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★**Artificial boundary method.**

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In many intriguing problems, we are interested in the solution of a partial differential equation posed in an unbounded domain. Except for some very special instances, such solutions cannot be written down in closed form, and one seeks a numerical approximation to the solution. For this, one must somehow reduce the original problem to one posed on a bounded domain. The design, analysis and implementation of reduction strategies has attracted a lot of scientific interest over the past many decades. How to effectively reduce a problem in an unbounded region to a bounded one, maintaining accuracy and computational tractability, is a question that arises in fields ranging from fluid mechanics to mathematical finance. As a rough indicator of the importance and relevance of the topic, a Google search at the time of writing of this review yielded nearly 3 million hits on 'Artificial Boundary Condition'.

One approach is to surround the computational region with an 'artificial absorbing layer'. In this layer, the coefficients of the PDE are changed such that the solution rapidly attenuates outside the region of interest. An example of such an approach is the 'perfectly matched layer' in electromagnetics [J.-P. Berenger, J. Comput. Phys. **114** (1994), no. 2, 185–200; MR1294924 (95e:78002)].

Another approach is to introduce an artificial boundary enclosing the region where the details of the solution are sought (e.g. in the vicinity of a scattering obstacle), and prescribe an *artificial boundary condition* on the artificial boundary. This results in a *reduced problem*, whose solution at least in the bounded domain should be close to that of the original problem.

The book under review is focused on this broad approach, and is a comprehensive guide to such artificial boundary conditions for many PDEs of interest. The authors have made many significant contributions in this area—indeed, their first joint article on the subject appeared in 1985 [J. Comput. Math. **3** (1985), no. 2, 179–192; MR0854359 (87k:65134)]. The present book compiles their results and also provides an excellent survey of the field. Parts of the text are based on courses on artificial boundary conditions (ABCs) taught by the authors in the mid-2000s. The treatment is comprehensive as well as accessible, and the book is therefore invaluable both as a reference text and as the basis for a graduate course.

The book is organized in nine chapters.

The first, and perhaps most accessible for students wishing to learn about the main ideas, is concerned with global ABCs for second-order elliptic problems.

Section 1.2 provides the overall strategy one must follow: introduce an artificial boundary, determine the 'exact' boundary condition consistent with the original problem (in this case, the Steklov-Poincaré map), apply an approximation to this boundary condition on the artificial boundary, ensure the reduced problem is well-posed, discretize, and perform an error analysis which accounts for the effects of the artificial boundary, the ABC, and the choice of discretization. This process is easy to follow in the simple instance of the exterior problem for the Laplacian. The artificial boundary chosen is a circle of radius R, on which the Steklov-Poincaré map can be explicitly written as a Fourier series. The ABC used is then a truncation of this series up to N terms. The well-posedness of the reduced problem with this ABC is established. The discretization method used is a finite element approach, and a careful error analysis is performed, documenting the contributions due to the mesh parameter, the size of the artificial boundary (R), and the accuracy with which the Steklov-Poincaré map is approximated (N). (A typical result is Equation 1.2.69.) This approach is repeated for the modified Helmholtz equation.

The effects of approximating the Steklov-Poincaré map are more involved for non-positive operators such as the Helmholtz operator. In this case, it is possible for the reduced problem to lose well-posedness (this is related to the existence of real eigenvalues of the Laplacian in the bounded region). However, if one uses a truncated Fourier series as an approximation to the Sommerfeld condition $\frac{\partial u}{\partial r} - iku$ on the artificial boundary, the resultant bounded-domain problem is shown to be well-posed for all wave numbers.

The first chapter demonstrates the many mathematical issues to be addressed when designing ABCs. The second chapter studies these questions for systems of PDEs (the Navier system for linear elasticity and the 2-D Stokes system for low-Reynolds viscous incompressible flows). Once again, the artificial boundary is a circle/sphere, and the ABC is based on a truncated series. This reviewer found the careful and detailed treatment in Section 2.5 (Global ABCs for the Exterior Problem of 3-D Navier System) particularly useful.

The introduction of time as an independent variable adds new complications. The ABC should reflect the time-evolving nature of the solution, and for some PDEs, this involves keeping track of the history of the solution. Designing an approximation which is computationally efficient and accurate without large memory requirements is a non-trivial task. The third chapter presents results in this direction for the heat and Schrödinger equation, and not surprisingly, the ABC involves a convolution in the time variable. The stability of the reduced problem is studied. A finite difference scheme for the 1-D Schrödinger equation using the proposed ABC is analyzed, documenting the additional effects introduced by discretizing the time-convolution.

The fourth chapter continues the study of ABCs for evolution equations for non-dispersive and dispersive waves. Once again, the ABC is applied on a circular (resp. spherical) artificial boundary, and is related to the truncation of a Fourier series. Readers may also be interested in the review [T. M. Hagstrom, in *Acta numerica, 1999*, 47–106, Acta Numer., 8, Cambridge Univ. Press, Cambridge, 1999; MR1819643 (2002c:35171)] and the book chapters [T. M. Hagstrom, in *Topics in computational wave propagation*, 1–42, Lect. Notes Comput. Sci. Eng., 31, Springer, Berlin, 2003; MR2032866; P. Joly and C. Tsogka, in *Effective computational methods for wave propagation*, 425–472, Numer. Insights, 5, Chapman & Hall/CRC, Boca Raton, FL, 2008; MR2404885 (2009f:65228)].

The first four chapters are based on 'global' ABCs: the value of the ABC at one point on the artificial boundary cannot be independently specified from that on the other points. However, from an efficient computational perspective, there is merit to ABCs which allow for decoupling (or low coupling) between parts of the boundary. These 'local' ABCs are studied in Chapter 5.

In Chapter 6, many of the problems studied are revisited, this time in the context of polygonal artificial boundaries. The ABCs no longer have simple expressions which result from truncation of a Fourier series. On the other hand, since many discretization strategies are based on polygonal approximations to the domain, the development and analysis of these discrete ABCs is important.

If a fundamental solution and representation formula for the solution of an exterior problem is available, then a truly elegant implementation of a boundary condition on an (almost arbitrarily shaped) artificial boundary is via boundary integral equations. The reduced problem then becomes a coupled PDE-integral equation. There are several ways to discretize the resulting system, for example the FEM-BEM coupling of M. Costabel and E. P. Stephan [SIAM J. Numer. Anal. **27** (1990), no. 5, 1212–1226; MR1061127 (92c:65125)]. The authors call the boundary-integral approach 'implicit', and devote Chapter 7 to it. An analysis is presented for several elliptic problems, and is based on well-known results from potential theory. A similar approach based on Kirchoff's formula for the wave equation is presented in Section 7.5.

Chapter 8 extends some of these ideas to nonlinear PDEs such as the KPZ equation and Burgers' equation. One strategy for constructing ABCs is to work with a linearization of the PDE operator. Another, discussed in this chapter, is based on the use of *nonlinear* ABCs. The authors examine, in detail, one such method, the 'split local absorbing boundary', based on operator splitting.

Chapter 9 examines the use of ABCs to treat problems with singularities such as cracks or reentrant corners. In principle, if the nature of the solution near a singularity is understood, one may explicitly include this in any approximation scheme. Section 9.4 describes how one may characterize the solution near a singularity by using ABCs.

As mentioned, the construction and analysis of reduction strategies for PDEs is a vast subject. The book under review is a substantial and well-written contribution to the area, and will be a valuable resource for both researchers and students. *Nilima Nigam*