SCATTERING OF ELASTIC WAVES BY A BURIED TUNNEL UNDER OBLIQUELY INCIDENT WAVES USING T MATRIX

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ABSTRACT

This paper first studies the transition matrix formulation for the analysis of responses of an elastic half-space with a buried tunnel subjected to obliquely incident waves. The basis functions are constructed using the moving P-, SV-, and SH-wave source potentials and to represent the scattered and refracted wave fields in series forms. The associated T-matrix expression of elastic inclusion is derived using Betti's third identity. Second, this study proposes a technique for calculating the integral representation of basis functions in the wave-number domain using the method of steepest descent. Finally, typical numerical results obtained under incident plane waves are presented for verification.

Keywords : T-matrix, Elastic wave, Steepest descent, Tunnel.

1. INTRODUCTION

The transition matrix (T-matrix) theory for the scattering of acoustic waves by a single scatterer in an infinite medium was first proposed by Waterman [1,2]. Initially, the theory was derived using Helmholtz integral formula. Pao [3,4] has shown that the derivation can be greatly simplified if Green's identity is applied to the acoustic wave problem and Betti's third identity is applied to the elastic wave problem. Since then, there have been numerous publications on the applications and extensions to electromagnetic and elastic wave propagation problems. Detailed references for electromagnetic and elastic wave propagation problems can be found in (Varatharajula and Pao [5]; and Varadan and Varadan [6]). Yeh and Pao [7] have proposed the T-matrix formulation for the scattering of acoustic waves by a multiple layered inclusion in an infinity medium. Chai, *et al*. [8] have successfully employed the cylindrical wave functions as basis functions for SH-wave scattering in a two-dimensional alluvial valley to study its resonant phenomenon according to the property of the T-matrix. It should be noted that only Neuman's boundary condition can be satisfied by the angular part of the cylindrical wave function.

As for research on dynamic responses of a buried layered inclusion (tunnel or pipeline) in an elastic semi-infinity medium subjected to incident plane waves, Lee and Trifunac [9] have studied the scattering of two-dimensional anti-plane problem using the method of series expansion. The cylindrical pipeline subjected to incident SH-waves was also analyzed and discussed. El-Akily and Datta [10] have used the Flügge shell theory to simplify the behavior of the cylindrical pipeline, and expand the scattering fields of the exterior region by circular cylindrical wave functions for their analyses. Wong *et al*. [11] combined the finite element method with wave function expansion to analyze the responses of a buried pipeline with non-circular cross-sections. Luco and de Barros [12] used the indirect boundary integral method to construct the scattering field and combined it with the Donell shell theory to study the responses of a cylindrical pipeline subjected to obliquely incident plane waves. Yeh *et al*. [13] have recently proposed a hybrid method for analyzing the dynamics responses of shells buried in an elastic half-plane. Recently, Chen *et al*. [14-16] used the boundary integral method with degenerate kernel of fundamental solutions to transform the improper boundary integrals to a series sum for study the exterior radiation and scattering problems with circular boundaries.

In this paper, Yeh and Pao's [7] concept is extended to analyze the scattering of elastic waves by a lined

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tunnel subjected to obliquely incident waves in an elastic half-space. The basis functions and its regular part are constructed using moving P-, SV- and SH-wave source potentials, Betti's third identity and orthogonality conditions for the elastic half-space. The T-matrix relating the coefficients of the scattered waves to those of the free field is developed among the basis functions. Although the theory presented herein is only for a tunnel buried in an elastic half-space, the extension theory for the multiple layered inclusion is straightforward and easy to derive.

This paper also proposes a technique for calculating the integral representation of basis functions in the wave-number domain using the method of steepest descent, with focus on the treatment for highly oscillating wave-number integration. The approach proposed here is developed from an integral representation of the complete response in terms of wave-number. There are a variety of techniques for calculating the integral. Apsel and Luco [17] have proposed a technique for evaluation of wave-number integral by replacing the integral function with a quadratic polynomial, thus obtaining in a Filon-type quadrature. Kundu and Mal [18] have proposed using an adaptive Gauss quadrature to accomplish the same task. Xu and Mal [19] have applied Clenshaw-Curtis approach in which the integrand function is approximated with Chebyshev polynomials and then integrated to produce a Filon-type quadrature. The technique that applies the steepest descent path to the wave-number integral was first proposed by Yeh *et al.* [20] for the analysis of plane-strain elastic-wave propagation problem. Liao *et al.* [21] extended this method to the analysis of wave propagation in a three-dimensional elastic half-space. In this paper, the modified steepest descent method is extended to analyze the problem of elastic half-space subjected to oblique incident waves. Compared with that in the previous works by Yeh *et al.* [20] and Liao *et al.* [21], the integrand in the integral of this study is quite different. The associated steepest descent path, location of the branch points and Rayleigh poles in the complex wave-number domain also appeared in a different configuration compared with those previously reported. It is also interesting to study the numerical integration analysis of the so-called 2.5-dimensional problems through the application of the modified steepest descent method. The basic theory of the modified method of steepest descent is to replace the original integral path with the steepest descent path (SDP) and let the wave- number integral yield a Gauss-Hermite type quadrature, so that the oscillating character of the original integral can be removed, leaving only the non-oscillating one to be evaluated. Finally, numerical results for the stress and displacement components of a buried tunnel within the elastic half-space with different obliquely incident waves are calculated.

2. PROBLEM STATEMENT AND BASIS FUNCTIONS

Consider a layered elastic inclusion (tunnel) embedded in an elastic half-space D as shown in Fig. 1, the interface between the elastic half-space and the inclusion is surface S_1. The elastic inclusion is divided by the surface S_2 into two parts denoted as D_1 and D_2, respectively. For the half-space D, the density and Lame' constants are ρ, λ and μ, respectively, and those for the domain D_j (j = 1, 2) are denoted as ρ_j, λ_j and μ_j, respectively. The elastic half-space is subjected to obliquely incident plane waves with harmonics to the time variable and to the space variable y. The vertical and horizontal incident angles of the incident waves are θ_v and θ_h, respectively. A factor $e^{i(\omega t - k_y y)}$ is assumed throughout this paper for the waves with specified circular frequency ω and wave-number k_y, where k_y is the y-component wave-number of the incident waves. The wave-number k_y for different types of incident waves are listed in the following

Incident P-wave: $k_y = k_p \sin\theta_\upsilon \sin\theta_h$

Incident S-wave: $k_y = k_s \sin\theta_\upsilon \sin\theta_h$

Incident Rayleigh wave: $k_y = k_R \sin\theta_\upsilon \sin\theta_h$

where $k_p = \omega/c_p$ and $k_s = \omega/c_s$ are the wave-number of the longitudinal and shear wave velocities $c_p = \sqrt{(\lambda+2\mu)/\rho}$ and $c_s = \sqrt{\mu/\rho}$, respectively, and $k_R = \omega/c_R$ is the wave-number of the Rayleigh wave velocity c_R.

As the incident wave impinges on the elastic inclusion, one part is transmitted into domains D_1 and D_2, and the other is reflected back into the half-space. Hence the wave field in the half-space comprises the free field $\mathbf{u}^f$ and the scattered field $\mathbf{u}^s$. Then the total displacement field for the half-space can be expressed as

$$\mathbf{u} = \mathbf{u}^f + \mathbf{u}^s, \quad \mathbf{x} \in D \tag{1}$$

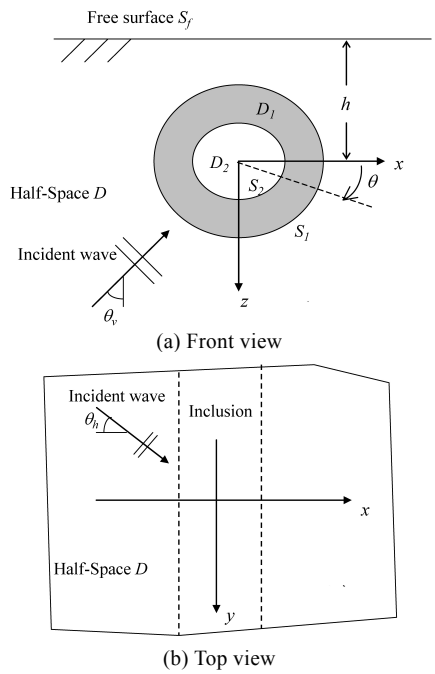

Fig. 1 Configuration and coordinate system

The scattered field can be represented by suitable independent sets of basis functions satisfying the free-surface and radiation conditions at infinity. The basis functions adapted in this study are singular solutions of the responses of a half-space subjected to moving P-, SV- and SH-wave source potentials and their high-order terms, and are expressed in integral form. The derivation of those basis functions are detailed in Appendix A. The scattered wave field of the exterior region can be expressed as

$$\mathbf{u}^s = \sum_{\alpha}\sum_{m=0}^{N} c_m^{\alpha}\,\mathbf{u}_m^{\alpha} = \sum_{m=0}^{N} c_m^{p}\,\mathbf{u}_m^{p} + \sum_{m=0}^{N} c_m^{\upsilon}\,\mathbf{u}_m^{\upsilon} + \sum_{m=0}^{N} c_m^{h}\,\mathbf{u}_m^{h} \tag{2}$$

where $\mathbf{u}_m^{\alpha}$ ($\alpha = p, v, h$) are the basis functions of the elastic half-space, and the superscript $\alpha = p$, v, and h denote the wave fields generated by moving P-, SV- and SH-wave source potentials, respectively. The subscript m ranges from 0 to N, and N is the approximated order. Similarly, the displacement field of the free field can be expanded into a series of regular basis function as

$$\mathbf{u}^f = \sum_{\alpha,m} Y_m^{\alpha}\,\hat{\mathbf{u}}_m^{\alpha} \tag{3}$$

where $\hat{\mathbf{u}}_m^{\alpha}$ is the regular part of the basis function $\mathbf{u}_m^{\alpha}$. For domains D_1 and D_2, the refracted wave fields can be expressed as

$$\mathbf{u}^{(1)} = \sum_{\alpha}\sum_{m=0}^{\infty} b_{m(1)}^{\alpha}\,\mathbf{u}_m^{\alpha(1)} + \sum_{\alpha}\sum_{m=0}^{\infty} b_{m(2)}^{\alpha}\,\hat{\mathbf{u}}_m^{\alpha(1)}\,, \quad \mathbf{x}\in D_1 \tag{4}$$

$$\mathbf{u}^{(2)} = \sum_{\alpha}\sum_{m=0}^{\infty} f_{m(1)}^{\alpha}\,\hat{\mathbf{u}}_m^{\alpha(2)}\,, \quad \mathbf{x}\in D_2 \tag{5}$$

where $\hat{\mathbf{u}}_m^{\alpha(j)}$ ($j = 1, 2$; $\alpha = p, v, h$) is the regular part of the basis function $\mathbf{u}_m^{\alpha(j)}$ of domain D_j. The basis function $\mathbf{u}_m^{\alpha}$ for the half-space should satisfy the traction-free conditions on the free surface, but the basis function $\mathbf{u}_m^{\alpha(j)}$ for the elastic inclusion need not satisfy the traction-free conditions. Hence, the basis functions for domains D_1 and D_2 include only the source term of the source potentials and have the following component form

$$u_{i(m)}^{\alpha(j)} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_m(k_x) H_i^{\alpha(j)} e^{-ikx} dk_x\,;$$
$$\mathbf{x}\in D_j\,,\ j = 1, 2\,,\ i = x, y, z \tag{6}$$

where $k_p^* = \sqrt{k_p^2 - k_y^2}$. For $\alpha = p$ (P-wave source potential), components of $H_i^{p(j)}$ are

$$H_x^p = \frac{-ik_x}{\upsilon} e^{-\upsilon|z|}$$
$$H_y^p = \frac{-ik_y}{\upsilon} e^{-\upsilon|z|}$$
$$H_z^p = -\mathrm{sgn}(z)\, e^{-\upsilon|z|} \tag{7}$$

For $\alpha = v$ (SV-wave source potential), components of $H_i^{v(j)}$ are

$$H_x^v = \mathrm{sgn}(z)\, e^{-\upsilon'|z|}$$
$$H_y^v = 0$$
$$H_z^v = \frac{-ik_x}{\upsilon'} e^{-\upsilon'|z|} \tag{8}$$

For $\alpha = h$ (SH-wave source potential), components of $H_i^{h(j)}$ are

$$H_x^h = \frac{-k_x k_y}{\upsilon'} e^{-\upsilon'|z|}$$
$$H_y^h = \frac{k_s^*}{\upsilon'} e^{-\upsilon'|z|}$$
$$H_y^h = \frac{ik_y \upsilon' \mathrm{sgn}(z)}{\upsilon'} e^{-\upsilon'|z|} \tag{9}$$

where $k_s^* = \sqrt{k_s^2 - k_y^2}$ and $S_m(k_x)$ is properly selected as

$$S_m(k_x) = \begin{cases} i^m\, T_n(k_x / k_p^*)\,, & n = m/2\,, \quad m = 0, 2, 4, \ldots \\ i^m\, \upsilon\, Q_l(k_x / k_p^*)\,, & l = (m+1)/2\,, \quad m = 1, 3, 5, \ldots \end{cases} \tag{10}$$

for P-wave source potential, and

$$S_m(k_x) = \begin{cases} i^m\, T_n(k_x / k_s^*)\,, & n = m/2\,, \quad m = 0, 2, 4, \ldots \\ i^m\, \upsilon' Q_l(k_x / k_s^*)\,, & l = (m+1)/2\,, \quad m = 1, 3, 5, \ldots \end{cases} \tag{11}$$

for SV-wave and SH-wave source potentials, and where T_n and Q_l are Chebyshev polynomials of the first and second kind, respectively. The basis functions $u_m^{\alpha(j)}$ ($j = 1, 2$; $\alpha = p, v, h$) can be easily evaluated using the method presented in Section 4.

By applying Betti's third identity of elasticity

$$\int_S [\mathbf{t}(\mathbf{u})\cdot\mathbf{v} - \mathbf{t}(\mathbf{v})\cdot\mathbf{u}]\, dS = \int_V \{[\nabla\cdot\boldsymbol{\sigma}(\mathbf{u})]\cdot\mathbf{v} - [\nabla\cdot\boldsymbol{\sigma}(\mathbf{v})]\cdot\mathbf{u}\}\, dV \tag{12}$$

where S is the surface of volume V, $\mathbf{t}(\mathbf{u})$ and $\mathbf{t}(\mathbf{v})$ are tractions of displacement fields $\mathbf{u}$ and $\mathbf{v}$, respectively. The orthogonality conditions for domain D are

$$\int_{S_+} [\mathbf{t}_m^{\alpha}\cdot\mathbf{u}_n^{\beta} - \mathbf{t}_n^{\beta}\cdot\mathbf{u}_m^{\alpha}]\, dS = 0 \tag{13}$$

$$\int_{S_+} [\hat{\mathbf{t}}_m^{\alpha}\cdot\hat{\mathbf{u}}_n^{\beta} - \hat{\mathbf{t}}_n^{\beta}\cdot\hat{\mathbf{u}}_m^{\alpha}]\, dS = 0 \tag{14}$$

$$\int_{S_+} [\mathbf{t}_m^{\alpha}\cdot\hat{\mathbf{u}}_n^{\beta} - \hat{\mathbf{t}}_n^{\beta}\cdot\mathbf{u}_m^{\alpha}]\, dS \equiv E_{nm}^{\beta\alpha} \tag{15}$$

where S_+ can be any arbitrary closed bounded surface that outside the domain surface D_1. Similarly, the orthogonality conditions for domain D_1 are

$$\int_{S_-} [\mathbf{t}_m^{\alpha(1)}\cdot\mathbf{u}_n^{\beta(1)} - \mathbf{t}_n^{\beta(1)}\cdot\mathbf{u}_m^{\alpha(1)}]\, dS = 0 \tag{16}$$

$$\int_{S_-} [\hat{\mathbf{t}}_m^{\alpha(1)} \cdot \hat{\mathbf{u}}_n^{\beta(1)} - \hat{\mathbf{t}}_n^{\beta(1)} \cdot \hat{\mathbf{u}}_m^{\alpha(1)}]\, dS = 0 \tag{17}$$

$$\int_{S_-} [\mathbf{t}_m^{\alpha(1)} \cdot \hat{\mathbf{u}}_n^{\beta(1)} - \hat{\mathbf{t}}_n^{\beta(1)} \cdot \mathbf{u}_m^{\alpha(1)}]\, dS \equiv E_{nm(1)}^{\beta\alpha} \tag{18}$$

where S_- can be any arbitrary closed surface inside domain D_1.

3. TRANSITION MATRIX FORMULISM

If the elastic half-space is assumed to be continuous with the elastic inclusion, the continuity conditions on the interface S_1 are

$$\begin{aligned} \mathbf{t}^+ &= \mathbf{t}_{(1)}^- \\ \mathbf{u}^+ &= \mathbf{u}_{(1)}^- , \quad \mathbf{x} \in S_1 \end{aligned} \tag{19}$$

where $\mathbf{u}^+$ and $\mathbf{u}_{(1)}^-$ are the total displacements on the interface S, approaching from the positive and negative direction of the unit normal, respectively. $\mathbf{t}^+$ ($t_i^+ = \sigma_{ij}^+ n_j$) and $\mathbf{t}_{(1)}^-$ ($t_{i(1)}^- = \sigma_{ij(1)}^- n_j$) are the associated tractions of $\mathbf{u}^+$ and $\mathbf{u}_{(1)}^-$, respectively. n_j is the component of unit outer normal vector $\mathbf{n}$ of surface S_1. For interface S_2 between domains D_2 and D_1, in the case that domain D_2 is an elastic medium and continuous with D_1, the boundary conditions on surface S_2 are

$$\begin{aligned} \mathbf{t}_{(1)}^+(\mathbf{x}) &= \mathbf{t}_{(2)}^-(\mathbf{x}) \\ \mathbf{u}_{(1)}^+(\mathbf{x}) &= \mathbf{u}_{(2)}^-(\mathbf{x}) , \quad \mathbf{x} \in S_2 \end{aligned} \tag{20}$$

In the case that domain D_2 is a cavity, then the boundary conditions are

$$\begin{aligned} \mathbf{t}_{(1)}^+(\mathbf{x}) &= 0 \\ \mathbf{t}_{(2)}^+(\mathbf{x}) &= 0 , \quad \mathbf{x} \in S_2 \end{aligned} \tag{21}$$

In this study, we focus only on the case that domain D_2 is a cavity, because it represents a buried lined tunnel or a pipeline. It is also meaningful and has more applications than domain D_2 as an elastic medium.

If the volume in Eq. (12) is the domain bounded by surfaces S_1 and S_+ shown in Fig. 2, and let $\mathbf{v} = \mathbf{u}_n^\beta$, then

$$\int_{S_1} [\mathbf{t}^+ \cdot \mathbf{u}_n^\beta - \mathbf{t}_n^\beta \cdot \mathbf{u}^+]\, dS = \int_{S_+} [\mathbf{t} \cdot \mathbf{u}_n^\beta - \mathbf{t}_n^\beta \cdot \mathbf{u}]\, dS \tag{22}$$

By substituting Eqs. (2) and (3) into the right-hand side of Eq. (22), and applying the orthogonality condition of Eq. (12), Eq. (21) can then be reduced to the following form

$$\int_{S_1} [\mathbf{t}^+ \cdot \mathbf{u}_n^\beta - \mathbf{t}_n^\beta \cdot \mathbf{u}^+]\, dS = \int_{S_+} [\mathbf{t}^f \cdot \mathbf{u}_n^\beta - \mathbf{t}_n^\beta \cdot \mathbf{u}^f]\, dS = a_n^\beta \tag{23}$$

Then from the continuity conditions of Eq. (19), the coefficient vector a_n^β can be expressed as

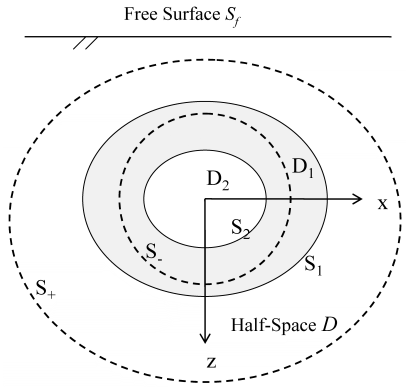

Fig. 2 Domains and relative surfaces of this problem

$$\begin{aligned} a_n^\beta &= \int_{S_1} [\mathbf{t}_{(1)}^- \cdot \mathbf{u}_n^\beta - \mathbf{t}_n^\beta \cdot \mathbf{u}_{(1)}^-]\, dS \\ &= \int_{S_1} \left[\left(\sum_{\alpha,m} b_{m(1)}^\alpha \mathbf{t}_m^{\alpha(1)} + \sum_{\alpha,m} b_{m(2)}^\alpha \hat{\mathbf{t}}_m^{\alpha(1)} \right) \cdot \mathbf{u}_n^\beta \right. \\ &\qquad \left. - \mathbf{t}_n^\beta \cdot \left(\sum_{\alpha,m} b_{m(1)}^\alpha \mathbf{u}_m^{\alpha(1)} + \sum_{\alpha,m} b_{m(2)}^\alpha \hat{\mathbf{u}}_m^{\alpha(1)} \right) \right] dS \\ &= \sum_{\alpha,m} P_{nm(1)}^{\beta\alpha} b_{m(1)}^\alpha + \sum_{\alpha,m} Q_{nm(1)}^{\beta\alpha} b_{m(2)}^\alpha \end{aligned} \tag{24}$$

where

$$P_{nm(1)}^{\beta\alpha} = \int_{S_1} [\mathbf{t}_m^{\alpha(1)} \cdot \mathbf{u}_n^\beta - \mathbf{t}_n^\beta \cdot \mathbf{u}_m^{\alpha(1)}]\, dS \tag{25}$$

$$Q_{nm(1)}^{\beta\alpha} = \int_{S_1} [\hat{\mathbf{t}}_m^{\alpha(1)} \cdot \mathbf{u}_n^\beta - \mathbf{t}_n^\beta \cdot \hat{\mathbf{u}}_m^{\alpha(1)}]\, dS \tag{26}$$

For the same domain bounded by surfaces S_1 and S_+, let $\mathbf{v} = \hat{\mathbf{u}}_n^\beta$ yields

$$\begin{aligned} &\int_{S_1} [\mathbf{t}^+ \cdot \hat{\mathbf{u}}_n^\beta - \hat{\mathbf{t}}_n^\beta \cdot \mathbf{u}^+]\, dS \\ &= \int_{S_+} \left[\left(\mathbf{t}^f + \sum_{\alpha,m} c_m^\alpha \mathbf{t}_m^\alpha \right) \cdot \hat{\mathbf{u}}_n^\beta - \hat{\mathbf{t}}_n^\beta \cdot \left(\mathbf{u}^f + \sum_{\alpha,m} c_m^\alpha \hat{\mathbf{u}}_m^\alpha \right) \right] dS \end{aligned} \tag{27}$$

By substituting the expansion form Eq. (3) of free-field displacement into the right-hand side of Eq. (27) and by using the orthogonality condition, Eq. (27) can be reduced to the following form

$$\begin{aligned} &\int_{S_1} [\mathbf{t}^+ \cdot \hat{\mathbf{u}}_n^\beta - \hat{\mathbf{t}}_n^\beta \cdot \mathbf{u}^+]\, dS \\ &= \sum_{\alpha,m} c_m^\alpha \int_{S_+} [\mathbf{t}_m^\alpha \cdot \hat{\mathbf{u}}_n^\beta - \hat{\mathbf{t}}_n^\beta \cdot \mathbf{u}_m^\alpha]\, dS = \sum_{\alpha,m} E_{nm}^{\beta\alpha} c_m^\alpha \end{aligned} \tag{28}$$

where the matrix $E_{nm}^{\beta\alpha}$ is defined as

$$E_{nm}^{\beta\alpha} = \int_{S_+} [\mathbf{t}_m^\alpha \cdot \hat{\mathbf{u}}_n^\beta - \hat{\mathbf{t}}_n^\beta \cdot \mathbf{u}_m^\alpha]\, dS \tag{29}$$

Furthermore, from the continuity conditions of Eq. (19), the matrix $E_{nm}^{\beta\alpha}$ is reduced to

$$\sum_{\alpha,m} E_{nm}^{\beta\alpha} c_m^{\alpha} = \int_{S_1} [\mathbf{t}_{(1)}^{-} \cdot \hat{\mathbf{u}}_n^{\beta} - \hat{\mathbf{t}}_{\alpha}^{\beta} \cdot \mathbf{u}_{(1)}^{-}]\, dS$$

$$= \int_{S_1} \left[\left(\sum_{\alpha,m} b_{m(1)}^{\alpha}\, \mathbf{t}_m^{\alpha(1)} + \sum_{\alpha,m} b_{m(2)}^{\alpha}\, \hat{\mathbf{t}}_m^{\alpha(1)} \right) \cdot \hat{\mathbf{u}}_n^{\beta} - \hat{\mathbf{t}}_n^{\beta} \cdot \left(\sum_{\alpha,m} b_{m(1)}^{\alpha}\, \mathbf{u}_m^{\alpha(1)} + \sum_{\alpha,m} b_{m(2)}^{\alpha}\, \hat{\mathbf{u}}_m^{\alpha(1)} \right) \right] dS$$

$$= \sum_{\alpha.m} \hat{P}_{nm(1)}^{\beta,\alpha}\, b_{m(1)}^{\alpha} + \sum_{\alpha,m} \hat{Q}_{nm(1)}^{\beta,\alpha}\, b_{m(2)}^{\alpha} \tag{30}$$

where

$$\hat{P}_{nm(1)}^{\beta\alpha} = \int_{S_1} [\mathbf{t}_m^{\alpha(1)} \cdot \hat{\mathbf{u}}_n^{\beta} - \hat{\mathbf{t}}_n^{\beta} \cdot \mathbf{u}_m^{\alpha(1)}]\, dS \tag{31}$$

$$\hat{Q}_{nm(1)}^{\beta\alpha} = \int_{S_1} [\hat{\mathbf{t}}_m^{\alpha(1)} \cdot \hat{\mathbf{u}}_n^{\beta} - \hat{\mathbf{t}}_n^{\beta} \cdot \hat{\mathbf{u}}_m^{\alpha(1)}]\, dS \tag{32}$$

Now consider volume V in Eq. (12) enclosed by surfaces S_2 and S_- as shown in Fig. 2, and let $\mathbf{v} = \mathbf{u}_n^{\beta(1)}$, then

$$\int_{S_2} [\mathbf{t}_{(1)}^{+} \cdot \mathbf{u}_n^{\beta(1)} - \mathbf{t}_n^{\beta(1)} \cdot \mathbf{u}_{(1)}^{+}]\, ds$$

$$= \int_{S_-} \left[\left(\sum_{\alpha,m} b_{m(1)}^{\alpha} \mathbf{t}_m^{\alpha(1)} + \sum_{\alpha,m} b_{m(2)}^{\alpha} \hat{\mathbf{t}}_m^{\alpha(1)} \right) \cdot \mathbf{u}_n^{\beta(1)} - \mathbf{t}_n^{\beta(1)} \cdot \left(\sum_{\alpha,m} b_{m(1)}^{\alpha} \mathbf{u}_m^{\alpha(1)} + \sum_{\alpha,m} b_{m(2)}^{\alpha} \hat{\mathbf{u}}_m^{\alpha(1)} \right) \right] dS \tag{33}$$

By using the orthogonality condition of Eq. (16), Eq. (33) can be reduced to

$$\int_{S_2} [\mathbf{t}_{(1)}^{+} \cdot \mathbf{u}_n^{\beta(1)} - \mathbf{t}_n^{\beta(1)} \cdot \mathbf{u}_{(1)}^{+}]\, dS$$

$$= \sum_{\alpha,m} b_{m(2)}^{\alpha} \int_{S_-} [\hat{\mathbf{t}}_m^{\alpha(1)} \cdot \mathbf{u}_n^{\beta(1)} - \mathbf{t}_n^{\beta(1)} \cdot \hat{\mathbf{u}}_m^{\alpha(1)}]\, dS$$

$$= -\sum_{\alpha,m} E_{mn(1)}^{\alpha\beta}\, b_{m(2)}^{\alpha} \tag{34}$$

where matrix $E_{mn(1)}^{\alpha\beta}$ is defined as

$$E_{mn(1)}^{\alpha\beta} = \int_{S_-} [\mathbf{t}_m^{\alpha(1)} \cdot \hat{\mathbf{u}}_n^{\beta(1)} - \hat{\mathbf{t}}_n^{\beta(1)} \cdot \mathbf{u}_m^{\alpha(1)}]\, dS \tag{35}$$

and Eq. (34) can be rewritten as

$$-\sum_{\alpha,m} E_{mn(1)}^{\alpha\beta}\, b_{m(2)}^{\alpha} = \int_{S_2} [\mathbf{t}_{(2)}^{-} \cdot \mathbf{u}_n^{\beta(1)} - \mathbf{t}_n^{\beta(1)} \cdot \mathbf{u}_{(2)}^{-}]\, dS \tag{36}$$

If domain D_2 is a cavity, from the boundary conditions of Eq. (21), we can obtain the relation between coefficients $b_{m(2)}^{\alpha}$ and f_m^{α} as

$$-\sum_{\alpha,m} E_{mn(1)}^{\alpha\beta}\, b_{m(2)}^{\alpha} = \int_{S_2} \left[-\mathbf{t}_n^{\beta(1)} \cdot \mathbf{u}_{(2)}^{-} \right] dS$$

$$= \sum_{\alpha,m} f_m^{\alpha} \int_{S_2} \left[-\mathbf{t}_n^{\beta(1)} \cdot \hat{\mathbf{u}}_m^{\alpha(2)} \right] dS$$

$$= \sum_{\alpha,m} Q_{nm(2)}^{\beta\alpha} f_m^{\alpha} \tag{37}$$

where

$$Q_{nm(2)}^{\beta\alpha} = \int_{S_2} [-\mathbf{t}_n^{\beta(1)} \cdot \hat{\mathbf{u}}_m^{\alpha(2)}]\, dS \tag{38}$$

Now consider the domain enclosed by surfaces S_- and S_2, and let $\mathbf{v} = \hat{\mathbf{u}}_n^{\beta(1)}$, similarly, we have

$$\int_{S_2} \left[\mathbf{t}_{(1)}^{+} \cdot \hat{\mathbf{u}}_n^{\beta(1)} - \hat{\mathbf{t}}_n^{\beta(1)} \cdot \hat{\mathbf{u}}_{(1)}^{+} \right] dS$$

$$= \sum_{\alpha,m} b_{m(1)}^{\alpha} \int_{S_-} \left[\mathbf{t}_m^{\alpha(1)} \cdot \hat{\mathbf{u}}_n^{\beta(1)} - \hat{\mathbf{t}}_n^{\beta(1)} \cdot \mathbf{u}_m^{\alpha(1)} \right] dS$$

$$= \sum_{\alpha,m} b_{m(1)}^{\alpha} E_{nm(1)}^{\beta\alpha} \tag{39}$$

From the boundary conditions of Eq. (21), Eq. (39) becomes

$$b_{m(1)}^{\alpha}\, E_{nm(1)}^{\beta\alpha} = \int_{S_2} -\hat{\mathbf{t}}_n^{\beta(1)} \cdot \mathbf{u}_{(2)}^{-}\, dS$$

$$= \sum_{\alpha,m} f_m^{\alpha} \int_{S_2} -\hat{\mathbf{t}} \cdot \hat{\mathbf{u}}_m^{\alpha(2)}\, dS$$

$$= \sum_{\alpha,m} \hat{Q}_{nm(2)}^{\beta\alpha} f_m^{\alpha} \tag{40}$$

where

$$\hat{Q}_{nm(2)}^{\beta\alpha} = \int_{S_2} -\hat{\mathbf{t}}_n^{\beta(1)} \cdot \hat{\mathbf{u}}_m^{\alpha(2)}\, dS \tag{41}$$

In summary, the relations between the unknown coefficients c_m^{α}, $b_{m(1)}^{\alpha}$, $b_{m(2)}^{\alpha}$ and f_m^{α} in Eqs. (24), (30), (37) and (40) can be expressed in matrix forms as in the following

$$[\mathrm{P}_1]\{\mathrm{b}_1\} + [\mathrm{Q}_1]\{\mathrm{b}_2\} = \{a\} \tag{42}$$

$$[\hat{\mathrm{P}}_1]\{\mathrm{b}_1\} + [\hat{\mathrm{Q}}_1]\{\mathrm{b}_2\} = [\mathrm{E}]\{\mathrm{c}\} \tag{43}$$

$$-[\mathrm{E}_1]^{\mathrm{T}}\{\mathrm{b}_2\} = [\mathrm{Q}_2]\{f\} \tag{44}$$

$$[\mathrm{E}_1]\{\mathrm{b}_1\} = [\hat{\mathrm{Q}}_2]\{f\} \tag{45}$$

Then the relation between the coefficient vector $\{f\}$ and vector $\{a\}$ is

$$\{a\} = [T]\{f\} = \{[P_1][\mathrm{E}_1]^{-1}[\hat{\mathrm{Q}}_2]\text{-}[\hat{\mathrm{Q}}_1][\mathrm{E}_1]^{\mathrm{T}-1}[\mathrm{Q}_2]\}\{f\} \tag{46}$$

where

$$[T] = [P_1][\mathrm{E}_1]^{-1}[\hat{\mathrm{Q}}_2]\text{-}[\hat{\mathrm{Q}}_1][\mathrm{E}_1]^{\mathrm{T}-1}[\mathrm{Q}_2] \tag{47}$$

is the T-matrix relating the coefficient of the scattered waves to that of the free field for the elastic half-space with layered inclusion.

4. NUMERICAL INTEGRATION OF BASIS FUNCTIONS

In view of the integral representation of the basis functions (displacements and the associated stress fields) shown in Appendix A, the integral for evaluation can be expressed as the general type

$$I = I_p + I_s \tag{48}$$

where

$$I_p = \int_{-\infty}^{\infty} \frac{E_p(k_x, \upsilon, \upsilon')}{F(k_x, \upsilon, \upsilon')} e^{-\upsilon(z+h)} e^{-ik_x x} dk_x \tag{49}$$

$$I_s = \int_{-\infty}^{\infty} \frac{E_s(k_x, \upsilon, \upsilon')}{F(k_x, \upsilon, \upsilon')} e^{-\upsilon'(z+h)} e^{-ik_x x} dk_x \tag{50}$$

and the integrals I_p and I_s mean the contribution from longitudinal wave and shear wave, respectively. From the representation of the radical functions υ and υ'

$$\upsilon = \sqrt{k_x^2 + k_y^2 - k_p^2}$$

$$\upsilon' = \sqrt{k_x^2 + k_y^2 - k_s^2} \tag{51}$$

The locations of branch points of radical function on the real axis or on the imaginary axis of k_x-plane vary with the value of k_y. For example, the branch points of radical function υ are located on the imaginary axis when $k_y > k_p$, and the branch points of radical function υ' are located on the imaginary axis when $k_y > k_s$. It is different from the in-plane strain case where the branch points are all located on the real axis. The locations of the branch points will affect the properties of the integrand and the selection of the integral path. Thus, the integral is classified into three cases according to the value of k_y as

Case I: $k_y < k_p < k_s$

Case II: $k_p < k_y < k_s$

Case III: $k_p < k_s < k_y$

and these cases will be discussed in detail in the following:

(1) For case I ($k_y < k_p < k_s$), we let

$$\upsilon = \sqrt{k_x^2 - k_p^{*2}} \;\; ; \;\; k_p^{*2} = k_p^2 - k_y^2$$

$$\upsilon' = \sqrt{k_x^2 - k_s^{*2}} \;\; ; \;\; k_s^{*2} = k_s^2 - k_y^2 \tag{52}$$

The coordinate system adopted here is shown in Fig. 3 and takes the following coordinate transformation

$$x = \bar{r} \cos \alpha$$
$$(z+h) = \bar{r} \sin \alpha \tag{53}$$

where

$$\bar{r} = \sqrt{x^2 + (z+h)^2} \tag{54}$$

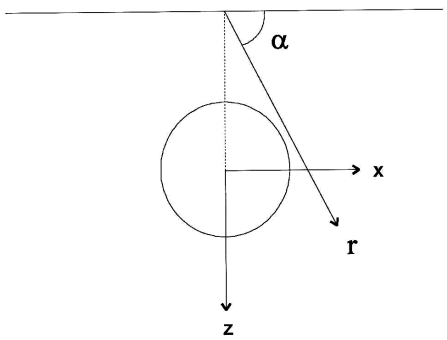

Fig. 3 The coordinate system of ($\bar{r}, \alpha$)

To illustrate the numerical procedure for calculating integrals I_p and I_s in detail, the integral I_p is first discussed. After defining the phase function $f_p(k_x)$ associated with the integral I_p

$$f_p(k_x) = ik_x \cos\theta + \sqrt{k_x^2 - k_p^{*2}} \sin\theta \tag{55}$$

Eq. (49) can thus be rewritten as

$$I_p = \int_{-\infty}^{\infty} \frac{E_p(k_x, k_y, \bar{\upsilon}, \bar{\upsilon}')}{F(k_x, k_y, \bar{\upsilon}, \bar{\upsilon}')} e^{-f(k_x)\bar{r}} dk_x \tag{56}$$

The integrand of the above integral I_p is same as that in the in-plane strain case, and can be treated using the method of steepest descent proposed by Yeh *et al.* [18] for the in-plane strain problem. Similarly, integral I_s can also be reduced to an in-plane strain problem. Thus, the numerical procedure for integrals I_p and I_s of this case will not be discussed in this paper.

(2) For case II ($k_p < k_y < k_s$), we let

$$\upsilon = \sqrt{k_x^2 + \bar{k}_p^2}$$
$$\upsilon' = \sqrt{k_x^2 - \bar{k}_s^{*2}} \tag{57}$$

where

$$\bar{k}_p^2 = k_y^2 - k_p^2 \tag{58}$$

From Eq. (57), we know that the branch points of radical functions υ and υ' are located at $\pm i\bar{k}_p$ and $\pm \bar{k}_s^*$, respectively. For the integral I_p, take the coordinate transform shown in Eq. (53) and let

$$f_p(k_x) = ik_x \cos\alpha + \sqrt{k_x^2 + \bar{k}_p^2} \sin\alpha \tag{59}$$

From the classical asymptotic analysis of the integral [22], the main contribution of the integral I_p comes from the saddle point k_{op}, which satisfies the condition

$$\left. \frac{df_p(k_x)}{dk_x} \right|_{k_x = k_{op}} = 0 \tag{60}$$

Solving the above equation, we can find that the saddle point k_{op} varies with the location of the field point and is given by

$$k_{op} = -i\bar{k}_p \cos\alpha \tag{61}$$

According to the theory of complex variables, the original path of integration along the axis of real k_x can be deformed into a special equivalent path, namely, the steepest descent path (SDP), which passes through the saddle point k_{op}. All points along the SDP in the complex k_x-plane satisfy the relation

$$f_p(k_x) - f_p(k_{op}) = t^2 \tag{62}$$

with

$$f_p(k_{op}) = \bar{k}_p \tag{63}$$

where t is a real variable. Then by solving Eq. (62), we can obtain the relation between wave-number k_x and real parameter t as

$$k_x = -i\cos\alpha(t^2 + \bar{k}_p) + t\sin\alpha\sqrt{t^2 + 2\bar{k}_p} \tag{64}$$

If the SDP does not encounter any branch cut or enclose the Rayleigh pole, according to the Cauchy theorem and Jordan's lemma, the evaluation of the original integral I_p can be changed into the following form

$$I_p = e^{-\bar{k}_p\bar{r}} \int_{-\infty}^{\infty} \frac{E(k_x(t))}{F(k_x(t))} e^{-\bar{r}t^2} \frac{dk_x}{dt} dt \tag{65}$$

in the above derivation. It can be observed that along the steepest descent path, the phase on each point remains stationary, and the integral I_p behaves like a decaying wave. The merits of the above procedure are described as follows.

(a) The oscillating character that arises in the original integral has been removed from the integral. This results in an integral of Hermite type, which converges faster than the original one in view of the weighting factor $e^{-\bar{r}t^2}$.

(b) The deformation of the original path into the steepest descent path is exact, and the only approximation comes from the truncation of the integral range, which varies with the value of $\bar{r}$.

(c) The above procedure is valid uniformly either in the case of near- to far-field or in the case of low to high frequency.

Unfortunately, from Eq. (64), we know that for any angle α, the SDP Γ_p associated with integral I_p will encounter the branch cuts at the saddle point $-ik_p\cos\alpha$ and enclose the Rayleigh pole k_R^* $(=\sqrt{k_R^2 - k_y^2})$. Therefore, the contribution of the Rayleigh pole should be included, and a return loop must be added to account for the contribution from the branch cut, which comes from the saddle point $-ik_p\cos\alpha$, passes around the branch point $\pm k_s^*$ in the proper manner, and returns to the saddle point. Figures 4 and 5 display the main integral path and the return loop of the deformed branch cut for $0 < \alpha < \pi/2$ and $\pi/2 < \alpha < \pi$, respectively. Then integral I_p is equivalent to

$$I_p = e^{-\bar{k}_p\bar{r}} \int_{\Gamma_p} \frac{E(k_x(t))}{F(k_x(t))} e^{-\bar{r}t^2} \frac{dk_x}{dt} dt + \int_{\Gamma_{PC}} \frac{E_p(k_x)}{F(k_x)} e^{-\upsilon(z+h)} e^{-ik_x x} dk_x$$
$$-\mathrm{sgn}(\cos\alpha)\, 2\pi i \left.\frac{E_p(k_x)}{F'(k_x)}\right|_{k_x = \mathrm{sgn}(\cos\alpha)\, k_R^*} e^{-\upsilon_R(z+h)} e^{-\mathrm{sgn}(\cos\alpha)\, ik_R^* x} \tag{66}$$

where

$$F' = \frac{dF(k_x)}{dk_x} = 8k_x(2k^2 - k_s^2 - \upsilon\upsilon') - 4k_x k^2 \left(\frac{\upsilon}{\upsilon'} + \frac{\upsilon'}{\upsilon}\right) \tag{67}$$

and

$$\upsilon_R = \sqrt{k_R^{*2} - k_p^{*2}} = \sqrt{k_R^2 - k_p^2} \tag{68}$$

and Γ_{pc} denotes the return loop along the branch cut of the radical function υ.

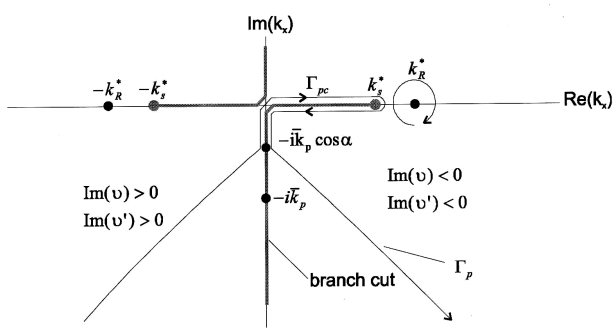

Fig. 4 Steepest descent path Γ_p of integral I_p for $\alpha < \pi/2$ and $k_p < k_y < k_s$

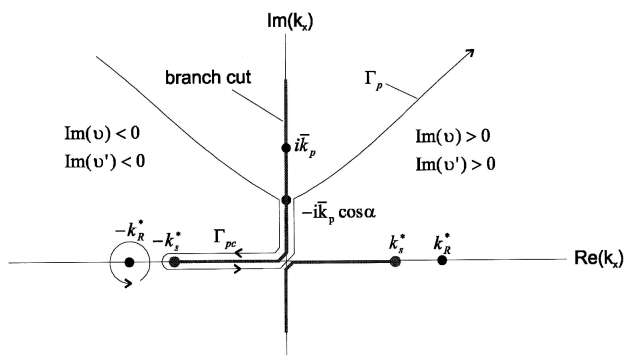

Fig. 5 Steepest descent path Γ_p of integral I_p for $\alpha > \pi/2$ and $k_p < k_y < k_s$

For integral I_s, the same consideration must be made and the phase function $f_s(k_x)$ associated with integral I_s is

$$f_s(k_x) = ik_x \cos\alpha + \sqrt{k_x^2 - k_s^{*2}} \sin\alpha \tag{69}$$

The original path is also deformed into the SDP Γ_s with respect to integral I_s, so all points along the SDP satisfy the relationship

$$f_s(k_x) - f_s(k_{os}) = t^2 \tag{70}$$

where $k_{os} = k_s^* \cos\alpha$ is the saddle point with respect to the phase function $f_s(k_x)$, and the relation between k_x and real variable t can be written as

$$k_x = -i\cos\alpha(t^2 + ik_s^*) + t\sin\alpha\sqrt{t^2 + 2ik_s^*} \tag{71}$$

The typical deformed SDP is shown in Figs. 6 and 7, path Γ_s intersects the imaginary k_x axis at point $-i\eta$, where η is a real positive number. The intersecting point $-i\eta$ could be employed to judge whether integral I_s should include the contribution from the branch cut or not. To obtain the closed-form expression of η, let $k_x = -i\eta$ and substituting it into Eq. (69), we have

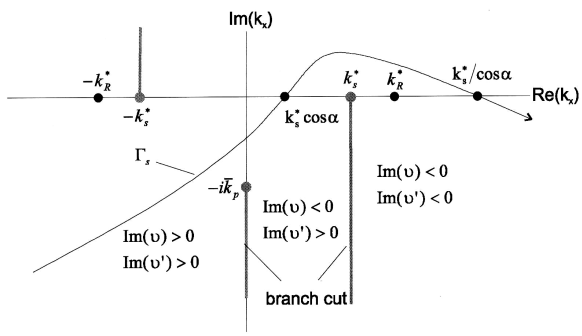

Fig. 6 Steepest descent path Γ_s of integral I_s for $\alpha < \pi/2$ and $k_p < k_y < k_s$

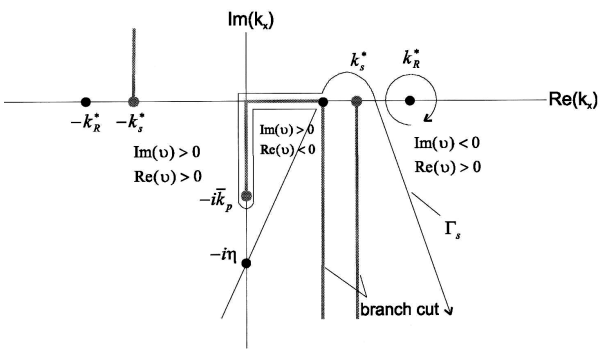

Fig. 7 Steepest descent path Γ_s of integral I_s for $\alpha < \pi/2$ and $k_p < k_y < k_s$. (The path including contribution from pole and branch cut)

$$\eta\cos\alpha + i\sqrt{\eta^2 + k_s^{*2}}\sin\alpha - ik_s^{*2} = t^2 \tag{72}$$

Solving the above equation then yields

$$\eta = \frac{k_s^*}{\tan\alpha} \tag{73}$$

The critical angle α_1 is defined as the angle α that path Γ_s just passes through the branch point $-i\bar{k}_p$, it corresponds to $\eta = \bar{k}_p$. Then the critical angle α_1 can be solved as

$$\alpha_1 = \cos^{-1}\left(\frac{\sqrt{k_y^2 - k_p^2}}{\sqrt{k_s^2 - k_p^2}}\right) \tag{74}$$

On the other side, path Γ_s intersects the real k_x axis at points $k_s^*\cos\alpha$ and $k_s^*/\cos\alpha$, the intersecting point $k_s^*/\cos\alpha$ could be employed to judge whether integral I_s should include the contribution from the Rayleigh pole or not. Similarly, the critical angle α_2 can be defined as the angle α that path Γ_s passes through the Rayleigh pole k_R^*, and expressed as

$$\alpha_2 = \cos^{-1}\left(\frac{\sqrt{k_s^2 - k_y^2}}{\sqrt{k_R^2 - k_y^2}}\right) \tag{75}$$

The equivalent integral I_s can be summarized as follows.

(a) For angle $\alpha > \max(\alpha_1, \alpha_2)$ and $\alpha < \min(\pi - \alpha_1, \pi - \alpha_2)$, integral I_s only includes the contribution along SDP Γ_s and can be expressed as

$$I_s = I_{pm} = e^{-ik_s^*\bar{r}} \int_{\Gamma_s} \frac{E_s(k_x(t))}{F(k_x(t))} e^{-\bar{r}t^2} \frac{dk_x}{dt} dt \tag{76}$$

the corresponding SDP Γ_s for this case is shown in Fig. 6.

(b) For angle $\alpha < \alpha_1$ or $\alpha > \pi - \alpha_1$, except the contribution along the SDP, integral I_s should additionally include the contribution from the return loop integral I_{sc} along the branch cut of ν, where

$$I_{sc} = \int_{\Gamma_{sc}} \frac{E_s(k_x)}{F(k_x)} e^{-\upsilon'(z+h)} e^{-ik_x x} dk_x \tag{77}$$

and

$$I_s = I_{pm} + I_{sc} \tag{78}$$

(c) For angle $\alpha < \alpha_2$ or $\alpha > \pi - \alpha_2$, integral I_s should additionally include the contribution I_{sR} from the Rayleigh pole, where

$$I_{sR} = -\mathrm{sgn}(\cos\alpha)\, 2\pi i \left.\frac{E_s(k_s)}{F'(k_x)}\right|_{k_x = \mathrm{sgn}(\cos\alpha)\, k_R^*} e^{-\upsilon_R'(z+h)} e^{-\mathrm{sgn}(\cos\alpha)\, ik_R^* x} \tag{79}$$

It should be noted that in the case of $\alpha < \alpha_1$ and $\alpha < \alpha_2$, integral I_s should include I_{sc} and I_{sR} simultaneously. The corresponding SDP Γ_s and return loop Γ_{sc} for this case are shown in Fig. 7.

(3) For case III ($k_p < k_s < k_y$), let

$$\upsilon = \sqrt{k_x^2 + \bar{k}_p^2}$$
$$\upsilon' = \sqrt{k_x^2 + \bar{k}_s^2} \tag{80}$$

where

$$\bar{k}_s = \sqrt{k_y^2 - k_s^2} \tag{81}$$

The SDP for integral I_p is the same as that defined by Eq. (64), and Fig. 8 shows the SDP for angle $\alpha < \pi/2$. Defining the critical angle α_3 as the angle α that SDP passes through the branch point $-i\bar{k}_s$ of radical function υ', and from the fact that the intersecting point of SDP and the imaginary k_x axis is $-i\bar{k}_p \cos\alpha$, we have

$$\alpha_3 = \cos^{-1}\left(\frac{\sqrt{k_y^2 - k_s^2}}{\sqrt{k_y^2 - k_p^2}}\right) \tag{82}$$

The critical angle α_3 is employed to judge whether integral I_p should include the contribution from the branch cut of υ' or not. The treatment of integral I_p can be summarized as follows:

(a) for angle $\pi - \alpha_3 > \alpha > \alpha_3$, integral I_p should include the contribution from the main integral I_{pm} along the SDP and from the Rayleigh pole I_{pR} as shown in Fig. 8, that is

$$I_p = I_{pm} + I_{pR} \tag{83}$$

where

$$I_{pm} = e^{-\bar{k}_p \bar{r}} \int_{\Gamma_p} \frac{E_p\left(k_x(t)\right)}{F\left(k_x(t)\right)} e^{-\bar{r}t^2} \frac{dk_x}{dt}\, dt \tag{84}$$

$$I_{pR} = -\mathrm{sgn}(\cos\alpha)\, 2\pi i \left.\frac{E_p(k_x)}{F'(k_x)}\right|_{k_x = \mathrm{sgn}(\cos\alpha)\, k_R^*} e^{-\upsilon R(z+h)} e^{-\mathrm{sgn}(\cos\alpha)\, i k_R^* x} \tag{85}$$

(b) for angle $\alpha < \alpha_3$ or $\alpha > \pi - \alpha_3$, integral I_p should additionally include the contribution from the return loop integral I_{pc} along the branch cut, that is

$$I_p = I_{pm} + I_{pR} + I_{pc} \tag{86}$$

$$I_{pc} = \int_{\Gamma_{pc}} \frac{E_p(k_x)}{F(k_x)} e^{-\upsilon(z+h)} e^{-ik_x x}\, dk_x \tag{87}$$

and the corresponding paths are shown in Fig. 9.

For integral I_s of this case, using similar procedures, we can obtain the deformed integral path as

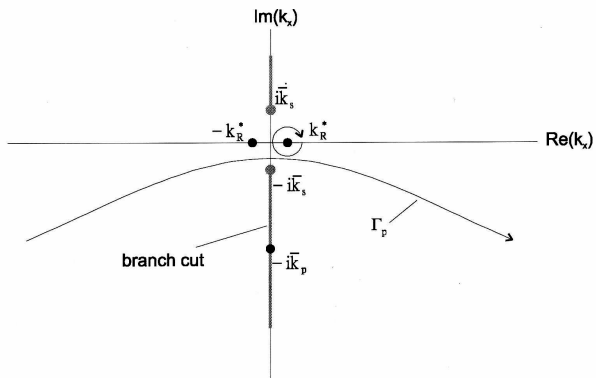

Fig. 8 Steepest descent path Γ_p of integral I_p for $\alpha_3 < \alpha < \pi/2$ and $k_p < k_s < k_y$

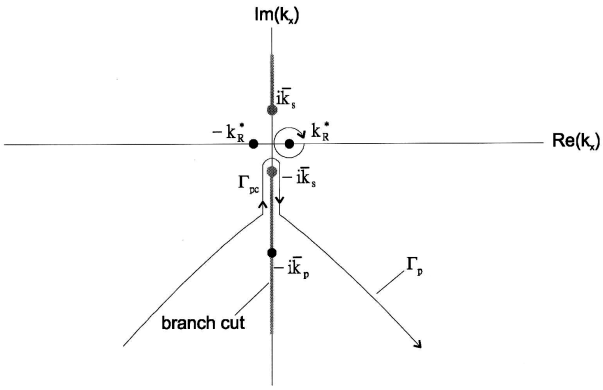

Fig. 9 Steepest descent path Γ_p of integral I_p for $\alpha < \alpha_3$ and $k_p < k_s < k_y$

$$k_x = -i\cos\alpha(t^2 + \bar{k}_s) + t\sin\alpha\sqrt{t^2 + 2\bar{k}_s} \tag{88}$$

The corresponding SDP is shown in Fig. 10. In this case, for any angle α, the SDP Γ_s will enclose the Rayleigh pole but the SDP never encounters the branch cuts. Then integral I_s is equivalent to

$$I_s = I_{sm} + I_{sR} \tag{89}$$

$$I_{sm} = e^{-\bar{k}_s \bar{r}} \int_{\Gamma_s} \frac{E_s\left(k_x(t)\right)}{F\left(k_x(t)\right)} e^{-\bar{r}t^2} \frac{dk_x}{dt}\, dt \tag{90}$$

$$I_{sR} = -\mathrm{sgn}(\cos\alpha)\, 2\pi i \left.\frac{E_s(k_x)}{F'(k_x)}\right|_{k_x = \mathrm{sgn}(\cos\alpha)\, k_R^*} e^{-\upsilon_R'(z+h)} e^{-\mathrm{sgn}(\cos\alpha) i k_R^* x} \tag{91}$$

5. NUMERICAL EXAMPLES

The problems of a cylindrical tunnel with $r_0/r_i = 1.1$ buried at different depths and subjected to incident plane waves are studied, where r_0 and r_i are the outer and inner radius of the tunnel, respectively. For all

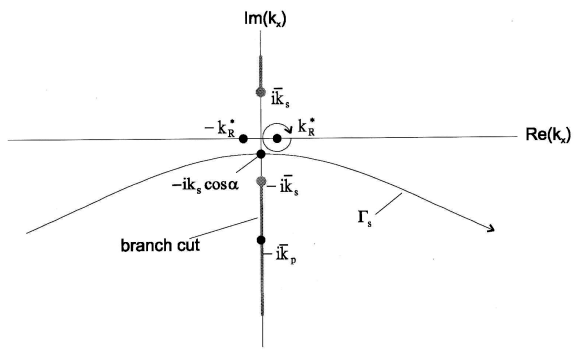

Fig. 10 Steepest descent path Γ_s of integral I_s for $\alpha < \pi/2$ and $k_p < k_s < k_y$

cases studied, the Poisson ratio for the tunnel and for the elastic half-space were 0.2 and 1/3, respectively, the ratio of density of the tunnel to the half-space was taken to be $\rho_1/\rho = 1.5$, and the ratio of shear modulus was set to be $\mu_1/\mu = 6$. The excitation frequency of incident wave was normalized as $\eta = \omega\gamma_1/\pi c_s$ in order to be consistent with other studies. The choice of approximate order N of wave series are base on the convergence of calculated results. In all numerical examples of this study, $N = 5$ is enough to have a satisfactory results as compared with those results with more high order terms. For the calculations of surface integrals in Eqs. (24), (31), (34), (37), and (40) are accomplished by Simpson's rule with integration points $M = 61$.

To verify the validity and accuracy of the T-matrix formulation and the numerical technique proposed in this research, the plane strain cases ($k_y = 0$) were first studied and the results obtained were then compared with those obtained by Yeh, *et al.* [23]. Figures 11 and 12 show the calculated displacement amplitudes and hoop stresses along the inner surface of the cylindrical tunnel buried at different buried depths subjected to incident P-wave and SV-wave ($\eta = 0.5$), respectively. The results determined by the least square method (Yeh, *et al.*, [23]) are denoted by circles in these figures, and it can be seen that the agreement is excellent. For the responses of a buried tunnel subjected to obliquely incident waves, Figs. 13 and 14 show the displacements and stresses along the inner surface of the cylindrical tunnel subjected to obliquely incident P-wave and SH-wave ($\eta = 0.5$, $\theta_h = 90°$, $\theta_v = 45°$) with buried depth $h/r_i = 2.0$ and $h/r_i = 5.0$, respectively. In these figures, the displacement $U_i = |u_i/A|$ ($i = x, y, z$) is normalized with respect to the amplitude A of incident wave, and the stress fields are normalized with respect to the factor $\rho\omega c_s$. The responses in Figs. 13 and 14 should be symmetric with respect to the vertical z-axis ($\theta = 90°$ and 270°) because the wave propagating in the direction of the axis of the tunnel. The effect of the embedment depth of the tunnel on the response also can be found in Figs. 13 and 14. Clearly, the embedment depth has a significant effect on the response and, particularly, on the case of incident P-wave.

6. CONCLUSIONS

The T-matrix, which relates the unknown scattering coefficients to the free-field coefficients, can be derived directly by the orthogonality conditions as well as the continuity conditions in the interface between the buried two-layered scatterer and the surrounding half-space. The basis functions and their regular parts are constructed in a systematic manner for the problems of obliquely incident waves, and their properties are discussed thoroughly. The results for wave scattering of a buried tunnel subjected to incident plane waves were calculated and verified. A numerical technique developed from the method of steepest descent was proposed and applied to the calculation of basis functions. Although only a few numerical results are presented here, applications of the proposed method to problems of wave scattering in multiple layered scatter, embedded cavities, poroelastic media [24], surface scatter and other problems of oblique incidence are also possible. In addition to elastic waves, the proposed T-matrix method could also be extended to coupled field problems, such as thermoelasticity and poroelasticity.

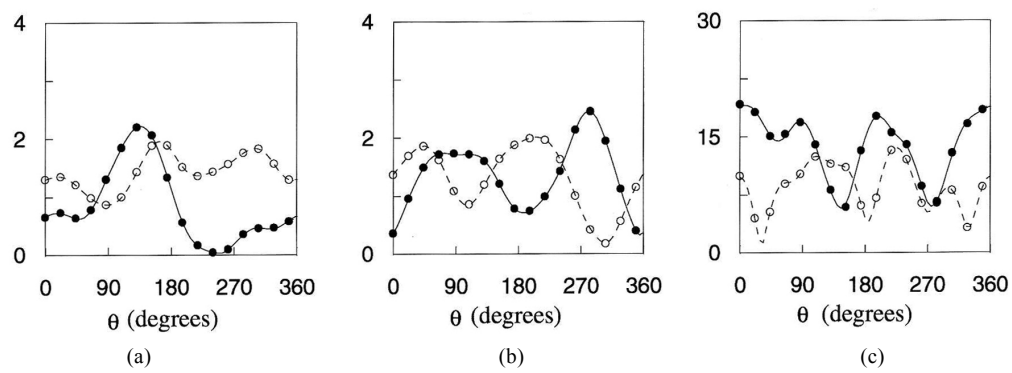

Fig. 11 Dynamic responses for (a) horizontal displacements, (b) vertical displacements, and (c) hoop stresses along inner surface of a buried tunnel subjected to incident P-wave with $\theta_v = 30°$ (solid line and dash line are results of $h/r_i = 1.5$ and $h/r_i = 5.0$, respectively. Circles are the results obtained by Yeh, *et al.* [22])

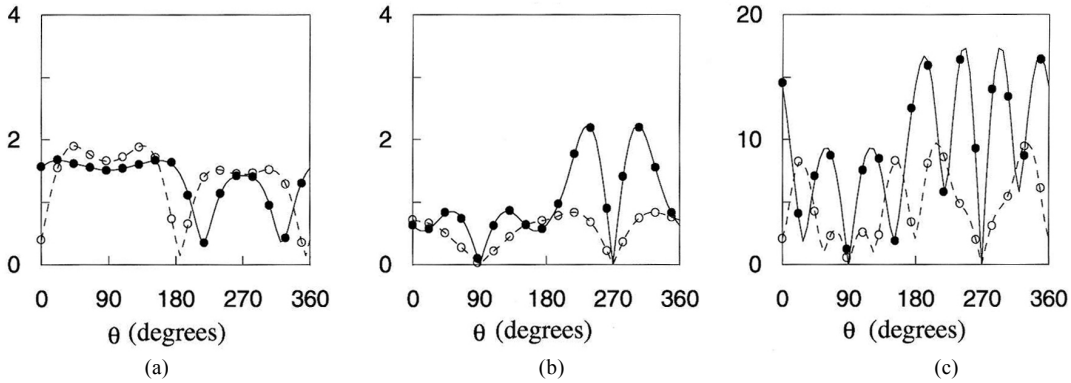

Fig. 12 Dynamic responses for (a) horizontal displacements, (b) vertical displacements, and (c) hoop stresses along inner surface of a buried tunnel subjected to incident SV-wave with $\theta_v = 0°$ (solid line and dash line are results of $h/r_i = 1.5$ and $h/r_i = 5.0$, respectively. Circles are the results obtained by Yeh, *et al*. [22])

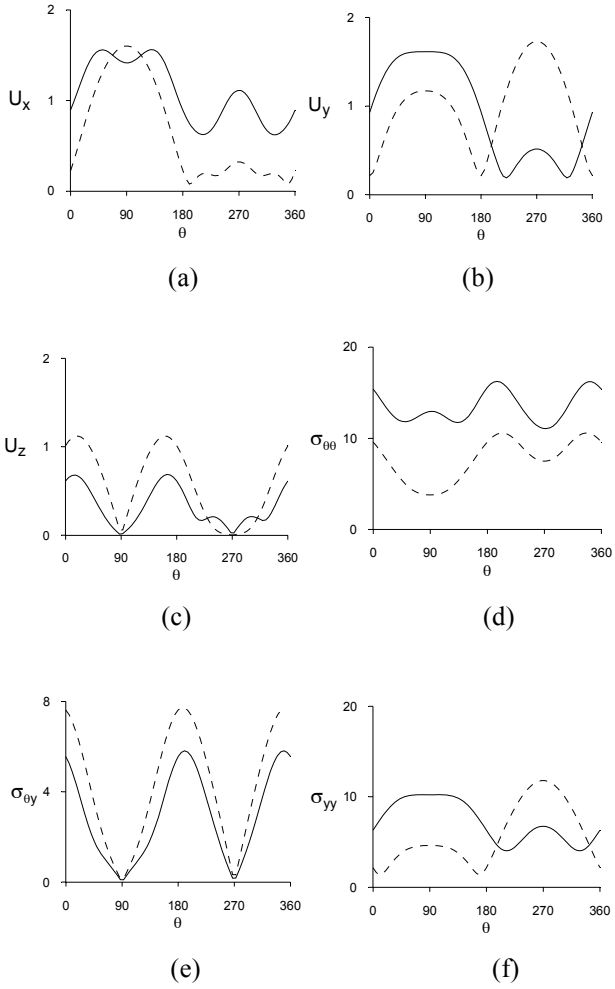

Fig. 13 Dynamic responses for (a)-(c) displacement amplitude, and (d)-(f) stress amplitudes along inner surface of a buried tunnel subjected to obliquely incident P-wave with $\theta_v = 45°$ and $\theta_h = 90°$ (solid line and dash line are results of $h/r_i = 1.5$ and $h/r_i = 5.0$, respectively)

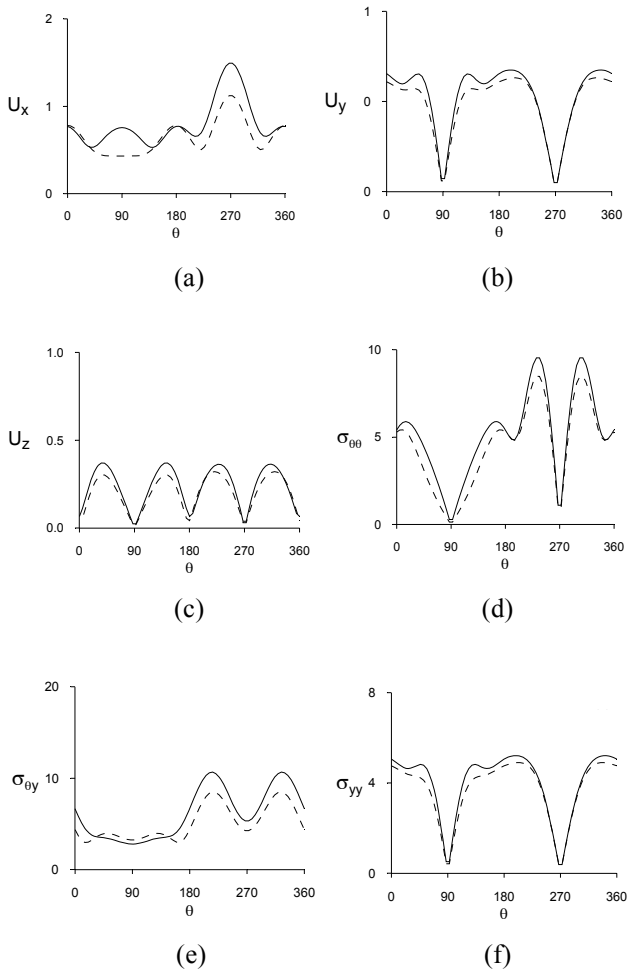

Fig. 14 Dynamic responses for (a)-(c) displacement amplitude, and (d)-(f) stress amplitudes along inner surface of a buried tunnel subjected to obliquely incident SH-wave with $\theta_v = 45°$ and $\theta_h = 90°$ (solid line and dash line are results of $h/r_i = 1.5$ and $h/r_i = 5.0$, respectively)

APPENDIX A
MOVING SOURCE POTENTIAL

The relation between the displacement field **u** and the wave potentials (ϕ, φ, χ) is

$$\mathbf{u} = \nabla\phi + \nabla\times(0, \varphi, 0) + \nabla\times\nabla\times(0, \chi, 0) \tag{A1}$$

where ϕ is the Longitudinal wave (P-wave) potential, φ and χ are the Shear wave (SV-wave and SH-wave) potentials, respectively. The governing equations for this study expressed in potential forms are

$$\nabla^2\phi + k_p^{*2}\phi = 0 \tag{A2}$$

$$\nabla^2\varphi + k_s^{*2}\varphi = 0 \tag{A3}$$

$$\nabla^2\chi + k_s^{*2}\chi = 0 \tag{A4}$$

For the scattering problem, the total wave field potentials (ϕ, φ, χ) of the elastic half-space is the sum of the free-field wave potentials (ϕ^f, φ^f, χ^f) and the scattering wave field potentials, that is

$$\begin{Bmatrix}\phi\\ \varphi\\ \chi\end{Bmatrix} = \begin{Bmatrix}\phi^f\\ \varphi^f\\ \chi^f\end{Bmatrix} + \begin{Bmatrix}\phi^s\\ \varphi^s\\ \chi^s\end{Bmatrix} \tag{A5}$$

For the potentials (ϕ^5, φ^5, χ^5) of scattering wave field of the elastic half-space should satisfy the following conditions (Gregory, [25,26]).

(1) Potentials (ϕ^5, φ^5, χ^5) should have second order continuity derivation and satisfy the governing equations.
(2) Traction free conditions on the free surface ($z = 0$).
(3) Potentials (ϕ^5, φ^5, χ^5) should be an outgoing wave and satisfy the radiation condition.

According to the conditions shown in above, the scattering wave fields can be the linear combination of three sets of moving source potentials, one longitudinal wave potential, and two shear wave potentials and their high order term. Then the scattering wave fields of the elastic half-space can be expanded by the following potential series

$$\begin{Bmatrix}\phi^S\\ \varphi^S\\ \chi^S\end{Bmatrix} = \sum_{n=0}^{N} P_n^p \begin{Bmatrix}\phi_n^p\\ \varphi_n^p\\ \chi_n^p\end{Bmatrix} + \sum_{n=0}^{N} P_n^v \begin{Bmatrix}\phi_n^\upsilon\\ \varphi_n^\upsilon\\ \chi_n^\upsilon\end{Bmatrix} + \sum_{n=0}^{N} P_n^h \begin{Bmatrix}\phi_n^h\\ \varphi_n^h\\ \chi_n^h\end{Bmatrix} \tag{A6}$$

where superscript p means the potential induced by P-wave source, and superscript v and h mean the potentials induced by SV-wave and SH-wave sources, respectively, and N is the approximated order. P_n^p, P_n^υ and P_n^h are the coefficients to be determined by boundary conditions.

For the P-wave point source potentials $(\phi_n^p, \varphi_n^p, \chi_n^p)$ with order n, the integral representations in wave-number k_x domain are

$$\phi_n^p = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\left(\frac{1}{\upsilon}e^{-\upsilon|z|} + R_{pp}e^{-\upsilon(z+h)}\right)e^{-ik_x x}\, dk_x \tag{A7}$$

$$\varphi_n^p = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\, R_{p\upsilon}e^{-\upsilon'(z+h)}e^{-ik_x x}\, dk_x \tag{A8}$$

$$\chi_n^p = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\, R_{ph}e^{-\upsilon'(z+h)}e^{-ik_x x}\, dk_x \tag{A9}$$

where $\upsilon = \sqrt{k^2 - k_p^2}$ and $\upsilon' = \sqrt{k^2 - k_s^2}$ are the radical functions associated with the P- and SV-wave, respectively, and $k^2 = k_x^2 + k_y^2$. The reflection coefficients R_{pp}, $R_{p\upsilon}$ and R_{ph} are

$$R_{pp} = \frac{-4k^2\upsilon'(2k^2 - k_s^2)\, e^{-\upsilon h}}{\upsilon F(k)}$$
$$R_{p\upsilon} = \frac{-4ik_x k_s^2(2k^2 - k_s^2)\, e^{-\upsilon h}}{k_s^{*2}F(k)}$$
$$R_{ph} = \frac{-4ik_y\upsilon'(2k^2 - k_s^2)\, e^{-\upsilon h}}{k_s^{*2}F(k)} \tag{A10}$$

where $k_s^{*2} = k_s^2 - k_y^2$, $F(k)$ is the Rayleigh wave function defined as

$$F(k) = (2k^2 - k_s^2)^2 - 4k^2\upsilon\upsilon' \tag{A11}$$

Similarly, solutions of n-th order SV-wave source potential $(\phi_n^\upsilon, \varphi_n^\upsilon, \chi_n^\upsilon)$ are

$$\phi_n^\upsilon = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\, R_{\upsilon p}e^{-\upsilon(z+h)}e^{-ik_x x}\, dk_x \tag{A12}$$

$$\varphi_n^\upsilon = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\left(\frac{e^{-\upsilon|z|}}{\upsilon'} + R_{\upsilon\upsilon}e^{-\upsilon'(z+h)}\right)e^{-ik_x x}\, dk_x \tag{A13}$$

$$\chi_n^\upsilon = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x) R_{\upsilon h}\, e^{-\upsilon'(z+h)}e^{-ik_x x}\, dk_x \tag{A14}$$

where the reflected coefficients are

$$R_{\upsilon p} = \frac{4ik(2k^2 - k_s^2)\, e^{-\upsilon' h}}{F(k)}$$
$$R_{\upsilon\upsilon} = \frac{\left(-4\upsilon\upsilon'(k_x^2k_s^2 + k_y^2\upsilon'^2) - k_s^{*2}(2k^2 - k_s^2)^2\right)e^{-\upsilon' h}}{\upsilon' k_s^{*2}F(k)}$$
$$R_{\upsilon h} = \frac{(-8k_x\, k_y\, \upsilon\upsilon')\, e^{-\upsilon' h}}{k_s^{*2}F(k)} \tag{A15}$$

For n-th order SH-wave source potentials, the solutions $(\phi_n^h, \varphi_n^h, \chi_n^h)$ are

$$\phi_n^h = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\, R_{hp}e^{-\upsilon(z+h)}\, e^{-ik_x x}\, dk_x \tag{A16}$$

$$\varphi_n^h = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\, R_{h\upsilon}\, e^{-\upsilon(z+h)}\, e^{-ik_x x}\, dk_x \tag{A17}$$

$$\chi_n^h = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\left(\frac{e^{-\upsilon'|z|}}{\upsilon'} + R_{hh} e^{-\upsilon'(z+h)}\right) e^{-ik_x x}\, dk_x \tag{A18}$$

where

$$R_{hp} = \frac{4ik_y \upsilon'(2k^2 - k_s^2)\, e^{-\upsilon' h}}{F(k)}$$

$$R_{h\upsilon} = \frac{(-8k_x k_y k_s^2 \upsilon\upsilon')\, e^{-\upsilon' h}}{k_s^{*2} F(k)}$$

$$R_{hh} = \frac{\left(-4\upsilon\upsilon'(k_x^2 k_s^2 + k_y^2 \upsilon'^2) + k_s^{*2}(2k^2 - k_s^2)^2\right) e^{-\upsilon' h}}{\upsilon' k_s^{*2} F(k)} \tag{A19}$$

From the potential-displacement relation, the integral representations of displacement fields can be obtained in the following:

$$u_{x(n)}^{\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\Big[H_x^{\alpha} - ik_x R_{\alpha p}\, e^{-\upsilon(z+h)} + \upsilon' R_{\alpha\upsilon}\, e^{-\upsilon'(z+h)} - k_x k_y R_{\alpha h}\, e^{-\upsilon'(z+h)}\Big] e^{-ik_x x} dk_x \tag{A20}$$

$$u_{y(n)}^{\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\Big[H_y^{\alpha} - ik_y R_{\alpha p}\, e^{-\upsilon(z+h)} + k_s^* R_{\alpha h}\, e^{-\upsilon'(z+h)}\Big] e^{-ik_x x}\, dk_x \tag{A21}$$

$$u_{z(n)}^{\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\Big[H_z^{\alpha} - \upsilon R_{\alpha p}\, e^{-\upsilon(z+h)} - ik_x R_{\alpha\upsilon}\, e^{-\upsilon'(z+h)} + ik_y \upsilon' R_{\alpha h}\, e^{-\upsilon'(z+h)}\Big] e^{-ik_x x}\, dk_x \tag{A22}$$

and the stress fields are

$$\sigma_{xx(n)}^{\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\Big[H_{xx}^{\alpha} + (2k_p^2 - 2k_x^2 - k_s^2) R_{\alpha p} e^{-\upsilon'(z+h)} + (-2ik_x \upsilon' R_{\alpha\upsilon} + 2ik_y k_x^2 R_{\alpha h})\, e^{-\upsilon(z+h)}\Big] e^{-ik_x x}\, dk_x \tag{A23}$$

$$\sigma_{yy(n)}^{\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\Big[H_{yy}^{\alpha} + (2k_p^2 - 2k_x^2 - k_s^2)\, R_{\alpha p} e^{-\upsilon'(z+h)} - 2ik_y k_s^* R_{\alpha h}\, e^{-\upsilon'(z+h)}\Big] e^{-ik_x x}\, dk_x \tag{A24}$$

$$\sigma_{zz(n)}^{\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\Big[H_{zz}^{\alpha} + (2k^2 - k_s^2)\, R_{\alpha p} e^{-\upsilon'(z+h)} + (+2ik_x \upsilon' R_{\alpha\upsilon} - 2ik_y \upsilon'^2 R_{\alpha h})\, e^{-\upsilon(z+h)}\Big] e^{-ik_x x}\, dk_x \tag{A25}$$

$$\sigma_{xy(n)}^{\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\Big[H_{xy}^{\alpha} - 2k_x\, 2k_y\, R_{\alpha p}\, e^{-\upsilon(z+h)} - (ik_y \upsilon' R_{\alpha\upsilon} - ik_x (k_y^2 - k_s^*)\, R_{\alpha h})\, e^{-\upsilon'(z+h)}\Big] e^{-ik_x x}\, dk_x \tag{A26}$$

$$\sigma_{xz(n)}^{\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\Big[H_{xz}^{\alpha} + 2ik_x\, \upsilon 2R_{\alpha p}\, e^{-\upsilon(z+h)} + \left((k_x^2 + \upsilon'^2)\, R_{\alpha\upsilon} + 2\upsilon' k_x k_y R_{\alpha h}\right) e^{-\upsilon'(z+h)}\Big] e^{-ik_x x}\, dk_x \tag{A27}$$

$$\sigma_{yz(n)}^{\alpha} = \frac{1}{2\pi}\int_{-\infty}^{\infty} S_n(k_x)\Big[H_{yz}^{\alpha} + 2ik_x\, \upsilon 2R_{\alpha p}\, e^{-\upsilon(z+h)} + \left(-ik_x k_y R_{\alpha\upsilon} - \upsilon'(k_s^2 - 2k_y^2)\, R_{\alpha h}\right) e^{-\upsilon'(z+h)}\Big] e^{-ik_x x}\, dk_x \tag{A28}$$

where $\alpha = p$, v or h, and terms H_{β}^{α} and $H_{\beta\gamma}^{\alpha}$ ($\beta, \gamma = x, y, z$) in Eqs. (A20)-(A28) represent the source term contribution. Coefficients H_{β}^{α} have been shown in Eqs. (7)-(9). For $\alpha = p$ (P-wave source), coefficients $H_{\beta\gamma}^{\alpha}$ are

$$H_{xx}^{p} = \frac{(2k_p^2 - 2k_x^2 - k_s^2)}{\upsilon} e^{-\upsilon|z|}, \quad H_{yy}^{p} = \frac{(2k_p^2 - 2k_y^2 - k_s^2)}{\upsilon} e^{-\upsilon|z|},$$
$$H_{zz}^{p} = \frac{(2k^2 - k_s^2)}{\upsilon} e^{-\upsilon|z|}$$
$$H_{xy}^{p} = \frac{-2k_x k_y}{\upsilon} e^{-\upsilon|z|}, \quad H_{xz}^{p} = 2ik_x\, \mathrm{sgn}(z)\, e^{-\upsilon|z|},$$
$$H_{yz}^{p} = 2ik_y\, \mathrm{sgn}(z)\, e^{-\upsilon|z|} \tag{A29}$$

for $\alpha = v$ (SV-wave source)

$$H_{xx}^{v} = -2ik_x\, \mathrm{sgn}(z)\, e^{-\upsilon'|z|}, \quad H_{yy}^{v} = 0,$$
$$H_{zz}^{v} = 2ik_x\, \mathrm{sgn}(z)\, e^{-\upsilon'|z|}$$
$$H_{xy}^{v} = -ik_y\, \mathrm{sgn}(z)\, e^{-\upsilon'|z|}, \quad H_{xz}^{v} = \frac{-(k_x^2 + \upsilon'^2)}{\upsilon'} e^{-\upsilon'|z|},$$
$$H_{xz}^{v} = \frac{-k_x k_y}{\upsilon'} e^{-\upsilon'|z|} \tag{A30}$$

and for $\alpha = h$ (SH-wave source)

$$H_{xx}^{h} = \frac{2ik_y k_x^2}{\upsilon'} e^{-\upsilon'|z|}, \quad H_{yy}^{h} = \frac{-2ik_y k_s^*}{\upsilon'} e^{-\upsilon'|z|},$$
$$H_{zz}^{h} = -2ik_y \upsilon' e^{-\upsilon'|z|}$$
$$H_{xy}^{h} = \frac{ik_x (k_y^2 - k_s^*)}{\upsilon'} e^{-\upsilon'|z|}, \quad H_{xz}^{h} = 2k_x k_y\, \mathrm{sgn}(z)\, e^{-\upsilon'|z|},$$
$$H_{yz}^{h} = -(k_s^2 - 2k_y^2)\, \mathrm{sgn}(z)\, e^{-\upsilon'|z|} \tag{A31}$$

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