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A novel numerical method for infinite domain potential problems

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The infinite domain potential problems arise in many branches of scientific and engineering fields, which by now still pose a great challenge to scientific computing community. This study proposes a novel meshless singular boundary method (SBM) to solve infinite domain potential problems. The SBM is mathematically simple, easy-to-program, meshless and integration-free. To guarantee the uniqueness of numerical solutions, this article adds a constant term into the SBM approximate representation. The efficiency and accuracy of the proposed technique are tested to the three infinite domain potential problems.

infinite domain, potential problem, singularity, fundamental solution, singular boundary method, meshless

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The infinite potential problems [1–3] have been observed in a wide variety of scientific and engineering fields, such as the ideal potential flow around a body, the electrostatic field and the steady temperature field, etc. However, the numerical solution to such problems still presents a great challenge so far to the communities of engineering simulation and scientific computing. Most of popular numerical methods such as finite element and finite volume methods need to truncate infinite domain to an artificial finite region with subtle artificial boundary conditions [4] or absorbing layers [5]. This truncation can be arbitrary largely based on trial-error experiences. On the other hand, the boundary element method (BEM) [6–8] appears very attractive to handle the unbounded domain problem because it applies the fundamental solution as the basis function, which satisfies the governing equation and the boundary condition at infinity. And no domain truncation is required. However, the singular or hyper-singular integrals [8] in the BEM are not mathematically simple and require additional computing costs.

To avoid the singularities of fundamental solutions, the method of fundamental solutions (MFS) [9–11] distributes the boundary knots on fictitious boundary which is outside the physical domain, and the location of fictitious boundary is vital for the accuracy and reliability of the MFS solution. However, despite great efforts for decades, the determination of fictitious boundary is still arbitrary and tricky, largely based on experiences. Recently, Young et al. [12] proposed an alternative meshless method, namely regularized meshless method (RMM) [13] to remedy this drawback of the MFS. By employing the desingularization of subtracting and adding-back technique, the RMM can place the source points on the real physical boundary. In addition, the ill-conditioned interpolation matrix of BEM and MFS is also circumvented in the RMM. However, the original RMM requires the uniform distribution of nodes and severely reduces its applicability to complex-shaped boundary problems. Similar to the RMM, Sarler [14] proposed the modified method of fundamental solution (MMFS) to solve the potential flow problems. However, the MMFS demands a complex calculation of the diagonal elements of interpolation matrix. It is worthy of noting that, unlike the mesh methods [4–8,15,16], the MFS, RMM and MMFS do not require any meshes and are all meshless [17] in nature.

Inspired by the pioneering work mentioned above, we propose a novel numerical method, called singular boundary

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method (SBM), to calculate infinite domain potential problems. The SBM is developed to overcome the major shortcomings in the MFS, RMM, and MMFS while retaining their merits. The key point of the SBM is to use a simple numerical approach to calculate diagonal elements when the collocation and source nodes are coincident and are all placed on the physical boundary. It is stressed that this article adds a constant term in the SBM approximate representation to guarantee the uniqueness and accuracy of the numerical solution. This study tests the efficiency and accuracy of the proposed technique to the three benchmark infinite domain potential problems.

1 Singular boundary method

This section introduces the singular boundary method (SBM) formulation of the two-dimensional unbounded domain potential problems defined below:

$$\nabla^2 u(x) = 0, \quad x \in \Omega^e \tag{1}$$

subjected to the boundary conditions:

$$u(x) = \bar{u}, \quad x \in \Gamma_{\mathrm{D}}, \tag{2a}$$

$$\frac{\partial u(x)}{\partial n} = \bar{t}, \quad x \in \Gamma_{\mathrm{N}}, \tag{2b}$$

$$\lim_{\|x\|_2 \to \infty} \left(u(x)\right) = \text{const}, \quad \|x\|_2 \to \infty, \tag{2c}$$

where ∇^2 represents Laplace operator, and Ω^e denotes a two-dimensional unbounded region. In eq. (2b), n is the unit inward normal on physical boundary. $\|x\|_2$ denotes the Euclidean distance, and const is a finite constant. Γ_{D} and Γ_{N} are the Dirichlet boundary (essential boundary) and Neumann boundary (natural boundary) parts, respectively, which construct the whole boundary of the physical domain Ω^e. It is noted that the solution $u(x)$ satisfies not only Dirichlet (2a) and Neumann (2b) boundary conditions but also the boundary condition at infinity (2c) [18].

The solution $u(x)$ to the potential problem (eqs. (1) and (2)) can be approximated by a linear combination of the two-dimensional fundamental solution G:

$$u(x_i) = \sum_{j=1}^{N} \beta_j G(x_i, s_j), \tag{3}$$

where N denotes the number of source points, β_j is the jth unknown coefficient, and the fundamental solution $G(x, s_j) = -\frac{1}{2\pi}\ln\left(\|x - s_j\|_2\right)$. We can find that the fundamental solution G satisfies both the governing eq. (1) and the boundary condition at infinity (2c). Thus, the formulation (3) does not require considering the boundary condition at infinity.

If the collocation points x_i and source points s_j coincide, i.e. $x_i=s_j$, we will encounter well-known singularity at origin, i.e. $G(x_i, s_j) = -\frac{1}{2\pi}\ln 0$. In order to remedy this troublesome problem, the MFS places the source nodes on an artificial boundary outside the physical domain while collocation nodes remain on the physical boundary. However, despite great efforts which have been made, the placement of this artificial boundary remains a perplexing issue when dealing with complex-shaped boundary or multiply-connected domain problems.

The SBM places all source and boundary collocation nodes on the same physical boundary. Moreover, the source points and the boundary collocation points are the same set of boundary nodes. The SBM formulation is given by

$$u(x_i) = \sum_{j=1}^{N} \alpha_j G(x_i, s_j), \; x_i \in \Omega^e, x_i \notin \Gamma_{\mathrm{D}}, \tag{4a}$$

$$u(x_i) = \sum_{j=1, j\neq i}^{N} \alpha_j G(x_i, s_j) + \alpha_i G_{ii}, \quad x_i \in \Gamma_{\mathrm{D}}, \tag{4b}$$

$$\frac{\partial u(x_i)}{\partial n} = \sum_{j=1, j\neq i}^{N} \alpha_j \frac{\partial G(x_i, s_j)}{\partial n} + \alpha_i \bar{G}_{ii}, \quad x_i \in \Gamma_{\mathrm{N}}, \tag{4c}$$

where α_j is the jth unknown coefficient, G_{ii} and $\bar{G}_{ii}$ are defined as the source intensity factors, namely, the diagonal elements of the SBM interpolation matrix. This study employs a simple numerical technique, called the inverse interpolation technique (IIT), to determine the source intensity factors. In the first step, the IIT requires choosing a known sample solution u_I to the Laplace potential problem and locating some sample points y_k inside the physical domain. It is noted that the sample points y_k do not coincide with the source points s_i, and the sample points number NK should not be fewer than the physical boundary source node number N. Using the interpolation formula (3), we can then determine the influence coefficients β_j by the following linear equations:

$$\left\{G(y_k, s_j)\right\}\left\{\beta_j\right\} = \left\{u_I(y_k)\right\}. \tag{5}$$

Replacing the sample points y_k with the boundary collocation points x_i, the SBM interpolation matrix of the potential problem (eqs. (1) and (2)) can be written as

$$\begin{Bmatrix} G_{ii} & G(x_i, s_j) \\ \dfrac{\partial G(x_i, s_j)}{\partial n} & \bar{G}_{ii} \end{Bmatrix} \left\{\beta_j\right\} = \begin{Bmatrix} u_I(x_i) \\ \dfrac{\partial u_I(x_i)}{\partial n} \end{Bmatrix}. \tag{6}$$

The source intensity factors can be calculated by the following formulations:

$$\bar{G}_{ii}=\frac{u_I\left(x_i\right)-\sum\limits_{j=1,s_j\neq x_i}^{N}\beta_j G\left(x_i,s_j\right)}{\beta_j}\qquad x_i=s_j,x_i\in\Gamma_{\mathrm{D}},\tag{7a}$$

$$\bar{G}_{ii}=\frac{\dfrac{\partial u_I\left(x_i\right)}{\partial n}-\sum\limits_{j=1,s_j\neq x_i}^{N}\beta_j\dfrac{\partial G\left(x_i,s_j\right)}{\partial n}}{\beta_j}\qquad x_i=s_j,x_i\in\Gamma_{\mathrm{N}}.\tag{7b}$$

It is stressed that the source intensity factors only depends on the distribution of the source points, the fundamental solution of the governing equation and the boundary conditions. Theoretically speaking, the source intensity factors remain unchanged with different sample solutions in the IIT. Therefore, by employing a novel inverse interpolation technique, we circumvent the singularity of the fundamental solution upon the coincidence of the source and collocation points.

It is noted that like the MFS, the SBM does not require considering the boundary condition at infinity (2c) and is a truly meshless numerical technique; unlike the MFS, the SBM avoids the perplexing issue of the fictitious boundary. Our numerical experiments find that the SBM formulation (4) performs well for some tested problems but that it can result in the wrong solution to the other problems, especially for the problem whose solution includes a constant potential. In order to remedy this drawback, we add a constant term to the SBM formulation (4) to guarantee the uniqueness of the SBM interpolation matrix. The modified SBM formulation with a constant term is given by

$$u\left(x_i\right)=\sum_{j=1}^{N}\alpha_j G\left(x_i,s_j\right)+\alpha_{N+1},\quad x_i\in\Omega^e,x_i\notin\Gamma_{\mathrm{D}},\tag{8a}$$

$$u\left(x_i\right)=\sum_{j=1,j\neq i}^{N}\alpha_j G\left(x_i,s_j\right)+\alpha_i G_{ii}+\alpha_{N+1},\quad x_i\in\Gamma_{\mathrm{D}},\tag{8b}$$

$$\frac{\partial u\left(x_i\right)}{\partial n}=\sum_{j=1,j\neq i}^{N}\alpha_j\frac{\partial G\left(x_i,s_j\right)}{\partial n}+\alpha_i\bar{G}_{ii},\quad x_i\in\Gamma_{\mathrm{N}}\tag{8c}$$

with the constraint

$$\sum_{j=1}^{N}\alpha_j=0.\tag{9}$$

It is noted that the additional constant term and the constraint (9) are also called as "moment condition" in literature. The above SBM formulation for the 2-dimensional unbounded Laplace potential problem is also applicable to the 3-dimensional cases.

2 Numerical results and discussion

In this section, the efficiency, accuracy and convergence of the SBM are tested to the infinite potential problems with circular and irregular domains. It is emphasized that the boundary conditions are discontinuous in Example 1. The performance of the SBM is compared with that of the exact solution and the MFS solution.

Rerr(u) represents the average relative error, which is defined as

$$\mathrm{Rerr}(u)=\sqrt{\frac{1}{NT}\sum_{i=1}^{NT}\left|\frac{u(i)-\bar{u}(i)}{\bar{u}(i)}\right|^2},\tag{10}$$

where $\bar{u}(i)$ and $u(i)$ are the analytical and numerical solutions at x_i, respectively, and NT is the total number of points in the interest domain which are used to test the solution accuracy. The sample solution is $u_I=r^2\cos 2\theta$ and the number of inner sample points is equal to the boundary knots, and the distribution of sample points depends on the shape of the physical domain. In the MFS, according to the boundary shape of the physical domain, we typically place the source points outside physical domain with a parameter d defined as

$$d=\frac{x_i-s_i}{x_i-op},\tag{11}$$

in which op is the geometric center, here namely origin point.

Example 1 Dirichlet problem with circular domain

We examine Laplace equation with the Dirichlet discontinuous BC:

$$u\left(1,\theta\right)=\begin{cases}1 & 0<\theta<\pi,\\ -1 & \pi<\theta<2\pi.\end{cases}\tag{12}$$

The exact solution is available as follows:

$$u\left(r,\theta\right)=\frac{2}{\pi}\arctan\left(\frac{2r\sin\theta}{r^2-1}\right).\tag{13}$$

Here the tested points (NT=820) are uniformly distributed on the upper half plane, which are outside the physical boundary, but inside the circle of radius 3. Table 1 shows the numerical results by applying the singular boundary method, the BEM and the MFS, in which the latter uses different off-set boundaries. We can see from Table 1 that the BEM has the lowest accuracy and the largest condition number. It is noted that the accuracy can be improved using the adaptive BEM; however, this may increase the computation cost. And the MFS with the fictitious boundary parameter d=0.2 cannot get the right solution when the number of boundary source nodes is more than 200. The reason is that the conditioning number of the MFS interpolation matrix increases exponentially with the increase of boundary node number N. This indicates that the arbitrary placing of the off-set boundary points may cause numerical stability issue.

It is noted from Table 1 that the SBM and MFS with the

Table 1 Numerical results for Example 1 with varying numbers of nodes

N	SBM		MFS (*d*=0.2)		MFS (*d*=0.01)		BEM	
	Rerr(*u*)	Cond(*A*)	Rerr(*u*)	Cond(*A*)	Rerr(*u*)	Cond(*A*)	Rerr(*u*)	Cond(*A*)
60	3.00×10^{-2}	35.1	3.00×10^{-2}	1.57×10^{7}	4.01×10^{-2}	38.5	1.21×10^{-1}	1.11×10^{3}
100	1.80×10^{-2}	45.3	1.80×10^{-2}	1.96×10^{11}	2.03×10^{-2}	1.10×10^{2}	7.35×10^{-2}	3.06×10^{3}
200	9.00×10^{-3}	72.1	wrong	wrong	9.10×10^{-3}	6.89×10^{2}	3.70×10^{-2}	1.22×10^{4}
300	6.00×10^{-3}	108	wrong	wrong	6.00×10^{-3}	2.95×10^{3}	2.47×10^{-2}	2.74×10^{4}

fictitious boundary parameter *d*=0.01 produce numerical solutions of similar accuracy, and their interpolation matrices also have the same level of conditioning number. It is stressed that the optimal value of the fictitious boundary parameter *d* is obtained by a trial-error approach. The determination of such a parameter *d* is very tricky and delicate in applications and particularly difficult for complex-shaped and multiply-connected domain problems.

Example 2 The example in [19]

We examine an infinite domain problem with the following analytical solution:

$$u(r,\theta)=\mathrm{e}^{\frac{\cos\theta}{r}}\cos\left(\frac{\sin\theta}{r}\right) \tag{14}$$

and the corresponding Dirichlet boundary condition can be easily derived from the exact solution. The amoeba-like irregular shape domain, as shown in Figure 1, is defined by

$$\rho(\theta)=\mathrm{e}^{\sin\theta}\sin^2(2\theta)+\mathrm{e}^{\cos\theta}\cos^2(2\theta). \tag{15}$$

Figure 2 plots relative errors along a circle of radius 3 using the SBM and MFS with 80 boundary knots. It is observed that the relative error of the SBM solution is in the order of 10^{-3}. In sharp contrast with case 1, the MFS with the off-set boundary parameter *d*=0.2 performs far better than the MFS with *d*=0.01. We can see that the fictitious boundary has a big influence on the MFS solution and its optimal value is problem-dependent. Thus, although the MFS with *d*=0.2 produce more accurate solution in this case than the SBM, the fictitious boundary severely damages its applicability to the real-world problems.

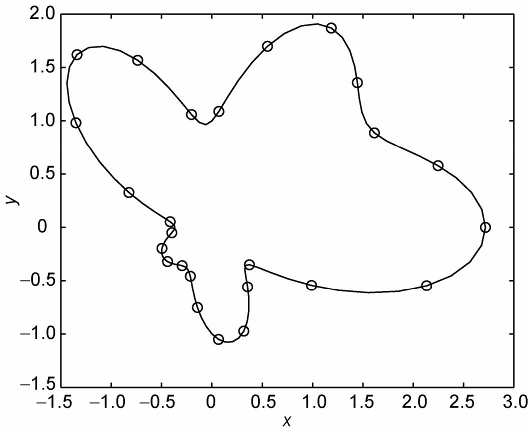

Figure 1 The shape of complex amoeba-like irregular domain and collocation nodes.

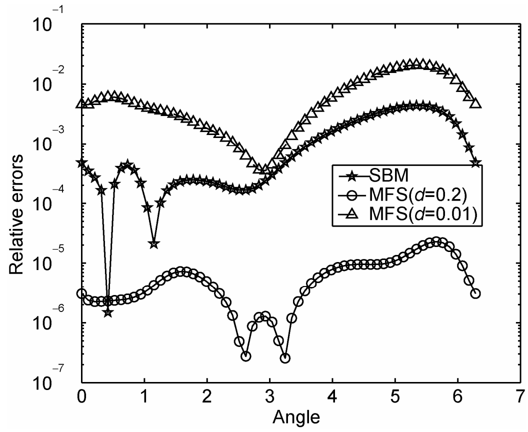

Figure 2 Relative errors along a circle with radius 3 of Example 2.

Example 3 Dirichlet problem with an epitrochoid boundary shape domain

We consider an epitrochoid boundary shape domain [19] defined by

$$\rho(\theta)=\sqrt{(a+b)^2+1-2(a+b)\cos(a\theta/b)} \tag{16}$$

with *a*=4 and *b*=1, which is plotted in Figure 3. And the exact solution of this case is the same as that of example, i.e. expression (15), and the corresponding Dirichlet BC can be easily derived by the exact solution. Here the tested points *NT*=1636 are uniformly distributed between the physical boundary and a square of boundary length 12.

Figure 4 shows the accuracy variation of Example 3 against the number of boundary nodes using the SBM and the MFS with *d*=0.01 and *d*=0.2. Figure 5 plots the corresponding conditioning numbers of interpolation matrix for Example 3 against the number of interpolation knots. It can be seen from Figure 4 that the SBM solutions and the MFS solutions with the fictitious boundary *d*=0.01 have similar accuracy using the same boundary knots.

To investigate the numerical stability, the artificial noisy boundary data are used to calculate the SBM solution which is generated by

$$\begin{aligned}\bar{u}_{\text{noise}}&=\bar{u}+\max|\bar{u}|\,\text{randn}(i)\,p, \quad x\in\Gamma_{\text{D}},\\ \bar{t}_{\text{noise}}&=\bar{t}+\max|\bar{t}|\,\text{randn}(i)\,p, \quad x\in\Gamma_{\text{N}},\end{aligned} \tag{17}$$

where $\bar{u}_{\text{noise}}$ and $\bar{t}_{\text{noise}}$ denote the noisy boundary condition. The random number randn(i) is chosen with a standard normal distribution, which is fixed at this example, and p denotes the noise level. In numerical experiments, we add p=5% noise on boundary value at 200 boundary knots. Figure 6 depicts the relative errors along a circle of radius 7. It can be observed that the SBM accuracy is comparable with that of the MFS with off-set boundary parameter d=0.01. It is seen from Figure 4 that the MFS solutions with d=0.2 have the best accuracy when boundary data are not contaminated with noise. However, it is noted from Figure 5 that its conditioning number grows far more rapidly with the increase of boundary node number N than those of the SBM and the MFS with d=0.01. Therefore, it is not surprising that the MFS with d=0.2 converges to wrong solutions when the noisy boundary data appear which are often encountered in inverse problems.

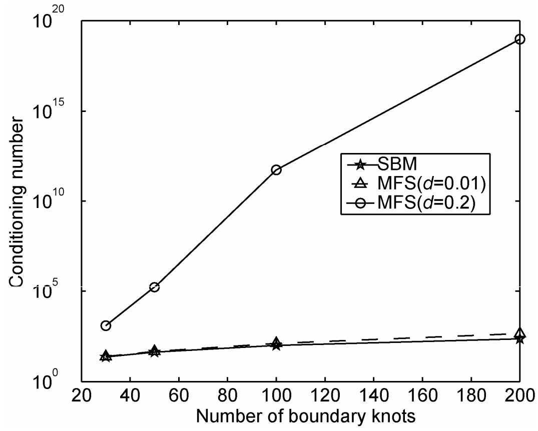

Figure 5 The conditioning number of interpolation matrix for Example 3 against the number of interpolation knots using SBM and MFS with d=0.01 and d=0.2.

3 Conclusions

We propose a novel singular boundary method formulation

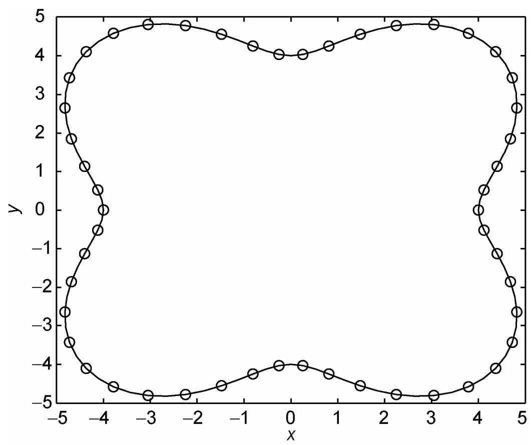

Figure 3 The shape of epitrochoid boundary domain and collocation nodes.

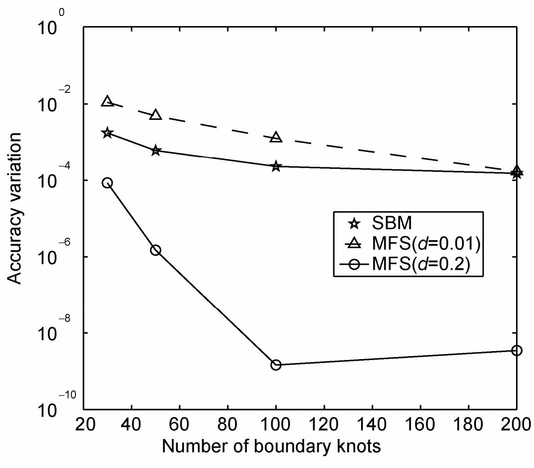

Figure 4 The accuracy variation of Example 3 against the number of interpolation knots using SBM and MFS with d=0.01 and d=0.2.

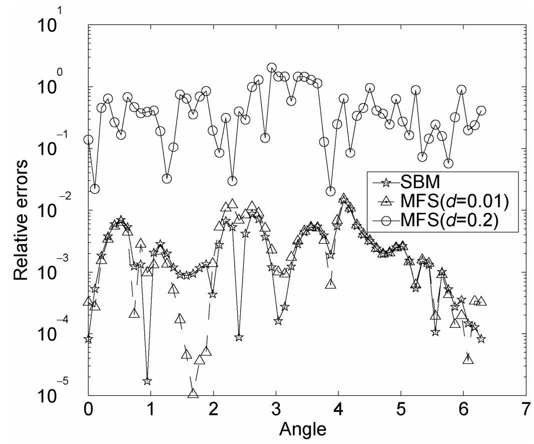

Figure 6 Relative errors along a circle with radius 7 of Example 3 with contaminated boundary data (p=5% noise).

to calculate the infinite domain potential problems. For guaranteeing the uniqueness of the SBM numerical solution, a constant term is added into the SBM approximate representation. Numerical results demonstrate that compared with other boundary-type methods, the present SBM is very competitive for infinite domain potential problems with complex boundary shape. Moreover, the SBM results in an interpolation matrix of small conditioning number and also performs very stably in the solution to the problems with noisy boundary data.

In addition, the present SBM is mathematically simple, easy-to-program, accurate, meshless and integration-free and avoids the controversy of the fictitious boundary in the MFS, the uniform boundary node requirement of the RMM, and the expensive calculation of diagonal elements in the MMFS. The application of the SBM to the infinite domain problems of the other types is still under study and will be reported in the subsequent paper.

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