

On Integral Equation Methods for Crack Problems\*

By Kazuo TAKAKUDA\*\*, Takashi KOIZUMI\*\* and Toshikazu SHIBUYA\*\*

In this paper, integral equation methods for boundary value problems of elastic bodies with cracks are investigated. We use Somigliana's identities to represent displacements in elastic bodies, and deduce integral equations from the boundary conditions on the crack surfaces. In these integral equations, unknown functions are displacement differences between the upper and the lower crack surfaces. Divergent integrals are contained and they are evaluated in the sense of Hadamard's finite parts. The integral equations are solved analytically for the case of a penny-shaped crack. Methods for numerical solutions of the equations are also studied.

Key Words : Elasticity, Crack Problem, Integral Equation Method, Elastic Potential, Numerical Analysis

1. Introduction

In connection with the fracture strength of materials, it is important to analyse the stresses induced in elastic bodies with cracks. However the analytical solutions are available only for such simple cases as a penny shaped crack in an infinite body. Hence several methods for numerical solutions have been developed, by which we can analyse elastic bodies with more complex configurations.

One method among them utilize Somigliana's identities and reduce elastic problems to integral equations<sup>(1)(2)</sup>, as the boundary integral equation methods (direct methods) do. This method has the following advantages: (i) it enables us to treat bodies with arbitrary shape if it is used together with the boundary integral equations (direct methods); (ii) unknown displacements or tractions on surfaces are obtained as the solutions of the integral equations. But it also has some difficulties, i.e., contained in the equations there are the tangential derivatives of the unknown displacement differences between the upper and the lower crack surfaces. In contrast, the body force technique<sup>(3)(4)</sup> has not such difficulty, no derivatives appearing as unknown functions wherein, and it is used effectively for crack problems.

In this paper, we remedy this shortcoming of the method for crack problems based on Somigliana's identities. We deduce integral equations for crack problems in which the unknown functions are the displacement differences between the upper and the lower crack surfaces, themselves. Divergent integrals contained, which have been avoided in previous studies and resulting in the appearance of the derivatives of unknowns, are evaluated in the sense of Hadamard's finite parts<sup>(5)</sup>. An example of the analytical

solutions of the integral equations is given, for the case of a penny-shaped crack in an infinite body. We also investigate the methods for numerical solutions of the integral equations.

2. Integral Equations for Crack Problems

We use Cartesian coordinates system  $x_i$ . Subscripts of English letters take the values 1, 2 or 3, and those of Greek letters, 1 or 2. Summation convention is assumed, and a subscript following a comma ( $i$ ) indicates differentiation ( $\partial/\partial x_i$ ) with respect to the coordinate variable of the point where the displacement is defined.

In general, the displacement  $u_i$  induced in an elastic body can be expressed in terms of the displacements  $u_i$  and the tractions  $t_i^{(n)}$  on the surface  $S$  of the elastic body, and the body forces  $f_i$  within the body  $D$ .

$$u_i(P) = \int_S t_j^{(n)}(Q) U_i^{(j)}(P, Q) ds(Q) - \int_S u_j(Q) T_i^{(j)}(P, Q) ds(Q) + \int_D f_j(Q) U_i^{(j)}(P, Q) dv(Q) \quad (P \in D) \dots (1)$$

This relation is known as Somigliana's identity<sup>(6)</sup>, in which  $U_i^{(j)}(P, Q)$  denotes the displacement in  $i$  direction at a point  $P$  in an infinite elastic body when unit concentrated body force in  $j$  direction is applied at a point  $Q$ . With the modulus of rigidity  $\mu$ , Poisson's ratio  $\nu$  and the distance  $R$  between points  $P$  and  $Q$ , it is represented in the form

$$U_i^{(j)}(P, Q) = \frac{1}{16\pi\mu} \frac{3-4\nu}{1-\nu} \frac{1}{R} \delta_{ij} + \frac{1}{16\pi\mu} \frac{1}{1-\nu} \frac{1}{R} R_i R_j \dots (2)$$

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\*\* Tokyo Institute of Technology, 2-12-1, Ookayama, Meguro-ku, Tokyo.

Further,  $T_i^{(j)}(P, Q)$  is the displacement in  $i$  direction at a point  $P$  in an infinite elastic body when an infinitesimal dislocation loop with unit magnitude in  $j$  direction exists at a point  $Q$  on a plane normal to a vector  $n$ . With the elastic constants tensor  $C_{ijkl}$

$$C_{ijkl} = \frac{2\nu\mu}{1-2\nu} \delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \mu\delta_{il}\delta_{jk} \dots (3)$$

and the unit outward normal vector  $n_i$  of the surface  $S$ , it is represented in the form

$$T_i^{(j)}(P, Q) = C_{mjkl} n_m(Q) U_{k,l}^{(j)}(Q, P) \dots (4)$$

Now we consider an infinite elastic body occupying the whole space  $x_1, x_2, x_3$  and containing a crack on the plane  $x_1, x_2$  as shown in Fig.1. External forces are applied on the crack surfaces. The displacements and the stresses decrease to zero at infinity. We regard the elastic body as either (i) an infinite body containing a cavity which collapses to a crack in the limit, or (ii) two semi-infinite bodies  $x_3 > 0$  and  $x_3 < 0$  which are joined over the plane  $x_3 = 0$  excluding the crack surface  $S_0$ . Anyway, Eq. (1) gives a representation formula for the displacements in this elastic body

$$\begin{aligned} u_i(P) = & \int_{S_0} [(t_j^{(n)}(Q))^+ + (t_j^{(n)}(Q))^-] \\ & \times U_i^{(j)}(P, Q) ds(Q) - \int_{S_0} [(u_j(Q))^+ \\ & - (u_j(Q))^-] [T_i^{(j)}(P, Q)]^+ ds(Q) \\ & + \int_D f_j(Q) U_i^{(j)}(P, Q) dv(Q) \quad (P \in D) \end{aligned} \dots (5)$$

where  $\{ \}^+ [ \{ \}^- ]$  indicates a value concerned with the upper [lower] crack surface  $x_3 = +0 [-0]$ .

We observe in the expression (5) that the first and the third terms of the right hand side are known. Hence if the displacement difference between the upper and the

For the first and the third terms in the above formula, we can easily obtain limits as the point  $P$  approaches the point  $P_0$  on the crack surface<sup>(7)</sup>. For the second term, however, the integrand

$$\begin{aligned} T_{i,k}^{(j)}(P, Q) = & \frac{1}{8\pi} \frac{1-2\nu}{1-\nu} \frac{1}{R^3} [(n_i\delta_{jk} - n_j\delta_{ik} + n_k\delta_{ij}) - 3(n_iR_jR_k - n_jR_iR_k) - 3n_iR_kR_i\delta_{ij}] \\ & + \frac{3}{8\pi} \frac{1}{1-\nu} \frac{1}{R^3} [n_kR_iR_j + n_iR_iR_j\delta_{ik} + n_iR_jR_i\delta_{ik} - 5n_iR_iR_jR_kR_i] \end{aligned} \dots (9)$$

has a singularity of  $O(R^{-3})$  as  $R \rightarrow 0$ . Therefore it is not possible to obtain a limit in the present form. We rewrite this term assuming that the displacement difference  $\Delta u_i(Q)$  is differentiable and can be expanded near a point  $P_0$  in the form

$$\Delta u_i(Q) = \Delta u_i(P_0) + \{x_\alpha(Q) - x_\alpha(P_0)\} \Delta u_{i,\alpha}(P_0) + \Delta u_{Ri}(Q, P_0) \dots (10)$$

where the function  $\Delta u_{Ri}(Q, P_0)$  is  $O(r^2)$  as  $r (= \overline{P_0Q}) \rightarrow 0$ .

Then the second term in the right hand side of Eq. (8) becomes

$$\begin{aligned} \int_{S_0} \Delta u_p(Q) C_{i3kl} \{T_k^{(p)}(P, Q)\}_i ds(Q) = & \int_{S_0} \Delta u_p(P_0) C_{i3kl} \{T_k^{(p)}(P, Q)\}_i ds(Q) \\ & + \int_{S_0} \{x_\alpha(Q) - x_\alpha(P_0)\} \Delta u_{p,\alpha}(P_0) C_{i3kl} \{T_k^{(p)}(P, Q)\}_i ds(Q) \\ & + \int_{S_0} \Delta u_{Rp}(Q, P_0) C_{i3kl} \{T_k^{(p)}(P, Q)\}_i ds(Q) \end{aligned} \dots (11)$$

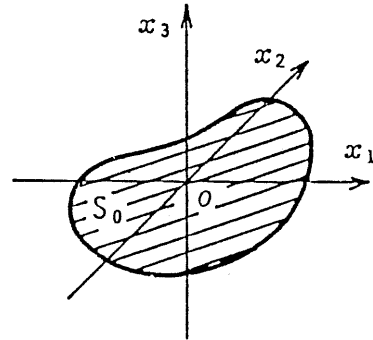


Fig.1 Plane crack in an infinite body

lower crack surfaces  $\{u_j(Q)\}^+ - \{u_j(Q)\}^-$  is determined such that the boundary condition on the crack surfaces is satisfied, then it may be substituted into Eq. (5) and give the displacement field within the elastic body. In order to realize this procedure, we require a representation formula for the tractions induced on the crack surfaces, accompanying the displacement field (5).

If we substitute Eq. (5) into the relation between displacements and stresses

$$\sigma_{ij}(P) = C_{ijkl} u_{k,l}(P) \dots (6)$$

then we have the stresses induced in the elastic body. With notations

$$\begin{aligned} \Sigma t_i(Q) = & \{t_i^{(m)}(Q)\}^+ + \{t_i^{(m)}(Q)\}^- \\ \Delta u_i(Q) = & \{u_i(Q)\}^+ - \{u_i(Q)\}^- \end{aligned} \dots (7)$$

the stress components  $\sigma_{i3}(P)$  concerned with surface tractions are expressed in the form

$$\begin{aligned} \sigma_{i3}(P) = & \int_{S_0} \Sigma t_p(Q) C_{i3kl} U_{k,l}^{(p)}(P, Q) ds(Q) \\ & - \int_{S_0} \Delta u_p(Q) C_{i3kl} \{T_k^{(p)}(P, Q)\}_i ds(Q) \\ & + \int_D f_p(Q) C_{i3kl} U_{k,l}^{(p)}(P, Q) dv(Q) \end{aligned} \dots (8)$$

( $P \in D$ )

First we calculate the limit of the first term in the right hand side of Eq. (11) as  $P \rightarrow P_0$ . We divide the integral region  $S_0$  into  $S_\epsilon$  and  $S_0 - S_\epsilon$ , where  $S_\epsilon$  is a circular region with center  $P_0$  and radius  $\epsilon$  as Fig.2 shows.

$$\begin{aligned} & \int_{S_0} \Delta u_p(P_0) C_{i3kl} \{ T_k^{(p)}(P, Q) \}_{,i} ds(Q) \\ &= \int_{S_0 - S_\epsilon} \Delta u_p(P_0) C_{i3kl} \{ T_k^{(p)}(P, Q) \}_{,i} ds(Q) \\ &+ \int_{S_\epsilon} \Delta u_p(P_0) C_{i3kl} \{ T_k^{(p)}(P, Q) \}_{,i} ds(Q) \\ & \dots \dots \dots (12) \end{aligned}$$

In the first term of the right hand side of the above expression, there appear definite integrals as  $\int_{S_\epsilon} 1/R^3 ds$ , each of which is evaluated in a closed form as

$$\begin{aligned} \int_{S_\epsilon} 1/R^3 ds &= 2\pi \int_0^\epsilon r/R^3 dr \\ &= 2\pi \int_\delta^{\sqrt{\epsilon^2 + \delta^2}} 1/R^2 dR = 2\pi (1/\delta - 1/(\epsilon^2 + \delta^2)^{1/2}) \end{aligned}$$

If these results are substituted into the first term of Eq.(12), then we see that the coefficients of the term  $1/\delta$  cancel out each other. So the first term itself turns

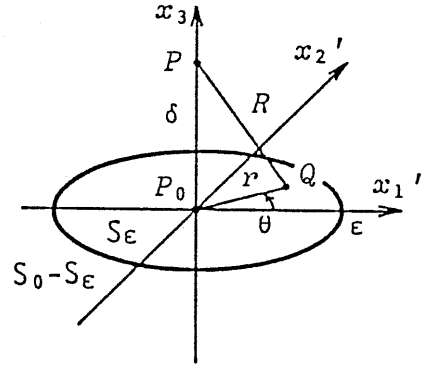


Fig.2 Limit values

out to be bounded and has a definite limit as  $P \rightarrow P_0 (\delta \rightarrow 0)$ . In this place, we have the following integral formulas deduced from the definition of the Pf integrals (Appendix 1) as:

$$Pf \int_{S_\epsilon} 1/r^3 ds = -2\pi/\epsilon$$

Hence the limit value of the first term as the point  $P$  approaches  $P_0$  is given by setting  $P = P_0$  in the integrand and evaluating the integrals in the sense of Pf integrals. On the other hand, the second term in the right hand side of Eq.(12) is a continuous function. Thus we have

$$\lim_{P \rightarrow P_0} \int_{S_0} \Delta u_p(P_0) C_{i3kl} \{ T_k^{(p)}(P, Q) \}_{,i} ds(Q) = Pf \int_{S_0} \Delta u_p(P_0) C_{i3kl} \{ T_k^{(p)}(P_0, Q) \}_{,i} ds(Q) \quad (P_0 \in S_0) \dots \dots \dots (13)$$

In a similar manner we have limits of the second and the third terms in the right hand side of Eq.(11).

$$\begin{aligned} & \lim_{P \rightarrow P_0} \int_{S_0} (x_\alpha(Q) - x_\alpha(P_0)) \Delta u_{p,\alpha}(P_0) C_{i3kl} \{ T_k^{(p)}(P, Q) \}_{,i} ds(Q) \\ &= Pf \int_{S_0} (x_\alpha(Q) - x_\alpha(P_0)) \Delta u_{p,\alpha}(P_0) C_{i3kl} \{ T_k^{(p)}(P_0, Q) \}_{,i} ds(Q) \quad (P_0 \in S_0) \dots \dots \dots (14) \end{aligned}$$

$$\begin{aligned} & \lim_{P \rightarrow P_0} \int_{S_0} \Delta u_{Rp}(Q, P_0) C_{i3kl} \{ T_k^{(p)}(P, Q) \}_{,i} ds(Q) \\ &= Pf \int_{S_0} \Delta u_{Rp}(Q, P_0) C_{i3kl} \{ T_k^{(p)}(P_0, Q) \}_{,i} ds(Q) \quad (P_0 \in S_0) \dots \dots \dots (15) \end{aligned}$$

Summing up the above results, the representation formula for the tractions on the crack surface is given by

$$\begin{aligned} (\sigma_{i3}(P_0))_i &= \mp \frac{1}{2} \sum t_i(P_0) + Pf \int_{S_0} \sum t_j(Q) \{ T_j^{(i)}(P_0, Q) \}_i ds(Q) \\ &- Pf \int_{S_0} \Delta u_p(Q) C_{i3kl} \{ T_k^{(p)}(P_0, Q) \}_{,i} ds(Q) + \int_{S_0} f_i(Q) \{ T_j^{(i)}(P_0, Q) \}_i dv(Q) \quad (P_0 \in S_0) \dots \dots \dots (16) \end{aligned}$$

In particular, in the cases that external forces act in opposite directions and have the same magnitudes on the upper and the lower crack surfaces  $[\sum t_i(P_0) = 0]$ , and the body forces are absent  $[f_i(P) = 0]$ , Eq.(16) becomes

$$\left. \begin{aligned} \sigma_{13}(P_0) &= \frac{1}{4\pi} \frac{\mu}{1-\nu} Pf \int_{S_0} \left[ \left( (1-2\nu) \frac{1}{R^3} + 3\nu \frac{1}{R^3} R_{,1} R_{,1} \right) \Delta u_1(Q) + 3\nu \frac{1}{R^3} R_{,1} R_{,2} \Delta u_2(Q) \right] ds(Q) \\ \sigma_{23}(P_0) &= \frac{1}{4\pi} \frac{\mu}{1-\nu} Pf \int_{S_0} \left[ 3\nu \frac{1}{R^3} R_{,1} R_{,2} \Delta u_1(Q) + \left( (1-2\nu) \frac{1}{R^3} + 3\nu \frac{1}{R^3} R_{,2} R_{,2} \right) \Delta u_2(Q) \right] ds(Q) \\ \sigma_{33}(P_0) &= \frac{1}{4\pi} \frac{\mu}{1-\nu} Pf \int_{S_0} \frac{1}{R^3} \Delta u_3(Q) ds(Q) \end{aligned} \right\} \dots \dots \dots (17)$$

where  $R$  is the distance between points  $P_0$  and  $Q$ . Consequently, if we determine  $\Delta u_i(P_0)$ , or the unknown displacement differences between the upper and the lower crack surfaces, such that Eq.(16) or (17) are satisfied for the given tractions  $\sigma_{i3}(P_0)$  on the crack surfaces, then we substitute it into Eq.(5) and obtain the solutions for the crack problems. From the uniqueness theorem for elastic problems, we see that the solutions  $\Delta u_i(P_0)$  of the integral equations (16) or (17) are also unique. Moreover,

the stress intensity factors can be obtained easily, since the relations between the displacement differences  $\Delta u_i$  and the stress intensity factors are expressed in the form

$$\Delta u_3 = 2(2/\pi)^{1/2}(1-\nu)/\mu \cdot K_{II} r^{1/2}, \quad \Delta u_n = 2(2/\pi)^{1/2}(1-\nu)/\mu \cdot K_{II} r^{1/2}, \quad \Delta u_i = 2(2/\pi)^{1/2}/\mu \cdot K_{III} r^{1/2} \dots (18)$$

where  $r$  denotes the distance from the crack edge,  $u_n[u_i]$  is the displacement in the direction parallel to the crack plane and perpendicular [parallel] to the crack edge.

In the investigations reported previously<sup>(1)(8)</sup>, the integral equations for crack problems are expressed in the forms

$$\left. \begin{aligned} \sigma_{\alpha 3}(P_0) &= \frac{1}{8\pi} \frac{\mu}{1-\nu} \int_{S_0} \frac{1}{R^2} [(1-2\nu)(R_{,r} \delta_{\alpha s} - R_{,s} \delta_{\alpha r}) + 3R_{,s} R_{,s} R_{,r}] \Delta u_{s,r}(Q) ds(Q) \quad (\alpha=1, 2) \\ \sigma_{33}(P_0) &= \frac{1}{4\pi} \frac{\mu}{1-\nu} \int_{S_0} \frac{1}{R^2} R_{,s} \Delta u_{3,s}(Q) ds(Q) \end{aligned} \right\} (P_0 \in S_0) \dots \dots \dots (19)$$

The equivalency of Eq.(17) to (19) can be proved by taking into account that  $\Delta u_i(P_0)$  has an expansion Eq.(10) and utilizing the integration formula (37) with the condition  $\Delta u_i(P_0)=0$  along the crack edge.

### 3. Penny-shaped Crack Subjected to Internal Pressure

In some cases we can obtain analytical solutions of the integral equation (17). Here we present an example for a problem of a penny-shaped crack in an infinite body subjected to arbitrary internal pressure.

We use cylindrical coordinates system  $(r, \theta, z)$ . The crack surface  $S_0$  is assumed to occupy the region  $(z=0, 0 \leq r < a, 0 \leq \theta < 2\pi)$ . Let  $p(r, \theta)$  denote internal pressure applied on the crack surface. Then the third one of the integral equation (17) is expressed in the form

$$\frac{1}{4\pi} \frac{\mu}{1-\nu} Pf \int_{S_0} \frac{1}{R^3} \Delta u_z(r_0, \theta_0) r_0 d\theta_0 dr_0 = -p(r, \theta) \quad (0 \leq r < a, 0 \leq \theta < 2\pi) \dots \dots \dots (20)$$

The second and the third ones of Eq.(17) are identically satisfied if we put  $\Delta u_r(r, \theta) = \Delta u_\theta(r, \theta) = 0$ .

Copson<sup>(9)</sup> has formulated a punch problem, for which a rigid circular punch indents an elastic half space, as an integral equation similar to Eq.(20). The equation was reduced to the form of Abel's type integral equation, and he could obtain a simple, clear solution. We expect that Eq.(20) for the crack problem can be solved similarly.

We begin with writing

$$p(r, \theta) = \sum_{n=0}^{\infty} p_n(r) \cos n\theta, \quad \Delta u_z(r, \theta) = \sum_{n=0}^{\infty} d_n(r) \cos n\theta \dots \dots \dots (21)$$

and substituting them into Eq.(20). Since the two-dimensional  $Pf$  integral can be evaluated by successive applications of the one-dimensional  $Pf$  integral with respect to  $r$  and  $\theta$ , Eq. (20) takes the form

$$\frac{\mu}{4\pi(1-\nu)} \sum_{n=0}^{\infty} Pf \int_0^a \left[ Pf \int_0^{2\pi} \frac{\cos n\theta_0 d\theta_0}{\{\sqrt{r^2 + r_0^2 - 2rr_0 \cos(\theta - \theta_0)}\}^3} \right] r_0 d_n(r_0) dr_0 = - \sum_{n=0}^{\infty} p_n(r) \cos n\theta \quad (0 \leq r < a, 0 \leq \theta < 2\pi)$$

Rewriting the inner integral in accordance with the integral formula (39) and comparing the coefficients of circular functions in both sides, we have

$$\begin{aligned} \frac{\mu}{\pi(1-\nu)} \frac{1}{r^n} \left[ Pf \int_0^a \left[ Pf \int_0^{r_0} \frac{t^{2n+2} dt}{\{\sqrt{(r^2 - t^2)(r_0^2 - t^2)}\}^3} \right] r_0^{1-n} d_n(r_0) dr_0 \right. \\ \left. + Pf \int_r^a \left[ Pf \int_0^r \frac{t^{2n+2} dt}{\{\sqrt{(r^2 - t^2)(r_0^2 - t^2)}\}^3} \right] r_0^{1-n} d_n(r_0) dr_0 \right] = p_n(r) \quad (n=0, 1, 2, \dots, 0 \leq r < a) \end{aligned}$$

Further, assuming that the order of integrations may be reversed, we have

$$Pf \int_0^r \frac{t^{2n+2}}{\{\sqrt{r^2 - t^2}\}^3} \left[ Pf \int_t^a \frac{r_0^{1-n} d_n(r_0)}{\{\sqrt{r_0^2 - t^2}\}^3} dr_0 \right] dt = \frac{\pi(1-\nu)}{\mu} r^n p_n(r) \quad (n=0, 1, 2, \dots, 0 \leq r < a) \dots \dots \dots (22)$$

The solution  $d_n(r_0)$  of this Abel's type integral equation is expressed in the form<sup>(10)</sup>

$$d_n(r_0) = \frac{4(1-\nu)}{\pi\mu} r_0^n \int_{r_0}^a \frac{t^{-2n}}{\sqrt{t^2 - r_0^2}} \left[ \int_0^t \frac{r^{n+1} p_n(r)}{\sqrt{t^2 - r^2}} dr \right] dt \quad (n=0, 1, 2, \dots) \dots \dots \dots (23)$$

The unknown displacement difference  $\Delta u_z$  is obtained after the substitution of Eq.(23) into the second of Eq.(21).

4. Methods for Numerical Solutions of Integral Equations

In general, we have to solve the integral equation (17) approximately. On that occasion, the Pf integrals must be evaluated numerically. Hence we first study the quadrature formula for the following Pf integral over the plane region S surrounded by the contour C as shown in Fig.3.

$$I = Pf \int_S \{R(P_0, Q)\}^{-3} f(Q) ds(Q) \quad (P_0 \in S) \dots \dots \dots (24)$$

When we derive the integral equation (17), we assume that the density function has an expansion as Eq.(10). In this place, we further assume that the density function is twice differentiable and can be expanded near a point P<sub>0</sub> in the form

$$f(Q) = f(P_0) + \{x_\alpha(Q) - x_\alpha(P_0)\} f_{,\alpha}(P_0) + \frac{1}{2} \{x_\alpha(Q) - x_\alpha(P_0)\} \{x_\beta(Q) - x_\beta(P_0)\} f_{,\alpha\beta}(P_0) + f_R(Q, P_0) \dots \dots \dots (25)$$

where function f<sub>R</sub>(Q, P<sub>0</sub>) is O(R<sup>3</sup>) as R → 0. Then the Pf integral (24) becomes

$$I = f(P_0) Pf \int_S \{R(P_0, Q)\}^{-3} ds(Q) + f_{,\alpha}(P_0) Pf \int_S \{x_\alpha(Q) - x_\alpha(P_0)\} \{R(P_0, Q)\}^{-3} ds(Q) + \frac{1}{2} f_{,\alpha\beta}(P_0) \int_S \{x_\alpha(Q) - x_\alpha(P_0)\} \{x_\beta(Q) - x_\beta(P_0)\} \{R(P_0, Q)\}^{-3} ds(Q) + \int_S f_R(Q, P_0) \{R(P_0, Q)\}^{-3} ds(Q) \dots \dots \dots (26)$$

The last term in the right hand side of Eq.(26) can be evaluated by the use of a common numerical quadrature formula. The first three integrals over the region S can be transformed into those along the contour C in the forms

$$Pf \int_S \{R(P_0, Q)\}^{-3} ds(Q) = \lim_{\epsilon \rightarrow 0} \left[ \int_0^{2\pi} \left[ \int_\epsilon^{r(\theta)} \{R(P_0, Q)\}^{-2} dR(Q) \right] d\theta(Q) - 2\pi\epsilon^{-1} \right] = - \int_0^{2\pi} \{r(\theta)\}^{-1} d\theta$$

$$Pf \int_S \{x_\alpha(Q) - x_\alpha(P_0)\} \{R(P_0, Q)\}^{-3} ds(Q) = \int_0^{2\pi} \{\cos \theta \delta_{\alpha 1} + \sin \theta \delta_{\alpha 2}\} \ln r(\theta) d\theta$$

$$\int_S \{x_\alpha(Q) - x_\alpha(P_0)\} \{x_\beta(Q) - x_\beta(P_0)\} \{R(P_0, Q)\}^{-3} ds(Q) = \int_0^{2\pi} \{\cos \theta \delta_{\alpha 1} + \sin \theta \delta_{\alpha 2}\} \{\cos \theta \delta_{\beta 1} + \sin \theta \delta_{\beta 2}\} r(\theta) d\theta \dots \dots \dots (27)$$

where r denotes the distance between the point P<sub>0</sub> and the contour C. It is easy to evaluate numerically the right hand sides of Eq.(27). In particular, when region S is a polygon and the contour C is composed of straight lines, they are obtained in closed forms.

Next we examine the strategy for discretizing the integral equation (17). A possible one may be as follows: we divide the crack surfaces into polygonal subregions, and assume that the displacement differences Δu<sub>i</sub> take constant values throughout these subregions. The approximate solutions are determined such that Eq.(17) is satisfied at suitably selected points. Although this method seems natural, it involves a difficulty that the surface tractions diverge to infinity near interboundaries between subregions. This comes from discontinuities of displacements. In Fig.4, we show an example of such distributions of surface tractions obtained from Eq.(17). Computations are carried out on the line OA of the penny-shaped crack of radius a subjected to constant internal

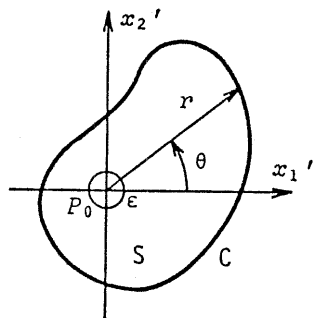


Fig.3 Numerical evaluations of Pf integrals

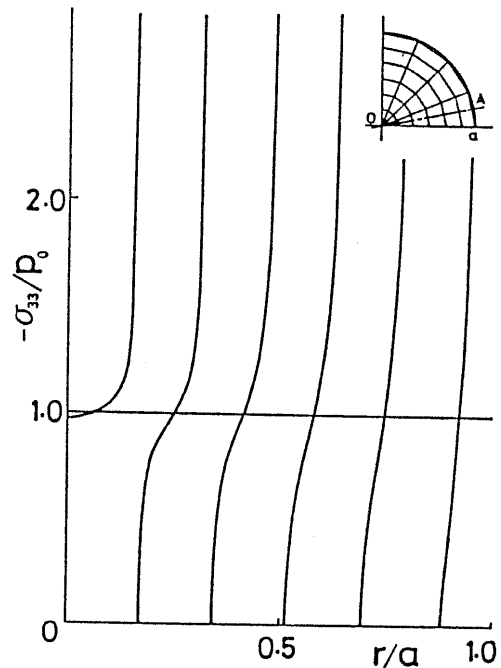


Fig.4 Surface tractions induced by constant density elements

pressure  $p_0$ . The surface of the crack is divided into 6 portions in radial direction and 16 in circumferential direction. Through-out each subregion, the displacement difference is assumed to take the exact values  $\Delta u_3 = 4(1-\nu)/\pi p_0/\mu \times (a^2 - r^2)^{1/2}$  at the centroids ( $r=r_c$ ). Since the distributions of tractions have abrupt variations within subregions, it may be expected that results utilizing constant density elements are seriously influenced by the choice of the collocation points.

As an alternative, we may utilize the  $C^1$  elements which have been developed in finite element methods<sup>(11)</sup>. It realizes continuous and differential displacement differences and the difficulty cited above will not take place. In this paper, however, we try to express the whole  $\Delta u_i$  over the crack surface by a polynomial. This is a natural extension of the method used in two-dimensional crack problems<sup>(12)</sup> in which the density of continuously distributed displacements are expressed by a series of Tchebysheff polynomials.

For the case of an elliptical crack subjected to constant internal pressure or shear<sup>(13)(14)</sup>, several calculations are carried out and a numerical solution of Eq.(17) is obtained. The elliptical crack with semi-axes  $a$  and  $b$  is represented as  $0 \leq \rho < 1$  and  $0 \leq \phi < 2\pi$ , where

$$x_1 = a\rho \cos \phi, \quad x_2 = b\rho \sin \phi \quad \dots\dots\dots(28)$$

Taking into account the behaviour of the solutions, we express the displacement differences  $\Delta u_i$  in the form

$$\Delta u_i(x_1, x_2) = \{1 - (x_1/a)^2 - (x_2/b)^2\}^{1/2} \times \sum_{n=0,2,\dots}^N \sum_{m=0,2,\dots}^M a_{nm}^i T_n(x_1/a) T_m(x_2/b) \dots\dots(29)$$

where  $a_{nm}^i$  is an unknown coefficient to be determined and  $T_n$  denotes the Tchebysheff polynomial of the first kind. The stress intensity factors can be obtained from  $a_{nm}^i$  using Eq.(18) and the relation

$$\begin{aligned} & \{1 - (x_1/a)^2 - (x_2/b)^2\}^{1/2} \\ & \rightarrow \{2/(ab)\}^{1/2} \{a^2 \sin^2 \phi + b^2 \cos^2 \phi\}^{1/4} r^{1/2} \\ & \quad (r \rightarrow 0) \quad \dots\dots\dots(30) \end{aligned}$$

where  $r$  denotes the distance from the crack edge.

Substituting Eq.(29) into the integral equation (17), we determine  $a_{nm}^i$  such that the boundary conditions are satisfied at the selected points. Since the linear dependency of simultaneous equations is expected, the number of collocation points is set larger than that of unknowns, and the least square methods are utilized to determine the coefficients. For evaluation of the Pf integrals (27), the crack edge ( $\rho=1, 0 \leq \phi < 2\pi$ ) is divided into intervals with  $\Delta\phi = 2\pi/96$  and the composite trapezoidal quadrature formula is applied. Further, for evaluation of integrals of continuous functions, the integral variables are changed into new variables

$$\begin{aligned} & \int_{S_0} f(x_1, x_2) dx_1 dx_2 \\ & = \int_0^{2\pi} d\phi \int_0^1 f(a\rho \cos \phi, b\rho \sin \phi) ab\rho d\rho \dots\dots(31) \end{aligned}$$

and the 40-points Gauss's quadrature formula for  $\rho$  and the 48 points trapezoidal quadrature formula for  $\phi$  are applied.

For aspect ratios  $b/a = 1.0, 0.9, \dots, 0.5$ , we show the coefficients obtained for the pressure case in Table 1, and those for the shear case in Table 2. Poisson's ratio is assumed to be 0.3; the parameters used are  $N=M=4$ : the locations of the collocation points are illustrated in Fig.5. The numerical solutions for coefficients other than those shown in these tables are order of  $10^{-3}$  for  $1.0 \geq b/a \geq 0.7$  and of  $10^{-2}$  for  $b/a = 0.6$  and  $0.5$ , whereas theoretical values for them are zero. The stress intensity factors are obtained to similar accuracy. Thus the numerical solutions obtained seem to have reasonable accuracies.

5. Conclusions

Crack problems are formulated in the

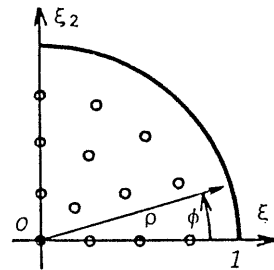


Fig.5 Collocation points

Table 1 Elliptic crack under pressure

$b/a$	num. $\mu\alpha_{00}^3/p_0$	theor. $\mu\alpha_{00}^3/p_0$
1.0	0.8917	0.8913
0.9	0.8442	0.8438
0.8	0.7901	0.7898
0.7	0.7278	0.7283
0.6	0.6545	0.6581
0.5	0.5660	0.5780

Table 2 Elliptic crack under shear

$b/a$	num. $\mu\alpha_{00}^1/\tau_0$	theor. $\mu\alpha_{00}^1/\tau_0$
1.0	1.0472	1.0486
0.9	1.0048	1.0067
0.8	0.9562	0.9572
0.7	0.9006	0.8983
0.6	0.8358	0.8278
0.5	0.7561	0.7429

form of integral equations, whose unknown functions are displacement differences between the upper and the lower crack surfaces. Divergent integrals are contained in the equations: they must be evaluated in the sense of Hadamard's finite parts (*Pf* integrals). The dominant features of the integral equations are as follows:

- (1) The unknown functions in the integral equations have physical meaning, i. e., the displacement differences between the upper and the lower crack surfaces.
- (2) The integral equation can be treated analytically or numerically with ease, since no derivatives of unknown functions are contained.
- (3) The stress intensity factors are directly determined from the solutions of the integral equations.
- (4) Amounts of computations required in numerical calculations are relatively little, since only the crack surfaces are considered.
- (5) Numerical evaluations of the finite parts of divergent integrals are as easy as those of Cauchy's principal values of integrals.

### Appendix 1. Finite Parts of Divergent Integrals

When integrands have strong singularities, the integrals diverge and have no meaning. In this case, however, we can evaluate them in the sense of Hadamard's finite part of divergent integrals (*Pf* integrals)<sup>(6)</sup>.

(i) Functions of one variable

Let  $F(x)$ , a function of  $x$ , be finite and integrable within the interval  $\epsilon \leq x \leq a$  and have an expansion near  $x=0$  in the form

$$F(x) = c_1 x^{-1} + \sum_{n=2}^N c_n x^{-n} + F_R(x) \dots \dots \dots (32)$$

where  $F(x)$  is a function integrable over  $0 \leq x \leq a$ . The *Pf* integral of  $F(x)$  from 0 to  $x$ ,  $Pf \int_0^a F(x) dx$ , is given by

$$Pf \int_0^a F(x) dx = c_1 \ln a - \sum_{n=2}^N c_n / (n-1) \cdot a^{-n+1} + \int_0^a F_R(x) dx \dots (33)$$

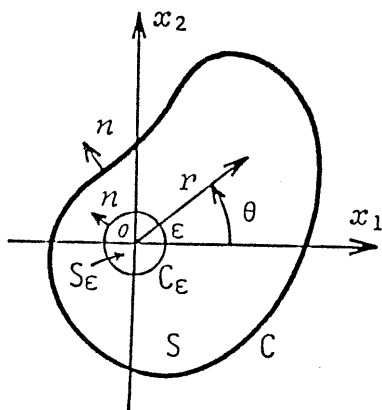


Fig.6 Integrations by parts

(ii) Function of two variables

Let  $F(r, \theta)$ , a function of  $r$  and  $\theta$  defined on the  $x_1 x_2$  plane [ $r = (x_1^2 + x_2^2)^{1/2}$ ,  $\theta = \tan^{-1}(x_2/x_1)$ ], be finite and integrable in the region ( $\epsilon \leq r \leq a$ ,  $0 \leq \theta < 2\pi$ ) and have an expansion near  $r=0$  in the form

$$F(r, \theta) = c_2(\theta)r^{-2} + \sum_{n=3}^N c_n(\theta)r^{-n} + F_R(r, \theta) \dots \dots \dots (34)$$

where  $c_2(\theta), c_3(\theta), \dots, c_N(\theta)$  are periodical and integrable functions of  $\theta$  ( $0 \leq \theta < 2\pi$ ), and  $F_R(r, \theta)$  is a function integrable in the region ( $0 \leq r \leq a, 0 \leq \theta < 2\pi$ ). The *Pf* integral of  $F(r, \theta)$  over the circular region  $D_a$  ( $0 \leq r \leq a, 0 \leq \theta < 2\pi$ ),  $Pf \int_{D_a} F(r, \theta) ds$ , is given by

$$Pf \int_{D_a} F(r, \theta) ds = \int_0^{2\pi} c_2(\theta) d\theta \cdot \ln a - \sum_{n=3}^N \int_0^{2\pi} c_n(\theta) d\theta / (n-2) \cdot a^{-n+2} + \int_{D_a} F_R(r, \theta) ds \dots \dots \dots (35)$$

If the integrals exist in the sense of Riemann or Cauchy's principal values, those values in the sense of *Pf* integrals coincide with them.

### Appendix 2. Integrations of Two-dimensional *Pf* Integrals by Parts

We derive a formula on integrations by parts for two-dimensional *Pf* integrals. Let functions  $F(r, \theta)$  and  $G(r, \theta)$  have singularities at origin only, and have expansions

$$\left. \begin{aligned} F(r, \theta) &= \sum_{n=-N}^{N+1} f_n(\theta) r^n + F_{N+2}(r, \theta) \\ G(r, \theta) &= \sum_{n=-N}^{N+1} g_n(\theta) r^n + G_{N+2}(r, \theta) \end{aligned} \right\} \dots (36)$$

where functions  $F_{N+2}(r, \theta)$  and  $G_{N+2}(r, \theta)$  are integrable in the regions  $S$  and  $O(r^{N+2})$  as  $r \rightarrow 0$ . Let  $S_\epsilon$  be a circular region with center at origin and radius  $\epsilon$ . There from it follows that

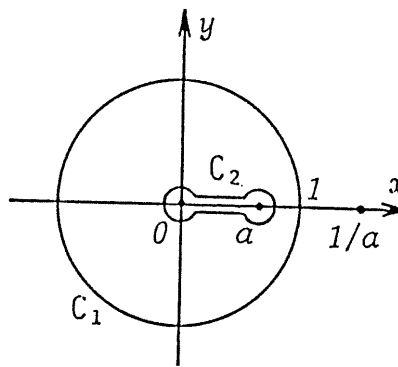


Fig.7 Integration contours

$$Pf \int_S \frac{\partial}{\partial x_a} F(r, \theta) \cdot G(r, \theta) ds = \int_{S-S_i} \frac{\partial}{\partial x_a} F(r, \theta) \cdot G(r, \theta) ds + \int_{S_i} \left[ \frac{\partial}{\partial x_a} F_{N+2}(r, \theta) \cdot G_{N+2}(r, \theta) + \frac{\partial}{\partial x_a} F_{N+2}(r, \theta) \cdot \left\{ \sum_{m=-N}^{N+1} g_m(\theta) r^m \right\} + \frac{\partial}{\partial x_a} \left\{ \sum_{n=-N}^{N+1} f_n(\theta) r^n \right\} \cdot G_{N+2}(r, \theta) \right] ds + Pf \int_{S_i} \frac{\partial}{\partial x_a} \left\{ \sum_{n=-N}^{N+1} f_n(\theta) r^n \right\} \cdot \left\{ \sum_{m=-N}^{N+1} g_m(\theta) r^m \right\} ds$$

If we integrate by parts the first and the second terms in the right hand side of the above equation, they are written in the form

$$\int_{C_i} n_a F(r, \theta) G(r, \theta) dl - \int_{C_i} n_a F(r, \theta) G(r, \theta) dl - \int_{S-S_i} F(r, \theta) \frac{\partial}{\partial x_a} G(r, \theta) ds + \int_{C_i} n_a [F_{N+2}(r, \theta) G_{N+2}(r, \theta) + F_{N+2}(r, \theta) \left\{ \sum_{m=-N}^{N+1} g_m(\theta) r^m \right\} + \left\{ \sum_{n=-N}^{N+1} f_n(\theta) r^n \right\} G_{N+2}(r, \theta)] dl - \int_{S_i} [F_{N+2}(r, \theta) \frac{\partial}{\partial x_a} G_{N+2}(r, \theta) + F_{N+2}(r, \theta) \frac{\partial}{\partial x_a} \left\{ \sum_{m=-N}^{N+1} g_m(\theta) r^m \right\} + \left\{ \sum_{n=-N}^{N+1} f_n(\theta) r^n \right\} \frac{\partial}{\partial x_a} G_{N+2}(r, \theta)] ds$$

where  $n_i$  denotes the  $x_i$  component of the unit outward normal to  $S$ . Further, if we rewrite the third term by carrying out the differentiation indicated, integrating with respect to  $r$ , integrating by parts with respect to  $\theta$  and transforming the terms containing  $\varepsilon$  into the integral forms with respect to  $r$ , then we have

$$\int_{C_i} n_a \left\{ \sum_{n=-N}^{N+1} f_n(\theta) r^n \right\} \left\{ \sum_{m=-N}^{N+1} g_m(\theta) r^m \right\} dl - \int_{S_i} \left\{ \sum_{n=-N}^{N+1} f_n(\theta) r^n \right\} \frac{\partial}{\partial x_a} \left\{ \sum_{m=-N}^{N+1} g_m(\theta) r^m \right\} ds - \sum_{n+m+1=0} \int_0^{2\pi} (\cos \theta \delta_{a1} + \sin \theta \delta_{a2}) f_n(\theta) g_m(\theta) d\theta$$

where the last term is a summation for every combination of  $n$  and  $m$  which satisfies the relation  $n+m+1=0$ . Summing up the above results, the formula on integrations by parts is expressed in the form

$$Pf \int_S \frac{\partial}{\partial x_a} F(r, \theta) \cdot G(r, \theta) ds = \int_{C_i} n_a F(r, \theta) G(r, \theta) dl - Pf \int_{S-S_i} F(r, \theta) \frac{\partial}{\partial x_a} G(r, \theta) ds - \sum_{n+m+1=0} \int_0^{2\pi} (\cos \theta \delta_{a1} + \sin \theta \delta_{a2}) f_n(\theta) g_m(\theta) d\theta \dots\dots\dots(37)$$

Appendix 3. Integral Formula

We rewrite the following integral ( $0 < a < 1$ ) into a more convenient form.

$$I = \int_0^{2\pi} \frac{e^{in\phi} d\phi}{(\sqrt{1+a^2-2a \cos \phi})^3} \dots\dots\dots(38)$$

First we change the integral variable  $\phi$  into a new variable  $z (= x + iy) = \exp(i\phi)$ . Then the integral is written in the form

$$I = \frac{1}{i} \int_{C_i} \frac{z^{n+2}}{(\sqrt{z(z-a)(1-az)})^3} dz$$

where the integral contour  $C_i$  is such as shown in Fig.7. Here such branch of the multi-valued function  $\sqrt{z(z-a)(1-az)}$  is chosen that the function is equal to  $1-a$  at  $z=1$ , with the branch cut  $[0, a]$  and  $[1/a, \infty)$ . If the integration contour  $C_i$  is transformed to  $C_2$  and degenerated to the line connecting points 0 and  $a$ , then we have the following expressions:

$$I = -2Pf \int_0^a \frac{r^{n+2}}{(\sqrt{r(a-r)(1-ar)})^3} dr = -4Pf \int_0^a \frac{a^{-n} t^{2n+2}}{(\sqrt{(a^2-t^2)(1-t^2)})^3} dt \dots\dots(39)$$

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