Journal of Advanced Research (2010) **1**, 227–232

Cairo University

Journal of Advanced Research

ORIGINAL ARTICLE

Numerical experiments using CHIEF to treat the nonuniqueness in solving acoustic axisymmetric exterior problems via boundary integral equations

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Received 13 May 2009; received in revised form 4 July 2009; accepted 15 November 2009
Available online 26 June 2010

KEYWORDS

Helmholtz equation;
Boundary integral equations;
Acoustic radiation and scattering;
Nonuniqueness;
CHIEF method

Abstract The problem of nonuniqueness (NU) of the solution of exterior acoustic problems via boundary integral equations is discussed in this article. The efficient implementation of the CHIEF (Combined Helmholtz Integral Equations Formulation) method to axisymmetric problems is studied. Interior axial fields are used to indicate the solution error and to select proper CHIEF points. The procedure makes full use of LU-decomposition as well as the forward solution derived in the solution. Implementations of the procedure for hard spheres are presented. Accurate results are obtained up to a normalised radius of ka = 20.983, using only one CHIEF point. The radiation from a uniformly vibrating sphere is also considered. Accurate results for ka up to 16.927 are obtained using two CHIEF points.

Introduction

Surface integral equation (SIE) treatment of exterior acoustic problems reduces the dimension of the problem by one and provides a direct implementation of the radiation and boundary conditions. However, the solution of SIEs is not unique at internal resonances [1]. Methods to modify or reformulate the solution procedures to ensure uniqueness over a range of wavenumbers or at all wavenumbers have been a topic of much theoretical and practical interest [2].

One of the main methods used to overcome the problem is the Combined Helmholtz Integral Equation Formulation (CHIEF) according to Schenck [3]. In addition to the N SIEs (N is the number of unknowns), Schenck imposed the internal field equations (CHIEF equations) at n points (CHIEF points). Potential problems with this approach include the choice of appropriate interior points and solving the resulting overdetermined system.

In this article we consider axisymmetric bodies and study a systematic and efficient procedure to select the interior points and augment the SIEs to solve the nonuniqueness (NU) problem. The efficient solution of the resulting system of equations is also addressed. Interior axial fields are used to indicate the solution error and to select proper CHIEF points based on their relative high value. The procedure makes full use of LU-decomposition as well as the forward solution derived in the solution. Thus if n CHIEF points are used, using Lagrange multipliers the overdetermined system of equations reduces to a square system of order $(N+n)$ and requires

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doi:10.1016/j.jare.2010.05.006

the evaluation of additional n rows of L and n columns of U. The solution of the new system utilises the previously computed forward solution which is common to both systems.

Tests are given for plane wave scattering by a hard sphere using Fredholm integral equations of the second kind. Accurate results are obtained up to ka = 20.983 using only one CHIEF point. The radiation from a uniformly vibrating sphere is also considered. Accurate results for ka up to 16.927 are obtained using two CHIEF points.

Integral representations of the solution

Let V_{i} denote a bounded domain in $\Re^3$ with a boundary Σ which is a closed surface. We denote the exterior of Σ by V_{o}. A suppressed time variation in the form $\exp(-i\omega t)$ is assumed. It is convenient to introduce the following notations:

$$S\{\phi\} \equiv \int_{\Sigma} \phi(q) G(p,q) ds_q \tag{1a}$$

$$D\{\phi\} \equiv \int_{\Sigma} \phi(q)\ \partial_{nq} G\ \ (p,q)\ \ ds_q \tag{1b}$$

$$K\{\phi\} \equiv \partial_{nq} S\{\phi\} \tag{1c}$$

$$M\{\phi\} \equiv \partial_{nq} D\{\phi\} \tag{1d}$$

where ∂_n denotes outward normal derivative in $\Re^3$, $G = \exp(ikR)/(4\pi kR)$ is the free space Green's function, R is the distance between a field and a source point. The capital P denotes a field point, while (q,p) denote a surface integration point and a general surface point respectively. $S\{.\}$ and $D\{.\}$ are the single and double layer operators, respectively.

Let U denote the scalar potential, applying Green's second identity we get the Helmholtz integral formula (HIF) [4]:

$$D\{u\} - S\{v\} = \begin{cases} U & P \in V_{\mathrm{o}} \\ c(p)u & P \in \Sigma \\ 0 & P \in V_{\mathrm{i}} \end{cases} \tag{2}$$

where u is the surface value of U, $v = \partial_n u$ and $c(p)$ is given by:

$$c(p) = 1 + \int_{\Sigma} \frac{\partial_{nq}(1/R) ds_q}{4\pi} \tag{2a}$$

This includes the possibility that the surface Σ may have a nonsmooth geometry at edges and corners. At smooth points $c(p) = 0.5$.

Upon invoking the appropriate boundary condition, we are led to an integral equation in the surface wave potential or its normal derivative. Thus for the Dirichlet problem with $u = f$ on Σ we have:

$$[c(p)I + K]\{v\} = \partial_{np}\{f\} \tag{3}$$

while for the Neumann problem with $v = g$, we have:

$$[-c(p)I + D]\{u\} = S\{g\} \tag{4}$$

For the case of scattering of a potential field U^i incident on a smooth Σ, we may write:

$$U = U^i + D\{u\} - S\{v\} \tag{5}$$

In the limit, this equation and its normal derivative yield on Σ:

$$\frac{u}{2} = u^i + D\{u\} - S\{v\} \tag{6}$$

$$\frac{v}{2} = v^i + M\{u\} - K\{v\} \tag{7}$$

The nonuniqueness problem

While the original boundary value problem has a unique solution, the corresponding SIEs may not be uniquely solvable at certain critical values of k corresponding to the adjoint interior problem. This gives rise to analytical complications and considerable difficulty in the numerical solution of the problem. While the integral equation fails only at a discrete point set of wavenumbers, the approximating linear equations become ill-conditioned in the vicinity of a critical value. Under these conditions, a severe loss of accuracy will be experienced. As k increases, so also does the density of critical values, and hence it becomes increasingly difficult to acquire an accurate solution.

The uniqueness of the solutions to the above equations has been discussed extensively by Burton [1]. Thus the solution of (3) is not unique if $k \in \{k_N\}$, where $\{k_N\}$ is the set of eigenvalues for the interior Neumann problem. At these wavenumbers the homogeneous equation adjoint to (3) has nontrivial solutions. On the other hand, the solution of (4) is not unique when $k \in \{k_D\}$, where $\{k_D\}$ is the set of eigenvalues of the interior Dirichlet problem.

Several approaches have been devised for surmounting these defects; these are discussed in [2,5] with references to previous contributions. These methods include the Burton and Miller (composite, combined, Helmholtz Gradient) field formulation, the combined source (mixed potential, modified Green's function) method, the use of interior Helmholtz integral relations and the source simulation (wave superposition) technique. Recent publications include the generalised combined field integral equations [6] and the boundary point method [7].

The NU problem may be detected via calculating the pivot ratio in Gauss elimination [8], monitoring the condition number of the resulting matrix [9], evaluating the minimum singular value decomposition (SVD) [10] or testing the level of interior fields [4].

In the present work we are mainly concerned with axisymmetric problems and the efficient implementation of the CHIEF method, which augments the SIE with few additional interior integral relations along the axis of symmetry. We adopt testing of the level of interior field as a reliable means to monitor the NU problem.

The use of interior Helmholtz integral relations

While the surface integral equations derived directly from Helmholtz formula suffer from NU, the interior integral relation:

$$U^i + D\{u\} - S\{v\} = 0, \quad \forall P \in V_{\mathrm{i}} \tag{8}$$

has a unique solution [11]. This is also called the extended integral equation (EIE). Copley [11] proved that for axisymmetric bodies it is sufficient to apply the above relation at all points along the axis of symmetry in V_{i}.

Schenck [3] augmented the boundary integral equation via forcing the interior integral relation at n number of points in V_{i}. The resolution of the resultant overdetermined $(N+n) \times N$ system can then be effected by means of a least-squares method. Implementations using Lagrange's multipliers are given in [12,13] to maintain a square $(N+n) \times (N+n)$ system. When $n \ll N$, the approach does not significantly add to the solution time. Schenck pointed out that only one proper interior point may be enough to establish a unique solution. The proper CHIEF point is required to be away from nodal surfaces. This was confirmed by Seybert and Rengarajan [12]. Chen et al [14] studied the problem in conjunction with SVD. They stressed that success depends on the number and location of chosen interior points. If properly chosen, only two interior points may be needed. In [15] they proposed a modification in which various first and second order derivatives of the interior equations are imposed. In [16], the interior equation and its first derivative are enforced in a weighted residual sense over a small interior volume. These methods add more equations for each interior point but make the proper selection of the CHIEF points less critical.

Although the use of interior integral relations has been shown to be useful for removing the resonant solutions, the arbitrariness of choosing the number and positions of the interior points causes

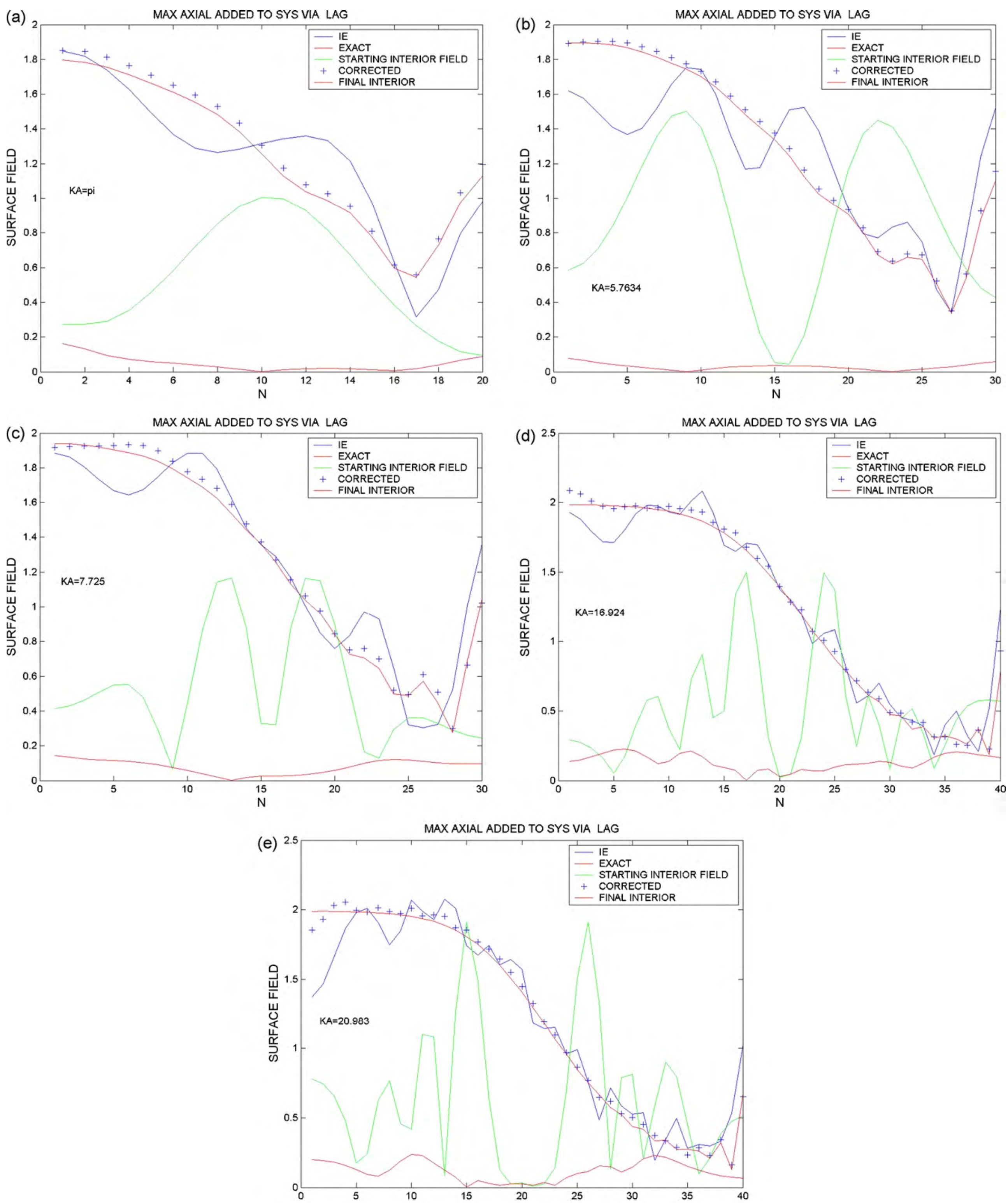

Fig. 1 Scattering: surface and axial fields before and after correction: (a) ka = pi; (b) ka = 5.7634; (c) ka = 7.725; (d) ka = 16.924 and (e) ka = 20.983.

some inconveniences. No criterion was generally given, except that these points must not be on the nodal surfaces of a modal field. However, these nodal surfaces are usually not known, so that the placing of the interior points has to be based on experience and intuition. On the other hand, the use of too many CHIEF points may not be computationally efficient. At fairly low frequency the method can be satisfactory because only a few critical wavenumbers have to be taken into account.

In the present work we are mainly concerned with axisymmetric problems and the efficient implementation of the CHIEF method, which augments the SIE with additional interior integral relations along the axis of symmetry. We adopt testing of the level of interior field as a reliable means to monitor the NU problem. Based on Copley's previous investigation [11], we use the axis of symmetry as the proper choice of interior points location. The level of the field at these points will indicate if there is a NU problem. In case one detects such problem, a proper choice of the CHIEF equation is recommended and full use of the previous solution can be made.

Methodology

The method is based on our previous investigations, detailed in [13,17,18]. However, we emphasise here selecting only one or two CHIEF points having the maximum error and demonstrating the range of applicability of the method. We further improve the previous method via storing and reusing the forward solutions besides the L and U decompositions. These solutions are efficiently reused in case a NU problem is detected. In this case the overdetermined system can be solved via a Lagrange multiplier approach requiring the solution of a maximum of $(N+2) \times (N+2)$ system. Following this an additional one or two rows of L and columns of U are required. The forward solution utilises the stored previously computed forward solution.

The main steps in the method can be summarised as follows:

1. Solve the SIE using LU-decomposition and store L and U as well as the forward solution $y\ (1{:}N)$ [Using MATLAB notation].
2. Using the obtained solution, calculate the interior field along the axis at a reasonable number of points.
3. Find the internal point of maximum error and take it as the CHIEF point if the magnitude of the error is larger than a preset value and go to 4, otherwise the solution is accepted and the calculation can be ended.
4. Calculate the extra row of L and column of U and solve the new system to find the corrected surface potential using the forward solution $y\ (1{:}N)$ which is common in both cases.

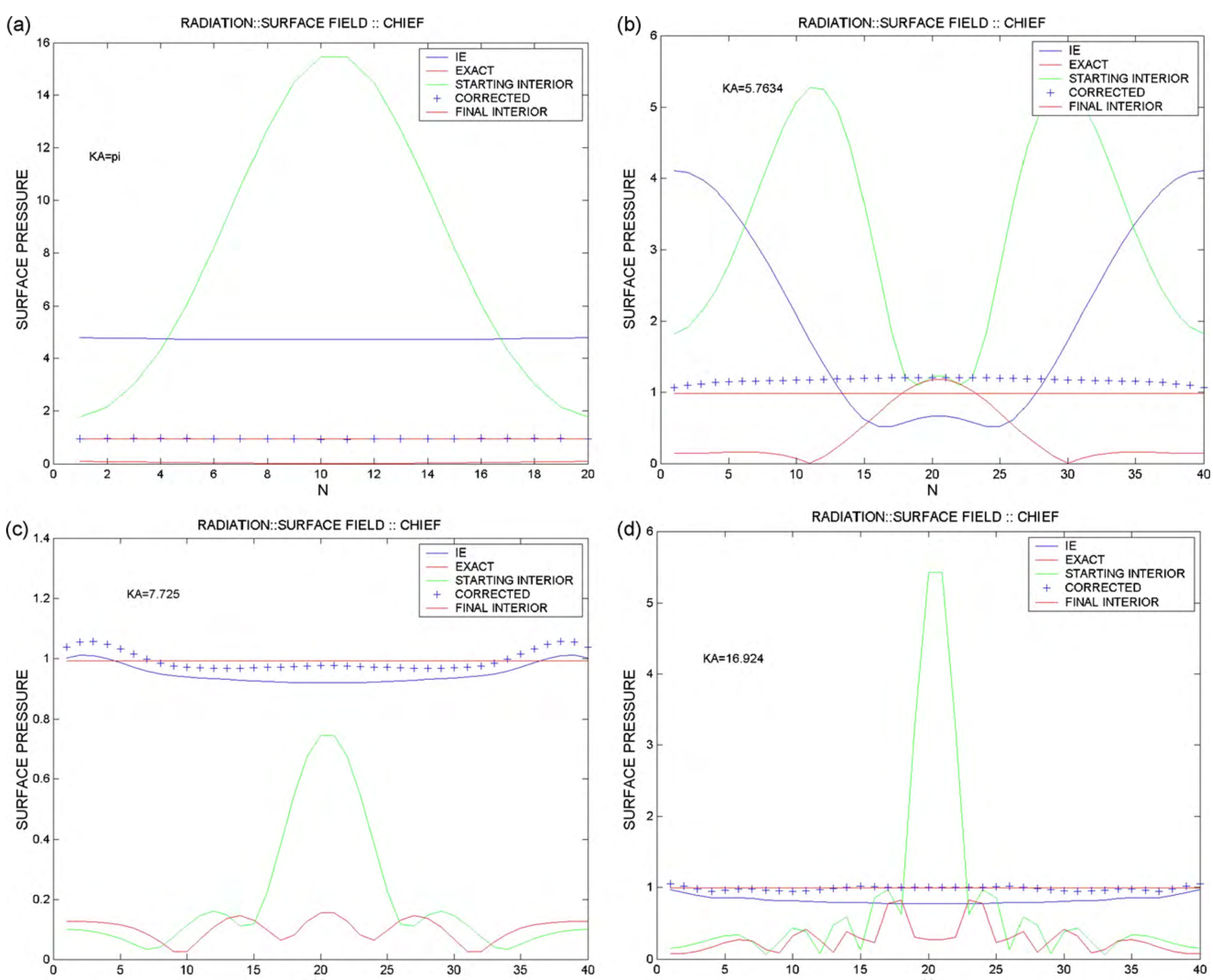

Fig. 2 Radiation: surface and axial fields before and after correction: (a) ka = pi; (b) ka = 5.7634; (c) ka = 7.725 and (d) ka = 16.924.

5. [If the accuracy is unsatisfactory repeat (3) and (4) using a second CHIEF point having the next maximum error].

Results

We first consider the scattering of a plane wave $U^i = \exp(-ikz)$ incident along the axial direction of a hard sphere of radius a, and normalised radius ka, whose centre is located at the origin. We use the integral equation of the second kind:

$$\left[\frac{I}{2} - D\right]\{u\} = u^i \tag{9}$$

We follow closely the treatment developed in [19] for axisymmetric problems. Using cylindrical coordinates (ρ,β,z), the surface integrals are reduced to one over β and another along the generating curve. Employing our method, the solution for ka = π compared to the exact solution is shown in Fig. 1(a). The figure also shows the interior fields before and after corrections. The nonuniqueness effect is evident in the high rise in the interior field up to one and the deviation of the surface field from the exact value. The implementation of CHIEF reduces the interior field to less than 0.2 and brings the surface field very close to the exact value. Fig. 1(b)–(e) shows similar results for ka = 5.7634, 7.725, 16.924 and 20.983, respectively. Only one CHIEF point was required in all cases. We note that while the interior field for ka = π demonstrates a single rise corresponding to a single interior resonance, the curves for higher ka demonstrate the effect of multiple interior resonances.

We next consider the radiation from a uniformly vibrating sphere [3]. Using Eq. (4) with known constant radial velocity v, we solve for the surface pressure. Employing our method, the solution for ka = π compared to the exact solution is shown in Fig. 2(a). The figure also shows the interior fields before and after corrections. Fig. 2(b)–(d) shows similar results for ka = 5.7634, 7.725 and 16.924, respectively. We note that ka = π required only one CHIEF point but the remaining cases required two CHIEF points.

Discussion

The problem of NU of the solution of acoustic problems via boundary integral equations is discussed. The efficient implementation of the CHIEF method to axisymmetric problems is studied. Interior axial fields are used to indicate the solution error and to select proper CHIEF points.

Our selection of the axial fields to indicate NU and to select the proper CHIEF points agrees with the recommendation in [11]. The studied method attempts to make full use of the previous matrix LU-decomposition and forward solution to estimate the interior field and to correct the solution. The figures show the nodal behaviour of the interior fields. Their symmetry demonstrates the independence of the exterior field. The effect of the correction on their level is evident.

The scattering by a hard sphere required only one CHIEF point at resonances up to ka = 20.983. The radiation problem which exhibits much higher internal fields generally requires more than one CHIEF point.

The frequency range around resonance over which the numerical solution is incorrect may be reduced using accurate quadrature schemes [20]. Besides this, the solver of the resulting system of equations should be properly chosen.

Chertock [21] emphasised that at high frequency (HF) it is not necessary to use the integral equation approach since accurate HF approximations may be utilised. Thus any method to handle NU need only be successful in the frequency range where HF approximations are not appropriate. In [16] an HF approach for ka > 8 was suggested. On the other hand, these HF approximations may be used to start iterative solutions of the integral equations as frequency increases.

Conclusion

In this article we considered axisymmetric bodies and presented a systematic and efficient procedure to detect NU, select the interior points and augment the SIEs to solve the NU problem. Also the efficient solution of the resulting system of equations was demonstrated. The extension of the procedure to more general shapes will be addressed in future studies.

Acknowledgement

The first author sincerely acknowledges the financial support of the AvH foundation.

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