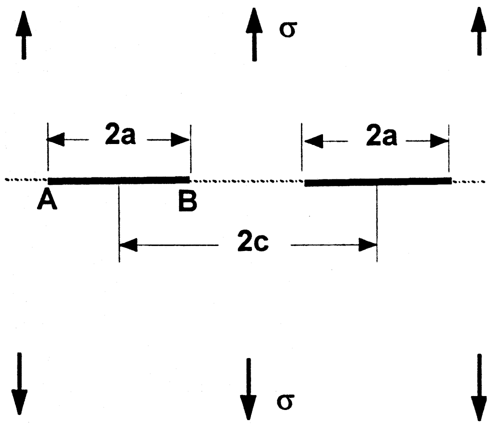

Fig. 10. An infinite plate containing two equal-length collinear cracks under far field tension σ along a direction perpendicular to the crack face.

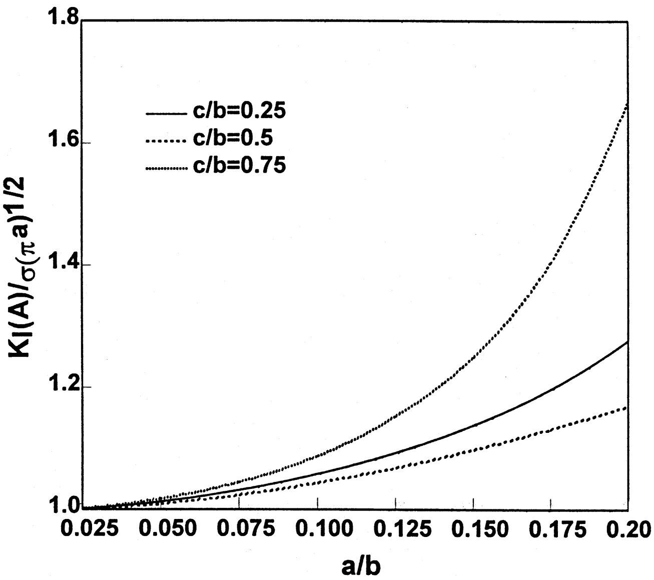

Fig. 11. Variations of the normalized SIFs with crack size a/b at the outer crack tip A of two-equal collinear cracks (see Fig. 9) in a finite rectangular plate ($b/h = 0.5$) for various values of c/b.

crack tip' is labelled as crack tip B, while the one farther away from the neighbouring crack or 'the outer crack tip' is labelled as crack tip A.

The BEM used are derived by satisfying the boundary integral eqn (46) and the compatibility eqn (68). Figures 11 and 12 plot the normalized Mode I stress intensity factors (SIF) $K_I(A)/[\sigma(\pi a)^{1/2}]$ and $K_I(B)/[\sigma(\pi a)^{1/2}]$ at the outer and inner crack tips vs the normalized crack length a/b for various crack spacing (i.e. $c/b = 0.25$, 0.5 and 0.75). The shape factor h/b of the rectangular plate is fixed

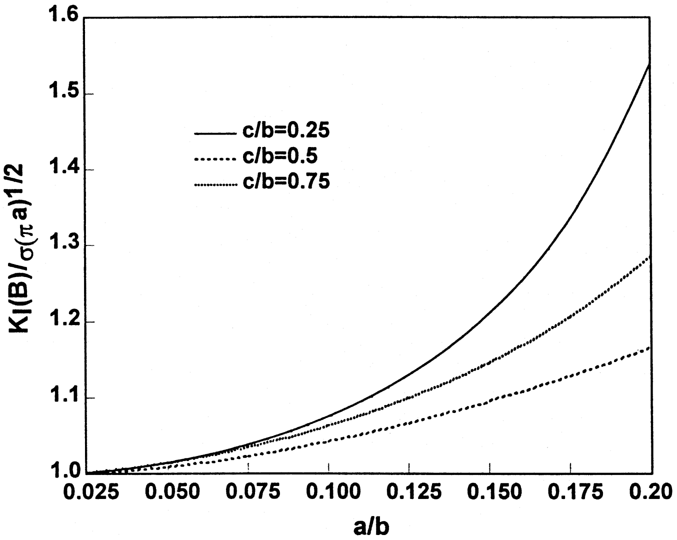

Fig. 12. Variations of the normalized SIFs with crack size a/b at the inner crack tip B of two-equal collinear cracks (see Fig. 9) in a finite rectangular plate ($b/h = 0.5$) for various values of c/b.

at 0.5 for all of the calculations shown. In general, all stress intensity factors, either at crack tip A or B, increase with the normalized crack length a/b. The crack interactions are not very significant for both crack tips A and B when $c/b = 0.5$. As discussed by Chandra et al. (1995), the interaction between the two cracks is more severe at the inner tip B when $c/b = 0.25$, whilst the crack-free edge interaction becomes more dominant at the outer tip A when $c/b = 0.75$. Our numerical results given in Figs 11 and 12 are virtually the same as those given in Fig. 5 of Chandra et al. (1995), in which a different BEM formulation was followed. To allow comparisons with other studies, the normalized SIFs are also compiled in Table 1 vs a/b for various values c/b. Therefore, we conclude that the BEM based upon the present formulation is as accurate as other numerical approaches.

10.2. Two interacting collinear cracks in an infinite plate

Our second example is basically the same as the first one, except that the dimension of the plate is now unbounded, as both b and h approaches infinity (see Fig. 10). The only controlling geometric parameter in this problem is a/c. The crack interactions between these collinear cracks can be calculated by following two different approaches: (1) applying the BEM for finite plates as described in Section 10.1 with a fixed, finite b/h and a very small a/b; and (2) applying a BEM, which is formulated based upon the boundary integral equation (93) and the compatibility condition (68).

Following the first approach, approximations for the normalized SIFs at crack tips A and B can be found and are summarized in Table 2 for various values of a/c (from 0.05 to 0.98) for the case of $b/h = 1$ and $a/b = 0.01$ (i.e. two small cracks in a large square plate). Alternatively, the BEM for an infinite domain can be also used (i.e. by employing (93) and (68) in the BEM formulation), and the results are summarized in Table 3 together with the numerical results obtained by Isida (cited in p. 195 of Murakami, 1987). As expected, the results of our two different approaches agree

Table 1
Variation of the normalized mode I stress intensity factors with crack size a/b at the outer and inner crack tips A, B (shown in Fig. 9) of two equal-length collinear cracks in a finite rectangular plate ($h/b = 0.5$) for various crack spacings c/b

	$K_I(A)/\sigma\sqrt{\pi a}$			$K_I(B)/\sigma\sqrt{\pi a}$		
a/b	$c/b = 0.25$	$c/b = 0.5$	$c/b = 0.75$	$c/b = 0.25$	$c/b = 0.5$	$c/b = 0.75$
0.025	1.0011	1.0001	1.0018	1.0013	1.0001	1.0015
0.050	1.0126	1.0087	1.0167	1.0145	1.0086	1.0140
0.075	1.0315	1.0230	1.0439	1.0383	1.0226	1.0343
0.100	1.0580	1.0428	1.0869	1.0753	1.0419	1.0626
0.125	1.0929	1.0676	1.1518	1.1298	1.0661	1.0992
0.150	1.1378	1.0973	1.2497	1.2100	1.0949	1.1458
0.175	1.1963	1.1313	1.4037	1.3325	1.1280	1.2054
0.200	1.2761	1.1693	1.6715	1.5394	1.1655	1.2860

Table 2
Variation of the normalized mode I stress intensity factors with crack spacing a/c at the outer and inner crack tips A, B (shown in Fig. 9) of two equal-length collinear cracks in a large rectangular plate ($h/b = 1.0$ and $a/b = 0.01$)

a/c	$K_I(A)/\sigma\sqrt{\pi a}$	$K_I(B)/\sigma\sqrt{\pi a}$
0.05	0.99838	0.99840
0.1	0.99930	0.99943
0.2	1.00274	1.00377
0.3	1.00828	1.01192
0.4	1.01597	1.02523
0.5	1.02605	1.04595
0.6	1.03902	1.07828
0.7	1.05594	1.13091
0.8	1.07914	1.22605
0.9	1.11547	1.44900
0.98	1.18183	2.33389

well with one another (comparing Tables 2 and 3); and, more importantly, they are virtually the same as those obtained by Isida following a different method, with a maximum error of about 0.54% for all the data shown in Table 3.

Despite the details of our formulation for the BEM are given elsewhere (Wang and Chau, 1997), the above two numerical examples indeed illustrate that our new boundary integral formulation

Table 3
Variation of the normalized mode I stress intensity factors with crack spacing a/c at the outer and inner crack tips A, B (shown in Fig. 10) of two equal-length collinear cracks in an infinite plate

a/c	Present BEM results		Isida[a] (Murakami, 1987)	
	$K_{\mathrm{I}}(A)/\sigma\sqrt{\pi a}$	$K_{\mathrm{I}}(B)/\sigma\sqrt{\pi a}$	$K_{\mathrm{I}}(A)/\sigma\sqrt{\pi a}$	$K_{\mathrm{I}}(B)/\sigma\sqrt{\pi a}$
0.05	0.99815	0.99816	1.00031	1.00032
0.1	0.99904	0.99916	1.00120	1.00132
0.2	1.00246	1.00349	1.00462	1.00566
0.3	1.00799	1.01164	1.01017	1.01383
0.4	1.01568	1.02494	1.01787	1.02717
0.5	1.02575	1.04565	1.02795	1.04796
0.6	1.03872	1.07797	1.04094	1.08040
0.7	1.05562	1.13057	1.05786	1.13326
0.8	1.07880	1.22566	1.08107	1.22894
0.9	1.11510	1.44851	1.11741	1.45387
0.98	1.18138	2.33300		

[a] Reported in Murakami, Y., 1987. Stress Intensity Factors Handbook. Vol. 1. Pergamon Press, Oxford, U.K., p. 195.

proposed here can provide a useful means to solve more complicated problems of crack–crack interactions.

11. Conclusions

A new formulation of boundary integral equations is proposed in this paper for a plane elastic body containing an arbitrary number of cracks and holes. The bodies can be either finite or infinite, the cracks can either be an isolated crack or an edge crack emanating from a hole, and the hole can be of arbitrary shapes. The formulation starts from Somigliana formula given by Cruse (1988), to which integration by parts is applied. The resulting traction boundary integral is then rewritten in terms of complex stress functions given by Muskhelishvili (1975), which are in the present context expressed in terms of Cauchy integral. The present formulation provides a formal link between the complex variable formulation by Muskhelishvili and the boundary integral formulation for bodies containing cracks and holes. Although the present formulation is motivated by its application to general numerical analysis, such as the boundary element method (e.g. Wang and Chau, 1997), our main focus here is on the application in obtaining exact analytic solutions. In particular, to verify our formulation we re-derive the stress concentration at a circular hole in an infinite body subject to: (i) far field biaxial compression; (ii) uniform internal pressure; and (iii) a concentrated point force normal to the hole's surface. For crack problems, the stress intensity factor at the tip of a circular-arc under far field uniaxial tension is reconsidered; and, as expected, the results agree with those given in the handbook of stress intensity factors (Tada et al., 1985).

The numerical implementation of the boundary element method (BEM) using the present formulation is out of the scope of this study and is presented elsewhere (Wang and Chau, 1997). As demonstrated by Wang and Chau (1997), its possible application in studying the interaction between hole and crack is illustrated by considering a kinked crack close to a circular hole. The accuracy of the BEM based on the present formulation has been verified when the crack is far away from the hole (Wang and Chau, 1997).

To provide further numerical verifications of the present formulation, the interaction problems of two equal-length collinear cracks in a finite rectangular plate and in an infinite plate under tension are solved by employing the BEM discussed by Wang and Chau (1997). As expected, the numerical results agree well with those by Chandra et al. (1995) for the case of finite plates and with those by Isida (cited in Murakami, 1987) for the case of infinite plates.

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References

Aliabadi, M.H., Brebbia, C.A., (eds.), 1993. Advances in Boundary Element Methods for Fracture Mechanics. Elsevier, London.

Aliabadi, M.H., Cartwright, D.J., Rooke, D.P., 1989. Fracture-mechanics weight–functions by the removal of singular fields using boundary element analysis. Int. J. Fract. 40, 271–284.

Ang, W.T., 1986. A boundary integral solution for the problem of multiple interacting cracks in an elastic material. Int. J. Fract. 31, 259–270.

Ang, W.T., 1987. A boundary integral equation for deformations of an elastic body with an arc crack. Quart. Appl. Math. 45, 131–139.

Ang, W.T., 1990. A finite-part boundary integral formulation of an elastic crack problem. J. Sci. Soc. Thailand 16, 141–151.

Ang, W.T., Clements, D.L., 1986. A boundary element method for determining the effect of holes on the stress distribution around a crack. Int. J. Numer. Meth. Eng. 23, 1727–1737.

Ang, W.T., Clements, D.L., 1987. A boundary integral equation method for the solution of a class of crack problems. J. Elast. 17, 9–21.

Blanford, G.E., Ingraffea, A.R., Liggett, J.A., 1991. Two-dimensional stress intensity factor computations using the boundary element method. Int. J. Numer. Meth. Eng. 17, 387–404.

Brebbia, C.A., 1984. The Boundary Element Method for Engineers. Pentech Press, London.

Brebbia, C.A., Dominguez, J., 1992. Boundary Elements An Introductory Course, 2nd ed. McGraw-Hill, New York.

Brebbia, C.A., Telles, J.C.F., Wrobel, L.C., 1984. Boundary Element Techniques: Theory and Applications in Engineering. Springer-Verlag, Berlin.

Bui, H.D., 1977. An integral equations method for solving the problem of a plane crack of arbitrary shape. J. Mech. Phys. Solids 25, 29–39.

Chandra, A., Huang, Y., Wei, X., Hu, K.X., 1995. A hybrid micro-macro BEM formulation for micro-crack clusters in elastic components. Int. J. Num. Meth. Eng., 38, 1215–1236.

Chang, C., Mear, M.E., 1995. A boundary element method for two dimensional linear elastic fracture analysis. Int. J. Fract. 74, 219–251.

Chen, W.H., Chen, T.C., 1995. An efficient dual boundary element technique for a two-dimensional fracture problem with multiple cracks. Int. J. Numer. Meth. Eng. 38, 1739–1756.

Chen, Y.Z., Hasebe, N., 1996. Hypersingular integral equation for a curved crack problem of circular region in antiplane elasticity. J. Appl. Mech. ASME 63, 845–849.

Crouch, S.L., 1976. Solution of plane elasticity problems by the displacement discontinuity method. Int. J. Numer. Meth. Eng. 10, 301–343.

Crouch, S.L.,Starfield, A.M., 1983. Boundary Element Methods in Solid Mechanics: With Applications in Rock Mechanics and Geological Engineering. Allen and Unwin, London.

Cruse, T.A., 1969. Numerical solutions in three dimensional elastostatics. Int. J. Solids Structures 5, 1259–1274.

Cruse, T.A., 1988. Boundary Element Analysis in Computational Fracture Mechanics. Kluwer Academic, Dordrecht.

Danson, D., 1983. Linear isotropic elasticity with body forces. In Brebbia, C.A. (Ed.), Progress in Boundary Element Methods, Vol. 2, Chap. 4, pp. 101–135.

Hong, H.-K., Chen, J.-T., 1988. Derivations of integral equations of elasticity. J. Eng. Mech. ASCE 114, 1028–1044.

Ioakimidis, N.I., 1982. Application of finite-part integral to the singular integral equations of crack problems in plane and three-dimensional elasticity. Acta Mechanica 45, 31–47.

Ioakimidis, N.I., 1983. A new singular integral equation for the classical crack problem in plane and antiplane elasticity. Int. J. Fract. 21, 115–122.

Ioakimidis, N.I., 1985. Exact expression for a two-dimensional finite-part integral appearing during the numerical solution of crack problems in three-dimensional elasticity. Comm. Appl. Numer. Meth. 1, 183–189.

Jiang, Z.Q., Chandra, A., Huang, Y., 1996. A hybrid micro-macro BEM with micro-scale inclusion-crack interactions. Int. J. Solids Structures 33, 2309–2329.

Kellogg, O.D., 1953. Foundations of Potential Theory. Dover, New York.

Kinoshita, N., Mura, T., 1956. On the boundary value problem of elasticity. Research Report, Faculty of Eng. Meiji University, No. 8, pp. 1–7.

Kupradze, V.C., 1953. Boundary value problems of steady-state elastodynamic problems. Uspekhi Mat. Nuak 8, 21–74.

Kupradze, V.C., 1965. Potential Methods in Theory of Elasticity. Israel Program Sci. Trans., Jerusalem.

Lavit, I.M., 1994. Boundary integral equation for a curvilinear boundary crack. J. Appl. Math. Mech. (translation of Prikl. Mat. Mekh.) 58, 161–170.

Lacht, J.C., Watson, J.O., 1976. Effective numerical treatment of boundary integral equations: a formulation for three-dimensional elastostatics. Int. J. Numer. Meth. Eng. 10, 991–1005.

Lee, J.C., Keer, L.M., 1986. Study of a three-dimensional crack terminating at an interface. J. Appl. Mech. ASME 53, 311–316.

Martin, P.A., Rizzo, F.J., 1989. On boundary integral equations for crack problems. Proc. R. Soc. Lond. A 421, 341–355.

Mikhlin, S.G., 1965. Multidimensional Singular Integral and Equations. Pergamon Press, New York.

Murakami, Y., 1987. Stress Intensity Factors Handbook, Vol. 1. Pergamon Press, Oxford, U.K.

Muskhelishvili, N.I., 1953. Singular Integral Equations, 2nd ed. (translated by Radok, J.R.M.). Noordhoff International Publishing, Leyden.

Muskhelishvili, N.I., 1975. Some Basic Problems of the Mathematical Theory of Elasticity, 4th ed (translated by Radok, J.R.M.). Noordhoff International Publishing, Leyden.

Nishimura, N., Kobayashi, S., 1988. An improved boundary integral equation method for crack problems. In: Cruse, T.A. (Ed.), Advanced Boundary Element Method. Springer-Verlag, Berlin.

Pan, E., Amadei, B., 1996. Fracture mechanics analysis of cracked 2-D anisotropic media with a new formulation of the boundary element method. Int. J. Fract. 77, 161–174.

Portela, A., Aliabadi, M.H., Rooke, D.P., 1992. The dual boundary element method: effective implementation for crack problems. Int. J. Numer. Meth. Eng. 33, 1269–1287.

Rizzo, F.J., 1967. An integral equation approach to boundary value problems of classical elastostatics. Quart. Appl. Math. 25, 83–91.

Saez, A., Gallego, R., Dominguez, J., 1995. Hypersingular quarter-point boundary elements for crack problems. Int. J. Numer. Meth. Eng. 38, 1681–1701.

Shou, K.J., Crouch, S.L., 1995. A higher order displacement discontinuity method for analysis of crack problems. Int. J. Rock Mech. Min. Sci. and Geomech. Abstr. 32, 49–55.

Sladek, V., Sladek, J., 1982. Three-dimensional crack analysis for an anisotropic body. Appl. Math. Modelling 6, 374–380.
Sladek, V., Sladek, J., 1990. On nonsingular boundary integral equations for crack problems. Mech. Comm. 17 (5), 281–289.
Stephan, E.P., 1986. A boundary integral equation method for three-dimensional crack problems in elasticity. Math. Meth. Appl. Sci. 8, 609–623.
Snyder, M.D., Cruse, T.A., 1975. Boundary-integral equation analysis of cracked anisotropic plates. Int. J. Fract. 11, 315–328.
Tada, H., Paris, P.C., Irwin, G.R., 1985. The Stress Analysis of Cracks Handbook, 2nd ed. Paris Production Incorporated, St. Louis.
Takakuda, K., Koizumi, T., Shibuya, T., 1985. On integral equation methods for crack problems. Bull. JSME 28, 217–224.
Tang, R., Wang, Y.B., 1986. On the problem of crack system with an elliptic hole. Acta Mechanica Sinica 2, 47–57.
Timoshenko, S.P., Goodier, J.N., 1951. Theory of Elasticity, 3rd ed. McGraw-Hill, New York.
Wang, Y.B., 1990. A boundary integral equation method for the Griffith crack problem under asymmetric loadings. J. Lanzhou University (Natural Science) 26, 35–39 (in Chinese).
Wang, Y.B., 1993. A new boundary integral equation method of three-dimensional crack analysis. Int. J. Fract. 63, 317–328.
Wang, Y.B., 1995. A boundary integral equation method of plane problems of interface cracks in elastic bimaterials. J. Lanzhou University (Natural Science) 31, 14–21 (in Chinese).
Wang, Y.B., Chau, K.T., 1997. A new boundary element method for plane elastic problems involving cracks and holes. Int. J. Fract., 87, 1–20.
Wang, Y.B., Chen, W.J., 1993. Interaction of two equal coplanar square cracks in three-dimensional elasticity. Int. J. Solids Structures 30, 3315–3320.
Wang, Y.B., Tang, R., 1988. The effect of a square hole on a crack tip's stress intensity factor. Acta Mechanica Solida Sinica 9, 260–264 (in Chinese).
Weaver, J., 1977. Three-dimensional crack analysis. Int. J. Solids Structures 13, 321–330.
Wendland, W.L., Stephan, E.P., 1990. A hypersingular boundary integral method for two-dimensional screen and crack problems. Arch. Rational Mech. Anal. 112, 363–390.
Zang, W.L., 1990. On modelling of piece-wise smooth cracks in two-dimensional finite bodies. Int. J. Fract. 46, 41–55.
Zang, W., Gudmundson, P., 1988. A boundary integral method for internal piece-wise smooth crack problems. Int. J. Fract. 38, 275–294.