Journal of Marine Science and Technology, Vol. 17, No. 3, pp. 164-172 (2009)

METHOD OF FUNDAMENTAL SOLUTIONS WITH EXTERNAL SOURCE FOR THE EIGENFREQUENCIES OF WAVEGUIDES

Chia-Ming Fan*, Der-Liang Young*, and Chia-Lin Chiu*

Key words: eigenfrequencies, waveguides, resonant excitation, method of fundamental solutions, method of particular solutions.

ABSTRACT

This study adopts a meshless numerical method, the combination of the method of fundamental solutions (MFS) and method of particular solutions (MPS) following the lead of Reutskiy, to determine the eigenfrequencies of four different waveguides, based on the principle of physical response of a system exposed to external source. The response amplitudes to determine the resonant frequencies for the eigenproblems are used. We use the MFS with external source (MFS-ES) and MPS to solve a sequence of inhomogeneous problems for the determination of the eigenfrequencies. This is an alternative to the typical methods of directly solving the homogeneous matrix system to search for the eigenvalues in an eigenproblem. The square, elliptic, concentric annular and eccentric annular waveguides are analyzed to demonstrate the capability and robustness of the present meshless numerical method. In the numerical experiments, the computational results are not sensitive at all to the locations of the external source. Furthermore, the spurious eigenfrequencies will not occur in this boundary-type meshless method which is different from other numerical methods.

I. INTRODUCTION

The waveguides, which guide the electromagnetic wave propagation, are important devices in optic and electronic applications, such as the optical fiber, microwave elements and components, etc. The determination of the eigenfrequencies of waveguides is a crucial topic when the electromagnetic waves with specific frequency have to propagate in the designed direction. In general the eigenproblems for the waveguides can be solved analytically for simple geometries or numerically for complex domains.

There are many numerical schemes proposed to deal with the eigenproblems and can be approximately classified as mesh-dependent and meshless or meshfree methods. The meshless numerical methods are more attractive to researchers than mesh-dependent methods, since the meshless numerical methods do not need the mesh generation and numerical quadrature. For example, Jiang *et al.* [3] used the radial basis functions collocation method (RBFCM), one of the popular meshless methods, to analyze the eigenproblem of the elliptic waveguides. The boundary nodes and the interior nodes are all required during the computation; hence the requirement of huge number of nodes in the calculation will limit the application of the RBFCM. Young *et al.* [12] in contrast adopted the method of fundamental solutions (MFS) with the singular value decomposition (SVD) technique to examine the eigenproblems of the waveguides. The MFS is a boundary-type meshless numerical method, since only the boundary nodes are required in the numerical procedure of the MFS. Therefore, the MFS is an ideal alternative in comparing with mesh-dependent and meshless domain-type methods.

The homogeneous partial differential equations and homogeneous boundary conditions are both essential in the eigenproblems, so the MFS usually needs to cooperate with the SVD [12] or direct determinant search method (DDSM) [10, 11] to find out the eigenvalues. The computations of determinant in the DDSM or singular values in the SVD are time-consuming. Reutskiy [6-9] recently proposed a novel numerical scheme utilizing the method of external sources (MES) and without using the SVD or DDSM to deal with the eigenproblems. This MES numerical scheme is based on the measurement of resonant response of the eigenproblem exposed to an external excitation. This method, marked here as the MFS-ES, is used in this paper to analyze the eigenfrequencies for different waveguides.

It is worthwhile to notice that the spurious eigenvalues will generally appear when either the MFS [11] or boundary element method (BEM) [1] is used to analyze the eigenproblems in multiply-connected domains. The mathematical derivation and rationale of the spurious eigenvalues in multiply-connected domains can be found in Ref. [1, 11]. Since the MFS-ES transfers the eigenproblems from a homogeneous problem into

Author for correspondence: D. L. Young (e-mail: dlyoung@ntu.edu.tw).
**Department of Civil Engineering & Hydrotech Research Institute, National Taiwan University, Taipei, Taiwan.*

a sequence of inhomogeneous problem, the spurious eigenvalues in multiply-connected domains will never happen and this is proven in our numerical experiments.

After briefly reviewing the numerical methods for waveguide eigenproblems, we delineate the governing equations and the present numerical procedures in the following sections. Then the numerical analyses of square, elliptic, concentric annular and eccentric annular waveguides are performed in the section of numerical results. The last section is the conclusions and discussions based on the numerical experiments.

II. GOVERNING EQUATIONS

The governing equations of the propagation of the electromagnetic waves are the well-known Maxwell's equations. In waveguide problems, some simplifications will reduce the governing equations into the Helmholtz equation, if harmonic waves are allowed,

$$\nabla^2\phi + k^2\phi = 0. \tag{1}$$

When $\phi = E_z$ it represents for the transverse magnetic (TM) waves, and $\phi = H_z$ accounts for the transverse electric (TE) waves. Here k is the cutoff wavenumber. The TM waves satisfy the following Dirichlet boundary condition,

$$E_z\big|_\Gamma = 0. \tag{2}$$

The TE waves satisfy the following Neumann boundary condition,

$$\left.\frac{\partial H_z}{\partial n}\right|_\Gamma = 0. \tag{3}$$

Here Γ is the boundary which surrounds the computational domain Ω.

The homogeneous Helmholtz equation, Eq. (1), with the homogeneous boundary condition, Eq. (2) or (3), forms the eigenproblems for the waveguides. The cutoff wavenumber must be analytically or numerically determined. Once the cutoff wavenumber is obtained, the cutoff wavelengths (λ) can be computed from the following formula [3, 12],

$$\lambda = \frac{2\pi}{k}. \tag{4}$$

III. NUMERICAL METHODS

We solve the eigenproblem with an external source instead of starting with the original homogeneous system. An external source with known strength can be located at any place other than the computational domain. The position of the external source is denoted as $\vec{x}_{ext} = (x_{ext}, y_{ext})$ and the problem becomes the following form of inhomogeneous partial differential equation,

$$\nabla^2\phi(\vec{x}) + k^2\phi(\vec{x}) = \delta\left(\left|\vec{x} - \vec{x}_{ext}\right|\right). \tag{5}$$

The boundary conditions remain the homogeneous Dirichlet or Neumann boundary condition. Since the MFS is only applicable to solve the homogeneous problem with inhomogeneous boundary conditions, we have to decompose (5) into a homogeneous solution and a particular solution as the following form,

$$\phi(\vec{x}) = \phi_h(\vec{x}) + \phi_p(\vec{x}). \tag{6}$$

The $\phi_p(\vec{x})$ is the particular solution which satisfies the inhomogeneous equation of (5) without boundary conditions from the method of particular solution (MPS). The particular solution is also the fundamental solution of the Helmholtz equation and can be obtained by using the Fourier transform theory and is shown as follows [11]:

$$\phi_p(\vec{x}) = \frac{i}{4} H_0^{(2)}\left(k\left|\vec{x} - \vec{x}_{ext}\right|\right). \tag{7}$$

Here $H_0^{(2)}(\bullet)$ is the Hankel function of the second kind of order zero. $\phi_h(\vec{x})$ is the homogeneous solution which satisfies the homogeneous equation and the modified inhomogeneous boundary conditions,

$$\nabla^2\phi_h(\vec{x}) + k^2\phi_h(\vec{x}) = 0, \tag{8}$$

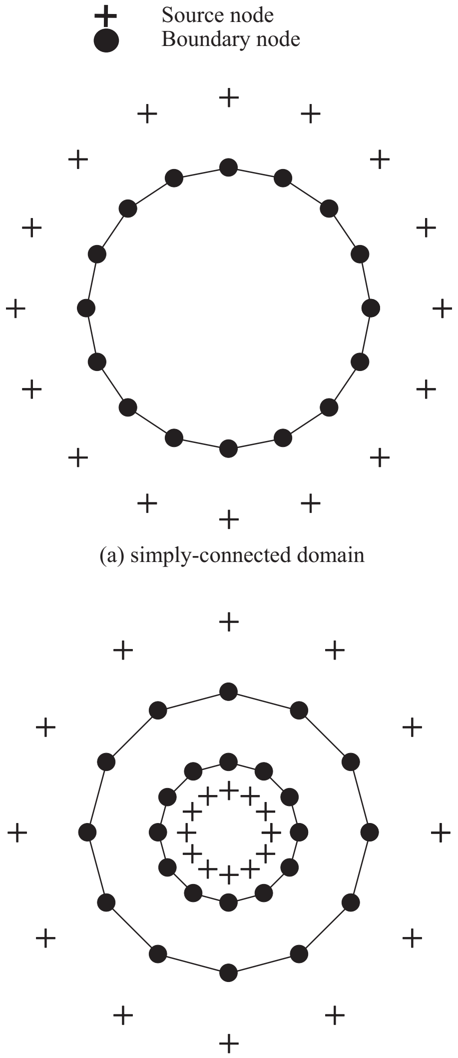
$$L_B\left[\phi_h(\vec{x})\right] = -L_B\left[\phi_p(\vec{x})\right]. \tag{9}$$

Here $L_B[\bullet]$ is the boundary partial differential operator which denotes the Dirichlet or the Neumann boundary condition. Now the eigenproblem is finally converted to the Helmholtz equation with inhomogeneous boundary conditions, Eq. (9). The MFS is very favorable to the solution of this system.

The MFS is capable to solve the sequence of the eigenproblems with inhomogeneous boundary conditions which correspond to different cutoff wavenumber. The MFS solutions can be expressed as a linear combination of fundamental solutions with unknown source strengths, α_j,

$$\phi_h(\vec{x}) = \sum_{j=1}^{N} \alpha_j G\left(k, \left|\vec{x} - \vec{\xi}_j\right|\right). \tag{10}$$

Here N is the number of source points and α_j are the unknown coefficients or source intensities which will be obtained by the method of collocation through the boundary conditions. $G\left(k, \left|\vec{x} - \vec{\xi}_j\right|\right)$ is the fundamental solution of the Helmholtz equation and it is identical to (7). $\vec{\xi}_j$ is the



(a) simply-connected domain

(b) multiply-connected domain

Fig. 1. The schematic diagrams of point distributions in the MFS for (a) simply-connected domain and (b) multiply-connected domain.

position of the j-th source node which is located outside the computational domain to avoid the singularity when the source points approach boundary or field points, as shown on Fig. 1.

By collocating the boundary conditions, a linear matrix equation is formed,

$$\left[A_{ij}\right]\left\{\alpha_j\right\}=\left\{b_i\right\}. \tag{11}$$

The components of $[A_{ij}]$ are constructed via the fundamental solution and $\{b_i\}$ is formulated by the inhomogeneous boundary conditions. After inverting the matrix system, the unknown coefficients can be obtained.

Since the numerical procedure is based on the response of the system to an external source, we have to introduce the norm of the solution as [6]:

$$F(k)=\sqrt{\frac{1}{N_t}\sum_{j=1}^{N_t}\left|\phi_h\left(\vec{x}_j\right)\right|^2}, \tag{12}$$

$$F_d(k)=F(k)/F(k_0). \tag{13}$$

Here $F_d(k)$ is a dimensionless value. k_0 is a reference wavenumber which is set as unit in our study. N_t is the number of measurement points randomly distributed inside the domain. We need to increase the wavenumber and solve for every eigenproblem which corresponds to different wavenumber. If the input wavenumber hits the exact eigenfrequency of the system, the norm will show a peak as depicted in the resonance curves of the following figures. Sometimes the figure may not be smooth; hence we have to introduce a smoothing procedure by a shift of the wavenumber [6-9]. The wavenumber of the external source will be shifted by a constant Δk. Now the particular solution becomes the following form,

$$\phi_p(\vec{x})=\frac{i}{4}H_0^{(2)}\left((k+\Delta k)\left|\vec{x}-\vec{x}_{ext}\right|\right). \tag{14}$$

Once the eigenfrequency is determined, the eigenfunction can be found by (6) which satisfies the original homogeneous equation and boundary conditions in the computational domain.

IV. NUMERICAL RESULTS

The distributions of nodes for simply- and multiply-connected domains are depicted respectively in Fig. 1, and it is noticed that the source nodes are located outside the computational domain to avoid the singularity. In this section we will investigate the square, elliptic, concentric annular and eccentric annular waveguides to test the feasibility of the adopted meshless numerical scheme. The following numerical experiments will also examine the influences of the location of the external source and the magnitude of the shift of wavenumber. Moreover, the spurious eigenvalues are not found in the concentric annular waveguide. By observing the numerical results, it can be shown that the present meshless method is a very powerful and stable numerical scheme for the determination of the eigenfrequencies for waveguides as comparing to other comparable numerical schemes.

1. Square Waveguide

The square waveguide is considered as the first validating eigenproblem and the corresponding resonance curve of the square waveguide is shown in Fig. 2. There are five peaks appeared in the range from zero to twelve so only the first five eigenfrequencies exist in the studying range. N and Δk are set as 40 and 0.1 respectively in this numerical experiment. The external source is located at (10, 10). The details of the comparison are illustrated in Table 1. In Table 1(a), the numerical

Table 1. Comparison of the first five eigenfrequencies for square waveguide.

(a) Comparison of the numerical solutions with analytical and other numerical results.

	Analytical Solution	MFS-ES (N = 24)	MFS-ES (N = 32)	MFS-ES (N = 40)	GDQ method (N = 144) [2]	GDQ method (N = 324) [2]
1	4.4429	4.4429	4.4429	4.4429	4.4429	4.4429
2	7.0248	7.0248	7.0248	7.0248	7.0248	7.0248
3	8.8858	8.8857	8.8858	8.8858	8.8857	8.8858
4	9.9346	9.9349	9.9346	9.9346	9.9469	9.9346
5	11.3237	11.3267	11.3237	11.3237	11.3448	11.3237

(b) Numerical solutions obtained by using different Δk.

	Analytical Solution	$\Delta k = 0.1$	$\Delta k = 1$	$\Delta k = 10$
1	4.4429	4.4429	4.4429	4.4429
2	7.0248	7.0248	7.0248	7.0248
3	8.8858	8.8858	8.8858	8.8858
4	9.9346	9.9346	9.9346	9.9346
5	11.3237	11.3237	11.3237	11.3237

(c) Numerical solutions obtained by different locations of external source.

	Analytical Solution	$\vec{x}_{ext} =$ (10, 10)	$\vec{x}_{ext} =$ (100, 100)	$\vec{x}_{ext} =$ (1000, 1000)
1	4.4429	4.4429	4.4429	4.4429
2	7.0248	7.0248	7.0248	7.0248
3	8.8858	8.8858	8.8858	8.8858
4	9.9346	9.9346	9.9346	9.9346
5	11.3237	11.3237	11.3237	11.3237

solutions with 24, 32 and 40 nodes are presented. Those results are compared well with analytical solutions and other numerical results obtained by the generalized differential quadrature (GDQ) method [2]. We can obtain the excellent solution by few collocating points when compared with the GDQ method, since the MFS-ES is a boundary-type numerical method. Tables 1(b) and 1(c) examine the influences of the shift of wavenumber and the locations of the external source on the numerical results. In our studying ranges, these two factors will not influence the accuracy of the solutions. Therefore, the numerical scheme is very stable and robust on determining the eigenfrequencies.

2. Elliptic Waveguide (eccentricity = e= 0.9)

Figure 3 shows the resonance curves for the TM mode and TE mode for the elliptic waveguide. The MFS computation with the external source located at (10, 10) only uses 24 nodes. N_t and Δk are set to be 20 and 0.1 respectively. Figures 4 and 5 respectively display the first four eigenmodes for the TM and TE waves of the elliptic waveguide. Table 2 illustrates the

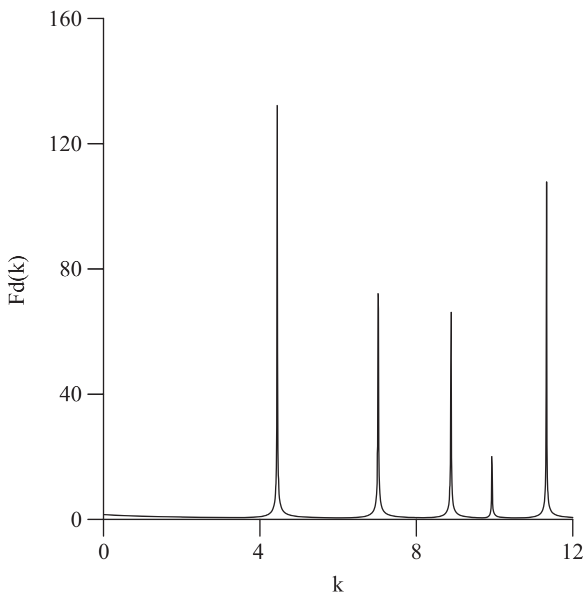


Fig. 2. Resonance curve for square waveguide.

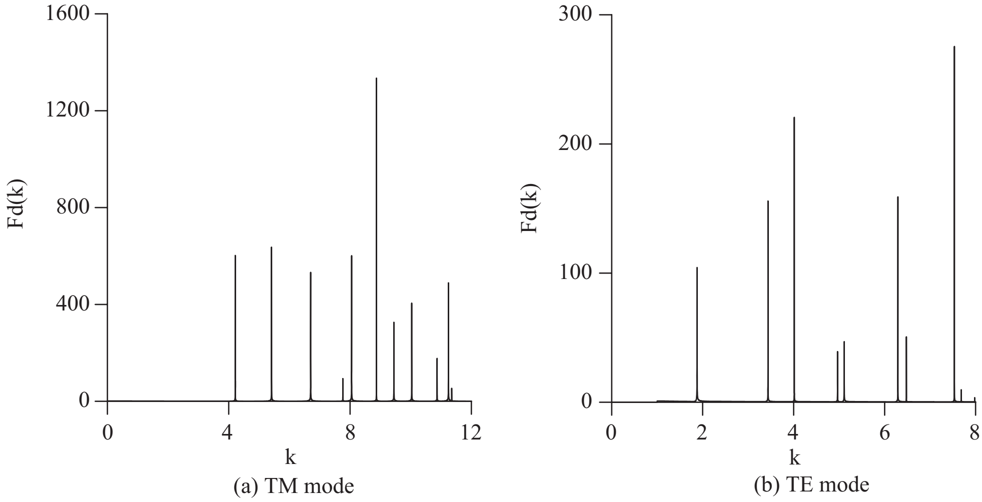


Fig. 3. Resonance curves for elliptic waveguide (a) TM mode and (b) TE mode.

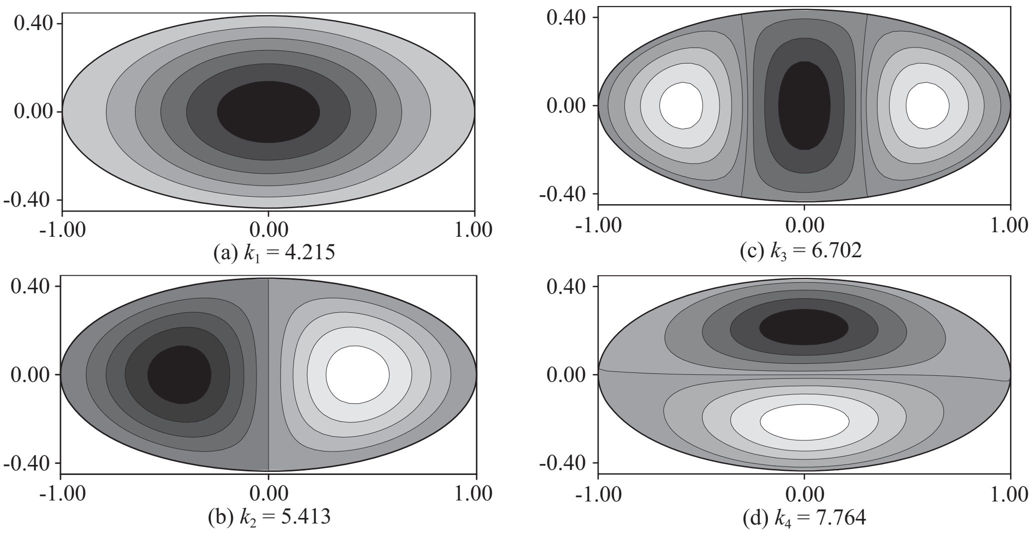


Fig. 4. The first four eigenmodes for elliptic waveguide (TM mode) .

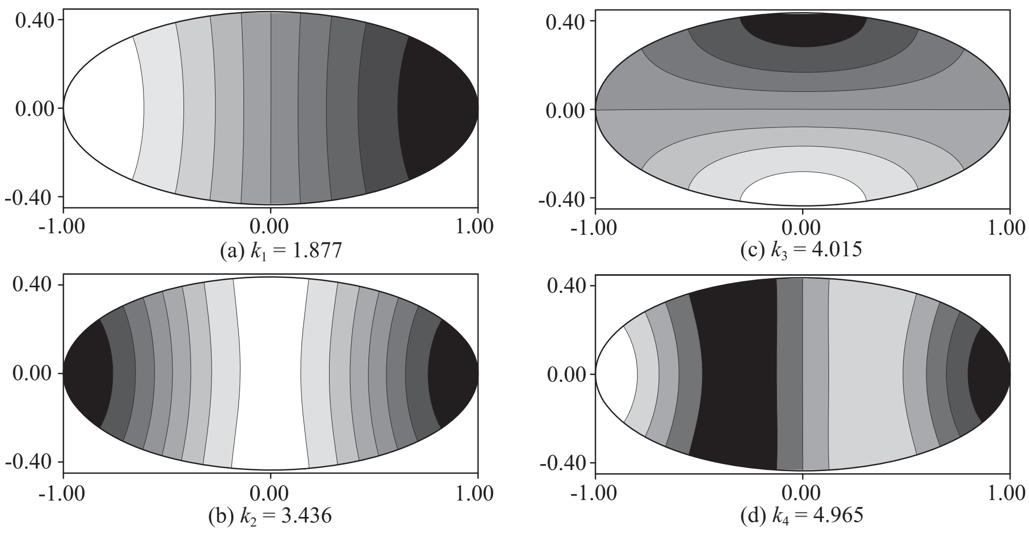


Fig. 5. The first four eigenmodes for elliptic waveguide (TE mode).

Table 2. Comparison of the first nine cutoff wavelengths for elliptic waveguide.

(a) TM mode

	Analytical Solution	RBFCM (N = 256) [3]	MFS-SVD (N = 20) [12]	MFS-ES (N = 20)	MFS-ES (N = 24)
1	1.4906	1.4906	1.4908	1.4906	1.4906
2	1.1607	1.1607	1.1607	1.1607	1.1607
3	0.9375	0.9376	0.9375	0.9375	0.9375
4	0.8093	0.8093	0.8093	0.8093	0.8093
5	0.7803	0.7804	0.7799	0.7803	0.7803
6	0.7083	0.7083	0.7071	0.7083	0.7083
7	0.6651	0.6649	0.6642	0.6652	0.6651
8	0.6262	0.6263	0.6260	0.6263	0.6262
9	0.5780	0.5781	0.5761	0.5779	0.5780

(b) TE mode

	Analytical Solution	RBFCM (N = 256) [3]	MFS-SVD (N = 20) [12]	MFS-ES (N = 16)	MFS-ES (N = 20)
1	3.3482	3.3479	3.3481	3.3482	3.3482
2	1.8287	1.8284	1.8287	1.8287	1.8287
3	1.5650	1.5650	1.5649	1.5650	1.5650
4	1.2654	1.2656	1.2654	1.2654	1.2654
5	1.2292	1.2293	1.2292	1.2292	1.2292
6	0.9986	0.9984	0.9986	0.9986	0.9986
7	0.9698	0.9750	0.9698	0.9696	0.9698
8	0.8340	0.8385	0.8340	0.8327	0.8340
9	0.8177	0.8180	0.8175	0.8177	0.8177

details of the comparisons of the elliptic waveguide. The numerical results compare well with the analytical solutions and other numerical results [3, 12].

3. Concentric Annular Waveguide

The radii of the inner and outer boundaries of the concentric annular waveguide are 0.5 and 2 respectively. The center of the inner and outer boundaries is (0, 0). The eigenfrequencies for the TM waves of the concentric annular waveguide can be obtained analytically, and it is proven that the spurious eigenvalues will exist if the inner sources are placed as a circle with radius R_1 [1, 8, 11],

$$J_n\left(kR_1\right) = 0. \tag{15}$$

Here $J_n(\bullet)$ is the Bessel function of the first kind of order n. In our experiment, R_1 is set as 0.4, hence the first spurious eigenvalue will be 6.012. By observing Fig. 6, the spurious eigenvalues do not occur. The existence of the spurious eigenvalues is due to direct solution of the homogeneous equation and boundary conditions [1, 8, 11]. Accordingly, it is

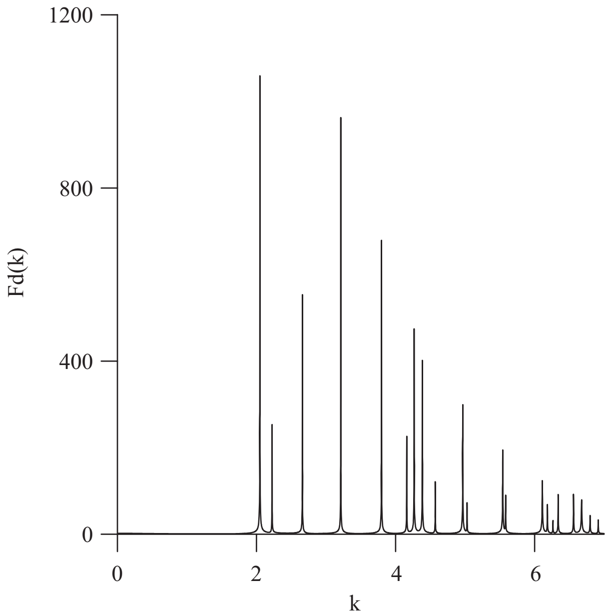


Fig. 6 Resonance curve for concentric annular waveguide.

Table 3. Comparison of the first five eigenfrequencies for concentric annular waveguide.

	Analytical Solution	FEM [1]	BEM [1]	MFS-DDSM [11]	MFS-ES (N = 40)
1	2.05	2.03	2.06	2.05	2.05
2	2.23	2.20	2.23	2.22	2.23
3	2.66	2.62	2.67	2.66	2.66
4	3.21	3.15	3.22	3.21	3.21
5	3.80	3.71	3.81	3.80	3.80

Table 4. Comparison of the first five eigenfrequencies for eccentric annular waveguide.

(a) TM mode

	Lin *et al.* (2001) [5]	Kuttler (1984) [4]	MFS-ES (N = 60)	MFS-ES (N = 100)	MFS-ES (N = 140)
1	4.8129	4.8119	4.8105	4.8106	4.8106
2	5.5252	5.5125	5.5112	5.5114	5.5114
3	6.2099	6.1735	6.1719	6.1724	6.1724
4	6.8375	6.8002	6.7989	6.7991	6.7991
5	7.4619	7.3957	7.3942	7.3945	7.3945

(b) TE mode

	Lin *et al.* (2001) [5]	Kuttler (1984) [4]	MFS-ES (N = 60)	MFS-ES (N = 100)	MFS-ES (N = 140)	MFS-ES (N = 200)
1	1.3614	1.3522	1.3535	1.3522	1.3522	1.3522
2	1.4122	1.4097	1.4092	1.4079	1.4079	1.4079
3	2.7155	2.6840	2.6852	2.6839	2.6839	2.6839
4	2.7331	2.6862	2.6874	2.6861	2.6861	2.6861
5	3.9723	3.9298	3.9308	3.9296	3.9295	3.9295

reasonable to expect that the spurious eigenvalues do not exist in this numerical scheme, since the homogeneous system has been converted into an inhomogeneous system with an external source. Table 3 shows the details of the comparison of the concentric annular waveguide with analytical and other numerical methods such as finite element method (FEM), BEM and MFS.

4. Eccentric Annular Waveguide

The computational domain of the eccentric annular waveguide is the same as that of Ref. [4, 5]. 100 nodes are used and the external source is located at (10, 10). N_t and Δk are set to be 20 and 0.1 respectively. Figure 7 depicts the TM and TE resonance curves of the eccentric annular waveguide. The first four eigenmodes for the TM and TE modes are displayed in Figs. 8 and 9, respectively. Table 4 lists the details of comparison of the eccentric annular waveguide with other numerical solutions. It is remarkable to observe the present algorithm will find the eigenfrequencies at very coarse collocating points such as less than 100 points as comparing to other numerical methods [4, 5].

V. CONCLUSIONS AND DISCUSSIONS

This article uses a novel meshless numerical method, the combination of the MFS and MPS following the lead of Reutskiy, to determine the eigenfrequencies of four different waveguides. The responses of a system exposed to an external source are recorded to estimate the cutoff wavenumber of the waveguides through the resonant excitation. Instead of adopting the homogeneous equation and boundary conditions in the eigenproblems as treated by other numerical methods such as the FEM, BEM, GDQ, RBFCM and MFS; the eigenproblems are instead converted to a sequence of inhomogeneous problems by adding an external source outside the computational domain so that we can employ the meshless MPS and MFS to solve the system. The square, elliptic, concentric annular and eccentric annular waveguides are examined to validate the capability and robustness of the present meshless numerical method. It is worthy to notice that the common spurious eigenvalues in multiply-connected domains suffered in analyzing eigenproblems by other boundary-type numerical methods will not happen in this meshless method. Moreover, our nu-

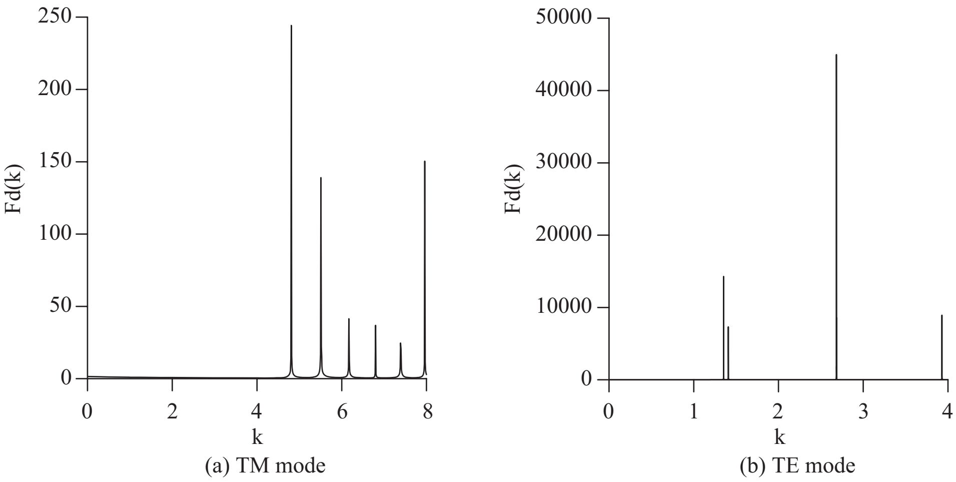


Fig. 7. Resonance curves for eccentric annular waveguide (a) TM mode and (b) TE mode.

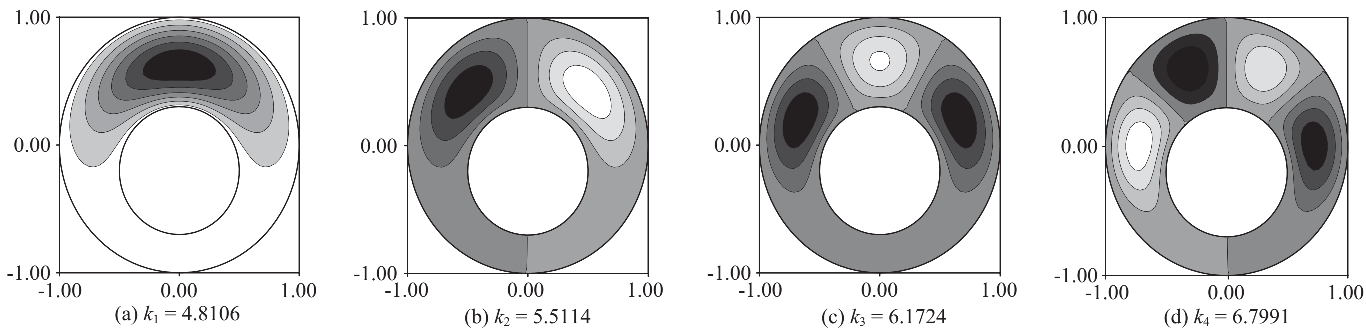


Fig. 8. The first four eigenmodes for eccentric annular waveguide (TM mode).

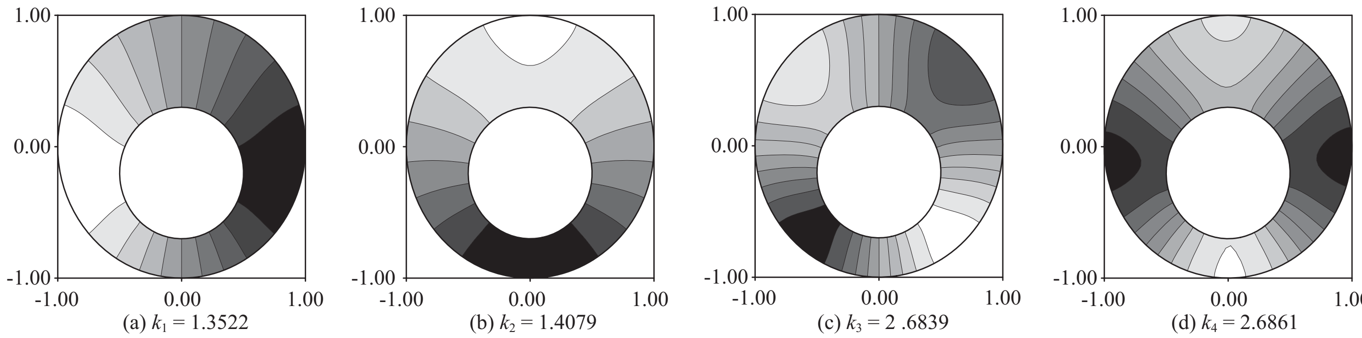


Fig. 9. The first four eigenmodes for eccentric annular waveguide (TE mode).

merical results compare well with analytical solutions and other numerical methods although very few collocating points are used in this study.

ACKNOWLEDGMENTS

The National Science Council of Taiwan is greatly appreciated for providing financial support of this research under the Grant No. NSC 95-2221-E-002-406.

REFERENCES

1. Chen, J. T., Liu, L. W., and Hong, H. K., "Spurious and true eigensolutions of Helmholtz BIEs and BEMs for a multiply connected problem," *Proceedings of the Royal Society A*, Vol. 459, pp. 1897-1924 (2003).
2. Dong, Y. C., Leong, M. S., Koo, P. S., Lam, K. Y., and Shu, C., "Computation of the propagation characteristics of TE and TM modes in waveguides with the use of the generalized differential quadrature method," *Microwave and Optical Technology Letters*, Vol. 14, No. 1, pp. 39-44 (1997).

3. Jiang, P. L., Li, S. Q., and Chan, C. H., "Analysis of elliptic waveguides by a meshless collocation method with the Wendland radial basis functions," *Microwave and Optical Technology Letters*, Vol. 32, No. 2, pp. 162-165 (2002).
4. Kuttler, J. R., "A new method for calculating TE and TM cutoff frequencies of uniform waveguides with lunar or eccentric annular cross section," *IEEE Transactions on Microwave Theory and Techniques*, Vol. 32, No. 4, pp. 348-354 (1984).
5. Lin, S. L., Li, L. W., Yeo, T. S., and Leong, M. S., "Analysis of metallic waveguides of a large class of cross sections using polynomial approximation and superquadric functions," *IEEE Transactions on Microwave Theory and Techniques*, Vol. 49, No. 6, pp. 1136-1139 (2001).
6. Reutskiy, S. Y., "The method of fundamental solutions for eigenproblems with Laplace and biharmonic operators," *Computers, Materials & Continua*, Vol. 2, No. 3, pp. 177-188 (2005).
7. Reutskiy, S. Y., "The method of fundamental solutions for Helmholtz eigenvalue problems in simply and multiply connected domains," *Engineering Analysis with Boundary Elements*, Vol. 30, pp. 150-159 (2006).
8. Reutskiy, S. Y., "The method of external sources (MES) for eigenvalues problems with Helmholtz equation," *CMES: Computer Modeling in Engineering & Sciences*, Vol. 12, No. 1, pp. 27-39 (2006).
9. Reutskiy, S. Y., "The method of fundamental solutions for problems of free vibrations of plates," *Engineering Analysis with Boundary Elements*, Vol. 31, pp. 10-21 (2007).
10. Tsai, C. C., Young, D. L., Chiu, C. L., and Fan, C. M., "Numerical analysis of acoustic modes using the linear least squares method of fundamental solutions," *Journal of Sound and Vibration*, Vol. 324, pp. 1086-1110 (2009).
11. Tsai, C. C., Young, D. L., Chen, C. W., and Fan, C. M., "The method of fundamental solutions for eigenproblems in domains with and without interior holes," *Proceedings of the Royal Society A*, Vol. 462, pp. 1443-1466 (2006).
12. Young, D. L., Hu, S. P., Chen, C. W., Fan, C. M., and Murugesan, K., "Analysis of elliptical waveguides by the method of fundamental solutions," *Microwave and Optical Technology Letters*, Vol. 44, No. 6, pp. 552-558 (2005).