Extended-Domain-Eigenfunction Method (EDEM) for solving elliptic boundary value problems with annular domains

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Abstract

Standard analytical solutions to elliptic boundary value problems on asymmetric domains are rarely, if ever, obtainable. In this paper, we propose a solution technique wherein we embed the original domain into one with simple boundaries where the classical eigenfunction solution approach can be used. The solution in the larger domain, when restricted to the original domain, is then the solution of the original boundary value problem. We call this the Extended-Domain-Eigenfunction Method (EDEM). To illustrate the method's strength and scope, we apply it to Laplace's equation on an annular-like domain.

1 Introduction

In many fields of science and technology one is required to solve a boundary value problem in order to describe, mathematically or physically, the behavior of a system or process. The classic partial differential equations (PDEs) of mathematical physics, including the scalar Poisson equation,

$$\Delta\psi = -\rho, \qquad x \in \Omega, \tag{1}$$

the Helmholtz equation,

$$\Delta\psi + \lambda\psi = 0, \qquad x \in \Omega, \tag{2}$$

and the scalar Laplace equation,

$$\Delta\psi = 0, \qquad x \in \Omega, \tag{3}$$

commonly occur in physical and engineering applications [1]. The domain, Ω, plays an important role, first, in establishing existence and uniqueness of solutions, and second, that which concerns us here, in the actual solution calculation process. Existence and uniqueness are dependent on the boundedness of Ω, the shape and regularity of its boundary Γ, and also on the conditions to be satisfied by some combination of the solution ψ and its normal derivative $\partial\psi/\partial n$ on Γ [1, 2, 3]. A similar story applies in hydrodynamics where steady laminar flow behavior of a bounded fluid is governed by the Stokes system of equations

$$-\Delta\mathbf{u} + \nabla p = 0, \qquad \nabla\cdot\mathbf{u} = 0, \qquad \mathbf{x} \in \Omega \tag{4}$$

whose solution $(\mathbf{u}, p)$ in $\mathbb{R}^\ell \times \mathbb{R}$ satisfies appropriate Dirichlet conditions

$$\mathbf{u} = \mathbf{v_0}, \qquad \mathbf{x} \in \Gamma \tag{5}$$

or stress (traction) conditions

$$p\mathbf{n} - \left(\nabla\mathbf{u} + (\nabla\mathbf{u})^T\right)\mathbf{n} = \tau, \qquad \mathbf{x} \in \Gamma \tag{6}$$

where $\mathbf{n}$ is the outward pointing unit normal vector to the boundary Γ.

The common feature of the partial differential equations (1)-(4) is that they are of elliptic type [2]. Existence and uniqueness of elliptic PDEs have been addressed in a great variety of situations [2, 3, 4, 5]. Knowing that a unique solution exists is seldom of direct practical use where interest focuses on the actual calculation of the solution. The present proposal is concerned with the practical task of explicitly evaluating the unique solutions to equations such as (1)-(4) in situations of practical relevance. In situations where the domain Ω has nice geometry (*e.g.*, it has spherical symmetry), it is often possible to obtain analytic or semi-analytic solutions by, say, the separation of variables technique. However, in many practical cases, the boundaries are not of simple geometric shape, thus preventing use of standard solution techniques.

Perturbation methods are useful when the shapes of the boundaries differ only slightly from regular geometries [6, 7, 8, 9, 10, 11]. However, their utility is limited as perturbation expansions often converge too slowly for very irregular shaped boundaries.

Currently, if one is to solve any of equations (1)-(4), or linear equations of the same type, in high dimensional situations involving domains with boundaries of complicated shape, then intense numerical techniques such as the finite element method [12, 13], the boundary element method (BEM) [13, 14, 15, 16, 17], the finite difference method [18], the boundary point method [19, 20, 21, 22, 23] or Monte Carlo techniques [18] represent the only alternative. Each of these must be implemented from the outset giving the scientist or engineer little direct physical insight into the problem. Also, such techniques are not always straightforward to implement for the non-numerically trained scientist. And, finally, the solution comes in the form of strictly numerical data, while the solution itself is not always the final object of study. In fact, determining the solution of the governing equations often represents the first stage in a chain of calculations toward a much more involved goal to show how the physical system behaves.

For these practical reasons it is important to consider alternate, semi-analytic approaches to solving elliptic boundary value problems (EBVPs), and such efforts addressing particular cases or complicated situations continually appear in the literature (*e.g.*, [24]). In particular, we make reference to the approaches known as Trefftz methods [25, 26, 27, 28, 29, 30, 31, 32, 33][1], named after the original proposer [25]. Although these methods have some commonality their definition is quite varied in the literature (see for example Ref. [26] wherein Herrera provides a formal definition). Notwithstanding this diversity, the method basically utilizes eigenfunctions of the differential operator in the construction of a finite sum approximation to the solution of an EBVP. Usually, the Trefftz method involves the breakup of the given (finite) domain into subdomains in which solutions, expressed in terms of eigenfunctions, are matched at the subdomain boundaries [26, 27, 28, 29, 30]. In this paper we present a theoretical methodology that overlaps in philosophy the general Trefftz method. Here, however, the approach is based on considering the actual domain as a subset of a larger region that possesses much greater symmetry. The boundary value problem is then extended to the larger domain allowing the use of eigenfunction expansions to represent the solution. The restriction of this solution to the original domain is then the solution of the original boundary value problem. Based on these designated steps the method we adopt will here be referred to as the Extended-Domain-Eigenfunction Method (EDEM) to distinguish it from the standard Trefftz approach. Unknown to us, a similar idea was suggested in [30], where nevertheless, issues of extend-

[1]We are grateful to an anonymous referee for pointing out in an earlier version of this paper the existence of these methods and accompanying literature, of which we were unaware.

ability were not discussed.

The rest of the paper is divided as follows. In Section 2 we provide a formal description of the problem, at a level of generality so as to allow us to cover all cases of interest. We also outline the main theoretical questions that need to be answered, establishing the formal relationship between the original and the extended problems, the importance of the region's boundary as well as the significance of the boundary data. These questions seem not to have been addressed in the presentations of the Trefftz method [26, 27, 28, 29, 30] where most attention has been paid to practical considerations. In Sections 3 and 4 we work out in detail all of the answers for the case of the Laplacian in two dimensions. Section 6 contains our final remarks.

2 The Extended-Domain-Eigenfunction Method

In this section we describe, and provide motivation for, the EDEM as a means of investigating and solving EBVPs, such as (1)-(4), with annular domains Ω, defined below. We formally set out the domain and operator conditions for the original and the extended boundary value problems. In particular, we describe necessary constraints to be satisfied on the boundary. We also identify the formal connection between the original and extended domain problems.

2.1 Domain boundary and the extended domain

Let $(\xi_1, \ldots, \xi_\ell) \equiv (\xi', \xi_\ell) \in \mathbb{R}^\ell$ be a suitably chosen orthogonal curvilinear coordinate system centered at the origin representing a point $x \in \mathbb{R}^\ell$. Consider two open domains Ω and Ξ with $\Omega \subset \Xi \subset \mathbb{R}^\ell$ on which the partial differential operators introduced below are defined.

Given constants a and A and two positive continuous functions, t_1 and t_2, defined on the open domain $\Omega' \subset \mathbb{R}^{\ell-1}$, *i.e.* $t_1 = t_1(\xi')$ and $t_2 = t_2(\xi')$ where $\xi' \in \Omega'$, we define Ω to be the *annular domain*

$$\Omega = \{(\xi', \xi_\ell) \mid 0 < a < t_1(\xi') < \xi_\ell < t_2(\xi') < A, \quad \xi' \in \Omega'\} .$$

All domains considered in this paper that are ℓ-dimensional subsets of $\mathbb{R}^\ell$ are annular.

With this definition, the annular domain Ω has an inner boundary Γ_1 given by $\xi_\ell = t_1(\xi')$ and an outer boundary Γ_2 given by $\xi_\ell = t_2(\xi')$. For our purposes, we can assume without loss of generality that Γ_2 coincides with the coordinate surface $\xi_\ell = A$. The closure of Ω is $\overline{\Omega} = \Omega \cup \Gamma_1 \cup \Gamma_2$. The set $\{\xi_\ell = 0\}$ is not in $\overline{\Omega}$, but is enclosed by the inner boundary component.

The smoothness of the boundary is determined by the smoothness of the functions t_1 and t_2, and the coordinate system ξ. We assume throughout that t_1 and t_2 are continuous and piecewise smooth, and that ξ is a smooth change of coordinates.

Consider the examples of annular domains shown in Figure 1.

In the first example, referring to polar coordinates (θ, r) in the plane, Ω is the domain bounded by the ellipse Γ_1: $x^2 + 4y^2 = 1$ and the circle Γ_2: $r = A = 3$. In polar coordinates, the equation of the ellipse is

$$r = t_1(\theta) = \frac{1}{\sqrt{1 + 3\sin^2\theta}}, \quad \theta \in \Omega' = [0, 2\pi)\,.$$

The expression for Γ_2 is $r = t_2(\theta) = 3$.

In the second example, we use cylindrical coordinates (z, θ, r). Here, Γ_1 is given by $r = t_1(z, \theta) = 1 + \frac{1}{2}\sin\theta\cos z$ and Γ_2 is the cylinder $r = t_2(z, \theta) = 5$ for $(z, \theta) \in \Omega' = \mathbb{R} \times [0, 2\pi)$.

For the final example shown in Figure 1, we use standard (x, y) coordinates in the plane. The boundary Γ_1 is given by $y = t_1(x) = 1 + \frac{1}{3}\sin x$ and Γ_2 is given by $y = t_2(x) = 3$ for $\xi' \in \Omega' = \mathbb{R}$.

As the examples will show, annular domains do not have to be bounded, and can indeed be simply connected.

We enclose the annular domain, Ω, in a larger annular domain, Ξ, as follows. Suppose t_1 and t_2 are the functions that define $\overline{\Omega}$. We choose $t_0 = t_0(\xi')$ and t_2 as the functions defining the larger region Ξ. In particular, choose $t_0 = a$ defining a second coordinate surface $\xi_\ell = a$ such that $t_0 \leq t_1$. We call Ξ an *extended domain* of Ω, with boundary components Γ_2 (as in $\overline{\Omega}$) and Γ_0 (given by t_0).

Note that by choosing t_0 and t_2 to be constant, the domain Ξ will have a simple, standard geometry, a property not possessed by Ω.

2.2 The operator L

Let L be a linear partial differential operator defined on the annular domain $\Xi = \{a < \xi_\ell < A\}$, written in divergence form as

$$Lu = L\left(x, \partial_x\right) u \equiv \sum_{i,j=1}^{\ell} \partial_{x_i}\left(\gamma_{i,j}\left(x\right) \partial_{x_j} u\right) + \varphi\left(x\right) u. \tag{7}$$

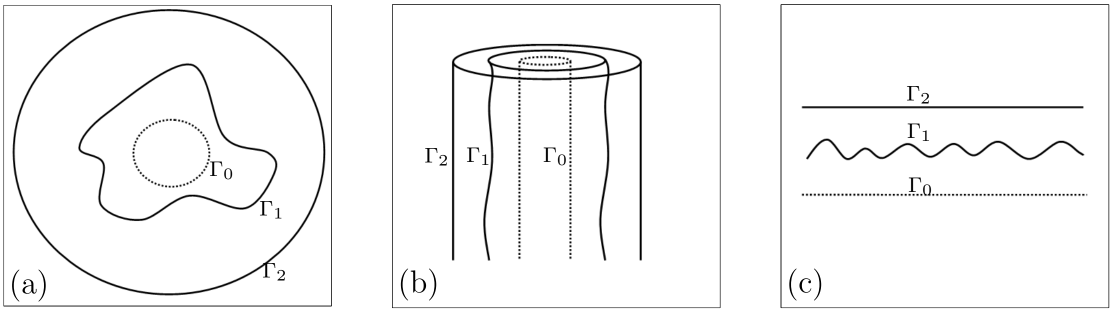

Figure 1: Schematic illustrations of some extended domains to which the EDEM can be applied: (a) a region between an ℓ-dimensional, closed smooth surface enclosed by an ℓ-dimensional sphere, (b) a region bounded by an undulating cylinder and a right-cylinder, and (c) a lamella domain bounded by an undulating membrane and a plane surface.

where we have used the standard Cartesian coordinate system $x = (x_1, ...x_l)$. We assume that functions $\gamma_{i,j}$ and φ are smooth and continuous up to the boundary, and that L satisfies the strong ellipticity condition

$$\sum_{i,j=1}^{\ell} \gamma_{i,j}(x)\zeta_i\zeta_j \geq C|\zeta|^2 \quad \forall\, x \in \Xi \quad \text{and} \quad \zeta \in \mathbb{R}^\ell$$

for some constant $C > 0$. The matrix $\{\gamma_{i,j}\}$ is taken to be symmetric. Our assumptions imply that any solution to $Lu = 0$ on Ξ is necessarily smooth (see [34]).

Under the change of coordinates $\xi = (\xi', \xi_\ell)$ it is assumed that the equation $Lw = 0$ may be written as $(L_{\xi'} + L_{\xi_\ell})\, w = 0$, with $L_{\xi'}$ independent of ξ_ℓ, and L_{ξ_ℓ} independent of ξ'. Thus, a separation of variables strategy with $w(\xi) = \widetilde{w}(\xi')\, \overline{w}(\xi_\ell)$, leads, at least formally, to

$$\frac{L_{\xi'}\widetilde{w}(\xi')}{\widetilde{w}(\xi')} + \frac{L_{\xi_\ell}\overline{w}(\xi_\ell)}{\overline{w}(\xi_\ell)} = 0.$$

A standard argument leads to the eigenvalue problems

$$L_{\xi'}\widetilde{w}(\xi') = -\lambda\widetilde{w}(\xi'), \qquad L_{\xi_\ell}\overline{w}(\xi_\ell) = \lambda\overline{w}(\xi_\ell). \tag{8}$$

The separation constant λ is determined from the first of these eigenvalue problems, and homogeneous boundary conditions. For each eigenvalue λ, the second order equation $L_{\xi_\ell}\overline{w}(\xi_\ell) = \lambda\overline{w}(\xi_\ell)$, can be solved upon applying some extra condition. A further degree of freedom is left to determine $\overline{w}$; that gap will only be filled after reconstructing the full solution w from its components, and matching it with some given data. By construction, *i.e.*, it is assumed that, the equation $L_{\xi'}\widetilde{w}(\xi') = -\lambda\widetilde{w}(\xi')$, together with the boundary conditions imposed by the context, constitute a self-adjoint problem with a countable number of eigenvalues, with no finite accumulation point. In this case, solutions to $Lw = 0$ may be written as

$$w(\xi) = \sum_{m=1}^{\infty} a_m\, \phi_m(\xi), \tag{9}$$

where

$$\phi_m(\xi) = \widetilde{\phi_m}(\xi')\,\overline{\phi_m}(\xi_\ell), \tag{10}$$

with $L_{\xi'}\widetilde{\phi_m}(\xi') = -\lambda_m\widetilde{\phi_m}(\xi')$, for some λ_m. The functions $\widetilde{\phi_m}$ may be taken to form an orthonormal basis of $L^2(\xi')$. Determination of the coefficients a_m depends on the given boundary data. We take $\overline{\phi_m}(A) = 0$ for all m, for the purpose of solving the problem posed in Section 2.3.

Some of the conditions imposed on the eigenvalues λ could either be changed significantly or relaxed completely. For a general treatment we would have to resort to abstract spectral properties of $L_{\xi'}$, but that is not our goal. We want to have as concrete a theory as possible, while still covering the most important practical examples.

2.3 The problem

We are interested in solving the following boundary value problem in the annular domain, Ω,

$$\text{problem } \mathrm{A}_0 \begin{cases} Lv = g, & x \in \Omega, \\ v\,|_{\Gamma_1} = \eta_1, & \\ v\,|_{\Gamma_2} = \eta_2, & \end{cases} \tag{11}$$

with $g \in C(\Omega)$ (say), $\eta_1 \in L^2(\Gamma_1)$ and $\eta_2 \in L^2(\Gamma_2)$. The choice to take our boundary data in $L^2(\Gamma_1)$ was deliberate. The proof of our main theorem is, however, sufficiently general for the latter to remain valid in more extensive settings. See Remark 3 immediately preceding Theorem 2.

The standard procedure is to find a particular solution $\tilde{v}$ of the equation $L\tilde{v} = g$, write $w = v - \tilde{v}$, and note that w solves the problem $Lw = 0$, $w\,|_{\Gamma_1} = \tilde{\eta}_1$,

$w\,|_{\Gamma_2}= \tilde{\eta_2}$, for some $\tilde{\eta_1},\ \tilde{\eta_2}$. The particular method for obtaining $\tilde{v}$ doesn't concern us here, and from now on we will simply take $g = 0$ in Problem A_0. For further convenience we will take η_2 to be zero as well, since it doesn't affect the argument. Problem A_0 becomes

$$\text{problem A}\ \begin{cases} Lv = 0, & x \in \Omega, \\ v\,|_{\Gamma_1}= \eta_1, & \\ v\,|_{\Gamma_2}= 0. & \end{cases} \tag{12}$$

Let Ξ be an extended domain for Ω, with inner boundary Γ_0 given by $0 < a = t_0(\xi') < \min t_1(\xi')$, and consider the problem

$$\text{problem B}\ \begin{cases} Lu = 0, & x \in \Xi, \\ u\,|_{\Gamma_0}= \eta_0, & \\ u\,|_{\Gamma_2}= 0, & \end{cases} \tag{13}$$

with $\eta_0 \in L^2(\Gamma_0)$. We will assume here that Problem B is well posed.

As outlined in Section 2.2, solutions to B have particularly good representations in series form. To solve Problem A given the information on g, Γ_1 and Γ_2, we require a boundary function η_0 defined on Γ_0 such that the solution u of Problem B, when restricted to Ω, is the solution of Problem A. That is, we require a η_0 such that

$$\begin{aligned} u\,|_{\Omega}&= v, \\ u\,|_{\Gamma_1}&= \eta_1. \end{aligned} \tag{14}$$

Essentially, our search involves the characterization of the operators

$$N : \eta_0 \mapsto \eta_1 \qquad \text{and its inverse} \qquad K : \eta_1 \mapsto \eta_0. \tag{15}$$

This characterization involves a description of their domains and boundedness properties. While not a limiting consideration, the characterization we follow here is in terms of series expansions in the space $L^2(\Omega')$.

2.4 Eigenfunction representation; operators N and K

Let V be the space of solutions to Problem B, for all $\eta_0 \in L^2(\Gamma_0)$. By construction, every $u \in V$ can be expressed as

$$u\,(x) = \sum_{m=1}^{\infty} a_m \phi_m\,(\xi) = \sum_{m=1}^{\infty} a_m \widetilde{\phi_m}(\xi')\,\overline{\phi_m}(\xi_\ell) \qquad x \in \Xi, \tag{16}$$

for some coefficients a_m to be determined. Here, x is identified by its ξ-coordinate representation. The boundary condition on Γ_0 gives us

$$u\mid_{\Gamma_0} = \eta_0\left(\xi'\right) = \sum_{m=1}^{\infty} a_m \phi_m\left(\xi', a\right) \equiv \sum_{m=1}^{\infty} a_m \overline{\phi_m}(a)\, \widetilde{\phi_m}(\xi'), \tag{17}$$

where we have taken advantage of the coordinate construction specifying Γ_0 to be the coordinate surface, $\xi_\ell = a$.

Equation (17) is a Fourier-like expansion for the boundary function $\eta_0 = \eta_0\left(\xi'\right)$ in terms of the orthonormal set $\left\{\widetilde{\phi_m}\right\}_{m=1}^{\infty}$, with coefficients $a_m\overline{\phi_m}(a)$. Recalling that $\Omega' \subset \mathbb{R}^{\ell-1}$ is the parameter space we are using to parametrize Γ_0, we write

$$\left\langle \eta_0, \widetilde{\phi_m} \right\rangle_{\Omega'} := a_m \overline{\phi_m}(a),$$

where $\langle\cdot,\cdot\rangle_{\Omega'}$ is an inner product defined over the subset Ω' of $\mathbb{R}^{\ell-1}$. In theory these values can be determined by integration once η_0 is specified. In practice, however, η_0 itself must be determined. This must be achieved using the remaining information regarding the given function η_1 on the boundary Γ_1 in Problem A. This takes the form

$$u\mid_{\Gamma_1} = \eta_1\left(\xi'\right) = \sum_{m=1}^{\infty} a_m \phi_m\left(\xi', t_1\left(\xi'\right)\right) = \sum_{m=1}^{\infty} \frac{\left\langle \eta_0, \widetilde{\phi_m} \right\rangle_{\Omega'}}{\overline{\phi_m}(a)} \phi_m\left(\xi', t_1\left(\xi'\right)\right). \tag{18}$$

Based on the Fourier-like interpretation of equation (17), we recognize that equation (18) defines a mapping $N : L^2\left(\Gamma_0\right) \to L^2\left(\Gamma_1\right)$, given explicitly as

$$\left(N\eta_0\right)\left(\xi'\right) = \sum_{m=1}^{\infty} \frac{\left\langle \eta_0, \widetilde{\phi_m} \right\rangle_{\Omega'}}{\overline{\phi_m}(a)} \phi_m\left(\xi', t_1\left(\xi'\right)\right) = \eta_1\left(\xi'\right). \tag{19}$$

The operator K, introduced in (15), would then effectively be identified as $K \equiv N^{-1}$.

Boundedness for the operator N is a consequence of our assumption that Problem B is well-posed and of the maximum principle applied to L (see [34]): if $(\eta_0)_k \geq 0$ is a sequence of boundary data converging to 0 in $L^2(\Gamma_0)$, then the corresponding solution u_k to Problem B converges to 0 uniformly on compact subsets of Ω. Since $(\eta_1)_k$ is the restriction to Γ_1 of u_k, we conclude that $(\eta_1)_k$ too goes to 0 in $L^2(\Gamma_1)$. Hence, N is bounded. The range of N is a much smaller subspace of $L^2(\Gamma_1)$; for example, as solutions to Problem B must be smooth, then η_1 must be at the very least continuous (in fact, its differentiability properties will be inherited from those of Γ_1 itself).

In general, operator K is discontinuous but, in the functional analysis sense, it is a closed operator. This can be argued directly from (18) and (19), but we will leave the discussion here in favor of investigating an explicit problem in the next section.

The fundamental proposal of the EDEM is to effect the inversion of equation (19) and extract the coefficients of the Fourier-like expansion of η_0. Having obtained the expansion coefficients for η_0, the solution of Problem A, equation (12), is obtained by restricting the eigenfunction expression, equation (16), to the domain Ω. One practical means of inverting the mapping N is described in a follow-up paper.

3 The extendability of solutions for the Laplace operator

Various approaches can be followed to analyze the invertibility of the operator N introduced in Section 2.3. The approach we take here is to address explicitly the equivalent question of extendability of Problem A for the Laplace operator in $\mathbb{R}^2$. The theory for the Laplace operator in $\mathbb{R}^n$, in particular for $n = 2$, already contains most of the features of the general case. As we develop this example, we shall point out differences from the general theory.

By scaling the domain if necessary, our Ω is bounded on the outside by the unit circle $\Gamma_2 = \{|x| = 1\}$. We adopt polar coordinates so that $\xi_1 = \xi' = \theta$, and $\xi_2 = \xi_\ell = r$. The inner boundary Γ_1 will be represented by the form $r = t(\theta) < 1$ where t is a given continuous, piecewise differentiable function, viewed either as defined on $\Gamma_1 \subset \mathbb{R}^2$, or on the parameter space $\Omega' = [0, 2\pi]$, where we identify the endpoints.

Take $0 < a < \min_\theta t(\theta)$. The solution to the Dirichlet problem on Ξ

$$\text{Problem B} \begin{cases} \Delta u(x) = 0; & 0 < a < |x| < 1; \\ u(x) = 0; & |x| = 1; \\ u(x) = \eta_0(\theta), & |x| = a, \end{cases} \tag{20}$$

where $\eta_0 \in L^2([0, 2\pi])$, may be written in polar coordinates as

$$u(r, \theta) = \sum_{n=-\infty}^{\infty} a_n(r) e^{in\theta}, \tag{21}$$

where

$$\begin{aligned} a_0(r) &= \frac{\widehat{\eta}_{0,0}}{\log a} \log r; \\ a_n(r) &= \frac{\widehat{\eta}_{0,n}}{a^n - a^{-n}} (r^n - r^{-n}), \quad n \neq 0, \end{aligned}$$

and the $\widehat{\eta}_{0,n}$ are the Fourier coefficients of the boundary data η_0:

$$\eta_0(\theta) = \sum_{n=-\infty}^{\infty} \widehat{\eta}_{0,n} e^{in\theta}. \tag{22}$$

It is well-known [2] that for any $\eta_0 \in L^2([0, 2\pi])$ there exists a unique function u such that $\Delta u = 0$ inside the annulus, u is continuous on $a < |x| \leq 1$, $u(x) = 0$ when $|x| = 1$, and such that

$$\lim_{r \to a^+} \int_0^{2\pi} |u(r, \theta) - \eta_0(\theta)|^2 \, d\theta = 0.$$

In particular, viewed as a function of θ for each r, the function $u(r, \cdot)$ converges to η_0 in $L^2([0, 2\pi])$, as r approaches a. Expression (21) gives an explicit formula for this solution u.

Now, consider the problem defined on $\Omega \subset \mathbb{R}^2$

$$\text{Problem A} \begin{cases} \Delta v(x) = 0; & 0 < t(\theta) < |x| < 1; \\ v(x) = 0; & |x| = 1; \\ v(x) = \eta_1(\theta), & |x| = t(\theta), \end{cases} \tag{23}$$

where η_1 is in $L^2([0, 2\pi])$. While problem (23) also has a unique solution [2], closed form expressions for such a solution are in general not forthcoming.

Definition 1. Let Ω be the annular domain bounded by $|x| = t(\theta)$ and $|x| = 1$, and Ξ its extended domain, as above. We define $E_a \subset L^2(\Gamma_1)$ to be the range of the operator N given in (19). In other words, $\eta_1 \in E_a$ if and only if there is some $\eta_0 \in L^2([0, 2\pi])$ such that whenever u solves (20) with data η_0, and v solves (23) with data η_1, then u agrees with v over Ω.

The set E_a contains boundary data for which solutions to (23) are *extendable* all the way down to radius a. Indeed, we will call such an a, η_1 *extendable*, or *a-extendable*. The main issue is to identify the conditions that extendable functions need to satisfy. We now prove the following principal result.

Theorem 1. *The boundary data η_1 is extendable if and only if we can write*

$$\eta_1(\theta) = \frac{c_0}{\log a} \log t(\theta) + \sum_{n \neq 0} \frac{c_n}{a^n - a^{-n}} \left(t^n(\theta) - t^{-n}(\theta)\right) e^{in\theta}, \tag{24}$$

where $c_n \in \mathbb{C}$ are such that $\sum_n |c_n|^2 < \infty$. In this case the function $\eta_0 = K\eta_1$ is such that $\widehat{\eta}_{0,n} = c_n$ for all n.

In other words, η_1 is extendable if and only if η_1 is in the linear span of the functions

$$\varphi_0(\theta) = \frac{\log t(\theta)}{\log a}; \tag{25}$$

$$\varphi_n(\theta) = \frac{t^n(\theta) - t^{-n}(\theta)}{a^n - a^{-n}} e^{in\theta}, \qquad n \neq 0, \quad n \in \mathbb{Z}, \tag{26}$$

with coefficients c_n that are square-summable.

Again, note that any solution to $\Delta u = 0$ on $a < |x| < 1$ must be smooth [4]; this shows that the data η_1 must be restricted somehow.

The proof of Theorem 1 will be broken up into two claims. First, we consider the necessary condition.

Claim 1. If $\eta_1 \in E_a$ it can be represented as

$$\eta_1(\theta) = \sum_{n=-\infty}^{\infty} c_n \varphi_n(\theta)$$

where $\sum_n |c_n|^2 < \infty$, with the series converging uniformly.

Proof. Suppose η_1 is extendable. By definition, there is some u given by (21) which, when restricted to Ω, provides the solution to (23). This u is necessarily smooth over Ξ, and series (21) converges uniformly on compact subsets of Ξ. If we evaluate u over the curve $r = t(\theta)$, we obtain (24), with $c_n = \widehat{\eta}_{0,n}$. This proves the claim. □

We consider next sufficiency.

Claim 2. Suppose η_1 is given by

$$\eta_1(\theta) = \sum_{n=-\infty}^{\infty} c_n \varphi_n(\theta),$$

where $\sum_n |c_n|^2 < \infty$. Then

1. $\eta_1 \in L^2([0, 2\pi])$ (in fact, η_1 is continuous);
2. Problem (23) admits a solution v, continuous on $\overline{\Omega}$, and $v(x) = \eta_1(\theta)$ for x on Γ_1;
3. There is a function $\eta_0 \in L^2([0, 2\pi])$ such that the solution u to Problem (20) coincides with v on Ω.

Proof. Let b be such that $0 < a < b = \min_\theta t(\theta) \le t(\theta) \le \max_\theta t(\theta) < 1$. We will write t for $t(\theta)$. For $n > 0$ we have

$$\begin{aligned} \left| \frac{t^n - t^{-n}}{a^n - a^{-n}} \right| &= \frac{t^{-n}}{a^{-n}} \left| \frac{1 - t^{2n}}{1 - a^{2n}} \right| \\ &\le \left(\frac{a}{t} \right)^n \left| \frac{1}{1 - a^{2n}} \right| \le 2 \left(\frac{a}{b} \right)^n, \end{aligned}$$

the last inequality is valid for sufficiently large n. A similar estimate can be constructed for $n < 0$. We conclude that the series defining η_1 converges uniformly. Moreover, as a uniform limit of continuous functions, we conclude that η_1 is continuous. This proves Points 1 and 2.

Now define

$$\eta_0(\theta) = \sum_{n=-\infty}^{\infty} c_n e^{in\theta}.$$

Given the condition on the coefficients c_n, we conclude that $\eta_0 \in L^2([0, 2\pi])$, and $c_n = \widehat{\eta}_{0,n}$. Use this η_0 as boundary data in (20); then the solution u must be given by (21), with the series converging uniformly in compact subsets of Ξ. Restricting u to Γ_1, we find that

$$u(\theta, t(\theta)) = \sum_{n=-\infty}^{\infty} \widehat{\eta}_{0,n} \varphi_n(\theta) = \eta_1(\theta).$$

We conclude that η_1 is extendable, finishing the proof. $\square$

Remarks. 1) We really only needed to show Point 3, since Points 1 and 2 are consequences of Point 3. However, we have shown a little bit extra in Points 1 and 2, namely that the terms in the series for η_1 have exponential decay.

2) Except for Points 1 and 2, we have not used any features that are specific to the Laplace operator in two dimensions. Thus, this proof generalizes to various other settings.

3) We defined the space $E_a([0, 2\pi])$ by requiring that the extension u has boundary value η_0 in $L^2([0, 2\pi])$. More generally, let $s > 0$ be a real number and define $E_{a,s}([0, 2\pi])$ to be the space of functions, η_1, such that η_0 is in the Sobolev space $H^s([0, 2\pi])$. (So, $E_a = E_{a,0}$.) If η_0, given by (22), is in H^s, then the coefficients c_n must have at most polynomial growth in n. Consequently, expression (21) is a classical solution of Problem B. Thus, the generalised version of Theorem 1 remains valid. That is, $\eta_1 \in E_{a,s}([0, 2\pi])$ if and only if η_1 is in the span of the φ_n, with coefficients satisfying the growth conditions for $H^s([0, 2\pi])$. It should also be pointed out that for $\eta_0 \in H^s([0, 2\pi])$, the sense in which η_0 is the boundary value of u is that $u(r, \cdot) \to \eta_0(\cdot)$ in the topology of $H^s([0, 2\pi])$, as $r \to a$. See also Remark 3 at the conclusion of Section 4.

The following theorem establishes the important fact that not all continuous and differentiable functions are in E_a for any a.

Theorem 2. *Suppose t has m continuous derivatives, where $m \in \{0,\ 1, \ldots, \infty\}$. Then there is a function $\eta_1 \in C^m(\Gamma_1)$ such that for all a with $0 < a < b = \min_\theta t(\theta)$, we have $\eta_1 \notin E_a$.*

Proof. Suppose m is finite. Define $d_0 = 0$, and for $n \neq 0$,

$$d_n = \frac{b^{|n|}}{|n|^{m+2}}.$$

Then

$$|d_n(t(\theta)^n - t(\theta)^{-n})| \leq \frac{1}{|n|^{m+2}}.$$

Since $m + 2 \geq 2$, the series

$$\eta_1(\theta) = \sum_{n \neq 0} d_n(t(\theta)^n - t(\theta)^{-n})e^{in\theta}$$

converges uniformly. We conclude that η_1 is continuous. The first derivative of the series expression for η_1 gives

$$\sum_{n \neq 0} d_n \cdot n \left[\left(\frac{t'(\theta)}{t(\theta)} \right) (t(\theta)^n + t(\theta)^{-n}) + i \right] e^{in\theta}.$$

which converges uniformly and so must equal $\eta_1'(\theta)$. Proceeding inductively, after m derivatives we obtain the series

$$\sum_{n\neq 0} d_n \cdot n^m \cdot H(n,m,\theta)\, e^{in\theta},$$

where H is a function coming from repeated use of the product rule. It can easily (but laboriously) be verified that $b^{|n|}H$ is bounded, and so this series represents $\eta_1^{(m)}(\theta)$, the m-th derivative of η_1. Hence, η_1 is of class C^m.

Suppose that for this η_1 we could extend the solution u to (23) to an annulus of inner radius $a < b$; that is, suppose $\eta_1 \in E_a$. Writing $c_n = d_n(a^n - a^{-n})$, we see that $\eta_1(\theta) = \sum_n c_n \varphi_n(\theta)$, and the coefficients c_n would have to be square-summable. But since $a < b$, it can easily be verified that c_n has exponential growth. We conclude that η_1 is not in E_a.

If $m = \infty$, we choose a sequence $k_n > 0$ that decays faster than any polynomial, but slower than any exponential, meaning that as $|n| \to \infty$ we have $k_n p(n) \to 0$ for any polynomial p, but $k_n c^{|n|} \to \infty$ for any $c > 1$. We set $d_n = b^{|n|}k_n$, and proceed inductively to check that η_1 has derivatives of all orders. (This is where we need k_n decaying faster than any polynomial.) On the other hand, for the problem to extend to $a < b$ we would need $k_n(b/a)^{|n|}$ to be square-summable, which it isn't. $\square$

As implied at the end of Section 2.4, E_a is a vector subspace of $L^2([0,2\pi])$ but *it is not a closed subspace*. Take, for example, the η_1 discussed in the preceding proof truncated to level N:

$$\eta_{1,N}(\theta) = \sum_{0<|n|\leq N} \frac{b^{|n|}}{|n|^{m+2}}(t(\theta)^n - t(\theta)^{-n})e^{in\theta}.$$

Then, $\eta_{1,N} \in E_a$ (since this is a finite sum), and $\eta_{1,N} \to \eta_1$ in L^2 as $N \to \infty$, but η_1 itself is not in E_a. In particular, we see that $K = N^{-1}$ is an unbounded operator. This indicates that the problem of finding η_0 given η_1 is ill-posed. The resolution of this issue is important to the numerical implementation of the EDEM and will therefore be discussed in a future numerical publication.

4 Density property of the set E_a

Theorem 1 gives an exact characterization of the extendable functions, while Theorem 2 shows us that even some very well-behaved functions may not be extendable at all (even if Γ_1 is a *circle*, there are *smooth* η_1 that are non-extendable).

What is important to establish now is the proposition that E_a is a dense subspace of $L^2 = L^2([0, 2\pi])$. If so, then any solution to problem (23) with data $\eta_1 \in L^2$ can be approximated by a solution to problem (20), for some $\eta_0 \in L^2$. This is the goal of EDEM. If the assertion were false, then EDEM would necessarily fail if we chose $\eta_1 \notin \overline{E}_a$, where $\overline{E}_a$ is the closure set of E_a in L^2.

Under the conditions and notation laid out in the preceding sections for problem (20) and problem (23), we establish the following theorem.

Theorem 3.

$$\overline{E}_a = L^2.$$

Proof. First assume that η_1 is in fact continuous. The idea of the proof is to obtain a harmonic function, defined on the punctured disc $0 < |x| < 1$, which approximates η_1 uniformly over Γ_1. The main claim then follows, because the continuous functions are dense in L^2.

Our proof uses Runge's Approximation Theorem from complex analysis (see [35]). In order to apply that theorem, we need first to enclose Ω in a larger domain.

Let λ be a parameter chosen close to 1, say $0 < \lambda \lessapprox 1$, and consider two auxiliary curves, Γ_λ and $\Gamma_{1/\lambda}$, defined by

$$\Gamma_\lambda = \lambda \Gamma_1, \qquad \Gamma_{1/\lambda} = \{|x| = \frac{1}{\lambda}\}.$$

In other words, Γ_λ is a slightly contracted version of Γ_1, and $\Gamma_{1/\lambda}$ is a circle of radius $1/\lambda > 1$. On Γ_1 we place boundary data η_1, *which we take to be continuous.* We transfer data η_1 from Γ_1 to Γ_λ, as follows:

$$\eta_{1,\lambda}(x) = \eta_1(x/\lambda), \quad x \in \Gamma_\lambda.$$

We define $\Omega_\lambda \supset \Omega$ to be the annular domain between Γ_λ and $\Gamma_{1/\lambda}$ and let w be the (unique) solution to the following harmonic problem in Ω_λ:

$$\begin{cases} \Delta w(x) = 0, & x \in \Omega_\lambda; \\ w(x) = \eta_{1,\lambda}(x), & x \in \Gamma_\lambda; \\ w(x) = 0, & x \in \Gamma_{1/\lambda}. \end{cases}$$

Observing that Γ_1 is contained in the open domain Ω_λ, we denote by w_Γ the restriction of w to Γ_1. We denote by w_0 the restriction of w to $|x| = 1$ (also contained in Ω_λ).

Clearly, by taking the limit $\lambda \to 1$, w_Γ becomes arbitrarily close to η_1 in the supremum norm (since η_1 is continuous). Simultaneously, w_0 gets arbitrarily close to 0 in the supremum norm. More rigorously, given $\varepsilon > 0$ we can choose λ sufficiently close to 1 such that for all θ we have

$$|\eta_1(\theta) - w(\theta, t(\theta))| < \varepsilon \quad \text{and} \quad |w(\theta, 1)| < \varepsilon. \tag{27}$$

At this stage, we need the following result.

Lemma 1. *There exists a real constant α, and a* ***holomorphic*** *function H defined on the annular domain Ω_λ, such that for all $x \in \Omega_\lambda$ we have*

$$w(x) = \alpha \log |x| + \Re H(x).$$

Here $\Re H(x)$ is the real part of H at x. A short proof of this fact can be found in the Appendix. Note that, if $|x| = 1$, then $|\Re H(x)| = |w(x)| < \varepsilon$.

By shrinking Ω_λ if necessary (but still containing Γ_1 and $|x| = 1$), we may assume that H is bounded on Ω_λ. In the same way we did for w, we denote by H_Γ the restriction of H to the curve Γ_1 and by H_0 the restriction of H to $|x| = 1$.

Finally, we invoke Runge's theorem (see [35]). The statement used here is adapted to our purposes.

Theorem 4 (Runge's Theorem). *Given $\varepsilon > 0$, there is a* ***rational function*** *$R(x)$, with poles only at $x = 0$ and $x = \infty$, such that for all $x \in \Gamma_1$, and all x with $|x| = 1$, we have*

$$|H(x) - R(x)| < \varepsilon.$$

Runge's theorem allows us to approximate H on Γ_1 and on $|x| = 1$, simultaneously, by a rational function with poles at 0 and ∞. Note that the approximation does not necessarily extend to the entire domain Ω_λ. With even more reason, the conclusion of the theorem holds for the real parts of H and R. With $q = \Re R$ and $h = \Re H$ we have, for all θ,

$$|q(\theta, t(\theta)) - h(\theta, t(\theta))| < \varepsilon \quad \text{and} \quad |q(\theta, 1) - h(\theta, 1)| < \varepsilon. \tag{28}$$

Note that $q = \Re R$ is defined on the punctured plane $\mathbb{R}^2 \setminus \{0\}$; in particular it is defined on $0 < |x| \le 1$. We compute

$$\begin{aligned} |q(\theta, 1)| &= |q(\theta, 1) - h(\theta, 1) + h(\theta, 1)| \\ &\le |q(\theta, 1) - h(\theta, 1)| + |h(\theta, 1)| < \varepsilon + \varepsilon = 2\varepsilon. \end{aligned} \tag{29}$$

Consider now the function $s = s(x)$ defined on $|x| \leq 1$, given by $\Delta s(x) = 0$ when $|x| < 1$, and $s(\theta, 1) = q(\theta, 1)$. Because of the maximum principle we have, for all $|x| \leq 1$, that

$$|s(x)| \leq \max_{\theta} |q(\theta, 1)| < 2\varepsilon.$$

Finally, consider the function $W(x) = \alpha \log |x| + q(x) - s(x)$. This function is defined on the punctured disc $0 < |x| \leq 1$, and it is harmonic on $0 < |x| < 1$. Moreover, $W(\theta, 1) = 0$, from the definition of s. Writing $x = (\theta, t(\theta)) \in \Gamma_1$, we estimate $|W(x) - \eta_1(\theta)|$.

$$\begin{aligned}
|W(x) - \eta_1(\theta)| &\leq |\alpha \log |x| + q(x) - \eta_1(\theta)| + |s(x)| \\
&\leq |\alpha \log |x| + h(x) - \eta_1(\theta)| + |q(x) - h(x)| + 2\varepsilon \\
&\leq |w(x) - \eta_1(\theta)| + \varepsilon + 2\varepsilon < 4\varepsilon.
\end{aligned}$$

We conclude that any continuous η_1 is arbitrarily close (in the uniform norm) to a function in E_a. Therefore, the closure of E_a in L^2 contains the continuous functions. Since the continuous functions form a dense set in L^2, we see that $L^2 \subset \overline{E}_a$. This concludes the proof. □

Remarks. 1) Because we used Runge's theorem, this proof is only valid in two dimensions. However, the result is valid for the Laplace operator in higher dimensions. The proof would then use the results contained in [36] to obtain the existence of an approximating function to replace our h. Our proof is more direct; in particular it shows a connection between the method proposed here and classical questions in other fields.

2) The question of whether the set of functions $\{\phi_m(\xi', t_1(\xi'))\}$ (see (19)) span L^2 is, of course, too difficult to solve in general, and the best we can do is to give sufficient conditions on $\phi_m(\xi', t_1(\xi'))$ for that to happen. We will develop this theme in a future publication, in connection with (2) and (4)-(6).

3) We have shown that $E_a([0, 2\pi])$ is dense in $L^2([0, 2\pi])$. Therefore, any $\eta_1 \in L^2$ is the limit of a sequence of functions in $E_a([0, 2\pi])$. In addition, it is well established that $L^2([0, 2\pi])$ is dense in $H^s([0, 2\pi])$ for $s < 0$, and contains $H^s([0, 2\pi])$ for $s \geq 0$. It is then an easy corollary that any function $\eta_1 \in L^2([0, 2\pi])$ may be approximated arbitrarily closely by a function in $E_{a,s}([0, 2\pi])$. It follows from our proof that if η_1 is continuous, then η_1 may be approximated uniformly by functions in $E_a([0, 2\pi])$ and $E_{a,s}([0, 2\pi])$.

5 EDEM for the Laplace operator on an annular domain

We now present a numerical implementation of the proposed Extended-Domain-Eigenfunction Method (EDEM) based on one means of inverting the operator N introduced above. The method we have implemented is not unique, but is straightforward and reliable. Coincidentally, its basis is the same as the numerical technique implemented in some of the Trefftz papers (see for example Liu [30]). For simplicity, we focus on the Dirichlet problem in $\mathbb{R}^2$ for the Laplace operator and consider a boundary value problem involving a circular outer boundary ($\Gamma_2 : r = t_2(\theta) = b$) and a square inner boundary, Γ_1. We shall restrict our numerical study to the case where the boundary data, η_1 and η_2, are continuous.

The square inner boundary is chosen to demonstrate that the EDEM gives good results even with complicated domains involving boundary corners. It is interesting to note that this combination of boundary shape and boundary data invalidates the definition of K since the function η_1 on the square boundary is not in any $E_a([0, 2\pi])$. Remarkably, despite this fact the EDEM can still be used to find a solution to this problem.

In a subsequent paper [37], we further illustrate the implementation of the EDEM using the Helmholtz operator, and compare the results with those obtained using standard boundary element methods (BEM) (for reasons of space, these comparisons have not been included here). The high accuracy and superior speed of the EDEM make it an excellent candidate as an alternative to the BEM.

5.1 Implementation of the EDEM for the Laplace Equation

We consider a Dirichlet problem for the Laplace equation defined in the region $\Omega \subset \mathbb{R}^2$. Both the outer circular boundary Γ_2 and the inner square boundary Γ_1 are centered at $(0, 0)$. The outer boundary has a prescribed value of $\eta_2 = 0$, while the inner boundary value is defined by the given function η_1.

Let problem B be defined on the extended domain $\Xi \supset \Omega$ bounded outside by the circular boundary Γ_2 of radius b and bounded inside by a circular boundary Γ_0 of radius a, with boundary value, η_0,

$$\text{problem B} \begin{cases} \Delta u = 0, & x \in \Xi, \\ u\mid_{\Gamma_0} = \eta_0, & \\ u\mid_{\Gamma_2} = 0, & \end{cases} \tag{30}$$

The most general solution to problem B satisfying homogeneous Dirichlet conditions on Γ_2 can be obtained by the separation of variables technique. The solution takes the form

$$u(r,\theta) = a_0 \log(r/b) + \sum_{m=1}^{\infty} (r^{-m} - r^m/b^{2m})(A_m \cos(m\theta) + B_m \sin(m\theta)) \tag{31}$$

with coefficients $\{a_0, \{A_m, B_m\}_{m=1}^{\infty}\}$ determined upon application of the boundary condition at Γ_0, as suggested by equation (17).

Inverting the operator N can be affected by first truncating the above expansion to a finite sum approximation

$$u(r,\theta) = a_0 \log(r/b) + \sum_{m=1}^{M} (r^{-m} - r^m/b^{2m})(A_m \cos(m\theta) + B_m \sin(m\theta)) \tag{32}$$

and then by considering a finite number, $2M+1$, of points on the inner boundary, Γ_0. Let us denote these by $\{y_j = (\theta_j, a)\}_{j=1}^{2M+1}$. By assumption, this identifies $2M+1$ unique points, $\{x_j = (\theta_j, t_1(\theta_j))\}_{j=1}^{2M+1}$, on the original boundary Γ_1 corresponding to those points chosen on Γ_0.

Imposing the boundary condition $u\mid_{\Gamma_1} = \eta_1$ on (32) at those $2M+1$ points generates $2M+1$ equations in terms of the unknown coefficients $\left\{a_0, \{A_m, B_m\}_{m=1}^{M}\right\}$, which can be written as the matrix equation $Az = B$, where

$$A = \begin{bmatrix} \log\left(\frac{t_1(\theta_1)}{b}\right) & \alpha_{11} & \cdots & \alpha_{M1} & \beta_{11} & \cdots & \beta_{M1} \\ \log\left(\frac{t_1(\theta_2)}{b}\right) & \alpha_{12} & \cdots & \alpha_{M2} & \beta_{12} & \cdots & \beta_{M2} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \log\left(\frac{t_1(\theta_{2M+1})}{b}\right) & \alpha_{1\ 2M+1} & \cdots & \alpha_{M\ 2M+1} & \beta_{1\ 2M+1} & \cdots & \beta_{M\ 2M+1} \end{bmatrix} \tag{33}$$

with

$$\alpha_{ij} = f_i(t_1(\theta_j)) \cos(i\theta_j), \qquad \beta_{ij} = f_i(t_1(\theta_j)) \sin(i\theta_j),$$

and $f_i(t_1(\theta_j)) = t_1^{-i}(\theta_j) - t_1^{i}(\theta_j)/b^{2i}$ for $i = 1, \cdots, M$, and where

$$z = \begin{bmatrix} a_0 \\ A_1 \\ \vdots \\ A_M \\ B_1 \\ \vdots \\ B_M \end{bmatrix}, \qquad B = \begin{bmatrix} \eta_1(t_1(\theta_1), \theta_1) \\ \eta_1(t_1(\theta_2), \theta_2) \\ \vdots \\ \eta_1(t_1(\theta_{2M+1}), \theta_{2M+1}) \end{bmatrix}. \tag{34}$$

The set of unknown coefficients are found by solving this system for the vector z by, say, Gaussian elimination. The solution within the domain Ω is then given by direct application of equation (32).

We note in passing that the above means of inverting N is not unique. Other approaches will be discussed in a follow-up paper.

5.2 Results

The square is centered on the origin with side length 2 and boundary value $\eta_1 = 50$. The outer boundary is a circle of radius $r = 3$, centered on the origin with boundary value $\eta_2 = 0$.

To simplify the calculations, we have exploited the symmetry of the problem. We focus on the sub-domain between angles $5\pi/4$ and $7\pi/4$ and apply a zero flux condition across the radial lines defining the extremes of this sub-region. The required derivative, $\partial v/\partial\theta$, derived directly from equation (32), will introduce a set of additional equations that can easily be incorporated into the matrix equation.

The problem is solved using the EDEM with 81 inner boundary points and 10 points along each radial line. The results of evaluating equation (32) are shown in Figure 2.

Qualitatively, the sequence of curves in Figure 2 is consistent with the behaviour expected of a solution to Laplace's equation, with the level curves becoming closer together as the inner boundary is approached, indicating a higher rate of change closer to that boundary. Moreover, the inner-most contours take on the shape of the square inner boundary and overall, the results are symmetric (within the subdomain $5\pi/4$ and $7\pi/4$).

Comparison of this solution with one obtained using the BEM, shows that solutions with a higher accuracy and greater symmetry are obtained with greater

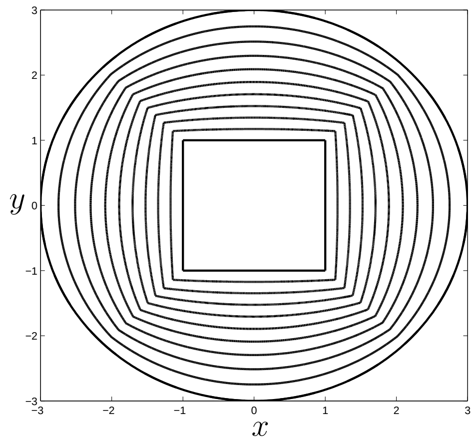

Figure 2: Contour (iso-potential) plots of the EDEM solution to the Dirichlet problem with a square inner boundary. Contours are evaluated in increments of 5 units, with inner contour at $v = 50$ and outer contour at $v = 0$.

speed and by using fewer boundary points with the EDEM.

6 Summary remarks

The necessity of providing efficient algorithms for the solution of linear partial differential equations for systems involving complicated boundaries is without dispute. Apart from full numerical approaches, analytical methods of solution are often restricted to domains that have little or no departure from simple geometries. By extending a complicated annular-like domain to one which has simple boundaries, we can use the classic eigenfunction solution approach as a means of producing a solution that can be written in a well recognised form. By restricting this solution to the original domain, we can obtain the solution to the original problem. This idea and the resulting solution algorithm has been referred to here as the Extended-Domain-Eigenfunction Method (EDEM). The Trefftz method is the methodology that lies closest to ours. However, our theoretical introduction to EDEM is a general argument and encompasses more than any one numerical implementation.

In a separate report [37], we illustrate the accuracy and efficiency of the EDEM in comparison with the well-known boundary element method (BEM). We have found that the numerical implementation of EDEM is able to produce accurate estimates of solutions to EBVPs, consistent with those found with BEM, but does so much more rapidly.

In future work, we will extend the development and analysis of this theory on several fronts. We aim to provide a rigorous discussion for more general elliptic differential operators on the theoretical questions surrounding EDEM. This includes the existence and uniqueness of operators K and N and conditions for which the known boundary data η_1 on Γ_1 and Γ_1 itself must satisfy to ensure that new data η_0 on the new boundary Γ_0 can be found. A numerical analysis of the method, including an investigation of the ill-posed nature of solving equation (19) for η_0, will be presented. We will also investigate the possibility of using the EDEM to solve problems involving nonlinear operators (*e.g.*, the nonlinear Poisson-Boltzmann operator [38]). An elliptic system of equations for a vector function appropriate for the Stokes problem, equations (4)-(6), on axisymmetrical and asymmetrical domains will be examined.

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Appendix

Proof of Lemma 1. Let's write $z = x + iy$, with x, y real. The function $\log|z|$ is the real part of the (multi-valued) complex function $\operatorname{Log} z = \log|z| + i \arg z$. We define a (possibly multi-valued) harmonic conjugate to w, as follows. Fix $z_0 \in \Omega_\lambda$, and for each $z \in \Omega_\lambda$, let $C \subset \Omega_\lambda$ be a smooth curve starting at z_0 and ending at z. We define

$$\widetilde{w}(z) = \int_C -w_y\, dx + w_x\, dy.$$

Note that by taking C to be the constant curve we get $\widetilde{w}(z_0) = 0$.

It is easy to see that w and $\widetilde{w}$ satisfy the Cauchy-Riemann equations. The definition of $\widetilde{w}$ may depend on the chosen C, but only up to a point: if C_1 is another curve starting at z_0 and ending at z, and if $C - C_1$ contracts to a point in Ω_λ, then they define the same value for $\widetilde{w}$, as can be readily seen by using Green's theorem.

Suppose now that C starts at z_0, ends at z_0, and winds once counterclockwise around the origin. Denote by $k = \widetilde{w}(z_0)$ the value obtained by adopting this curve C. Since C does not contract to a point, we may have $k \neq 0$. Now notice that the harmonic function $\arg z$ increases its value by 2π when we go once around the origin along C. Therefore we choose $\alpha = \frac{k}{2\pi}$, and define

$$H(z) = [w(z) - \alpha \log |z|] + i[\widetilde{w}(z) - \alpha \arg(z)].$$

It is clear that the real and imaginary parts of H are each continuous, not multi-valued, and satisfy the Cauchy-Riemann equations. $\square$

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