A higher order asymptotic approximation for the fundamental frequency of a multiply connected membrane

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Abstract

The fundamental frequency of a fixed membrane is the square root of the lowest eigenvalue of negative Laplace operator with Dirichlet boundary conditions. A multiply connected membrane with inner cores of vanishing maximal dimensions $2c_j$ is considered in the present article. The modified perturbation method developed for a doubly connected membrane is extended to provide a general formula for the fundamental frequency of the multiply connected membrane. A higher order asymptotic approximation (as $c_j \to 0$) for the fundamental frequency of a membrane with inner circular cores of radius $c_j$ is specified. It is an excellent extension of the results in the literature. Moreover, a second-order asymptotic approximation (as $c \to 0$) for the fundamental frequency of a circular membrane of radius 1 with finitely many inner circular cores of small radius $c$ is found and computed explicitly. The effects of the positions of the inner cores on the second-order asymptotic approximation are investigated. The accuracy of the second-order asymptotic approximation is also shown by the comparisons among the asymptotic approximations and the numerical values computed by other investigators.

1. Introduction

The square root of the lowest eigenvalue of negative Laplace operator with Dirichlet boundary conditions in two dimensions represents the fundamental frequency of a fixed membrane. The determination of the fundamental frequency is important in the studies of acoustics and electromagnetism [1]. It is also important in the study of the human eardrum and in the design of engineering devices such as microphones, loudspeakers, pumps, compressors, pressure regulators, antennae for space communications [1,2]. The human eardrum is a membrane with a stirrup which acts as a rigid core. However, related articles [3–7] on the fundamental frequency of a multiply connected membrane are few. None of these articles except Wang's [7] was concerned about the asymptotic case in which the maximal dimensions of the inner cores of a multiply connected membrane are vanishing.

In the present article, a multiply connected membrane with inner cores of vanishing maximal dimensions $2c_j$ is concerned. The modified perturbation method [8] developed by the author for the fundamental frequency of a doubly connected membrane is extended to provide a general formula for the fundamental frequency of the
## Nomenclature

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
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<tbody>
<tr>
<td>$B_j$</td>
<td>the inner core enclosed by $S_j$</td>
</tr>
<tr>
<td>$B_{0j}, B_{mj}, A_{mj}$</td>
<td>the constant coefficients in the expression of $W_0$ as in Eq. (27)</td>
</tr>
<tr>
<td>$B_0(N,p), B_{m0}(N,p), A_{m0}(N,p)$</td>
<td>the constant coefficients of $U_{N,p}(r, \theta)$ as in Eq. (34)</td>
</tr>
<tr>
<td>$c$</td>
<td>the radius of inner circular cores</td>
</tr>
<tr>
<td>$2c_j$</td>
<td>the maximal dimension of $B_j$</td>
</tr>
<tr>
<td>$D_{0j}, C_{mj}, D_{mj}$</td>
<td>the constant coefficients of $V_{2j}^k(r, \theta)$ as in Eqs. (41)–(44)</td>
</tr>
<tr>
<td>$J_n$</td>
<td>the $n$th order Bessel function</td>
</tr>
<tr>
<td>$K$</td>
<td>the fundamental frequency of $R$</td>
</tr>
<tr>
<td>$K_N$</td>
<td>eigenvalue on $R_0$</td>
</tr>
<tr>
<td>$K_{s,t}$</td>
<td>the $t$th zero of $J_s$</td>
</tr>
<tr>
<td>$K_0$</td>
<td>the fundamental frequency of $R_0$</td>
</tr>
<tr>
<td>$l(N)$</td>
<td>the number of the eigenfunctions to $K_N$</td>
</tr>
<tr>
<td>$M$</td>
<td>the number of inner cores</td>
</tr>
<tr>
<td>$P_j$</td>
<td>the center of $j$th inner circular core</td>
</tr>
<tr>
<td>$(r, \theta)$</td>
<td>the polar coordinates with the origin at the center of a circular membrane of radius 1</td>
</tr>
<tr>
<td>$(r_j, \theta_j)$</td>
<td>the polar coordinates with the origin at $P_j$</td>
</tr>
<tr>
<td>$(r_{0j}, \theta_{0j})$</td>
<td>the polar coordinates of $P_j$ in $(r, \theta)$</td>
</tr>
<tr>
<td>$R$</td>
<td>multiply connected membrane</td>
</tr>
<tr>
<td>$R_0$</td>
<td>simply connected membrane</td>
</tr>
<tr>
<td>$(R_{ji}, \phi_{ji})$</td>
<td>the polar coordinates of $P_i$ in $(r_j, \theta_j)$</td>
</tr>
<tr>
<td>$S_j$</td>
<td>the inner boundary of $R$</td>
</tr>
<tr>
<td>$S_0$</td>
<td>the boundary of $R_0$</td>
</tr>
<tr>
<td>$U_{N,p}$</td>
<td>corresponding eigenfunction to $K_N$</td>
</tr>
<tr>
<td>$V_{2j} = V_{2j}^1 + V_{2j}^2$</td>
<td></td>
</tr>
<tr>
<td>$W$</td>
<td>corresponding eigenfunction to $K$</td>
</tr>
<tr>
<td>$W_0$</td>
<td>corresponding eigenfunction to $K_0$</td>
</tr>
<tr>
<td>$Y_n$</td>
<td>the $n$th order Neumann function</td>
</tr>
<tr>
<td>$\gamma$</td>
<td>$\approx 0.5772$</td>
</tr>
</tbody>
</table>

A higher order asymptotic approximation (as $c_j \to 0$) for the fundamental frequency of a membrane with $M$ inner circular cores of radius $c_j$ is specified by correcting the boundary conditions, by using the results [8] for a membrane with an inner circular core, by using the translational addition theorems for circular cylindrical wave functions [4,9], and by applying the generalized Green’s function [10]. It is found to be an excellent extension of the results [7,8] in the literature. Moreover, by using the generalized Green’s function [8] for a circular membrane of radius 1, by applying the results [8] for a circular membrane of radius 1 with an inner circular core of small radius $c$, and by employing the translational addition theorems for circular cylindrical wave functions [4,9], a second-order asymptotic approximation (as $c \to 0$) for the fundamental frequency of a circular membrane of radius 1 with $M$ inner circular cores of small radius $c$ is found and computed explicitly. The effects of the positions of the inner cores on the second-order asymptotic approximation are investigated and the comparisons among the asymptotic approximations and the numerical values [4,5,11] computed by other investigators are made.

### 2. Perturbation formulation

Let $R_0$ be a simply connected membrane with boundary $S_0$ and $R$ be a multiply connected membrane with the outer boundary $S_0$ and $M$ inner boundaries $S_j$ enclosing $M$ inner cores $B_j$ of vanishing maximal dimensions $2c_j$, $j = 1, 2, \ldots, M$. All lengths have been normalized by a characteristic length $L$. The governing Helmholtz equation is

$$\Delta W + K^2 W = 0,$$

where $W$ is the normalized vertical displacement and $K$ is the normalized vibrational frequency, $K = \text{frequency} \cdot \sqrt{\text{density/tension per length}}$. Consider the eigenvalue problems on $R$ and $R_0$, respectively,

$$\Delta W + K^2 W = 0 \text{ in } R,$$
\[ W = 0 \text{ on } S_0 \cup S_1 \cup \cdots \cup S_M \]  
(3)

and

\[ \Delta \hat{W} + \hat{K}^2 \hat{W} = 0 \text{ in } R_0, \]  
(4)

\[ \hat{W} = 0 \text{ on } S_0. \]  
(5)

Let \( K_0^2 \) be the smallest eigenvalue of the problem on the membrane \( R_0 \) and \( W_0 \) be the corresponding eigenfunction. By extending the modified perturbation method \([8]\) developed by the author for a doubly connected membrane, a general formula for the fundamental frequency \( K \) of the multiply connected membrane \( R \) is derived below.

The fundamental frequency \( K \) of the multiply connected membrane \( R \) and its corresponding eigenfunction \( W \) can be expressed as

\[ K = K_0 + F_1 + \cdots + F_m + \cdots \]  
(6)

and

\[ W = \left( \sum_{j=1}^{M} V_{0j} + W_0 \right) + \left( \sum_{j=1}^{M} V_{1j} + W_1 \right) + \cdots + \left( \sum_{j=1}^{M} V_{mj} + W_m \right) + \cdots, \]  
(7)

where \( V_{0j} = 0 \) on \( R_0 \setminus B_j \),

\[ \Delta W_0 + K_0^2 W_0 = 0 \text{ in } R_0, \]  
(8)

\[ W_0 = 0 \text{ on } S_0, \]  
(9)

\[ \Delta V_{1j} + K_0^2 V_{1j} = 0 \text{ in } R_0 \setminus B_j, \]  
(10)

\[ V_{1j} = -W_0 \text{ on } S_j, \]  
(11)

\[ \Delta W_1 + K_0^2 W_1 = -2K_0 F_1 W_0 \text{ in } R_0, \]  
(12)

\[ W_1 = -\sum_{j=1}^{M} V_{1j} \text{ on } S_0, \]  
(13)

\[ \Delta V_{2j} + K_0^2 V_{2j} = -2K_0 F_1 V_{1j} \text{ in } R_0 \setminus B_j, \]  
(14)

\[ V_{2j} = -W_1 - \sum_{l=1, l \neq j}^{M} V_{1l} \text{ on } S_j, \]  
(15)

\[ \Delta W_2 + K_0^2 W_2 = -2K_0 F_1 W_1 - 2K_0 F_2 W_0 - F_1^2 W_0 \text{ in } R_0, \]  
(16)

\[ W_2 = -\sum_{j=1}^{M} V_{2j} \text{ on } S_0, \]  
(17)

\[ \Delta V_{mj} + K_0^2 V_{mj} = -2K_0 \sum_{p=1}^{m} F_p V_{(m-p)j} - \sum_{s=1}^{m-1} \sum_{l=1}^{m-s} F_s F_l V_{(m-s-l)j} \text{ in } R_0 \setminus B_j, \]  
(18)

\[ V_{mj} = -W_{m-1} - \sum_{l=1, l \neq j}^{M} V_{(m-1)l} \text{ on } S_j, \text{ } m = 3, 4, 5, \ldots, \]  
(19)
\[ \Delta W_m + K_0^2 W_m = -2K_0 \sum_{p=1}^{m} F_p W_{m-p} - \sum_{s=1}^{m-1} \sum_{l=1}^{m-s} F_s F_l W_{m-s-l} \text{ in } R_0, \] (20)

\[ W_m = -\sum_{j=1}^{M} V_{mj} \text{ on } S_0, \ m = 3, 4, 5, \ldots . \] (21)

The conditions for the existence of \( W_n, n = 1, 2, 3, \ldots, \) are provided by Fredholm Alternative theorem [10]. These conditions give formulas to the correction terms of the fundamental frequency \( K \). The formulas are as follows:

\[ F_1 = \frac{\sum_{j=1}^{M} \int_{S_0} \frac{\partial W_0}{\partial n} V_{1j} \, ds}{-2K_0 \int_{R_0} W_0^2 \, dA}, \] (22)

\[ F_2 = \frac{\int_{R_0} (2K_0 F_1 W_1 + F_1^2 W_0) W_0 \, dA + \sum_{j=1}^{M} \int_{S_0} \frac{\partial W_0}{\partial n} V_{2j} \, ds}{-2K_0 \int_{R_0} W_0^2 \, dA}, \] (23)

\[ F_m = \frac{\int_{R_0} (2K_0 \sum_{p=1}^{m-1} F_p W_{m-p}) W_0 \, dA + \sum_{j=1}^{M} \int_{S_0} \frac{\partial W_0}{\partial n} V_{mj} \, ds}{-2K_0 \int_{R_0} W_0^2 \, dA} \frac{\int_{R_0} (\sum_{s=1}^{m-1} \sum_{l=1}^{m-s} F_s F_l W_{m-s-l}) W_0 \, dA}{2K_0 \int_{R_0} W_0^2 \, dA}, \ m = 3, 4, 5, \ldots , \] (24)

where the derivative on \( S_0 \) is the outward normal derivative.

Hence, a general formula for the fundamental frequency \( K \) of the multiply connected membrane \( R \) is found to be the formula Eq. (6) with Eqs. (22)–(24). However, Eqs. (22)–(24) will not be specified without the specifications of \( W_n \)’s and \( V_{mj} \)’s. The specification for \( W_n \)’s will be done by correcting the boundary conditions on the inner boundaries \( S_j, j = 1, 2, \ldots, M \), and the specification for \( V_{mj} \)’s will be done by correcting the boundary conditions on the outer boundary \( S_0 \). This will be shown in the next section.

### 3. A membrane with \( M \) inner circular cores of small radius \( c_j \)

A higher order asymptotic approximation (as \( c_j \to 0 \)) for the fundamental frequency \( K \) of a membrane with \( M \) inner circular cores of small radius \( c_j \) centered at \( P_j \) is specified in this section.

By correcting the boundary condition for \( W_0 + V_{1j} \) on the outer boundary \( S_0 \) to \( O(1/|\ln c_j|) \), by using the results [8, Eqs. (54) and (55)] for a membrane with an inner circular core, and by using Eq. (22), \( V_{1j} \) and the first correction term \( F_1 \) are specified as

\[ V_{1j}(r_j, \theta_j) = \frac{-B_{0j} J_0(K_0 c_j)}{Y_0(K_0 c_j)} Y_0(K_0 r_j) \]

\[ -\sum_{m=1}^{\infty} \frac{J_m(K_0 c_j)}{Y_m(K_0 c_j)} Y_m(K_0 r_j)(A_{mj} \sin(m\theta_j) + B_{mj} \cos(m\theta_j)) \] (25)

and

\[ F_1 = \sum_{j=1}^{M} \left( \frac{\pi B_{0j}^2}{K_0 \int_{R_0} W_0^2 \, dA} \right) \frac{1}{|\ln c_j|} + \sum_{j=1}^{M} \left( \frac{\pi(K_0 + \gamma - \ln 2)B_{0j}^2}{K_0 \int_{R_0} W_0^2 \, dA} \right) \frac{1}{|\ln c_j|^2} + \cdots , \] (26)
where $B_{0j}$, $B_{mj}$, $A_{mj}$ are the constant coefficients in the expression of $W_0$:

$$W_0(r_j, \theta_j) = B_{0j}J_0(K_0r_j) + \sum_{n=1}^{\infty} J_n(K_0r_j)(A_{mj}\sin(m\theta_j) + B_{mj}\cos(m\theta_j)),$$

(27)

$(r_j, \theta_j)$ is the polar coordinates with the origin at $P_j$, $\gamma \approx 0.5772$, $J_n$ is the $n$th order Bessel function, $Y_n$ is the $n$th order Neumann function, and [12]

$$J_n(z) = \sum_{l=0}^{\infty} \frac{(-1)^l}{l!(n+l)!} \left(\frac{z}{2}\right)^{n+2l}, \quad n = 0, 1, 2, \ldots,$$

(28)

$$Y_0(z) = \frac{2}{\pi} \left(\ln \frac{z}{2} + \gamma\right)J_0(z) - \frac{2}{\pi} \sum_{l=1}^{\infty} \frac{(-1)^l}{(l-1)!} \left(1 + 1 + \ldots + \frac{1}{l}\right) \left(\frac{z}{2}\right)^{2l},$$

(29)

$$Y_m(z) = \frac{2}{\pi} \left(\ln \frac{z}{2}\right)J_m(z) - \frac{1}{\pi} \sum_{l=0}^{m-1} \frac{(-1)^l}{l!(m+l)!} \left(\psi(m+l+1) + \psi(l+1)\right) \left(\frac{z}{2}\right)^{2l},$$

$$\psi(1) = -\gamma, \quad \psi(m+1) = \left(1 + \frac{1}{2} + \ldots + \frac{1}{m}\right) - \gamma, \quad m = 1, 2, 3, \ldots.$$  

(30)

Notice that $V_{1j}$ has $O(1/|\ln c_j|)$ boundary value on the inner boundaries $S_i$ for all $i \neq j$ because the translational addition theorems for circular cylindrical wave functions [4,9] give

$$Y_0(K_0r_i) = \sum_{l=-\infty}^{\infty} Y_l(K_0R_{ji})J_l(K_0r_j)\cos(l\phi_{ji} - l\theta_j),$$

(31)

$$Y_p(K_0r_i)\sin(p\theta_i) = \sum_{l=-\infty}^{\infty} Y_{l+p}(K_0R_{ji})J_{l+p}(K_0r_j)\sin((l+p)\phi_{ji} - l\theta_j) \times (-1)^p,$$

(32)

$$Y_p(K_0r_i)\cos(p\theta_i) = \sum_{l=-\infty}^{\infty} Y_{l+p}(K_0R_{ji})J_{l+p}(K_0r_j)\cos((l+p)\phi_{ji} - l\theta_j) \times (-1)^p,$$

(33)

where $(R_{ji}, \phi_{ji})$ is the polar coordinates representation of the center point $P_i$ in the polar coordinates $(r_j, \theta_j)$ and $p = 1, 2, 3, \ldots$.

To specify the second correction term $F_2$, the specifications of $W_1$ and $V_{2j}$’s are required. Let $U_{N,p}$, $p = 1, 2, 3, \ldots, l(N)$, be the corresponding eigenfunctions to the eigenvalues $K_N(\neq K_0)$, $N = 1, 2, 3, \ldots$, of the eigenvalue problem on the simply connected membrane $R_0$, then $U_{N,p}$ can be expressed as

$$U_{N,p}(r_j, \theta_j) = B_{0j}(N,p)J_0(K_Nr_j) + \sum_{m=1}^{\infty} J_m(K_Nr_j)(A_{mj}(N,p)\sin(m\theta_j) + B_{mj}(N,p)\cos(m\theta_j)),$$

(34)

with appropriate constant coefficients $B_{0j}(N,p)$, $B_{mj}(N,p)$, and $A_{mj}(N,p)$ determined by the boundary condition on $S_0$, Eq. (5). Thus, by using Green’s second identity [10], by using the generalized Green’s function $G$ [10] for the boundary value problem consisting of Eqs. (8) and (9), and by correcting the boundary condition for $W_0 + \sum_{i=1}^{M} V_{1i} + W_1$ on the inner boundary $S_j$ to $\sum_{i=1}^{M} O(1/|\ln c_i|)$, $j = 1, 2, \ldots, M$, $W_1$ is
specified as

\[ W_1 = \sum_{i=1}^{M} \int_{S_0}^{} \frac{\partial G}{\partial n} V_{1i} \, ds \]

\[ = \sum_{N=1}^{h(N)} \sum_{p=1}^{U_{N,p}} \frac{U_{N,p}}{\|U_{N,p}\|^2} \left( \sum_{i=1}^{M} \int_{S_0}^{} \frac{\partial U_{N,p}}{\partial n} V_{1i} \, ds \right). \]  

(35)

(36)

To specify \( V_2 \)'s, let \( V_2 = V_2^i + V_2^h \) with

\[ \Delta V_2^i + K_0^2 V_2^i = -2K_0F_1 V_1 \text{ in } R_0 \setminus B_j, \]

\[ \Delta V_2^h + K_0^2 V_2^h = 0 \text{ in } R_0 \setminus B_j, \]

\[ V_2^h = -W_1 - \sum_{i=1, \ l \neq j}^{M} V_{1i} - V_2^i \text{ on } S_j, \]

then

\[ V_2^j(r_j, \theta_j) = - F_1 \left[ \frac{B_0 J_0(K_0c_j)}{Y_0(K_0c_j)} r_j Y_0(K_0r_j) + \sum_{m=1}^{\infty} \frac{J_m(K_0c_j)}{Y_m(K_0c_j)} r_j Y_m(K_0r_j) \right] \]

\[ \times (A_{mj} \sin(m\theta_j) + B_{mj} \cos(m\theta_j)) \]

(40)

and

\[ V_2^h(r_j, \theta_j) = \tilde{D}_0 Y_0(r_j) \left( J_0(K_0r_j) - J_0(K_0c_j) \right) Y_0(K_0r_j) + \sum_{m=1}^{\infty} \tilde{C}_{mj} Y_0(K_0r_j) \]

\[ \times \sin(m\theta_j) + \tilde{D}_{mj} \cos(m\theta_j) \left( J_m(K_0r_j) - J_m(K_0c_j) \right) Y_m(K_0r_j) \]

\[ + D_0 Y_0(K_0r_j) + \sum_{m=1}^{\infty} Y_m(K_0r_j) (C_{mj} \sin(m\theta_j) + D_{mj} \cos(m\theta_j)), \]

(41)

where

\[ D_0 = \frac{-1}{2\pi Y_0(K_0c_j)} \int_0^{2\pi} \left( W_1 + \sum_{i=1, \ l \neq j}^{M} V_{1i} \right) (c_j, \theta_j) \, d\theta_j \]

\[ + \frac{B_0 F_1 c_j J_0(K_0c_j) Y_0(K_0c_j)}{Y_0^2(K_0c_j)}, \]

(42)

\[ C_{mj} = \frac{-1}{\pi Y_m(K_0c_j)} \int_0^{2\pi} \sin(m\theta_j) \left( W_1 + \sum_{i=1, \ l \neq j}^{M} V_{1i} \right) (c_j, \theta_j) \, d\theta_j \]

\[ + \frac{A_{mj} F_1 c_j J_m(K_0c_j) Y_m(K_0c_j)}{Y_m^2(K_0c_j)}, \]

(43)

\[ D_{mj} = \frac{-1}{\pi Y_m(K_0c_j)} \int_0^{2\pi} \cos(m\theta_j) \left( W_1 + \sum_{i=1, \ l \neq j}^{M} V_{1i} \right) (c_j, \theta_j) \, d\theta_j \]

\[ + \frac{B_{mj} F_1 c_j J_m(K_0c_j) Y_m(K_0c_j)}{Y_m^2(K_0c_j)}. \]

(44)
Moreover, the specification for $V^h_{2j}$ is made to the following. The asymptotic expansion of $J_0(K_0c_j)/Y_0(K_0c_j)$ as $c_j \to 0$ is

$$
\frac{J_0(K_0c_j)}{Y_0(K_0c_j)} = -\frac{\pi}{2} \frac{1}{\ln c_j} + \frac{\pi}{2} \left( \ln 2 - \gamma - \ln K_0 \right) \frac{1}{(\ln c_j)^2} + \cdots,
$$

(45)

where $\gamma \approx 0.5772$. Thus,

$$
\int_0^{2\pi} W_1(c_j, \theta_j) \, d\theta_j = \sum_{l=1}^{M} \sum_{N=1}^{\infty} \int_0^{2\pi} \frac{\partial U_{N,\ell}(r_l, \theta_l)}{\partial n} Y_0(K_0r_l) \, ds \left\{ \frac{1}{|\ln c_l|} + \cdots, \right\}
$$

(46)

$$
\int_0^{2\pi} \sin(m\theta_j) W_1(c_j, \theta_j) \, d\theta_j = \sum_{l=1}^{M} \sum_{N=1}^{\infty} \int_0^{2\pi} \frac{\partial U_{N,\ell}(r_l, \theta_l)}{\partial n} Y_0(K_0r_l) \, ds \left\{ \frac{1}{|\ln c_l|} + \cdots, \right\}
$$

(47)

$$
\int_0^{2\pi} \cos(m\theta_j) W_1(c_j, \theta_j) \, d\theta_j = \sum_{l=1}^{M} \sum_{N=1}^{\infty} \int_0^{2\pi} \frac{\partial U_{N,\ell}(r_l, \theta_l)}{\partial n} Y_0(K_0r_l) \, ds \left\{ \frac{1}{|\ln c_l|} + \cdots, \right\}
$$

(48)

and by Eqs. (31)–(33),

$$
\sum_{l=1, l \neq j}^{M} \int_0^{2\pi} V_1(c_j, \theta_j) \, d\theta_j = \sum_{l=1, l \neq j}^{M} \frac{\pi^2 B_0}{K_0c_j} Y_0(K_0r_l) \frac{1}{|\ln c_l|} + \cdots,
$$

(49)

$$
\sum_{l=1, l \neq j}^{M} \int_0^{2\pi} \sin(m\theta_j) V_1(c_j, \theta_j) \, d\theta_j = \sum_{l=1, l \neq j}^{M} 0 \frac{1}{|\ln c_l|} + \cdots,
$$

(50)

$$
\sum_{l=1, l \neq j}^{M} \int_0^{2\pi} \cos(m\theta_j) V_1(c_j, \theta_j) \, d\theta_j = \sum_{l=1, l \neq j}^{M} 0 \frac{1}{|\ln c_l|} + \cdots.
$$

(51)

Green’s second identity [10] gives

$$
\oint_{S_0} \frac{\partial W_0}{\partial n} V_2 j \, ds = \oint_{S_0} \frac{\partial W_0}{\partial n} V_2 j \, ds - \oint_{S_0} \frac{\partial W_0}{\partial n} \left( V^h_{2j} - \frac{\partial V^h_{2j}}{\partial n} W_0 \right) \, ds.
$$

(52)

Also,

$$
\oint_{S_0} \frac{\partial W_0}{\partial n} V_2 j \, ds = - F_1 \left\{ J_0(K_0c_j) \oint_{S_0} \frac{\partial W_0}{\partial n} r_j Y_0'(K_0r_j) \, ds \right. \\
+ \sum_{m=1}^{\infty} J_m(K_0c_j) \oint_{S_0} \frac{\partial W_0}{\partial n} (A_m \sin(m\theta_j) + B_m \cos(m\theta_j)) r_j Y_0'(K_0r_j) \, ds \right\}
$$

$$
= \sum_{l=1}^{M} \left\{ \frac{\pi^2 B_0}{2K_0} \oint_{R_0} \frac{\partial W_0}{\partial n} r_j Y_0'(K_0r_j) \, ds \right\} \frac{1}{|\ln c_l| |\ln c_j|} + \cdots
$$

(53)
and Wronskian \( W \) of \( J_m(z) \) and \( Y_m(z) \) [12], \( W(J_m(z), Y_m(z)) = 2/\pi z \), gives

\[
\int_{S_1} \left( \frac{\partial W_0}{\partial n} V^h_{2j} - \frac{\partial V^h_{2j}}{\partial n} W_0 \right) ds
\]

\[
= \int_0^{2\pi} \left( -\frac{\partial W_0(r_j, \theta_j)}{\partial r_j} \bigg|_{r_j=c_j} V^h_{2j}(c_j, \theta_j) + \frac{\partial V^h_{2j}(r_j, \theta_j)}{\partial r_j} \bigg|_{r_j=c_j} W_0(c_j, \theta_j) \right) c_j d\theta_j
\]

\[
= -4B_{0j}D_{0j}(c_j) \frac{J_0(K_0 c_j)}{Y_0(K_0 c_j)} - 2 \sum_{m=1}^{\infty} (A_{mj} \tilde{C}_{mj}(c_j) + B_{mj} \tilde{D}_{mj}(c_j)) \frac{J_m(K_0 c_j)}{Y_m(K_0 c_j)}
\]

\[
+ 4B_{0j}D_{0j} + 2 \sum_{m=1}^{\infty} (A_{mj} C_{mj} + B_{mj} D_{mj}).
\] (54)

Thus, by correcting the boundary condition for \( W_0 + \sum_{i=1}^{M} V_{i1} + V_{i2} \) on the outer boundary \( S_0 \) to \( \sum_{i=1}^{M} O(1/|\ln c_i||\ln c_j|) \) and by Eq. (23), \( V^h_{2j} \) and \( F_2 \) are specified as

\[
V^h_{2j}(r_j, \theta_j) = D_{0j} Y_0(K_0 r_j) + \sum_{m=1}^{\infty} Y_m(K_0 r_j)(C_{mj} \sin(m\theta_j) + D_{mj} \cos(m\theta_j))
\] (55)

and

\[
F_2 = \sum_{j=1}^{M} \sum_{i=1}^{M} \left[ \frac{\pi^2 B_{0j}^2 B_{0j}^2}{-2K_0^3 \left( \int_{R_0} W_0^2 dA \right) 4} + \frac{\pi^2 B_{0j}^2 B_{0j}^2}{4} \int_{S_0} \frac{\partial W_0}{\partial n} r_j Y_0''(K_0 r_j) ds \right]
\]

\[
+ \sum_{N=1}^{(N)} \sum_{p=1}^{(N)} \left( \frac{\pi^2 B_{0j}^2 B_{0j}^2}{2K_0^2 (K_N^2 - K_0^2)} U_{N,p}^2 \int_{R_0} W_0^2 dA \right) \int_{S_0} \frac{\partial U_{N,p}}{\partial n} Y_0(K_0 r_j) ds \times \frac{1}{\ln c_i \ln c_j} + \sum_{j=1}^{M} \sum_{i=1, i \neq j}^{M} \left( \frac{\pi^2 B_{0j}^2 B_{0j}^2}{2K_0^2 \int_{R_0} W_0^2 dA} \right) \frac{1}{\ln c_i \ln c_j} + \cdots
\] (56)

Therefore, a higher order asymptotic approximation (as \( c_j \to 0 \)) for the fundamental frequency \( K \) of a membrane with \( M \) inner circular cores of small radius \( c_j \) centered at \( P_j \) is specified as

\[
K = K_0 + \sum_{j=1}^{M} \left( \frac{\pi B_{0j}^2}{K_0 \int_{R_0} W_0^2 dA} \right) \frac{1}{\ln c_j} + \sum_{j=1}^{M} \left( \frac{\pi (\ln K_0 + \gamma - \ln 2) B_{0j}^2}{K_0 \int_{R_0} W_0^2 dA} \right) \frac{1}{\ln c_j^2}
\]

\[
+ \sum_{j=1}^{M} \sum_{i=1}^{M} \left[ \frac{\pi^2 B_{0j}^2 B_{0j}^2}{-2K_0^3 \left( \int_{R_0} W_0^2 dA \right) 4} + \frac{\pi^2 B_{0j}^2 B_{0j}^2}{4} \int_{S_0} \frac{\partial W_0}{\partial n} r_j Y_0''(K_0 r_j) ds \right]
\]

\[
+ \sum_{N=1}^{(N)} \sum_{p=1}^{(N)} \left( \frac{\pi^2 B_{0j}^2 B_{0j}^2}{2K_0^2 (K_N^2 - K_0^2)} U_{N,p}^2 \int_{R_0} W_0^2 dA \right) \int_{S_0} \frac{\partial U_{N,p}}{\partial n} Y_0(K_0 r_j) ds \times \frac{1}{\ln c_i \ln c_j} + \sum_{j=1}^{M} \sum_{i=1, i \neq j}^{M} \left( \frac{\pi^2 B_{0j}^2 B_{0j}^2}{2K_0^2 \int_{R_0} W_0^2 dA} \right) \frac{1}{\ln c_i \ln c_j} + \cdots
\] (57)

This higher order asymptotic approximation agrees with the result [8, Eq. (85)] derived by the author for a membrane with a small inner circular core, and extends the lower order asymptotic approximation [7, Eq. (8)] derived by Wang for a membrane with \( M \) small inner circular cores.
4. A circular membrane of radius 1 with $M$ inner circular cores of small radius $c$

Now, consider a circular membrane of radius 1 with $M$ inner circular cores of small radius $c$ centered at $P_j$. Let $(r, \theta)$ be the polar coordinates with the origin at the center of the circular membrane without the inner cores and $P_j = (r_0,j, \theta_0)$. By the results [8, Eqs. (96)–(98), (101)–(104), (106)–(108)] derived by the author for a circular membrane of radius 1 with an inner circular core of small radius $c$, a second-order asymptotic approximation (as $c \to 0$) for the fundamental frequency $K$ of a circular membrane of radius 1 with $M$ inner circular cores of small radius $c$ centered at $(r_0,j, \theta_0)$ is found explicitly to be

$$K = K_0 + \sum_{j=1}^{M} \left( \frac{J_0(K_0r_0j)}{K_0J_1(K_0)} \right) \frac{1}{|\ln c|} + \sum_{j=1}^{M} \sum_{i=1}^{M} \left[ \frac{J_0(K_0r_0j)}{2K_0J_1(K_0)} \right]$$

$$= \sum_{j=1}^{M} \sum_{i=1}^{M} \left[ (\ln K_0 + \gamma - \ln 2)J_0(K_0r_0j) - J_0(K_0r_0j) \right] + \sum_{j=1}^{M} \sum_{i=1}^{M} \left[ \frac{J_0^2(K_0r_0j)}{2K_0J_1^2(K_0)} \right]$$

$$= \sum_{j=1}^{M} \sum_{i=1}^{M} \left[ \frac{J_0^2(K_0r_0j)}{2K_0J_1^2(K_0)} \right] 1 + r_0^2 - 2r_0 \cos \theta Y_1(K_0r_0j) + r_0^2 \cos \theta \right] \right]$$

$$= \sum_{j=1}^{M} \sum_{i=1}^{M} \left[ \frac{J_0^2(K_0r_0j)}{2K_0J_1^2(K_0)} \right] 1 + r_0^2 - 2r_0 \cos \theta Y_1(K_0r_0j) + r_0^2 \cos \theta \right] \right]$$

$$= K_0 \left( \sum_{j=1}^{M} \left( \frac{J_0(K_0r_0j)}{K_0J_1(K_0)} \right) \frac{1}{|\ln c|} \right) + \sum_{j=1}^{M} \sum_{i=1}^{M} \left[ \frac{J_0^2(K_0r_0j)}{2K_0J_1^2(K_0)} \right]$$

$$= \left\{ \begin{array}{l}
- \frac{1}{4} \left[ J_0(K_0r_0j)Y_0(K_0\tilde{r}) \right. \\
+ \frac{Y_0(K_0)J_0(K_0r_0j)J_0(K_0\tilde{r})}{K_0J_1(K_0)} \\
+ \frac{J_0(K_0r_0j)Y_1(K_0) + \tilde{r}J_0(K_0r_0j)Y_0(K_0)J_1(K_0)}{J_1(K_0)} \end{array} \right\}$$

$$= \left\{ \begin{array}{l}
- \frac{1}{4} \left[ J_0(K_0r_0j)Y_0(K_0\tilde{r}) \\
+ \frac{Y_0(K_0)J_0(K_0r_0j)J_0(K_0\tilde{r})}{K_0J_1(K_0)} \\
+ \frac{J_0(K_0r_0j)Y_1(K_0) + \tilde{r}J_0(K_0r_0j)Y_0(K_0)J_1(K_0)}{J_1(K_0)} \end{array} \right\}$$

$$= \frac{1}{2} \sum_{m=1}^{M} \cos(m(\theta - \tilde{\theta})) \left[ J_m(K_0r_0j)Y_m(K_0\tilde{r}) \\
- \frac{J_m(K_0r_0j)Y_m(K_0\tilde{r})}{J_m(K_0r_0j)} \right], \quad r \leq \tilde{r}, \quad r \geq \tilde{r},$$

the translational addition theorems for circular cylindrical wave functions [4,9] give

$$Y_0(K_0r_0j) = \sum_{l=-\infty}^{\infty} J_l(K_0r_0j)Y_l(K_0r) \cos(l - \theta_0),$$

$$Y_m(K_0r_0j) \sin(m\theta) = \sum_{l=-\infty}^{\infty} J_{l-m}(K_0r_0j)Y_l(K_0r) \sin(l - (l - m)\theta_0),$$

where $\gamma \approx 0.5772$, $R_{\beta}$ is the distance between the center points $P_j$ and $P_{\beta}$, $J_s$ is the $s$th order Bessel function, $Y_s$ is the $s$th order Neumann function, $K_{s,1}$ is the $t$th zero of $J_s$, $s = 0, 1, 2, 3, \ldots$, $t = 1, 2, 3, \ldots$, and $K_0 = K_{0,1} \approx 2.4048$.

Moreover, the generalized Green’s function $G$ has a compact expression for a circular membrane of radius 1, a more compact expression for the second-order asymptotic approximation Eq. (58), is found below.

The compact expression of the generalized Green’s function $G$ for a circular membrane of radius 1 [8] is
The translational addition theorems for circular cylindrical wave functions \([4,9]\) give

\[
Y_m(K_0 r_j) \cos(m \theta_j) = \sum_{l=-\infty}^{\infty} J_{l-m}(K_0 r_0) Y_l(K_0 r_j) \cos(l \theta - (l-m) \theta_0),
\]

\(m = 1, 2, 3, \ldots\), and \(J_{-m}(z) = (-1)^m J_m(z)\), \(Y_{-m}(z) = (-1)^m Y_m(z)\) \([12]\), \(i = 1, 2, 3, \ldots\). Hence,

\[
W_1(\tilde{r}, \tilde{\theta}) = \sum_{i=1}^{M} \int_{0}^{2\pi} \frac{\partial G(r, \theta; \tilde{r}, \tilde{\theta})}{\partial r} \, d\theta = \sum_{i=1}^{M} \left\{ -\frac{1}{4} \left[ J_0(K_0 \tilde{r}) Y_0'(K_0)K_0 - \frac{J_0(K_0 \tilde{r}) J_0'(K_0)K_0}{J_1(K_0)} \right. \\
- \frac{Y_0(K_0) J_0'(K_0) J_0(K_0) - \tilde{r} Y_0(K_0) J_1'(K_0) J_0(K_0)}{J_1(K_0)} \\
- \frac{J_0(K_0 \tilde{r}) J_0'(K_0) K_0 Y_1(K_0) + J_0(K_0 \tilde{r}) J_0(K_0)(J_1(K_0)+J'_1(K_0)K_0)}{J_1(K_0)} \right. \\
\times \left[ -\frac{B_0 J_0(K_0 \tilde{r})}{Y_0(K_0)} \right. \\
\left.- \sum_{m=1}^{\infty} \frac{J_m(K_0 \tilde{r})}{Y_m(K_0 \tilde{r})} 2\pi A_{m} j_{-m}(K_0 r_0) Y_0(K_0) \sin(m \theta_0) \right. \\
- \sum_{m=1}^{\infty} \frac{J_m(K_0 \tilde{r})}{Y_m(K_0 \tilde{r})} 2\pi B_{m} j_{-m}(K_0 r_0) Y_0(K_0) \cos(m \theta_0) \right. \\
- \sum_{m=1}^{\infty} \frac{\pi}{2} \left[ J_m(K_0 \tilde{r}) Y'_m(K_0) K_0 - \frac{J_m(K_0 \tilde{r}) Y_m(K_0) J'_m(K_0) K_0}{J_m(K_0)} \right] \\
\times \left[ -\frac{B_0 J_0(K_0 \tilde{r})}{Y_0(K_0)} \right. \\
\left.- \sum_{m=1}^{\infty} \frac{J_m(K_0 \tilde{r})}{Y_m(K_0 \tilde{r})} 2\pi A_{m} j_{-m}(K_0 r_0) Y_0(K_0) \cos(m \tilde{\theta} - m \theta_0) \right. \\
+ \sum_{p=1}^{\infty} A_{1-p} \frac{J_p(K_0 \tilde{r})}{Y_p(K_0 \tilde{r})} (J_{m-p}(K_0 r_0) Y_m(K_0) \sin(m \tilde{\theta} - (m-p) \theta_0)) \\
+ J_{-m-p}(K_0 r_0) Y_{-m}(K_0) \sin(m \tilde{\theta} - (m+p) \theta_0)) \\
\left. - \sum_{p=1}^{\infty} B_{1-p} \frac{J_p(K_0 \tilde{r})}{Y_p(K_0 \tilde{r})} (J_{m-p}(K_0 r_0) Y_m(K_0) \cos(m \tilde{\theta} - (m-p) \theta_0)) \right. \\
+ J_{-m-p}(K_0 r_0) Y_{-m}(K_0) \cos(m \tilde{\theta} - (m+p) \theta_0) \right\}.
\]

The translational addition theorems for circular cylindrical wave functions \([4,9]\) give

\[
J_0(K_0 \tilde{r}) = \sum_{l=-\infty}^{\infty} J_l(K_0 r_0) J_l(K_0 \tilde{r}) \cos(l \theta - l(\theta_0 + \pi)),
\]

\[
J_p(K_0 \tilde{r}) \sin(p \tilde{\theta}) = \sum_{l=-\infty}^{\infty} J_{l-p}(K_0 r_0) J_l(K_0 \tilde{r}) \sin(l \theta - (l-p)(\theta_0 + \pi)),
\]

\[
J_p(K_0 \tilde{r}) \cos(p \tilde{\theta}) = \sum_{l=-\infty}^{\infty} J_{l-p}(K_0 r_0) J_l(K_0 \tilde{r}) \cos(l \theta - (l-p)(\theta_0 + \pi)),
\]
\( p = 1, 2, 3, \ldots \). Hence, by Law of cosine, \( J'_0(z) = -J_1(z), \ Y'_0(z) = -Y_1(z) \), Wronskian \( \mathcal{W} \) of \( J_m(z) \) and \( Y_m(z) \) [12] which is \( \mathcal{W}(J_m(z), Y_m(z)) = 2/\pi z \), and \( B_{0i} = J_0(K_0r_{0i}) \) [8, Eq. (96)],

\[
\frac{-1}{2\pi Y_0(K_0c)} \int_{0}^{2\pi} W_1(r_j, \theta_j) \left| W_0(r_j, \theta_j) \right| \left( \frac{\pi r_0 J_0(K_0r_{0j}) Y_0^2(K_0)}{2} - \frac{\pi r_0 J_0(K_0r_{0j}) J_1(K_0r_{0j}) Y_0^2(K_0)}{4} \right.
+ \frac{\pi r_0 J_0(K_0r_{0j}) J_0(K_0r_{0j}) Y_0^2(K_0) J_1'(K_0)}{4J_1(K_0)}
- \frac{\sum_{m=1}^{\infty} J_m(K_0r_{0j}) J_m(K_0) \cos(m\theta_0 - m\theta_j)}{J_m(K_0)} \right) \frac{1}{|\ln c|^2} + \cdots .
\]

(67)

Therefore, following the derivation for the second correction term \( F_2 \) in Section 3, a more compact expression for the second-order asymptotic approximation Eq. (58) is found to be

\[
K = K_0 + \sum_{j=1}^{M} \left( \frac{J_0'(K_0r_{0j})}{K_0 J_1^2(K_0)} \right) \frac{1}{|\ln c|} + \left\{ \sum_{j=1}^{M} \left( \frac{\ln K_0 + \gamma - \ln 2)J_0'(K_0r_{0j})}{K_0 J_1^2(K_0)} \right) \right. + \sum_{j=1}^{M} \sum_{i=1}^{M} \left[ \frac{-J_0'(K_0r_{0j}) J_0'(K_0r_{0i})}{2K_0 J_1^2(K_0)} \right.
+ \frac{\pi^2 J_0^2(K_0r_{0j}) J_0^2(K_0r_{0j}) Y_0^2(K_0)}{2K_0 J_1^2(K_0)} + \frac{\pi^2 J_0^2(K_0r_{0j}) J_0'(K_0r_{0j}) Y_0^2(K_0) J_1'(K_0)}{4J_1(K_0)}
- \frac{\pi r_0 J_0(K_0r_{0j}) J_0(K_0r_{0j}) J_1(K_0r_{0j}) Y_0^2(K_0)}{4J_1^2(K_0)}
- \frac{\pi J_0(K_0r_{0j}) J_0(K_0r_{0j}) \sum_{m=1}^{\infty} J_m(K_0r_{0j}) J_m(K_0r_{0j}) Y_m(K_0) \cos(m\theta_0 - m\theta_j)}{K_0 J_1^2(K_0)} \left( \frac{1}{|\ln c|^2} + \cdots ,
\right.
\]

(68)

where \( \gamma \approx 0.5772 \), \( R_p \) is the distance between the centers of the inner circular cores \( B_j \) and \( B_i \), whose center points are \( P_j = (r_{0j}, \theta_{0j}) \) and \( P_i = (r_{0i}, \theta_{0i}) \), respectively. \( J_j \) is the \( j \)th order Bessel function, \( Y_j \) is the \( j \)th order Neumann function, and \( K_0 = K_{0,1} \approx 2.4048 \) is the first zero of \( J_0 \).

### Table 1

The fundamental frequency \( K \) for a circular membrane of radius 1 with an eccentric circular core of radius 0.1 centered at \((r, \theta) = (r_0, 0)\)

<table>
<thead>
<tr>
<th>( r_0 )</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>Lin</td>
<td>3.325</td>
<td>3.328</td>
<td>3.188</td>
<td>3.083</td>
<td>2.980</td>
<td>2.868</td>
<td>2.780</td>
<td>2.700</td>
<td>2.620</td>
<td>2.500</td>
</tr>
<tr>
<td>The first-order approximation</td>
<td>3.075</td>
<td>3.056</td>
<td>3.001</td>
<td>2.917</td>
<td>2.814</td>
<td>2.706</td>
<td>2.603</td>
<td>2.516</td>
<td>2.453</td>
<td>2.416</td>
</tr>
<tr>
<td>The second-order approximation</td>
<td>3.299</td>
<td>3.261</td>
<td>3.157</td>
<td>3.014</td>
<td>2.864</td>
<td>2.731</td>
<td>2.622</td>
<td>2.536</td>
<td>2.470</td>
<td>2.424</td>
</tr>
<tr>
<td>The exact value</td>
<td>3.314</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>
There are also some existing results on the fundamental frequency of a circular membrane with inner circular cores. In 1977, Nagaya [11] considered a circular membrane with an eccentric inner circular core. The author used the exact solution of the equation of motion which satisfies the inner boundary condition and adopted the Fourier expansion method on the outer boundary condition to calculate the fundamental frequency of the membrane. In 1981, Lin [4] considered an equivalent problem to the fundamental frequency of a circular membrane with inner circular cores, in which both the equation of motion and the boundary conditions were exactly satisfied and the technique of transformation of cylindrical wave functions was then used to determinate the fundamental frequency of the membrane. In 1989, Nagaya and Poltorak [5] considered a circular membrane with a number of eccentric inner circular cores. The authors used the point-matching approach to treat the inner boundary conditions and presented the expression to find the fundamental frequency of the membrane. The comparisons among the asymptotic approximations Eq. (68) and the numerical values computed by the above investigators are shown in Tables 1–5.

Table 2
The fundamental frequency $K$ for a circular membrane of radius 1 with two eccentric circular cores of radius 0.1 centered at $(r, \theta) = (r_0, 0)$ and $(r, \theta) = (r_0, \pi)$, respectively

<table>
<thead>
<tr>
<th>$r_0$</th>
<th>0</th>
<th>0.08</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>The second-order approximation</td>
<td>3.299</td>
<td>3.571</td>
<td>3.909</td>
<td>3.792</td>
<td>3.511</td>
<td>3.190</td>
<td>2.907</td>
<td>2.692</td>
<td>2.540</td>
<td>2.443</td>
</tr>
<tr>
<td>The exact value</td>
<td>3.314</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
<td>—</td>
</tr>
</tbody>
</table>

Table 3
The fundamental frequency $K$ for a circular membrane of radius 1 with two eccentric circular cores of radius $c$ centered at $(r, \theta) = (0.5, 0)$ and $(r, \theta) = (0.5, \pi)$, respectively

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nagaya and Poltorak</td>
<td>3.295</td>
<td>3.882</td>
<td>4.548</td>
<td>5.194</td>
</tr>
<tr>
<td>The first-order approximation</td>
<td>3.006</td>
<td>3.265</td>
<td>3.555</td>
<td>3.916</td>
</tr>
<tr>
<td>The second-order approximation</td>
<td>3.190</td>
<td>3.641</td>
<td>4.226</td>
<td>5.074</td>
</tr>
</tbody>
</table>

Table 4
The fundamental frequency $K$ for a circular membrane of radius 1 with four eccentric circular cores of radius 0.1 centered at $(r, \theta) = (r_0, [(j - 1)\pi]/2), j = 1, 2, 3, 4$, respectively

<table>
<thead>
<tr>
<th>$r_0$</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>The first-order approximation</td>
<td>4.788</td>
<td>4.453</td>
<td>4.043</td>
<td>3.608</td>
<td>3.196</td>
<td>2.850</td>
<td>2.597</td>
<td>2.450</td>
</tr>
<tr>
<td>The second-order approximation</td>
<td>4.452</td>
<td>4.975</td>
<td>4.700</td>
<td>4.102</td>
<td>3.491</td>
<td>3.011</td>
<td>2.683</td>
<td>2.482</td>
</tr>
</tbody>
</table>

Table 5
The fundamental frequency $K$ for a circular membrane of radius 1 with four eccentric circular cores of radius $c$ centered at $(r, \theta) = (0.5, [(j - 1)\pi]/2), j = 1, 2, 3, 4$, respectively

<table>
<thead>
<tr>
<th>$c$</th>
<th>0.05</th>
<th>0.1</th>
<th>0.15</th>
<th>0.2</th>
<th>0.25</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nagaya and Poltorak</td>
<td>3.906</td>
<td>4.655</td>
<td>5.614</td>
<td>6.710</td>
<td>7.620</td>
</tr>
<tr>
<td>The first-order approximation</td>
<td>3.329</td>
<td>3.608</td>
<td>3.865</td>
<td>4.126</td>
<td>4.403</td>
</tr>
<tr>
<td>The second-order approximation</td>
<td>3.621</td>
<td>4.102</td>
<td>4.593</td>
<td>5.137</td>
<td>5.766</td>
</tr>
</tbody>
</table>
5. Results and discussion

The fundamental frequency of a multiply connected membrane with inner cores of vanishing maximal dimensions is concerned in the present article. A general formula for the fundamental frequency is derived by the extension of the modified perturbation method [8] and found to be the formula Eq. (6) with Eqs. (22)–(24). A higher order asymptotic approximation (as \( c_j \to 0 \)) for the fundamental frequency of a membrane with inner circular cores of radius \( c_j \) is specified as the approximation Eq. (57). It is an excellent extension for the second-order result [8, Eq. (85)] derived by the author for a doubly connected membrane with a vanishing inner circular core and an excellent extension for the lower order result [7, Eq. (8)] derived by Wang for a multiply connected membrane with vanishing inner circular cores.

Moreover, a second-order asymptotic approximation (as \( c \to 0 \)) for the fundamental frequency of a circular membrane of radius 1 with \( M \) inner circular cores of small radius \( c \) is found explicitly as the approximation Eq. (68). Observing from the second-order asymptotic approximation Eq. (68), it is found that the positions of the inner cores related to the membrane affect the approximation starting at the first correction term, while the inter-positions of the inner cores affect the approximation starting at the second correction term. Also, the comparisons among the asymptotic approximations and the numerical values [4,5,11] computed by other investigators are made and shown in Tables 1–5. Observing from the tables, it is found that the second-order asymptotic approximation achieves more accuracy than the first-order asymptotic approximation does. Moreover, Nagaya and Poltorak [5] pointed out that the fundamental frequency decreases as the eccentricity increases for a circular membrane with an inner circular core and the fundamental frequency increases first and then decreases as the eccentricity increases for a circular membrane with more than one inner circular cores. The second-order asymptotic approximation depicts this phenomenon, while the first-order asymptotic approximation does not.

References