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Modified Tikhonov regularization method for the Cauchy problem of the Helmholtz equation

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ABSTRACT

In this paper, the Cauchy problem for the Helmholtz equation is investigated. By Green's formulation, the problem can be transformed into a moment problem. Then we propose a modified Tikhonov regularization algorithm for obtaining an approximate solution to the Neumann data on the unspecified boundary. Error estimation and convergence analysis have been given. Finally, we present numerical results for several examples and show the effectiveness of the proposed method.

1. Introduction

The Helmholtz equation arises in many areas, especially in practical physical applications, such as acoustic, wave propagation and scattering, vibration of structures, electromagnetic field and so on, see [3,4,8,9]. The direct problems, i.e. Dirichlet, Neumann or mixed boundary value problems for the Helmholtz equation have been studied extensively in the past century. However, in some practical problems, the boundary data on the whole boundary cannot be obtained. We only know the noisy data on a part of the boundary or at some interior points of concerned domain. This is called an inverse problem. The Cauchy problem for the Helmholtz equation is an inverse problem and is severely ill-posed. That means the solution does not depend continuously on the given Cauchy data and any small change in the given data may cause a large change in the solution [10,21]. The determination of sources was discussed in [17]. The reconstruction of the radiation field was discussed in [18]. Several numerical methods have been proposed to solve this problem. These include, alternating iterative algorithm based on the Landweber method in conjunction with the boundary element method (BEM) [14], the conjugate gradient method with the BEM [15], the singular value decomposition method (SVD) [2] and the method of fundamental solution (MFS) [12,16,23]. In paper [11], the boundary knot method was applied to solve the Cauchy problem of the inhomogeneous Helmholtz equation. Recently, Teresa et al. in paper [19] used a wavelet method to solve the Cauchy problem of the Helmholtz equation. In this paper, we propose a regularization method for dealing with this problem in a special domain. The main idea is to transform the Cauchy problem into a moment problem and then use a Tikhonov type regularization method to solve the corresponding moment problem. Convergence analysis and numerical verification will be presented.

The paper is organized as follows. In Section 2, we formulate the problem and transform the Cauchy problem into a moment problem according to the idea in [5]. In Section 3, a regularization algorithm is proposed to solve the moment problem. In Section 4, the error estimate and convergence result are given. In Section 5, we give four numerical examples to demonstrate the effectiveness of our proposed method. Finally we give the conclusion in Section 6.

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2. Formulation of the problem and transformation to a moment problem

Let Ω be a simply connected and bounded domain in $\mathbb{R}^2$ with sufficiently regular boundary $\partial\Omega$ and Γ be an open part of boundary $\partial\Omega$. Without loss of generality, we assume that Γ is connected.

Consider the following Cauchy problem:

$$\Delta u(x,y) + k^2 u(x,y) = 0, \quad (x,y) \in \Omega, \tag{2.1}$$

$$u(x,y) = f(x,y), \quad (x,y) \in \Gamma, \tag{2.2}$$

$$\frac{\partial u(x,y)}{\partial n} = g(x,y), \quad (x,y) \in \Gamma, \tag{2.3}$$

where $f \in H^{3/2}(\Gamma)$, $g \in H^{1/2}(\Gamma)$, n is the outer unit normal with respect to $\partial\Omega$ and constant $k > 0$ is the wave number. In this paper we assume that $-k^2$ is not an eigenvalue of the Laplacian operator with Neumann boundary condition.

Suppose the Cauchy problem (2.1)–(2.3) has a solution u in $H^2(\Omega)$, then for any $\phi \in H^1(\Omega)$, we know that u satisfies the following formulation

$$\int_\Omega \nabla u \nabla \phi \mathrm{d}x\mathrm{d}y - k^2 \int_\Omega u\phi \mathrm{d}x\mathrm{d}y = \int_\Gamma g\phi \mathrm{d}s + \int_{\partial\Omega\setminus\Gamma} \frac{\partial u}{\partial n}\phi \mathrm{d}s, \quad \forall \phi \in H^1(\Omega), \tag{2.4}$$

where ds is the curve element.

For any $q \in L^2(\Gamma)$, let $v_q \in H^1(\Omega)$ be a weak solution of the following problem

$$\Delta v(x,y) + k^2 v(x,y) = 0, \quad (x,y) \in \Omega, \tag{2.5}$$

$$\frac{\partial v(x,y)}{\partial n} = 0, \quad (x,y) \in \partial\Omega \setminus \Gamma, \tag{2.6}$$

$$\frac{\partial v(x,y)}{\partial n} = q, \quad (x,y) \in \Gamma, \tag{2.7}$$

then by Theorem A.5 in Appendix, v_q exists and satisfies

$$\int_\Omega \nabla v_q \nabla \phi \mathrm{d}x\mathrm{d}y - k^2 \int_\Omega v_q\phi \mathrm{d}x\mathrm{d}y = \int_\Gamma q\phi \mathrm{d}s, \quad \forall \phi \in H^1(\Omega). \tag{2.8}$$

Denote

$$\mathcal{H} = \{v(x,y) \in H^1(\Omega) | v \text{ satisfies (2.8) for all } q \in L^2(\Gamma)\}.$$

For any $v \in \mathcal{H}$, take $\phi = v$ in (2.4) and $\phi = u$ in (2.8) with $v_q = v$, minus (2.4) by (2.8), note that $u|_\Gamma = f$, then we have the following equation

$$\int_{\partial\Omega\setminus\Gamma} v\frac{\partial u}{\partial n}\mathrm{d}s = \int_\Gamma \left(f\frac{\partial v}{\partial n} - gv\right)\mathrm{d}s. \tag{2.9}$$

Proposition 2.1. *If the Cauchy problem (2.1)–(2.3) has a solution $u \in H^2(\Omega)$ such that $\frac{\partial u}{\partial n}|_{\partial\Omega\setminus\Gamma} \in H^{\frac{1}{2}}(\partial\Omega \setminus \Gamma)$, then $\beta = \frac{\partial u}{\partial n}|_{\partial\Omega\setminus\Gamma}$ satisfies the following moment problem:*

$$\int_{\partial\Omega\setminus\Gamma} v\beta \mathrm{d}s = \int_\Gamma \left(f\frac{\partial v}{\partial n} - gv\right)\mathrm{d}s \equiv \mu_v(f,g), \tag{2.10}$$

where $v \in \mathcal{H}$.

Conversely if $\beta \in L^2(\partial\Omega \setminus \Gamma)$ is the solution of (2.10), then there exists a solution $u \in H^1(\Omega)$ of the Cauchy problem (2.1)–(2.3) such that $\frac{\partial u}{\partial n}|_{\partial\Omega\setminus\Gamma} = \beta$.

Proof. From the above deduction, we know that if u is a solution of the Cauchy problem (2.1)–(2.3) in $H^2(\Omega)$ and $\frac{\partial u}{\partial n}|_{\partial\Omega\setminus\Gamma} \in H^{\frac{1}{2}}(\partial\Omega \setminus \Gamma)$, then $\beta = \frac{\partial u}{\partial n}|_{\partial\Omega\setminus\Gamma}$ is a solution of the moment problem (2.10).

In the following, we verify that if $\beta \in L^2(\partial\Omega \setminus \Gamma)$ is a solution of the moment problem (2.10), then we can get a solution for the Cauchy problem (2.1)–(2.3) in $H^1(\Omega)$. Consider the following Neumann boundary value problem:

$$\Delta w + k^2 w = 0, \quad \text{in } \Omega, \tag{2.11}$$

$$\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega\setminus\Gamma} = \beta, \tag{2.12}$$

$$\left.\frac{\partial w}{\partial n}\right|_\Gamma = g. \tag{2.13}$$

By Theorem A.5 in Appendix, we know that there exists a unique weak solution $w \in H^1(\Omega)$ for the Neumann boundary value problem (2.11)–(2.13) when $g \in H^{1/2}(\Gamma) \subset L^2(\Gamma)$ and $\beta \in L^2(\partial\Omega \setminus \Gamma)$. In the following, we will show that $w|_\Gamma = f$.

By Definition A.1 in Appendix, we know

$$\int_\Omega \nabla w \nabla \phi \mathrm{d}x\mathrm{d}y - k^2 \int_\Omega w\phi \mathrm{d}x\mathrm{d}y = \int_\Gamma g\phi \mathrm{d}s + \int_{\partial\Omega\setminus\Gamma} \beta\phi \mathrm{d}s, \quad \forall \phi \in H^1(\Omega). \tag{2.14}$$

For any $v \in \mathcal{H}$, we have

$$\int_\Omega \nabla v \nabla \varphi \mathrm{d}x\mathrm{d}y - k^2 \int_\Omega v\varphi \mathrm{d}x\mathrm{d}y = \int_\Gamma \frac{\partial v}{\partial n}\varphi \mathrm{d}s, \quad \forall \varphi \in H^1(\Omega). \tag{2.15}$$

Let $\phi = v$ in (2.14) and $\varphi = w$ in (2.15), minus (2.15) by (2.14), it is easy to get

$$\int_{\partial\Omega\setminus\Gamma} \beta v \mathrm{d}s = \int_\Gamma \left(w\frac{\partial v}{\partial n} - gv \right) \mathrm{d}s. \tag{2.16}$$

Since β is a solution of the moment problem (2.10), by (2.16), we have

$$\int_\Gamma (w - f)\frac{\partial v}{\partial n} \mathrm{d}s = 0. \tag{2.17}$$

Now by Theorem A.5, there exists a function $v \in H^1(\Omega)$ satisfying (2.8) with

$$\left.\frac{\partial v}{\partial n}\right|_\Gamma = w - f. \tag{2.18}$$

Hence (2.17) becomes

$$\int_\Gamma (w - f)^2 \mathrm{d}s = 0. \tag{2.19}$$

Hence $w|_\Gamma = f$ and w is a solution of the Cauchy problem (2.1)–(2.3). The proof is completed. □

In the following, we choose $\{v_n\}_{n=1}^\infty \subset \mathcal{H}$, such that

$$\overline{\mathrm{span}\{v_n|_{\partial\Omega\setminus\Gamma}\}_{n=1}^\infty} = L^2(\partial\Omega \setminus \Gamma). \tag{2.20}$$

Then the moment problem (2.10) becomes

$$\int_{\partial\Omega\setminus\Gamma} v_n \frac{\partial u}{\partial n} \mathrm{d}s = \int_\Gamma \left(f\frac{\partial v_n}{\partial n} - v_n g \right) \mathrm{d}s := \mu_n, \quad n = 1, 2, \ldots, \tag{2.21}$$

where μ_n is determined by f, g, v_n. It is noted that there is at most one solution to the moment problem (2.10).

3. A modified Tikhonov regularization method for solving the moment problem

In this section, we choose a basis of $L^2(\partial\Omega \setminus \Gamma)$ in space $\mathcal{H}$ for a special domain and then the moment problem (2.10) will become a Hausdorff moment problem. Further, we use Tikhonov type regularization method to solve it. The error estimate and convergence analysis will be given in Theorems 4.2 and 4.3.

Let $\Omega \subset \mathbb{R}^2$ be a simply connected and bounded domain hereafter and $\partial\Omega \setminus \Gamma = \{(x, y)|y = 0, 0 \le x \le 1\}$ and Γ be a smooth curve in half plane $\{(x, y)|y \ge 0\}$ which connects two points $(0, 0)$ and $(1, 0)$, see [5,19].

Choose a basis of $L^2(\partial\Omega \setminus \Gamma)$ in space $\mathcal{H}$ as the following

$$v_n(x, y) = \frac{1}{n^2k^2} \cos(\sqrt{n^2 + 1}ky)\mathrm{e}^{nkx}, \quad n = 1, 2, \ldots. \tag{3.1}$$

It is easy to verify that v_n satisfy

$$\Delta v_n(x, y) + k^2 v_n(x, y) = 0, \quad (x, y) \in \mathbb{R}^2, \tag{3.2}$$

$$\frac{\partial v_n(x, 0)}{\partial y} = 0, \quad x \in \mathbb{R}. \tag{3.3}$$

Then the Cauchy problem of the Helmholtz equation can be transformed to be the following moment problem:

$$\int_0^1 \frac{1}{n^2k^2} \mathrm{e}^{nkx} \beta(x) \mathrm{d}x = \mu_n, \quad n = 1, 2, \ldots, \tag{3.4}$$

where

$$\mu_n = \int_\Gamma \left(f\frac{\partial v_n}{\partial n} - gv_n \right) \mathrm{d}s. \tag{3.5}$$

Assume that $z=\frac{e^{kx}-1}{e^k-1}$, then the moment problem (3.4) becomes

$$\int_0^1 \frac{e^k-1}{n^2k^3}\left(1+(e^k-1)z\right)^{n-1}\beta\left(\frac{\ln(1+(e^k-1)z)}{k}\right)dz=\mu_n,\quad n=1,2,\ldots, \tag{3.6}$$

furthermore, we have

$$\sum_{m=0}^{n-1}\frac{1}{n^2k^3}C_{n-1}^m(e^k-1)^{(m+1)}\int_0^1 z^m\beta\left(\frac{\ln(1+(e^k-1)z)}{k}\right)dz=\mu_n,\quad n=1,2,\ldots. \tag{3.7}$$

Remark 3.1. If $\overline{\text{span}\{v_n(x,0)\}_{n=1}^{\infty}}\neq L^2(0,1)$, then there exists a function $\beta_0(x)\in L^2(0,1)$ and $\beta_0(x)\neq 0$, satisfying

$$\int_0^1 \frac{1}{n^2k^2}e^{nkx}\beta_0(x)dx=0,\quad n=1,2,\ldots. \tag{3.8}$$

From (3.7), it is easy to know

$$\int_0^1 z^m\beta_0\left(\frac{\ln(1+(e^k-1)z)}{k}\right)dz=0,\quad m=1,2,\ldots. \tag{3.9}$$

Note that $\beta_0(x)\in L^2(0,1)$, then $\beta_0\left(\frac{\ln(1+(e^k-1)z)}{k}\right)\in L^2(0,1)$, due to $\overline{\text{span}\{1,z,z^2,\ldots\}}=L^2(0,1)$, we know that $\beta_0\left(\frac{\ln(1+(e^k-1)z)}{k}\right)=0$, further $\beta_0(x)=0$, which leads to a contradiction. Thus $\overline{\text{span}\{v_n(x,0)\}_{n=1}^{\infty}}=L^2(0,1)$.

In the following, we consider a finite moment problem for (3.6), i.e. take index n from 1 to $N+1$. Then we obtain a linear system of equations,

$$Ba=\mu, \tag{3.10}$$

where B is a matrix $B=(b_{i,j})_{N+1,N+1}$ with the (i,j) element

$$b_{i,j}=\begin{cases}\dfrac{C_{i-1}^{j-1}(e^k-1)^j}{i^2k^3}, & i\geq j,\\ 0, & i<j,\end{cases} \tag{3.11}$$

and μ is a vector

$$\mu=(\mu_1,\mu_2,\ldots,\mu_{N+1})^T;$$

a is a vector to be determined by solving (3.10)

$$a=(a_1,a_2,\ldots,a_{N+1})^T$$

with $a_j=\int_0^1 z^{j-1}\beta(\frac{\ln(1+(e^k-1)z)}{k})dz$.

Denote $\rho(z)=\beta(\frac{\ln(1+(e^k-1)z)}{k})$, then we get a finite Hausdorff moment problem as follows:

$$\int_0^1 z^{j-1}\rho(z)dz=a_j,\quad j=1,2,\ldots,N+1. \tag{3.12}$$

The numerical computation for the Hausdorff moment problem (HMP) has been proposed in [1,20,22]. In this paper, we use a Tikhonov type regularization method to solve (3.12) and the basic idea comes from paper [24].

Denote the finite Hausdorff moment problem (3.12) as an operator equation:

$$A\rho=a, \tag{3.13}$$

where

$$A\rho=((A\rho)_0,(A\rho)_1,\ldots,(A\rho)_N)^T$$

with

$$(A\rho)_i=\int_0^1 z^i\rho(z)dz,\quad i=0,1,\ldots,N.$$

It is easy to see that A is a linear and bounded operator from $L^2(0,1)$ into $\mathbb{R}^{N+1}$ and $\|A\|\leq\sqrt{\pi}$, refer to paper [20].

Due to the ill-posedness of the Cauchy problem for the Helmholtz equation, we need to assume that Cauchy data f and g contain some noises. Let f_δ and g_δ be the measured noisy data satisfying

$$\|f-f_\delta\|_{L^2(\Gamma)}+\|g-g_\delta\|_{L^2(\Gamma)}\leq\delta. \tag{3.14}$$

Moments corresponding to f_δ and g_δ in (3.5) are

$$\mu_n^\delta = \int_\Gamma \left(f_\delta \frac{\partial v_n}{\partial n} - g_\delta v_n \right) \mathrm{d}s, \quad n = 1, 2, \ldots \tag{3.15}$$

and the operator Eq. (3.13) becomes

$$A\rho = a^\delta, \tag{3.16}$$

where the right-hand side a^δ is the solution of equations $Ba^\delta = \mu^\delta$.

In the following, we propose a stable method to find an approximate solution for the operator Eq. (3.16).

Define a Tikhonov functional on $H_0^1(0, 1)$ by

$$F_\alpha(\rho) = \|A\rho - a^\delta\|_2^2 + \alpha\|\rho\|^2_{H_0^1(0,1)}, \tag{3.17}$$

where $\|\cdot\|_2$ is Euclidian norm in $\mathbb{R}^{N+1}$ and α is a regularization parameter.

Since $F_\alpha(\rho) \geq 0$, there exists $\eta \geq 0$ such that $\eta = \inf_{\rho\in H_0^1(0,1)} F_\alpha(\rho)$. Consider that $\delta^2 > 0$, then there exists $\rho^\delta_{\alpha,N}$ such that $F_\alpha(\rho^\delta_{\alpha,N}) \leq \eta + \delta^2$. The function $\rho^\delta_{\alpha,N}$ is called a regularized solution of the moment problem (3.12).

Let $z_i = \frac{i}{m}$, $i = 0, 1, \ldots, m$ and X_m be a finite-dimensional subspace of $H^1(0, 1)$ with $X_m = \mathrm{span}\{d_1(z), \ldots, d_{m-1}(z)\}$, where $d_i(z)$ are piecewise linear functions given by

$$d_i(z_j) = \delta_{ij} = \begin{cases} 1, & j = i, \\ 0, & j \neq i. \end{cases} \tag{3.18}$$

It is known that for any function $\phi \in H_0^1(0, 1)$, its interpolation function $\phi_m(z) = \sum_{i=1}^{m-1} \phi(z_i)d_i(z)$ converges to $\phi(z)$ as $m \to \infty$ in $L^2(0, 1)$ norm.

One approximation to $\rho^\delta_{\alpha,N}$ can be obtained by solving the following optimization problem in the finite-dimensional subspace X_m.

$$\min_{\rho\in X_m}\{\|A\rho - a^\delta\|_2^2 + \alpha\|\rho\|^2_{H^1(0,1)}\}. \tag{3.19}$$

By a simple process, the minimizer of functional (3.19) is given by

$$\rho^{\delta,m}_{\alpha,N} = \sum_{i=1}^{m-1} c_i d_i(z), \tag{3.20}$$

where $c = (c_1, \ldots, c_{m-1})^{\mathrm{T}}$ is a solution of the linear system

$$(W + \alpha L)c = b^\delta, \tag{3.21}$$

where matrixes $W = (w_{i,j})_{m-1,m-1}$, $L = (l_{i,j})_{m-1,m-1}$ with the (i, j) element respectively are

$$w_{i,j} = \sum_{k=1}^{N+1} \left(\int_0^1 z^{k-1} d_i(z)\mathrm{d}z \int_0^1 z^{k-1} d_j(z)\mathrm{d}z \right),$$
$$l_{i,j} = (d_i, d_j)_{H^1(0,1)},$$

and

$$b^\delta = (b_1^\delta, b_2^\delta, \ldots, b_{m-1}^\delta)^{\mathrm{T}}$$

with $b_j^\delta = \sum_{k=1}^{N+1} \left(a_k^\delta \int_0^1 z^{k-1} d_j(z)\mathrm{d}z \right)$.

Therefore, we get an approximate solution to β as follows

$$\beta^{\delta,m}_{\alpha,N}(x) = \rho^{\delta,m}_{\alpha,N}\left(\frac{\mathrm{e}^{kx} - 1}{\mathrm{e}^k - 1} \right). \tag{3.22}$$

4. Convergence results

Denote $\beta^\delta_{\alpha,N}(x) = \rho^\delta_{\alpha,N}(\frac{\mathrm{e}^{kx}-1}{\mathrm{e}^k-1})$, in this section, we will give an error estimate for $\|\beta - \beta^\delta_{\alpha,N}\|$ and obtain a convergence result while choosing a suitable regularization parameter α and a value of N.

From (3.5), (3.14) and (3.15), by the Hölder inequality, we can obtain

$$|\mu_n^\delta - \mu_n| \leq \left[\int_\Gamma \left(v_n^2 + \left(\frac{\partial v_n}{\partial n} \right)^2 \right) \mathrm{d}s \right]^{\frac{1}{2}} \delta. \tag{4.1}$$

By the definition of function v_n in (3.1), we have

$$|v_n(x,y)| \le \frac{e^{nkx}}{n^2k^2} \le \frac{(M^k)^n}{k^2},$$

where $M = \sup_{(x,y)\in\Omega} |e^x| > 1$ is a constant depending on Ω. Similarly, we can estimate $|\frac{\partial v_n}{\partial x}| \le \frac{(M^k)^n}{k}$, $|\frac{\partial v_n}{\partial y}| \le \frac{\sqrt{2}(M^k)^n}{k}$. Then we can obtain that

$$\sum_{n=1}^{N+1} |\mu_n^\delta - \mu_n|^2 \le \tau (M^k)^{2N+2}\delta^2,$$

where $\tau > 0$ is a constant which depends on Ω and k.

According to (3.10), we know

$$a^\delta = B^{-1}\mu^\delta, \tag{4.2}$$

where

$$\mu^\delta = (\mu_1^\delta, \ldots, \mu_{N+1}^\delta)^{\mathrm{T}}, \qquad a^\delta = (a_1^\delta, \ldots, a_{N+1}^\delta)^{\mathrm{T}}.$$

Therefore, we have the following proposition:

Proposition 4.1. *The difference between a^δ and a in 2-norm is bounded by*

$$\|a^\delta - a\|_2 \le \begin{cases} \dfrac{\sqrt{\tau}(M^k)^{N+1}k^3 2^{N^2+N+1}e^{N+1}}{(e^k-1)^{N+1}}\delta, & 0 < k < \ln 2, \\ \sqrt{\tau}(M^k)^{N+1}k^3 2^{N^2+N+1}e^{N+1}\delta, & k \ge \ln 2. \end{cases} \tag{4.3}$$

Proof. The difference between a^δ and a in 2-norm is bounded by

$$\|a^\delta - a\|_2 = \|B^{-1}(\mu^\delta - \mu)\|_2 \le \|B^{-1}\|_2\|\mu^\delta - \mu\|_2 \le \sqrt{\tau}\|B^{-1}\|_2 (M^k)^{N+1}\delta. \tag{4.4}$$

In the following, we estimate $\|B^{-1}\|_2$. By (3.11), we note that the matrix

$$B = \frac{1}{k^3}QDP,$$

where diagonal matrixes $Q = (q_{ij})_{N+1,N+1}$ and $P = (p_{ij})_{N+1,N+1}$, matrix $D = (d_{ij})_{N+1,N+1}$ with the (i,j) element respectively are

$$q_{ij} = \begin{cases} \dfrac{1}{i^2}, & i = j, \\ 0, & i \ne j; \end{cases} \tag{4.5}$$

$$p_{ij} = \begin{cases} (e^k - 1)^i, & i = j, \\ 0, & i \ne j; \end{cases} \tag{4.6}$$

$$d_{ij} = \begin{cases} C_{i-1}^{j-1}, & i \ge j, \\ 0, & i < j. \end{cases} \tag{4.7}$$

The inverse matrix of B is then

$$B^{-1} = k^3 P^{-1}D^{-1}Q^{-1}. \tag{4.8}$$

It is not hard to obtain the inverse matrixes of P and Q, and the 2-norm of matrixes P^{-1} and Q^{-1} as follows

$$\|P^{-1}\|_2 = \begin{cases} (e^k-1)^{-(N+1)}, & 0 < k < \ln 2, \\ (e^k-1)^{-1}, & k \ge \ln 2, \end{cases} \tag{4.9}$$

$$\|Q^{-1}\|_2 = (N+1)^2. \tag{4.10}$$

In the following, we estimate $\|D^{-1}\|_2$. Consider the linear system of equations

$$D\xi = \gamma, \tag{4.11}$$

where

$$\xi = (\xi_1, \ldots, \xi_{N+1})^{\mathrm{T}}, \qquad \gamma = (\gamma_1, \ldots, \gamma_{N+1})^{\mathrm{T}},$$

and

$$D=\begin{pmatrix} C_0^0 & 0 & 0 & \cdots & 0 \\ C_1^0 & C_1^1 & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ C_N^0 & C_N^1 & C_N^2 & \cdots & C_N^N \end{pmatrix}.$$

Since $C_N^0+C_N^1+\cdots+C_N^{N-1}+C_N^N=2^N$, the maximum element d in matrix D is bounded by

$$1\le d\le 2^N. \tag{4.12}$$

According to (4.11), it is easy to see

$$\xi_{i+1}=\gamma_{i+1}-C_i^0\xi_1-C_i^1\xi_2-\cdots-C_i^{i-1}\xi_i,\quad i=0,1,\ldots,N. \tag{4.13}$$

Note that for every γ_i, $|\gamma_i|\le\|\gamma\|_2$, $i=1,2,\ldots,N+1$. For $i=0$, $\xi_1=\gamma_1$, thus we have $|\xi_1|\le\|\gamma\|_2$. Suppose that the inequality

$$|\xi_i|\le(1+d)^{i-1}\|\gamma\|_2 \tag{4.14}$$

is satisfied, then we can prove

$$|\xi_{i+1}|\le\|\gamma\|_2+d(1+d)^0\|\gamma\|_2+\cdots+d(1+d)^{i-1}\|\gamma\|_2=(1+d)^i\|\gamma\|_2.$$

Therefore, by the induction, the estimate (4.14) is satisfied for all $i=1,2,\ldots,N+1$.

By (4.11), we have

$$\xi=D^{-1}\gamma. \tag{4.15}$$

According to (4.14) and (4.15), we can obtain

$$\|\xi\|_2^2=\sum_{i=1}^{N+1}\xi_i^2\le\left(\sum_{i=1}^{N+1}(1+d)^{2(i-1)}\right)\|\gamma\|_2^2\le 2^{2(N+1)}d^{2N}\|\gamma\|_2^2, \tag{4.16}$$

thus

$$\|D^{-1}\|_2\le 2^{N+1}d^N. \tag{4.17}$$

Further, consider (4.12), it can be obtained

$$\|D^{-1}\|_2\le 2^{N^2+N+1}. \tag{4.18}$$

Therefore, by (4.8)–(4.10) and (4.18), we have

$$\|B^{-1}\|_2=\|k^3P^{-1}D^{-1}Q^{-1}\|_2\le k^3\|P^{-1}\|_2\|D^{-1}\|_2\|Q^{-1}\|_2 \tag{4.19}$$

$$\le\begin{cases}\dfrac{k^3(N+1)^2 2^{N^2+N+1}}{(e^k-1)^{N+1}}, & 0<k<\ln 2,\\[2ex] \dfrac{k^3(N+1)^2 2^{N^2+N+1}}{(e^k-1)}, & k\ge\ln 2.\end{cases} \tag{4.20}$$

Further,

$$\|B^{-1}\|_2\le\begin{cases}\dfrac{k^3 2^{N^2+N+1}e^{N+1}}{(e^k-1)^{N+1}}, & 0<k<\ln 2,\\[2ex] k^3 2^{N^2+N+1}e^{N+1}, & k\ge\ln 2.\end{cases} \tag{4.21}$$

Hence by (4.4), we have

$$\|a^\delta-a\|_2\le\begin{cases}\dfrac{\sqrt{\tau}(M^k)^{N+1}k^3 2^{N^2+N+1}e^{N+1}}{(e^k-1)^{N+1}}\delta, & 0<k<\ln 2,\\[2ex] \sqrt{\tau}(M^k)^{N+1}k^3 2^{N^2+N+1}e^{N+1}\delta, & k\ge\ln 2.\end{cases} \tag{4.22}$$

In the following, the right-hand side terms in (4.21) are denoted by F_N, i.e

$$F_N = \begin{cases} \dfrac{k^3 2^{N^2+N+1} e^{N+1}}{(e^k-1)^{N+1}}, & 0 < k < \ln 2, \\ k^3 2^{N^2+N+1} e^{N+1}, & k \geq \ln 2. \end{cases} \tag{4.23}$$

By (4.3), we have

$$\|a^\delta - a\|_2 \leq \sqrt{\tau} F_N (M^k)^{N+1} \delta. \tag{4.24}$$

Denote $K_N = \sqrt{\tau} F_N (M^k)^{N+1}$, we have

$$\|a^\delta - a\|_2 \leq K_N \delta. \tag{4.25}$$

In the following, we give a convergence result.

Theorem 4.2. *Suppose that the solution u of the Cauchy problem (2.1)–(2.3) of the Helmholtz equation satisfies $\frac{\partial u}{\partial n}|_{\partial\Omega\setminus\Gamma} \in H_0^1(\partial\Omega \setminus \Gamma)$ and there exists a constant $E > 0$, such that $\|\frac{\partial u}{\partial n}\|_{H_0^1(\partial\Omega\setminus\Gamma)} \leq E$. Denote*

$$S(N) = \sqrt{\frac{4 + (N+2)^2}{(N+1)^3 2^{2N+2}}}.$$

Choose $\alpha = \delta^2$ and

$$N = \begin{cases} \left[\left(\dfrac{\ln \frac{1}{\delta}}{2k \ln M + 2 + 4\ln 2 - 2\ln(e^k - 1)} \right)^{\frac{1}{2}} \right], & 0 < k < \ln 2, \\ \left[\left(\dfrac{\ln \frac{1}{\delta}}{2k \ln M + 2 + 4\ln 2} \right)^{\frac{1}{2}} \right], & k \geq \ln 2, \end{cases}$$

then there exists a constant $C > 0$ which depends on E, Ω and k such that

$$\left\| \frac{\partial u}{\partial n}(x, 0) - \beta^\delta_{\alpha,N}(x) \right\|_{L^2(0,1)} \leq \frac{C}{\log \frac{1}{\nu(\sqrt{2}+\sqrt{4+2E^2})\delta^{\frac{1}{2}} + S(N)}}, \tag{4.26}$$

where $\nu > 0$ is a constant and $[\cdot]$ denotes the nearest integer towards minus infinity of a real number.

Proof. Denote $z = \frac{e^{kx}-1}{e^k-1}$ and

$$\rho_0(z) = \frac{\partial u}{\partial n}\left(\frac{1}{k}\ln(1 + (e^k - 1)z), 0\right), z \in [0, 1].$$

Let $\rho^\delta_{\alpha,N}(z)$ be defined by (3.17), then from Theorem 3.4.1 in [24] and (4.25), we have

$$\|\rho^\delta_{\alpha,N}(z) - \rho_0(z)\|_{L^2(0,1)} \leq \frac{C_1}{\log \frac{1}{(\sqrt{2}+\sqrt{4+2E^2})\delta K_N + S(N)}}.$$

Denote $\beta^\delta_{\alpha,N}(x) = \rho^\delta_{\alpha,N}(\frac{e^{kx}-1}{e^k-1})$, therefore

$$\begin{aligned} \left\| \frac{\partial u}{\partial n}(x, 0) - \beta^\delta_{\alpha,N}(x) \right\|_{L^2(0,1)} &\leq C_2 \|\rho^\delta_{\alpha,N}(z) - \rho_0(z)\|_{L^2(0,1)} \\ &\leq \frac{C}{\log \frac{1}{(\sqrt{2}+\sqrt{4+2E^2})\delta K_N + S(N)}}, \end{aligned}$$

where C depends on Ω, E and k.

For $0 < k < \ln 2$,

$$\begin{aligned} K_N &= \frac{\sqrt{\tau}(M^k)^{N+1} k^3 2^{N^2+N+1} e^{N+1}}{(e^k - 1)^{N+1}} \\ &= \tau_1 \frac{(M^k)^N 2^{N^2+N} e^N}{(e^k - 1)^N} \\ &\leq \tau_1 \frac{(M^k)^{N^2} 2^{2N^2} e^{N^2}}{(e^k - 1)^{N^2}}, \end{aligned}$$

where $\tau_1 = 2\sqrt{\tau} M^k k^3 e (e^k - 1)^{-1}$.

Let

$$\frac{(M^k)^{N^2} 2^{2N^2} e^{N^2}}{(e^k - 1)^{N^2}} = \delta^{-\frac{1}{2}},$$

then we can choose

$$N = \left[\left(\frac{\ln \frac{1}{\delta}}{2k \ln M + 2 + 4 \ln 2 - 2 \ln(e^k - 1)} \right)^{\frac{1}{2}} \right].$$

For this case, $K_N \delta \le \tau_1 \delta^{1/2}$.

In the case of $k \ge \ln 2$, we have

$$\begin{aligned} K_N &= \sqrt{\tau} (M^k)^{N+1} k^3 2^{N^2+N+1} e^{N+1} \\ &= \tau_2 (M^k)^N 2^{N^2+N} e^N \\ &\le \tau_2 (M^k)^{N^2} 2^{2N^2} e^{N^2}, \end{aligned}$$

where $\tau_2 = 2\sqrt{\tau} M^k k^3 e$.
Let

$$(M^k)^{N^2} 2^{2N^2} e^{N^2} = \delta^{-\frac{1}{2}},$$

then

$$N = \left[\left(\frac{\ln \frac{1}{\delta}}{2k \ln M + 2 + 4 \ln 2} \right)^{\frac{1}{2}} \right].$$

For this choice of N, we have $K_N \delta \le \tau_2 \delta^{1/2}$.

For the choice of N above, $K_N \delta \le \nu \delta^{\frac{1}{2}}$ in which $\nu = \max\{\tau_1, \tau_2\}$, note that $S(N) \to 0$, as $\delta \to 0$, hence we have a convergence estimate

$$\left\| \frac{\partial u}{\partial n}(x, 0) - \beta^{\delta}_{\alpha,N}(x) \right\|_{L^2(0,1)} \le \frac{C}{\log \frac{1}{\nu(\sqrt{2}+\sqrt{4+2E^2})\delta^{\frac{1}{2}} + S(N)}}.$$

The proof is completed. □

Consider the following Neumann boundary value problem:

$$\Delta u_N^{\delta} + k^2 u_N^{\delta} = 0, \quad \text{in } \Omega, \tag{4.27}$$

$$\left. \frac{\partial u_N^{\delta}}{\partial n} \right|_{\Gamma} = g_{\delta}, \tag{4.28}$$

$$\left. \frac{\partial u_N^{\delta}}{\partial n} \right|_{\partial\Omega \setminus \Gamma} = \beta^{\delta}_{\delta^2,N}, \tag{4.29}$$

where we assume that $g_{\delta} \in L^2(\Gamma)$.

Suppose that u is a solution of the Cauchy problem (2.1)–(2.3). By Theorem A.7 in the Appendix, the following error estimate is satisfied

$$\begin{aligned} \|u_N^{\delta} - u\|_{L^2(\Omega)} &\le C \left(\left\| \frac{\partial u}{\partial n}(x, 0) - \beta^{\delta}_{\delta^2,N}(x) \right\|_{L^2(\partial\Omega\setminus\Gamma)} + \|g - g_{\delta}\|_{L^2(\Gamma)} \right) \\ &\le C \left(\left\| \frac{\partial u}{\partial n}(x, 0) - \beta^{\delta}_{\delta^2,N}(x) \right\|_{L^2(\partial\Omega\setminus\Gamma)} + \delta \right), \end{aligned}$$

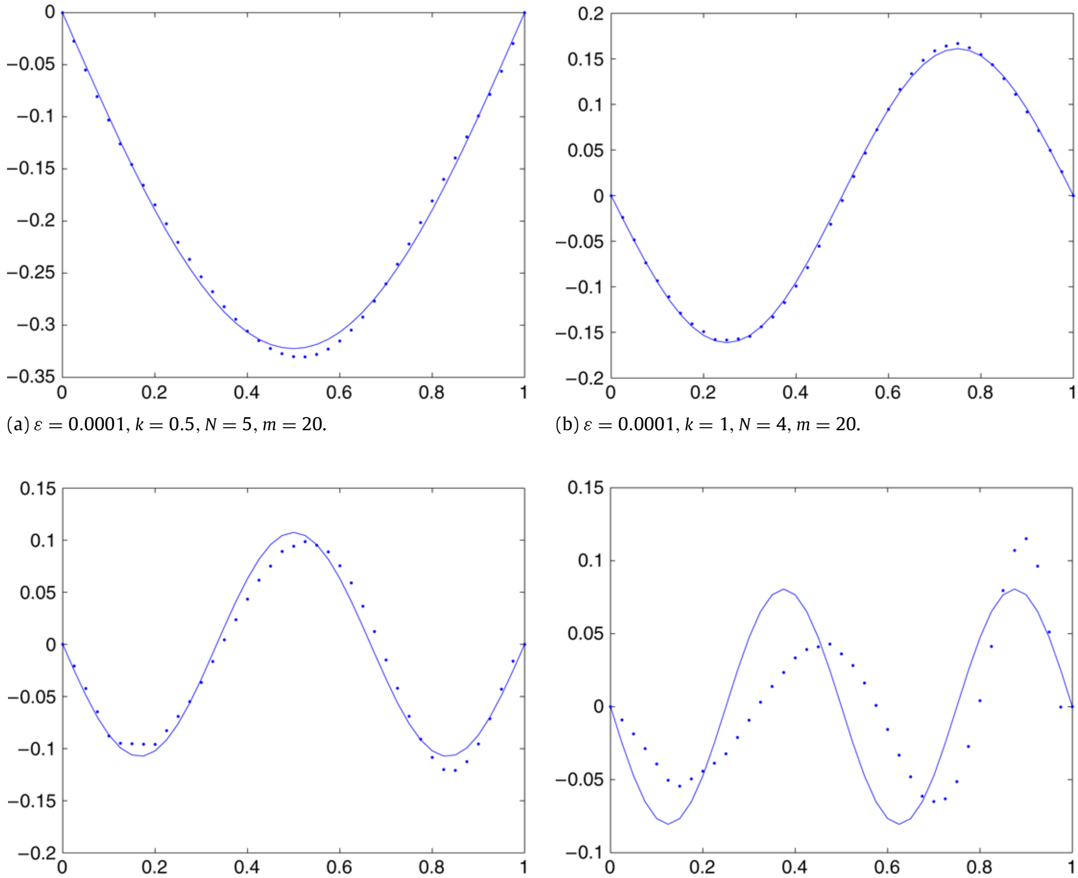
where $C > 0$ is a constant depending on Ω, Γ and k.

Therefore we have the following main result.

Theorem 4.3. *Under the assumptions given in Theorem 4.2, we have the following convergence estimate*

$$\|u_N^{\delta} - u\|_{L^2(\Omega)} \le C \left(\frac{1}{\log \frac{1}{\nu(\sqrt{2}+\sqrt{4+2E^2})\delta^{\frac{1}{2}} + S(N)}} + \delta \right),$$

where constant $C > 0$ depends on Ω, Γ, E and k.



(a) $\varepsilon = 0.0001$, $k = 0.5$, $N = 5$, $m = 20$.

(b) $\varepsilon = 0.0001$, $k = 1$, $N = 4$, $m = 20$.

(c) $\varepsilon = 0.0001$, $k = 1.5$, $N = 4$, $m = 20$.

(d) $\varepsilon = 0.0001$, $k = 2$, $N = 4$, $m = 20$.

Fig. 1. The exact solution β (solid lines) and its approximation $\beta^{\delta,m}_{\delta^2,N}$ (dotted lines) by using the noisy data $\delta = 0.0004$.

5. Numerical examples

Let $\Omega = \{(x, y)|0 \le x \le 1, 0 \le y \le 1\}$ and $\partial\Omega \setminus \Gamma = \{(x, y) \mid y = 0, 0 \le x \le 1\}$.

We choose $u(x, y) = \frac{1}{2(4\pi^2-1)k^2}\sin(2k\pi x)(e^{k\sqrt{4\pi^2-1}y} - e^{-k\sqrt{4\pi^2-1}y})$ as the exact solutions of (2.1)–(2.3) for $k = 0.5, 1, 1.5, 2$. The numerical results for the approximate solution $\beta^{\delta,m}_{\delta^2,N}(x)$ and the exact solution $\frac{\partial u}{\partial n}|_{\partial\Omega\setminus\Gamma}(x)$ for noisy level $\delta = 0.0004$ and $\delta = 0.004$ are shown in Figs. 1 and 2 in which the solid line represents the exact solution and the dotted line is its approximation. Noisy Cauchy data are generated by $f^\delta = f + \varepsilon e^x \sin y$ and $g^\delta = g + \varepsilon e^x \cos y$ with $\varepsilon = 0.0001$, $\varepsilon = 0.001$ respectively. The parameters N and m are shown in the caption of each figure. It is observed that our proposed algorithm is effective and stable to noises.

6. Conclusions

In this paper, we propose a numerical method for solving the Cauchy problem of the Helmholtz equation. We firstly transform the Cauchy problem into a moment problem by using Green's formula, then we make use of a modified Tikhonov regularization method to solve the Hausdorff moment problem. The error estimate and convergence analysis have been presented. The numerical results demonstrate that our proposed method is accurate and efficient.

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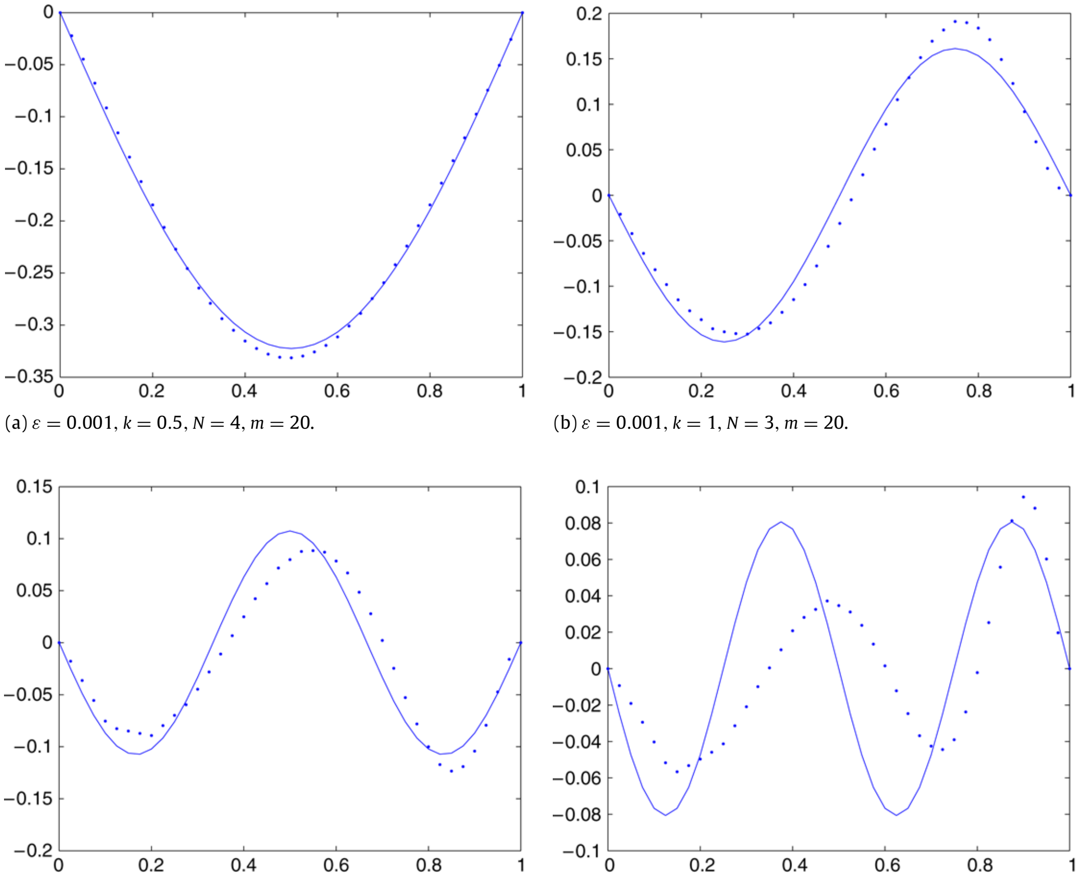

(a) $\varepsilon = 0.001, k = 0.5, N = 4, m = 20$.

(b) $\varepsilon = 0.001, k = 1, N = 3, m = 20$.

(c) $\varepsilon = 0.001, k = 1.5, N = 3, m = 20$.

(d) $\varepsilon = 0.001, k = 2, N = 3, m = 20$.

Fig. 2. The exact solution β (solid lines) and its approximation $\beta^{\delta,m}_{\delta^2,N}$ (dotted lines) by using the noisy data $\delta = 0.004$.

Appendix

Let Ω be a simply connected and bounded open set in $\mathbb{R}^2$ with a sufficiently regular boundary $\partial\Omega$.

Definition A.1. Suppose $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$, the weak solution of the Neumann boundary value problem

$$-\Delta u + cu = f, \quad \text{in } \Omega, \tag{A.1}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g, \tag{A.2}$$

is defined as a solution of the following variational problem:

$$u \in H^1(\Omega), \quad \int_\Omega \nabla u \nabla v \mathrm{d}x + \int_\Omega cuv \mathrm{d}x = \int_\Omega fv \mathrm{d}x + \int_{\partial\Omega} gv \mathrm{d}s, \ \forall v \in H^1(\Omega), \tag{A.3}$$

where c is a real number.

Proposition A.2. *The variational problem (A.3) with $c > 0$ has a unique solution in $H^1(\Omega)$.*

Proof. Define $a(u, v) = \int_\Omega (\nabla u \nabla v + cuv)\mathrm{d}x$, $\ell(v) = \int_\Omega fv\mathrm{d}x + \int_{\partial\Omega} gv\mathrm{d}s$. Then (A.3) becomes $a(u, v) = \ell(v), \forall v \in H^1(\Omega)$. By the Lax–Milgram Theorem from Chap. VII, Section 1 of book [6], the variational problem (A.3) has a unique solution $u \in H^1(\Omega)$. □

Define

$$M = \left\{u \in H^1(\Omega); -\Delta u \in L^2(\Omega) \text{ and } \left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0\right\}.$$

Proposition A.3. *For $g = 0$, $c = 1$ and any $f \in L^2(\Omega)$, the Neumann boundary value problem (A.1)–(A.2) admits a unique weak solution $u \in M$. Further, we have*

$$\|u\|_{H^1(\Omega)} \le C\|f\|_{L^2(\Omega)}, \tag{A.4}$$

where $C > 0$ is a constant.

Proof. See Page 96 in Chapter VIII of book [7] and Pages 70–78 in Chapter IV of book [13]. □

Furthermore, we have the following proposition.

Proposition A.4. *The boundary value problem*

$$-\Delta u - k^2 u = f, \quad \text{in } \Omega, \tag{A.5}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \tag{A.6}$$

has a unique weak solution $u \in M$ for each $f \in L^2(\Omega)$ if and only if $-k^2$ is not the eigenvalue of the Laplacian operator with the homogeneous Neumann boundary value problem.

Proof. From Proposition A.3, we know that $L := (-\Delta + I)^{-1} : L^2(\Omega) \mapsto M \subset H^1(\Omega)$ is a bounded linear operator. Note that $H^1(\Omega) \hookrightarrow L^2(\Omega)$ is compactly embedded. Thus, L is a linear compact operator from $L^2(\Omega) \to L^2(\Omega)$.

Note that if $\varphi = -\Delta u + u$, $\psi = -\Delta v + v$, with $\forall u, v \in M$, we have

$$\begin{aligned}(L\varphi, \psi)_{L^2(\Omega)} &= (L(-\Delta + I)u, -\Delta v + v)_{L^2(\Omega)} \\ &= (u, -\Delta v + v)_{L^2(\Omega)} = (\nabla u, \nabla v)_{L^2(\Omega)} + (u, v)_{L^2(\Omega)} \\ &= (-\Delta u + u, v)_{L^2(\Omega)} = (\varphi, L\psi)_{L^2(\Omega)}.\end{aligned}$$

Thus, $L = L^*$, i.e. L is self-adjoint.

The boundary value problem (A.5)–(A.6) is equivalent to

$$-\Delta u + u = (k^2 + 1)u + f, \quad \text{in } \Omega, \tag{A.7}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0. \tag{A.8}$$

Thus, we can rewrite (A.5)–(A.6) as

$$u - (k^2 + 1)Lu = Lf. \tag{A.9}$$

According to the Fredholm alternative theorem from Chap. VIII, Section 2 of book [7], the boundary value problem (A.9) has a solution in $L^2(\Omega)$ for every $f \in L^2(\Omega)$ if its homogeneous problem $v - (k^2 + 1)Lv = 0$ has a unique solution $v = 0$. Further, there exists a set of real numbers $\Lambda = \{k_1, k_2, \ldots\}$ where $\frac{1}{k_j^2+1}$ are the eigenvalues of problem $\lambda v - Lv = 0$. For $k \notin \Lambda$, the problem $v - (k^2 + 1)Lv = 0$ has a unique solution $v = 0$. Thus, the problem (A.9) has a unique solution in $L^2(\Omega)$ if $k \notin \Lambda$.

Let $u_j \in L^2(\Omega)$ be the eigenfunction of problem $\lambda v - Lv = 0$ corresponding to eigenvalue $\frac{1}{k_j^2+1}$, i.e.,

$$u_j - (k_j^2 + 1)Lu_j = 0.$$

Note that $Lu_j \in M$, so we know $u_j \in M$. Further, we have $(-\Delta u_j + u_j) - (k_j^2 + 1)u_j = 0$, i.e., $\Delta u_j = -k_j^2 u_j$. Therefore, $-k_j^2$ is the eigenvalue of the Laplacian operator with the homogeneous Neumann boundary value problem. Thus, the proof is completed. □

Theorem A.5. *The boundary value problem*

$$-\Delta u - k^2 u = f, \quad \text{in } \Omega, \tag{A.10}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g, \tag{A.11}$$

admits a unique weak solution in $H^1(\Omega)$ provided that $f \in L^2(\Omega)$, $g \in L^2(\partial\Omega)$ and $-k^2$ is not the eigenvalue of the Laplacian operator with the homogeneous Neumann boundary value problem.

Proof. For $g \in L^2(\partial\Omega)$, by Proposition A.2, there exists a unique weak solution $w \in H^1(\Omega)$ for the following Neumann boundary value problem

$$-\Delta w + w = 0, \quad \text{in } \Omega, \tag{A.12}$$

$$\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = g, \tag{A.13}$$

i.e. w satisfies the following variational problem

$$\int_{\Omega} \nabla w \nabla \mu \mathrm{d}x + \int_{\Omega} w\mu \mathrm{d}x = \int_{\partial\Omega} g\mu \mathrm{d}s, \quad \forall \mu \in H^1(\Omega). \tag{A.14}$$

The variational formulation of the problem (A.10)–(A.11) is

$$\int_{\Omega} \nabla u \nabla \mu \mathrm{d}x - \int_{\Omega} k^2 u\mu \mathrm{d}x = \int_{\Omega} f\mu \mathrm{d}x + \int_{\partial\Omega} g\mu \mathrm{d}s, \quad \forall \mu \in H^1(\Omega). \tag{A.15}$$

Let $v = w - u$, from (A.14) and (A.15), we have

$$\int_{\Omega} \nabla v \nabla \mu \mathrm{d}x - \int_{\Omega} k^2 v\mu \mathrm{d}x = -\int_{\Omega} f\mu \mathrm{d}x - \int_{\Omega} (k^2+1) w\mu \mathrm{d}x, \quad \forall \mu \in H^1(\Omega). \tag{A.16}$$

Note that (A.16) is the variational formulation of the following Neumann boundary value problem

$$-\Delta v - k^2 v = -(1+k^2)w - f, \quad \text{in } \Omega, \tag{A.17}$$

$$\left.\frac{\partial v}{\partial n}\right|_{\partial\Omega} = 0. \tag{A.18}$$

Then, by Proposition A.4, the problem (A.17)–(A.18) has a unique solution $v \in M$ provided that $-k^2$ is not an eigenvalue of the Laplacian operator with Neumann boundary condition. Hence, $u = w - v \in H^1(\Omega)$ is the unique solution of problem (A.10)–(A.11) if $-k^2$ is not an eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition. □

Lemma A.6. *Let $-k^2$ not be an eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition and $u \in M$ be the unique weak solution of problem*

$$-\Delta u - k^2 u = g, \quad \textit{in } \Omega, \tag{A.19}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = 0, \tag{A.20}$$

where $g \in L^2(\Omega)$, then there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \le C\|g\|_{L^2(\Omega)}.$$

Proof. If the statement is not true, there exist sequences $\{g_j\}_{j=1}^{\infty} \subset L^2(\Omega)$ and $\{u_j\}_{j=1}^{\infty} \subset M$ are the weak solutions of problems

$$-\Delta u_j - k^2 u_j = g_j, \quad \text{in } \Omega, \tag{A.21}$$

$$\left.\frac{\partial u_j}{\partial n}\right|_{\partial\Omega} = 0, \tag{A.22}$$

with $\|u_j\|_{L^2(\Omega)} = 1$ and

$$\|u_j\|_{L^2(\Omega)} \ge j\|g_j\|_{L^2(\Omega)}, \quad j = 1, 2, \ldots.$$

Then $g_j \to 0$ in $L^2(\Omega)$ when $j \to \infty$. The problem (A.21)–(A.22) has the following variational formulation:

$$\int_{\Omega} (\nabla u_j \nabla v + u_j v)\mathrm{d}x = \int_{\Omega} ((k^2+1)u_j v + g_j v)\mathrm{d}x, \quad \text{for } \forall v \in H^1(\Omega). \tag{A.23}$$

Choose $v = u_j$, then

$$\|u_j\|^2_{H^1(\Omega)} \le (k^2+1)\|u_j\|^2_{L^2(\Omega)} + \|u_j\|_{L^2(\Omega)}\|g_j\|_{L^2(\Omega)} = (k^2+1) + \|g_j\|_{L^2(\Omega)},$$

i.e. $\{u_j\}_{j=1}^{\infty}$ is bounded in $H^1(\Omega)$. Then, there exist subsequences $\{u_{j_m}\}_{m=1}^{\infty} \subset \{u_j\}_{j=1}^{\infty}$ such that $u_{j_m} \rightharpoonup u_0$ weakly in $H^1(\Omega)$, hence $u_{j_m} \to u_0$ in $L^2(\Omega)$.

Let $m \to \infty$, then from (A.23), we obtain

$$\int_{\Omega} (\nabla u_0 \nabla v + u_0 v)\mathrm{d}x = \int_{\Omega} (k^2+1)u_0 v \mathrm{d}x, \quad \text{for } \forall v \in H^1(\Omega).$$

Therefore, u_0 is a weak solution of the following Neumann boundary value problem

$$-\Delta u_0 = k^2 u_0, \quad \text{in } \Omega, \tag{A.24}$$

$$\left.\frac{\partial u_0}{\partial n}\right|_{\partial\Omega} = 0. \tag{A.25}$$

Since $-k^2$ is not an eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition, we have $u_0 \equiv 0$. This leads to a contraction with $\|u_0\|_{L^2(\Omega)} = 1$. □

Theorem A.7. *Let $-k^2$ not be an eigenvalue of the Laplacian operator with the homogeneous Neumann boundary condition and $u \in H^1(\Omega)$ be the unique weak solution of the problem*

$$-\Delta u - k^2 u = 0, \quad \text{in } \Omega, \tag{A.26}$$

$$\left.\frac{\partial u}{\partial n}\right|_{\partial\Omega} = g, \tag{A.27}$$

where $g \in L^2(\partial\Omega)$, then there exists a constant $C > 0$ such that

$$\|u\|_{L^2(\Omega)} \le C\|g\|_{L^2(\partial\Omega)}.$$

Proof. Consider the following Neumann boundary value problem

$$-\Delta w + w = 0, \quad \text{in } \Omega, \tag{A.28}$$

$$\left.\frac{\partial w}{\partial n}\right|_{\partial\Omega} = g. \tag{A.29}$$

By Proposition A.2, there exists a unique weak solution $w \in H^1(\Omega)$ such that

$$\int_\Omega \nabla w \nabla v \mathrm{d}x + \int_\Omega wv \mathrm{d}x = \int_{\partial\Omega} gv \mathrm{d}s, \quad \forall v \in H^1(\Omega). \tag{A.30}$$

Choose $v = w$, we have

$$\|w\|^2_{H^1(\Omega)} \le \|g\|_{L^2(\partial\Omega)} \|w\|_{L^2(\partial\Omega)} \le C_1 \|g\|_{L^2(\partial\Omega)} \|w\|_{H^1(\Omega)}. \tag{A.31}$$

Consequently,

$$\|w\|_{H^1(\Omega)} \le C_1 \|g\|_{L^2(\partial\Omega)}. \tag{A.32}$$

Note that the problem (A.26)–(A.27) has the following variational formulation :

$$u \in H^1(\Omega), \int_\Omega \nabla u \nabla v \mathrm{d}x - \int_\Omega k^2 uv \mathrm{d}x = \int_{\partial\Omega} gv \mathrm{d}s, \quad \forall v \in H^1(\Omega). \tag{A.33}$$

Let $\tilde{u} = u - w$, by (A.30) and (A.33), we have

$$\int_\Omega \nabla \tilde{u} \nabla v \mathrm{d}x - \int_\Omega k^2 \tilde{u} v \mathrm{d}x = \int_\Omega (k^2 + 1) wv \mathrm{d}x, \forall v \in H^1(\Omega), \tag{A.34}$$

which is the variational formulation of the following Neumann boundary value problem

$$-\Delta \tilde{u} - k^2 \tilde{u} = (1 + k^2) w, \quad \text{in } \Omega, \tag{A.35}$$

$$\left.\frac{\partial \tilde{u}}{\partial n}\right|_{\partial\Omega} = 0. \tag{A.36}$$

Then, by Proposition A.4, there exists a unique solution $\tilde{u} \in H^1(\Omega)$ for the problem (A.35)–(A.36) provided that $-k^2$ is not an eigenvalue of the Laplacian operator with Neumann boundary condition. By Lemma A.6, we have

$$\|\tilde{u}\|_{L^2(\Omega)} \le C_2 \|w + k^2 w\|_{L^2(\Omega)} \le C_3 \|w\|_{L^2(\Omega)},$$

where $C_3 > 0$ is a constant which depends on Ω, $\partial\Omega$ and k.

Since $\tilde{u} = u - w$, then

$$\|u\|_{L^2(\Omega)} \le \|w\|_{L^2(\Omega)} + \|\tilde{u}\|_{L^2(\Omega)} \le C_4 \|w\|_{L^2(\Omega)},$$

combining (A.32), we have $\|u\|_{L^2(\Omega)} \le C\|g\|_{L^2(\partial\Omega)}$, where the constant $C > 0$ depends on Ω, $\partial\Omega$ and k. □

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