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Solution of Poisson's equation by analytical boundary element integration

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ABSTRACT

The solution of Poisson's equation is essential for many branches of science and engineering such as fluid-mechanics, geosciences, and electrostatics. Solution of two-dimensional Poisson's equations is carried out by BEM based on Galerkin Vector Method in which the integrals that appear in the boundary element method are expressed by analytical integration. In this paper, the Galerkin vector method is developed for more general case than presented in the previous works. The integrals are computed for constant and linear elements in BEM. By employing analytical integration in BEM computation, the numerical schemes and coordinate transformations can be avoided. The presented method can also be used for the multiple domain case. The results of the analytical integration are employed in BEM code and the obtained analytical expression will be applied to several examples where the exact solution exists. The produced results are in good agreement with the exact solution.

1. Introduction

Solution of Poisson's equation by boundary element method (BEM) is investigated in many engineering studies. Some authors have used dual reciprocity BEM for solving the Poisson's equation [1,2]. The Monte Carlo method is also introduced by other authors for solution of Poisson's equation [3].

In this paper, the Galerkin Vector Method is employed for solving the poison equation. In Ref. [3], the authors have focused on the Galerkin Vector Method which is applicable just for the particular case when the source term μ satisfies Laplace's equation. Authors in Ref. [4] would extend this approach for another particular case when the source expression satisfies the condition:

$$\nabla^2\mu = \varepsilon = \text{constant}.$$

In the current work, a more general case is investigated where $\nabla^2\varepsilon = 0$ which implies $\nabla^4\mu = 0$. The integrals that appear in second green's identity are calculated by analytical integration. To be more precise, this paper focuses on analytical evaluation of the boundary element integrals for solving the problems governed by the Poisson's equation.

Riccardella [5] were the first to provide the solution for the *G* matrix terms for the constant element with on-diagonal element case for the two-dimensional Laplace's equation. Brebbia and Dominguez [6] also presented the solution for the *G* matrix terms based on the continuous linear element with the on-diagonal element. Almeida and Pina [7] offered solutions for the *G* matrix terms for the constant and linear elements with the on-diagonal element. Fratantonio and Rencis [8] presented analytical solutions of the boundary element integral coefficients for the *H* and *G* influence matrices that appeared in

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solution of two-dimensional Laplace's equation using the constant, linear and quadratic elements. Zhang and Zhang [9] derived the exact integration of the integrals in the boundary element analysis of two-dimensional elasto-static. They considered constant, discontinuous linear and discontinuous quadratic elements. In a different work, the exact integration of the integrals of the discontinuous cubic and quadratic boundary element analysis of two-dimensional elasto-static problems was also derived by Zhang and Zhang [10]. In that study, the derived exact integrations made the evaluation of the non-singular and singular integrals possible in the same way, making special treatment of the singular integrals unnecessary. A new completely analytical integral algorithm was proposed and applied to evaluate the nearly singular integrals in the BEM for two-dimensional orthotropic potential problems of thin bodies by Zhou et al. [11]. The completely analytical integral formulas were derived with integration by parts for the linear boundary interpolation. Their presented algorithm made use of the obtained analytical formulas to deal with the nearly singular integrals. Fedotov and Spevak [12] proposed an approach to the derivation of the analytical formulae for exact integration in the boundary element solution of two-dimensional elasticity problems. In their approach, the integration over an arbitrary boundary element reduced to the integration over a specific element. It is noteworthy that behavior of singular integrals is also a main issue in the BEM. Accordingly, these singular integrals must be evaluated accurately. Much work has been performed in this regard to remove the singularities appearing in the boundary element integrals [13–18].

Again, all of these latter works, with the exception of the work cited in Ref. [11], were related to the solution of the Laplace's equation. However, the focus of the current study is the solution of Poisson's equation and the adopted approach combines the method implemented by Fratantonio et al. and the Galerkin Vector Method. As a result of implementing this scheme, the expressions that appear in the solution of Poisson's equation include six different integrals on the boundary which must be solved. In obtaining the solutions, two different approaches can be applied. These approaches include manual calculation with the help of Mathematical Tables [19] and/or the Matlab symbolic solver. Finally, the results of the analytical integration are employed in BEM code and subsequently the computed analytical expression will be implemented for several examples where the exact solution exists.

Once again, the aim of this work is to solve Poisson's equation by expanding analytical integration of the integrals that appear in BEM formulation in which the Galerkin Vector Method is extended for more general case where $\nabla^2\varepsilon = 0$ or $\nabla^4\mu = 0$.

2. Galerkin vector method

In view of a two-dimensional domain Γ with boundary $\partial\Gamma$, the discretized boundary-integral equation for the two-dimensional Poisson's equation may be readily written as follows:

$$c_i u_i = \sum_{j=1}^{n} \int_{\Gamma_j} q^* u \, d\Gamma - \sum_{j=1}^{n} \int_{\Gamma_j} u^* q \, d\Gamma + \int\int_{\Gamma} \mu u^* \, d\Gamma, \tag{1}$$

where u^*, the fundamental solution of the Laplace equation in two dimensions, is

$$u^* = \frac{-1}{2\pi} \ln r. \tag{2}$$

Details of Galerkin Vector approach can be seen in Ref. [3]. Based on this very useful approach, for the mentioned general case of $\nabla^2\varepsilon = 0$ or $\nabla^4\mu = 0$, the last integral in Eq. (1) converts to:

$$c_i u_i = \sum_{j=1}^{n} \int_{\Gamma_j} q^* u \, d\Gamma - \sum_{j=1}^{n} \int_{\Gamma_j} u^* q \, d\Gamma + \sum_{j=1}^{n} \int_{\Gamma_j} \mu \frac{\partial w}{\partial n} \, d\Gamma - \sum_{j=1}^{n} \int_{\Gamma_j} w \frac{\partial \mu}{\partial n} \, d\Gamma + \sum_{j=1}^{n} \int_{\Gamma_j} \frac{\partial w^1}{\partial n} \, d\Gamma \sum_{j=1}^{n} \int_{\Gamma_j} w^1 \frac{\partial \varepsilon}{\partial n} \, d\Gamma, \tag{3}$$

where w and w^1 may be calculated as follows:

$$w = \frac{r^2}{8\pi}\left(\ln\frac{1}{r} + 1\right), \tag{4}$$

$$w^1 = \frac{1}{256\pi} r^4 (3 - 2\ln r). \tag{5}$$

3. Analytical solution of integrals

3.1. Basic definition

Eq. (3) is a set of six integrals. Each integral will be evaluated separately. The six boundary integrals associated with the G, H, A, B, C, D matrices in Eq. (3) may be acquired from the following integrals:

$$G \rightarrow \int_{\Gamma_j} u^* q \, d\Gamma, \tag{6}$$

$$H \rightarrow \int_{\Gamma_j} q^* u \, d\Gamma, \tag{7}$$

$$A \to \int_{\Gamma_j} w \frac{\partial \mu}{\partial n} d\Gamma, \tag{8}$$

$$B \to \int_{\Gamma_j} \mu \frac{\partial w}{\partial n} d\Gamma, \tag{9}$$

$$C \to \int_{\Gamma_j} w^1 \frac{\partial \varepsilon}{\partial n} d\Gamma, \tag{10}$$

$$D \to \int_{\Gamma_j} \varepsilon \frac{\partial w^1}{\partial n} d\Gamma. \tag{11}$$

Here, constant and discontinuous linear elements are investigated. The element geometry is defined by linear shape function because the geometry of each element is straight.

$$x = \frac{1}{2}(1-\zeta)x_1 + \frac{1}{2}(1+\zeta)x_2, \quad y = \frac{1}{2}(1-\zeta)y_1 + \frac{1}{2}(1+\zeta)y_2. \tag{12}$$

The local coordinate system ζ is defined as shown in Fig. 1, and as a result we have

$$d\Gamma = \frac{Lj}{2} d\zeta,$$

where Lj is the length of jth element.

Distance between the field point and the source point is introduced as r. Since term r^2 appears successively in the integration process, r^2 may be written as a quadratic polynomial in terms of the local coordinate system ζ as follows:

$$r' = r^2 = (x - x_1)^2 + (y - y1)^2 = a + b\zeta + c\zeta^2. \tag{13}$$

Substituting Eq. (12) into Eq. (13) yields in

$$\begin{aligned} a &= \frac{1}{4}[(x_1 + x_2 - 2x_i)^2 + (y_1 + y_2 - 2y_i)^2], \\ b &= \frac{1}{4}[(x_1 + x_2 - 2x_i)(x_2 - x_1) + (y_1 + y_2 - 2y_i)(y_2 - y_1)], \\ c &= [(x_2 - x_1)^2 + (y_2 - y_1)^2] = \frac{Lj^2}{4}. \end{aligned} \tag{14}$$

The polynomial form is used because the computation of the integrations $\frac{1}{r^2}$ and $\ln r^2$ will be possible easily.

Some terms that appear in Eqs. (6)–(11) may also be evaluated as

$$\frac{\partial r}{\partial n} = \nabla r \cdot n = \frac{\partial r}{\partial x} n_x + \frac{\partial r}{\partial y} n_y,$$
$$\frac{\partial r}{\partial x} = \frac{x - x_i}{r}, \frac{\partial r}{\partial y} = \frac{y - y_i}{r}.$$

Also,

$$n_x = \frac{y_2 - y_1}{Lj}, \quad n_y = -\frac{x_2 - x_1}{Lj}.$$

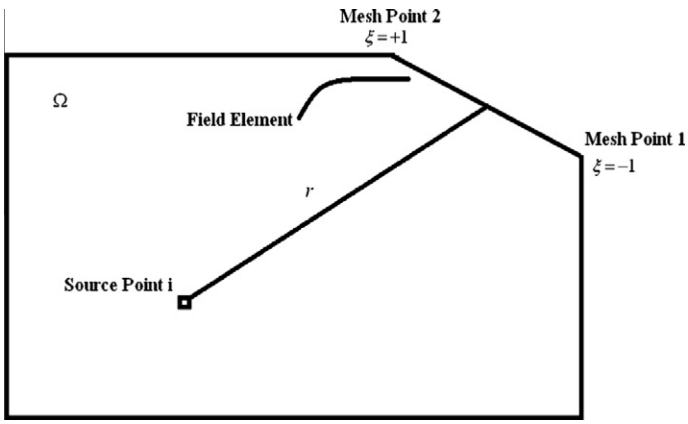

Fig. 1. Element definition for analytical integration.

Therefore, we conclude

$$\frac{\partial r}{\partial n} = \frac{-d}{rLj}, \tag{15}$$

where $d = (x_2y_1 - x_1y_2 + x_iy_2 - x_iy_1 + x_1y_i - x_2y_i)$. The fundamental solution may be written as

$$u^* = \frac{-1}{2\pi}\ln r = \frac{-1}{4\pi}\ln r^2$$

which yields in the following equations:

$$\begin{aligned}
&\frac{\partial u^*}{\partial r} = \frac{-1}{2\pi r}, \quad \frac{\partial u^*}{\partial n} = \frac{\partial u^*}{\partial r}\frac{\partial r}{\partial n} = \frac{d}{4\pi r^2\sqrt{c}},\\
&\frac{\partial w}{\partial r} = \frac{r}{8\pi}(1 - \ln r^2), \quad \frac{\partial w}{\partial n} = \frac{d}{8\pi Lj}(\ln r^2 - 1),\\
&\frac{\partial w^1}{\partial r} = \frac{2r^3}{256\pi}(5 - 4\ln r), \quad \frac{\partial w^1}{\partial n} = \frac{\partial w^1}{\partial r}\frac{\partial r}{\partial n} = \frac{2r^2 d}{256\pi Lj}(4\ln r - 5).
\end{aligned} \tag{16}$$

Now, the evaluation of integrations in Eqs. (6)–(11) may be performed for constant and linear element.

4. Constant element

First, the analytical boundary element integrations for constant element are found. The integrations are carried out using manual integration and the symbolic solvers in Matlab software. In constant element definition, the boundary condition u and q are considered to be constant along each element j.

Now the boundary element integration is carried out in two positions, off-diagonal and on-diagonal integrations (Fig. 2). The boundary integrals in the G, H, A, B, C, D matrices, i.e. Eqs. (6)–(11), that are applied to calculate the nodal potentials and fluxes are outlined for the constant element. The off-diagonal integrations will be determined first followed by the on-diagonal integrations.

4.1. Off-diagonal integrations

In this particular case, the source point lies outside the j-element which means that the distance r does not vanish and, consequently, the integral is non-singular. The location of the source point is fixed and the field point is varying since the element is being integrated. The integration is defined as off-diagonal because the source point is not located on element. The G, H, A, B, C, and D matrices terms may be found by evaluating the following off-diagonal integrals that are raised from Eqs. (6)–(11) as

$$G_{off-c_j} = \int_{\Gamma_j} u^*\, d\Gamma = \frac{-Lj}{8\pi}\int_{-1}^{+1} \ln r^2\, d\zeta, \tag{17}$$

$$H_{off-c_j} = \int_{\Gamma_j} q^*\, d\Gamma = \frac{d}{4\pi}\int_{-1}^{+1} \frac{1}{r^2}\, d\zeta, \tag{18}$$

$$A_{off-c_j} = \int_{\Gamma_j} w\, d\Gamma = \frac{Lj}{16\pi}\int_{-1}^{+1} r^2\left[1 - \left(\frac{1}{2}\ln r^2\right)\right] d\zeta, \tag{19}$$

$$B_{off-c_j} = \int_{\Gamma_j} \frac{\partial w}{\partial n}\, d\Gamma = \frac{d}{16\pi}\int_{-1}^{+1} [(\ln r^2) - 1]\, d\zeta, \tag{20}$$

$$C_{off-c_j} = \int_{\Gamma_j} w^1\, d\Gamma = \frac{Lj}{512\pi}\int_{-1}^{+1} r^4[3 - \ln r^2]\, d\zeta, \tag{21}$$

$$D_{off-c_j} = \int_{\Gamma_j} \frac{\partial w^1}{\partial n}\, d\Gamma = \frac{d}{256\pi}\int_{-1}^{+1} r^2[(2\ln r^2) - 5]\, d\zeta, \tag{22}$$

where $off - c_j$ expresses off-diagonal integrals over a constant element and j expresses element j, and the integration limits are from -1 to +1 in the local coordinate systems.

As the roots of quadratic equation which are considered for r^2 may have one of three solutions, they must be considered when integrating Eqs. (17)–(22). The discriminant (Δ) of the quadratic is given as

$$\Delta = -\frac{d^2}{4}. \tag{23}$$

Obviously, the sign of Δ is always less than or equal to zero (Fig. 3). Thus, the off-diagonal integrals must be evaluated for each case.

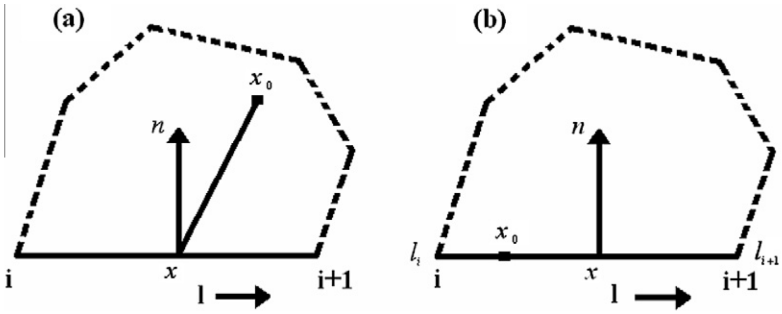

Fig. 2. Evaluation of the influence coefficient over (a) an off-diagonal and (b) on-diagonal straight element. The unit normal vector points into the solution domain.

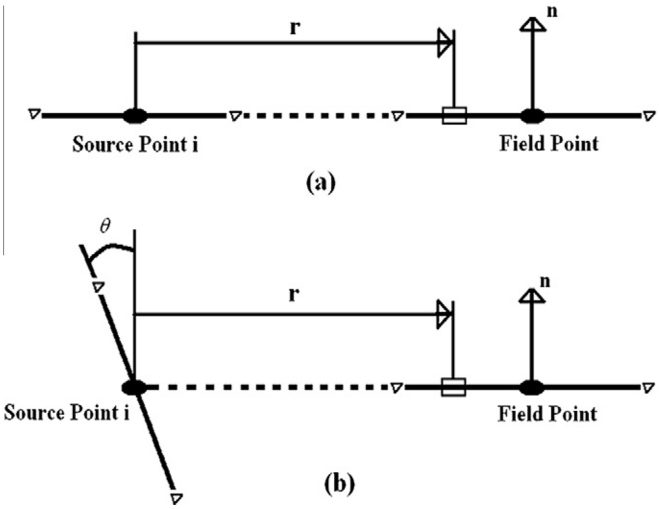

Fig. 3. (a) The value of Δ equals zero when elements are collinear and (b) the node point is collinear with the element, Δ vanishes.

In the first case, Δ is less than zero. Therefore, the analytical solutions for off-diagonal integrals are obtained from Eqs. (17)–(22) as follows:

$$G_{off-c_j} = \frac{-Lj}{8\pi}[2\ln r^2 - 4 + e(t-t')], \tag{24}$$

$$H_{off-c_j} = \frac{d}{2\pi\sqrt{-\Delta}}(t-t'), \tag{25}$$

$$A_{off-c_j} = \frac{Lj}{16\pi}\left[g(t-t') - \left(a+\frac{c}{3}\right)\ln r^2 + f\right], \tag{26}$$

$$B_{off-c_j} = \frac{d}{16\pi}\left[2\ln r^2 - 6 + \frac{\sqrt{-\Delta}}{c}(t-t')\right], \tag{27}$$

$$C_{off-c_j} = \frac{Lj}{512\pi}\left[k\ln r^2 + \frac{(t-t')i}{\sqrt{-\Delta}} + h\right], \tag{28}$$

$$D_{off-c_j} = \frac{d}{256\pi}\left[\left(\frac{4c}{3}+4a\right)\ln r^2 + \frac{(t-t')m}{\sqrt{-\Delta}} + n\right], \tag{29}$$

where the constants are defined as

$$t = \tan^{-1}\left(\frac{b+2c}{\sqrt{-\Delta}}\right), \quad t' = \tan^{-1}\left(\frac{b-2c}{\sqrt{-\Delta}}\right), \quad e = \frac{4a-b^2}{\sqrt{-\Delta}}, \quad f = \left(\frac{10a}{3} - \frac{b^2}{6c} + \frac{8c}{9}\right),$$
$$f = \left(\frac{10a}{3} - \frac{b^2}{6c} + \frac{8c}{9}\right), \quad i = \left(\frac{-32a^3}{15} + \frac{8a^2b^2}{5c} - \frac{2ab^4}{5c^2} + \frac{b^6}{30c^3}\right),$$
$$k = \left(-2a^2 - \frac{2b^2}{3} - \frac{2c^2}{5} - \frac{4ca}{3}\right), \quad h = \left(\frac{208ac}{45} + \frac{122a^2}{15} - \frac{3ab^2}{5c} + \frac{b^4}{15c^2} + \frac{101b^2}{45} + \frac{34c^2}{25}\right), \tag{30}$$
$$m = \left(\frac{16a^2}{3} - \frac{8ab^2}{3c} + \frac{b^4}{3c^2}\right), \quad n = \left(\frac{2b^2}{3c} - \frac{46a}{3} - \frac{38c}{9}\right).$$

When the source point and the field element are located on the same line, the Δ is vanished and the analytical expressions for off-diagonal integrals are given as

$$G_{off-c_j} = \frac{-Lj}{8\pi}\left[4\ln\frac{c}{2} + \ln\left(\frac{b+2c}{b-2c}\right)^2 + \frac{b}{c}\ln\left(\frac{b+2c}{b-2c}\right) - 4\right], \tag{31}$$

$$A_{off-c_j} = \frac{Ljc}{16\pi}\left[o + p + \left(\frac{2b^2}{3c^2} + \frac{8}{9}\right) + \frac{b^3}{24c^3}(\ln\left(\frac{b+2c}{b-2c}\right)\right], \tag{32}$$

$$C_{off-c_j} = \frac{Ljc^2}{512\pi}\left[q + \frac{b^3}{12c^3}\ln\left(\frac{b-2c}{b+2c}\right)\right]. \tag{33}$$

Also,

$$H_{off-c_j} = B_{off-c_j} = D_{off-c_j} = 0, \tag{34}$$

where the constants are,

$$o = \left(\frac{-b^2}{8c^2} - \frac{b}{4c} - \frac{1}{6}\right)\left(\ln\left(\frac{b+2c}{c}\right)^2 - 2\ln 2\right), \quad p = \left(\frac{-b^2}{8c^2} + \frac{b}{4c} - \frac{1}{6}\right)\left(\ln\left(\frac{b-2c}{c}\right)^2 - 2\ln 2\right),$$
$$q = \left(\frac{b^2}{4c} + \frac{b}{2c} + \frac{1}{3}\right)\left(\frac{11}{3} - \ln\left(\frac{b+2c}{c}\right)^2 + 2\ln 2\right) + \left(\frac{b^2}{4c} - \frac{b}{2c} + \frac{1}{3}\right)\left(\frac{11}{3} - \ln\left(\frac{b-2c}{c}\right)^2 + 2\ln 2\right). \tag{35}$$

4.2. On-diagonal element

In this case, the source point coincides with field point and r lies on the element. Consequently, we have:

$$G_{on-c_j} = \int_{\Gamma_j}\frac{1}{2\pi}\ln r\, d\Gamma_j = 2\int_0^{l_j/2}\frac{1}{2\pi}\ln r dr = \frac{1}{\pi}\frac{l_j}{2}[\ln(l_j/2) - 1],$$
$$H_{on-c_j} = \int_{\Gamma_j} q^*\, d\Gamma = \int_{\Gamma_j}\frac{\partial u^*}{\partial r}\frac{\partial r}{\partial n}\, d\Gamma = 0,$$
$$A_{on-c_j} = \frac{Ljc}{16\pi}\left(\frac{8}{9} - \frac{1}{3}\ln c\right), \tag{36}$$
$$B_{on-c_j} = 0,$$
$$C_{on-c_j} = \frac{Ljc^2}{768\pi}\left[\left(\frac{11}{3} + 2\ln 2 - \ln(4c)\right)\right],$$
$$D_{on-c_j} = 0,$$

where $on - c_j$ indicates the on-diagonal integrals over a constant element and "j" indicates element j.

5. Linear element

For the case of linear element, the analytical boundary element integration is obtained. These boundary integrals will be carried out for off-diagonal and on-diagonal integrations.

Linear element approximates the geometry of the boundary by straight lines and the boundary quantity by a linear function on each element, as follows (Fig. 4):

$$f(\xi) = \psi_1(\xi)f_1 + \psi_2(\xi)f_2, \tag{37}$$

$$\psi_1(\xi) = \left(\frac{\xi_2 - \xi}{\xi_2 - \xi_1}\right), \quad \psi_2(\xi) = \left(\frac{\xi - \xi_1}{\xi_2 - \xi_1}\right), \tag{38}$$

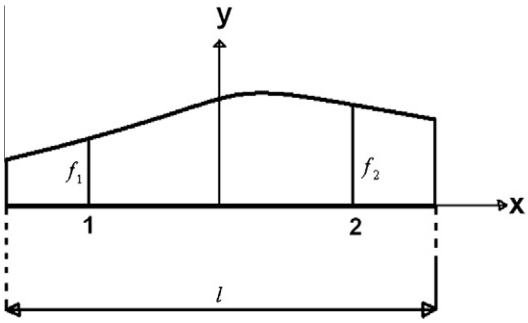

Fig. 4. General linear element definition.

where subscripts 1 and 2 denote the local node numbers. The descretization of the integral is represented by

$$\begin{aligned}\frac{1}{2}\varphi_i = &\sum_{j=1}^{n}\left(\frac{\xi_2}{\xi_2-\xi_1}H^0_{ij}-\frac{1}{\xi_2-\xi_1}H^1_{ij}\right)\varphi^1_j+\left(\frac{1}{\xi_2-\xi_1}H^1_{ij}-\frac{\xi_1}{\xi_2-\xi_1}H^0_{ij}\right)\varphi^2_j-\sum_{j=1}^{n}\left(\frac{\xi_2}{\xi_2-\xi_1}G^0_{ij}-\frac{1}{\xi_2-\xi_1}G^1_{ij}\right)\varphi^1_{nj}\\ &-\left(\frac{1}{\xi_2-\xi_1}G^1_{ij}-\frac{\xi_1}{\xi_2-\xi_1}G^0_{ij}\right)\varphi^2_{nj}+\sum_{j=1}^{n}\left(\frac{\xi_2}{\xi_2-\xi_1}B^0_{ij}-\frac{1}{\xi_2-\xi_1}B^1_{ij}\right)b^1_j+\left(\frac{1}{\xi_2-\xi_1}B^1_{ij}-\frac{\xi_1}{\xi_2-\xi_1}B^0_{ij}\right)b^2_j\\ &-\sum_{j=1}^{n}\left(\frac{\xi_2}{\xi_2-\xi_1}A^0_{ij}-\frac{1}{\xi_2-\xi_1}A^1_{ij}\right)b^1_{nj}+\left(\frac{1}{\xi_2-\xi_1}A^1_{ij}-\frac{\xi_1}{\xi_2-\xi_1}A^0_{ij}\right)b^2_{nj}+\sum_{j=1}^{n}\left(\frac{\xi_2}{\xi_2-\xi_1}D^0_{ij}-\frac{1}{\xi_2-\xi_1}D^1_{ij}\right)a^1_j\\ &+\left(\frac{1}{\xi_2-\xi_1}D^1_{ij}-\frac{\xi_1}{\xi_2-\xi_1}D^0_{ij}\right)a^2_j-\sum_{j=1}^{n}\left(\frac{\xi_2}{\xi_2-\xi_1}C^0_{ij}-\frac{1}{\xi_2-\xi_1}C^1_{ij}\right)a^1_{nj}+\left(\frac{1}{\xi_2-\xi_1}C^1_{ij}-\frac{\xi_1}{\xi_2-\xi_1}C^0_{ij}\right)a^2_{nj}.\end{aligned} \tag{39}$$

Here, $H^0_{ij}, G^0_{ij}, B^0_{ij}, A^0_{ij}, D^0_{ij}, C^0_{ij}$ are the same as those appearing in the constant element case.

$$G^1_{off-l_j} = \int_{\Gamma_j}\xi u^*\,d\Gamma, \tag{40}$$

$$H^1_{off-l_j} = \int_{\Gamma_j}\xi q^*\,d\Gamma, \tag{41}$$

$$B^1_{off-l_j} = \int_{\Gamma_j}\xi\frac{\partial w}{\partial n}\,d\Gamma, \tag{42}$$

$$A^1_{off-l_j} = \int_{\Gamma_j}\xi w\,d\Gamma, \tag{43}$$

$$D^1_{off-l_j} = \int_{\Gamma_j}\xi\frac{\partial w^1}{\partial n}\,d\Gamma, \tag{44}$$

$$C^1_{off-l_j} = \int_{\Gamma_j}\xi w^1\,d\Gamma. \tag{45}$$

5.1. Boundary integrations

The boundary integrals in the *G*, *H*, *A*, *B*, *C*, *D* matrices that are used to determine the nodal potentials and fluxes will be developed for the general linear element. First, the outside integrations will be determined and then the inside integrations.

5.2. Outside integration

The integrals that exist in Eqs. (40)–(45) may be evaluated analytically for two cases. In the first case, Δ is less than zero and for another case is zero. The integrals for the first case are as follows:

$$H^1_{off-l_j} = \frac{d}{4\pi}\left[\frac{b}{c\sqrt{-\Delta}}(t'-t)\right], \tag{46}$$

$$G^1_{off-l_j} = \frac{-L_j}{8\pi}\left[\frac{b\sqrt{-\Delta}}{2c^2}(t'-t)+\frac{b}{c}\right], \tag{47}$$

$$B^1_{off-l_j} = \frac{d}{16\pi}\left[\frac{b\sqrt{-\Delta}}{2c^2}(t'-t)+\frac{b}{c}\right], \tag{48}$$

$$A^1_{off-l_j} = \frac{L_j}{16\pi}\left[s(t-t') - \frac{b}{c}\ln r^2 + ss\right] \tag{49}$$

$$D^1_{off-l_j} = \frac{d}{256\pi}\left[\left(\frac{4b}{3}\ln r^2\right) + ff(t-t') + ee\right], \tag{50}$$

$$C^1_{off-l_j} = \frac{L_j}{512\pi}\left[\left(\frac{-4ab}{3} - \frac{4bc}{5}\right)\ln r^2 + kk + ll(t-t')\right]. \tag{51}$$

The constants which appear in the above integrals are:

$$\begin{aligned}
&s = \frac{1}{\sqrt{-\Delta}}\left(\frac{2ba^2}{3c} - \frac{ab^3}{3c^2} + \frac{b^5}{24c^3}\right), \quad ll = \frac{1}{\sqrt{-\Delta}}\left(\frac{16ba^3}{15c} - \frac{4a^2b^3}{9c^2} - \frac{b^7}{60c^4} + \frac{ab^5}{5c^3}\right),\\
&ff = \frac{1}{\sqrt{-\Delta}}\left(\frac{-8ba^2}{3c} + \frac{4ab^3}{3c^2} - \frac{b^5}{6c^3}\right), \quad ee = \frac{5ab}{3c} - \frac{35b}{9} - \frac{b^3}{3c^2},\\
&kk = \left(\frac{196ab}{45} - \frac{b^3}{90c} - \frac{b^5}{30c^3} + \frac{199bc}{75} - \frac{11ba^2}{15c} + \frac{3ab^2}{10c^2}\right), \quad ss = \frac{29b}{36} + \frac{b^3}{12c^2} - \frac{5ab}{12c}.
\end{aligned} \tag{52}$$

For the second case where Δ is zero, the integrals may be computed as

$$G^1_{off-l_j} = \frac{-L_j}{8\pi}\left[\frac{1}{2}\ln\left(\frac{b+2c}{b-2c}\right)^2 + \frac{b}{c} - \frac{b^2}{4c^2}\ln\frac{b+2c}{b-2c}\right], \tag{53}$$

$$A^1_{off-l_j} = \frac{L_jc}{16\pi}\left[\left(\frac{-1}{8} - \frac{b}{6c} - \frac{b^2}{16c^2}\right)\ln\frac{(b+2c)^2}{4c} + \frac{b^4}{192c^4}\ln\frac{b+2c}{b-2c} - \left(\frac{-1}{8} + \frac{b}{6c} - \frac{b^2}{16c^2}\right)\ln\frac{(b-2c)^2}{4c} + \frac{29b}{36c} + \frac{b^3}{48c^3}\right], \tag{54}$$

$$C^1_{off-l_j} = \frac{L_jc^2}{512\pi}\left[gg + \frac{b^6}{960c^6}\ln\frac{b+2c}{b-2c} + \frac{199b}{75c} + \frac{97b^3}{90c^3} - \frac{b^5}{240c^5}\right], \tag{55}$$

$$H^1_{off-l_j} = B^1_{off-l_j} = D^1_{off-l_j} = 0 \tag{56}$$

and

$$gg = \left(\frac{-1}{6} - \frac{2b}{5c} - \frac{3b^2}{8c^2} - \frac{b^3}{6c^3} - \frac{b^4}{32c^4}\right)\ln\frac{(b+2c)^2}{4c} - \left(\frac{-1}{6} + \frac{2b}{5c} - \frac{3b^2}{8c^2} + \frac{b^3}{6c^3} - \frac{b^4}{32c^4}\right)\ln\frac{(b-2c)^2}{4c}. \tag{57}$$

5.3. Inside integration

In this case, the source lies on the element over which the integration is performed. As the integration point runs along the whole element, it will coincide inevitably with the source point. Therefore, the distance r vanishes and the integrands of Eqs. (40)–(45) exhibit a singular behavior. These integrals are known as singular integrals. Their values exist and are determined through special integration technique. Therefore in this case, we introduce

$$a = \frac{\xi_0^2L_j^2}{4}, \quad b = \frac{-\xi_0L_j^2}{4}, \quad c = \frac{L_j^2}{4} \tag{58}$$

and as a result, we have

$$G^1_{on-l_j} = \frac{-L_j}{8\pi}\left[-2\xi_0 + (1-\xi_0^2)\ln\left(\frac{1-\xi_0}{-1-\xi_0}\right)^2\right], \tag{59}$$

$$A^1_{on-l_j} = \frac{L_j}{16\pi}\left[\frac{-2b}{3}\ln c + \frac{\xi_0^2L_j^3}{24} - \frac{29\xi_0L_j^2}{72} + aa\right], \tag{60}$$

$$C^1_{on-l_j} = \frac{L_j}{512\pi}\left[\left(\frac{L_j^4\xi_0}{10} + \frac{L_j^4\xi_0^3}{6}\right)\ln c - \left(\frac{199L_j^4\xi_0}{600} + \frac{97L_j^4\xi_0^3}{180} - \frac{L_j^4\xi_0^5}{120}\right) + bb\right], \tag{61}$$

$$H^1_{on-l_j} = B^1_{on-l_j} = D^1_{on-l_j} = 0. \tag{62}$$

Here the constants are:

$$aa = \left(\frac{a(1-\xi_0^2)}{4} - \frac{b(1-\xi_0^3)}{6} - \frac{c(1-\xi_0^4)}{8}\right)\ln(1-\xi_0^2) + \left(\frac{-a(-1-\xi_0^2)}{4} + \frac{b(-1-\xi_0^3)}{6} + \frac{c(-1-\xi_0^4)}{8}\right)\ln(-1-\xi_0)^2$$
$$bb = \left(\frac{L_j^4\xi_0(1-\xi_0^5)}{20} + \frac{L_j^4\xi_0^3(1-\xi_0^3)}{12}\right)\ln(k-1)^2 - \left(\frac{L_j^4\xi_0(-1-\xi_0^5)}{20} + \frac{L_j^4\xi_0^3(-1-\xi_0^3)}{12}\right)\ln(k+1)^2. \tag{63}$$

Next, we will apply the results of exact integration in the numerical computation of BEM.

6. Numerical examples

The mentioned computed integrals are applied to four different examples where the closed analytical solution exists for the sake of comparison.

Example 1 (*Poisson's problem on a square domain*). In this particular case, for simplicity, the domain was taken as a square with $a = b = 2.00$ and the boundary may uniformly be divided into 32, 64, and 128 segments, as in Fig. 5. In order to demonstrate the potential ability of the presented scheme, the Dirichlet boundary condition is employed on the computational domain and as a result, flux will be determined. The accuracy of the computation is shown at a corner point where the computational error may potentially occur and also at a point in the middle of one side of the square, as shown in Table 1. The Dirichlet boundary condition is taken as,

$$u = x^4.$$

Therefore, the analytical solution for the flux may be calculated readily.

Example 2 (*Poisson's Problem on a circular domain*). This example studies the influence of curved boundaries. A circle with $R = 2$ is considered in Fig. 6. The boundary of the circle is divided into 32, 64 and 128 segments. The Dirichlet boundary condition is used in the same manner as in Example 1 and the exact solution can be produced from it. The results of the computation are shown in Table 2.

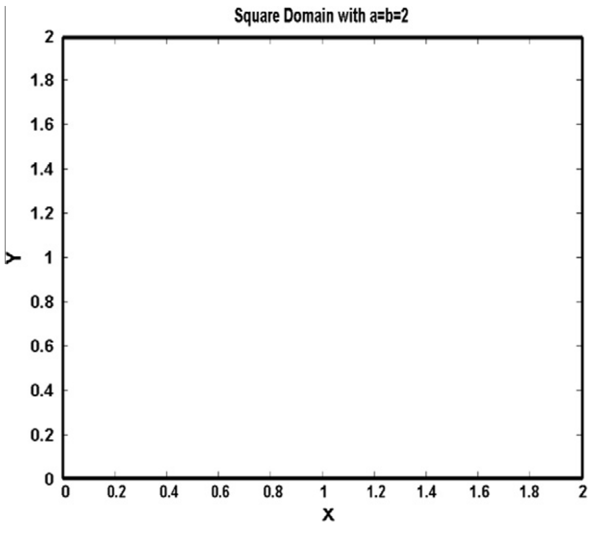

Fig. 5. Geometry of the computational domain.

Table 1
The comparisons of results of Example 1 by numerical and analytical integration in BEM solution and exact solution.

Square domain with $a = b = 2.00$	Location of nodes (x,y)	Flux		
Number of nodes on square		Analytical	Numerical	Exact
32	2.0000, 1.3700	31.77	31.76	32.00
64	2.0000, 1.0312	32.01	32.01	32.00
128	2.0000, 0.6718	32.00	32.00	32.00
32	2.0000, 1.8700	35.43	35.27	32.00
64	2.0000, 1.9375	35.51	35.53	32.00
128	2.0000, 1.9531	31.02	30.94	32.00

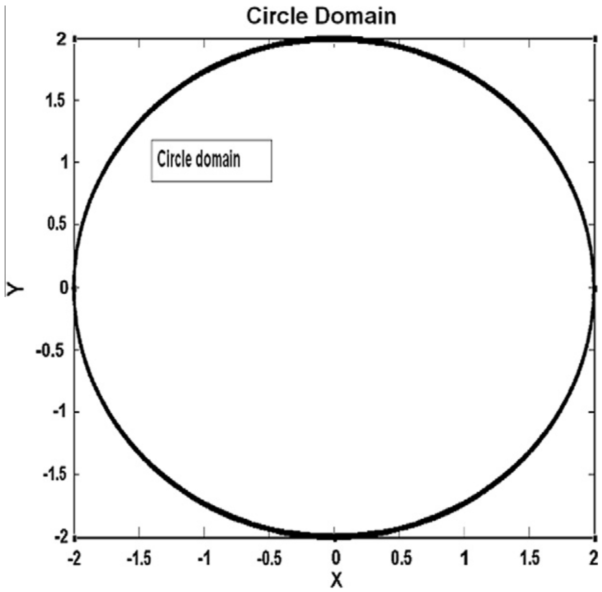

Fig. 6. Geometry of the circular domain.

Example 3 (*Uniform incompressible viscous fluid*). The equation of motion of a uniform incompressible viscous fluid in steady state one-directional (in the z direction) flow may be written as [20]

$$-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=0,$$

where μ is the viscosity of the fluid, $\frac{\partial p}{\partial z}=-G$ is a constant pressure gradient and u is the velocity component in the z direction. This equation can be rewritten as

$$\nabla^2 u=-\frac{G}{\mu}.$$

For an elliptical cross-section, the velocity distribution is of the form

$$u=\frac{G}{2\mu(a^{-2}+b^{-2})}\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right),$$

where a and $b$are the semi axes of ellipse. Taking the value of the constant $G/\mu = 2$ and the semi axes $a = 2$ and $b = 2$, the problem that must be solved is

$$\nabla^2 u=-2$$

and the boundary condition is

$$u=0 \quad \text{on} \quad \Gamma.$$

The exact solution can be written in this case as follows:

$$u=2\left(1-\frac{x^2}{4}-\frac{y^2}{4}\right).$$

Table 2
The comparisons of results of Example 2 by numerical and analytical integration in BEM solution and exact solution.

Circular domain with $R = 2$	Location of nodes (x,y)	Flux		
Number of nodes		Constant	Linear	Exact
16	1.9238, −0.3826	28.42	29.65	29.61
32	1.9807, −0.1950	31.12	31.38	31.38
64	1.9951, −0.0980	31.78	31.84	31.84
128	1.9987, −0.0490	31.94	31.96	31.96
256	1.9996, −0.0245	31.99	31.99	31.99

Table 3
The comparisons of results of Example 3 by numerical and analytical integration in BEM solution and exact solution.

Elliptical (circle) domain with $a = b = 2$ ($r = 2$)	Location of nodes (x,y)	Flux		
Number of nodes		Presented	Numerical	Exact
16	1.9238, −0.3826	−1.95	−1.95	−2.00
64	1.9951, −0.0980	−1.9907	−1.9907	−2.00
256	1.9996, −0.0245	−1.9998	−1.9998	−2.00

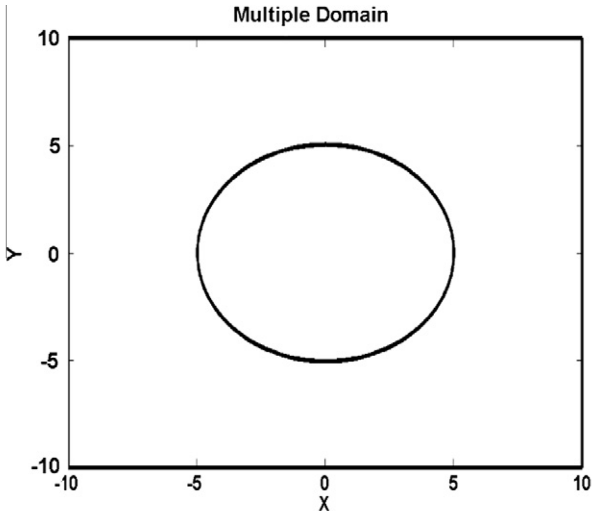

Fig. 7. Geometry of multi-domain.

Table 4
The comparisons of results of Example 4 by numerical and analytical integration in BEM solution and exact solution.

Multiple domain with $a = b = 20$ and $R = 5$	Location of nodes (x,y)	Flux		
Number of nodes on square and circle		Analytical	Numerical	Exact
32	8.7500, 10.0000 (square)	−10.32	−10.31	−10.00
128	9.6875, 10.0000 (square)	−10.38	−10.40	−10.00
256	−9.8437, −10.0000 (square)	−10.44	−10.49	−10.00
32	−4.9519, 0.4877 (circle)	5.01	5.01	5.00
128	−4.9609, 0.6118 (circle)	5.00	5.00	5.00
256	2.5175, −4.3195 (circle)	5.00	5.00	5.00
32	1.2500, 10.0000 (square)	−9.95	−9.95	−10.00
128	4.6875, 10.0000 (square)	−10.00	−10.00	−10.00
256	3.2812, 10.0000 (square)	−10.00	−10.00	−10.00

Table 3 illustrates the results of this example. Good agreement is observed between the presented results and the numerical and exact solutions, albeit the accuracy of the solution may be affected by the number of nodes on the boundary surface.

Example 4 (*Poisson's Problem on a multiple domain*). The presented method can also be used for the multiple domain case (Fig. 7). This is shown in Table 4 where it can clearly be seen that the accuracy of the presented solution is quite comparable with the numerical computation.

7. Conclusion

The analytical boundary element integration was carried out in this paper for the solution of Poisson's equation for the first time, by considering constant and linear elements without any domain integration. This was done by extending the Galerkin Vector Method. Six integrals were analytically determined and applied to solve the Poisson's equation. The analytical integration reduces the resulted error where the computational domain has straight boundaries. By employing analytical integration in BEM computation, the numerical schemes and coordinate transformations can be avoided. In general, there are three major concerns in solution of a problem by a numerical method which are accuracy, efficiency, and simplicity. The authors believe that the application of the current analytical integration in BEM will be comparable with the numerical method in simplicity and efficiency, but accuracy is only satisfied in some applications. Generally, the suitable application of the analytical integrations for error reduction in BEM depends only on the shape of the boundary. However, the accuracy of solution may be affected by the type of the boundary conditions. The presented formula can also be employed to solve multi-domain problem which was demonstrated by an example. Comparison of the findings with the exact and numerical values indicated good accuracy.

Based on the presented results achieved in this paper, the analytical integration may potentially be used in BEM formulation of many other significant application problems in the future.

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